

On monophonic position sets of Cartesian and lexicographic products of graphs

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Abstract

The general position problem in graph theory asks for the number of vertices in a largest set S of vertices of a graph G such that no shortest path of G contains more than two vertices of S . The analogous monophonic position problem is obtained from the general position problem by replacing “shortest path” by “induced path.” This paper studies monophonic position sets in the Cartesian and lexicographic products of graphs. Sharp lower and upper bounds for the monophonic position number of Cartesian products are established, along with several exact values. For the lexicographic product, the monophonic position number is determined for arbitrary graphs.

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1 Introduction

The general position problem for graphs originates in Dudeney's "Puzzle with Pawns" in his book "Amusements in Mathematics" [7] from 1917. This problem asked the reader to find the largest number of pawns that can be placed on a chessboard without any three pawns lying on a common straight line. This problem was generalised to the setting of graph theory independently in [4, 17] as follows: a set of vertices S in a graph G is in *general position* if no shortest path of G contains more than two vertices of S . The problem then consists of finding the largest set of vertices in general position for a given graph G . The structure of general position sets in graphs was described in [1].

The general position problem has been the subject of intensive research, with some 48 papers appearing on the subject since 2018. For some recent developments see [2, 3, 5, 10, 11, 19, 20, 21, 25, 27, 29]. One case of particular interest is the general position number of product graphs. The general position problem for Cartesian products has been treated in many papers, including [12, 14, 15, 16, 24, 26]. General position numbers of strong and lexicographic products were investigated in [6].

Several variations on the general position problem have also been considered. For example, we can replace 'shortest path' in the definition of the problem by some other family of paths. In [13] the authors restrict attention to shortest paths of bounded length, whilst [9] considers the problem for the widest possible family, all paths. Another important family is the *induced* or *monophonic* paths. Partially inspired by the extensive literature on monophonic convexity (a recent example is [18]), the *monophonic position problem* was introduced in [22]. Some extremal problems for monophonic position sets are discussed in [28], smallest maximal monophonic position sets are treated in [5] and graph colourings in which every colour class is in monophonic position are explored in [20]. In the present article we explore the monophonic position problem for various graph products.

The plan of this paper is as follows. In the following section, we provide the necessary definitions required for later use. In Section 3, we conduct a structural analysis of monophonic position sets in Cartesian products through a series of lemmas. Additionally, we establish precise lower and upper bounds for the monophonic position number of Cartesian products and present several exact values. Then, in

Section 4, we turn our attention to lexicographic products and determine the monophonic position number of arbitrary lexicographic products.

2 Preliminaries

We now define the terminology that will be used in this paper. By a graph $G = (V(G), E(G))$ we mean a finite, undirected, simple graph. We will write $u \sim v$ if vertices u and v are adjacent. The *open neighbourhood* $N(u)$ of $u \in V(G)$ is $\{v \in V(G) : u \sim v\}$, whilst the *closed neighbourhood* $N[v]$ is defined by $N[v] = N(u) \cup \{u\}$. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u, v -path in G , and any such shortest u, v -path is a *geodesic*. A path P in G is *induced* or *monophonic* if G contains no chords between non-consecutive vertices of P . For two distinct vertices u, v of a graph G , the *monophonic interval* $J_G[u, v]$ is the set of all vertices lying on at least one monophonic u, v -path.

Recall that a set S of vertices in a graph G is a *general position set* if no shortest path in G contains more than two vertices of S . The number of vertices in a largest general position set of G is called the *general position number* of G and is denoted by $\text{gp}(G)$. A set $M \subseteq V(G)$ is a *monophonic position set* of G if no three vertices of M lie on a common monophonic path in G . The *monophonic position number* or *mp-number* $\text{mp}(G)$ of G is the number of vertices in a largest monophonic position set of G . Observe that every monophonic position set of a graph G is also in general position, and hence $\text{mp}(G) \leq \text{gp}(G)$. Any pair of vertices is in monophonic position, so for graphs with order $n(G) \geq 2$ we have $2 \leq \text{mp}(G) \leq n(G)$. Trivially $\text{mp}(G) = n(G)$ holds for a connected graph G if and only if G is a complete graph. It was shown in [22, 28] that for any $2 \leq a \leq b$ there exists a graph G with $\text{mp}(G) = a$ and $\text{gp}(G) = b$. Interestingly the largest possible number of edges in a graph with order n and monophonic position number a is quadratic in a , whereas for fixed general position number the largest size is linear in n , see [28]. Thus the general and monophonic position problems are intrinsically different.

We will also make use of the following terminology to simplify our arguments: a monophonic path containing three vertices of a set S is *S -bad*. If S is clear from the context, we will simply write *bad*. Hence, if S is assumed to be in monophonic position, then the appearance of a bad path constitutes a contradiction.

We will denote the subgraph of G induced by a subset $S \subseteq V(G)$ by $G[S]$. A vertex is *simplicial* if its neighbourhood induces a clique. The *clique number* $\omega(G)$ of G is the number of vertices in a maximum clique in G and the *independence number* $\alpha(G)$ is the number of vertices of a maximum independent set. The path of

order ℓ will be written as P_ℓ and the cycle of length ℓ as C_ℓ . For any positive integer k , we fix $[k] = \{1, 2, \dots, k\}$.

We will need the following result on monophonic position sets.

Lemma 2.1. [22, Lemma 3.1] *Let G be a connected graph and $M \subseteq V(G)$ be a monophonic position set. Then $G[M]$ is a disjoint union of k cliques $G[M] = \bigcup_{i=1}^k W_i$. If $k \geq 2$, then for each $i \in [k]$ any two vertices of W_i have a common neighbour in $G - M$.*

Let G and H be graphs. In this paper we discuss the monophonic position number of the *Cartesian product* $G \square H$ and the *lexicographic product* $G \circ H$. Both products have the vertex set $V(G) \times V(H)$. Let $(g_1, h_1), (g_2, h_2) \in V(G) \times V(H)$. In the Cartesian product $G \square H$ the vertices (g_1, h_1) and (g_2, h_2) are adjacent if (i) $g_1 \sim g_2$ in G and $h_1 = h_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$ in H . In the lexicographic product $G \circ H$ these vertices are adjacent if (i) $g_1 \sim g_2$ or (ii) $g_1 = g_2$ and $h_1 \sim h_2$.

If $* \in \{\square, \circ\}$, then the *projection mappings* $\pi_G : G * H \rightarrow G$ and $\pi_H : G * H \rightarrow H$ are given by $\pi_G(u, v) = u$ and $\pi_H(u, v) = v$, respectively. If $S \subseteq V(G \square H)$, then the set $\{g \in V(G) : (g, h) \in S \text{ for some } h \in V(H)\}$ is the *projection* $\pi_G(S)$ of S on G . The projection $\pi_H(S)$ of S on H is defined analogously. We adopt the following conventions. If $v_0, v_1, \dots, v_{\ell-1}, v_\ell$ is a path Q in H , then by ${}_u Q$ we denote the path

$$(u, v_0), (u, v_1), \dots, (u, v_{\ell-1}), (u, v_\ell)$$

in $G * H$. Similarly, if $u_0, u_1, \dots, u_{\ell-1}, u_\ell$ is a path in G , then in $G * H$, the path

$$(u_0, v), (u_1, v), \dots, (u_{\ell-1}, v), (u_\ell, v)$$

is denoted by P_v . Furthermore, a tilde over a path will denote that it is traversed in the opposite direction; for example, if P is the path u_0, u_1, \dots, u_ℓ , then \tilde{P} is the path u_ℓ, \dots, u_1, u_0 . Finally, for $u \in V(G)$ and $v \in V(H)$ we define ${}^u H$ to be the subgraph of $G * H$ induced by $\{u\} \times V(H)$, which we call a *H-layer*, whilst the *G-layer* G^v is the subgraph induced by $V(G) \times \{v\}$.

3 Cartesian products

In this section we investigate the monophonic position number of the Cartesian product of connected graphs. In the main result (Theorem 3.13) we give sharp lower and upper bounds for $\text{mp}(G \square H)$ in general, and the exact formula for the case when factors do not contain simplicial vertices. For the case when factors are bipartite, we also give in Theorem 3.14 an upper bound, which is of a different nature.

Note that if $S \subseteq V(G \square H)$, then due to the commutativity of the Cartesian product, any result that holds for $\pi_G(S)$ also holds for $\pi_H(S)$ and vice versa.

Let's begin with the following straightforward lower bound for $\text{mp}(G \square H)$.

Observation 3.1. *If G and H are graphs, then $\text{mp}(G \square H) \geq \max\{\omega(G), \omega(H)\}$.*

Proof. The observation follows because for any graph X we have $\text{mp}(X) \geq \omega(X)$, and because $\omega(G \square H) = \max\{\omega(G), \omega(H)\}$. \square

In order to be able to prove the main result of this section, Theorem 3.13, we need a sequence of lemmas.

Lemma 3.2. *If S is a monophonic position set of $G \square H$, then $\pi_H(S)$ is a monophonic position set of H .*

Proof. Suppose for a contradiction that $\pi_H(S)$ is not in monophonic position in H . Then there exists a set $S' = \{v_1, v_2, v_3\} \subseteq \pi_H(S)$ such that there is an induced v_1, v_3 -path P that passes through v_2 . Since $\{v_1, v_2, v_3\} \subseteq \pi_H(S)$, there exist (not necessarily distinct) vertices u_1, u_2, u_3 of G such that (u_1, v_1) , (u_2, v_2) and (u_3, v_3) belong to S ; we will derive a contradiction by constructing a monophonic path in $G \square H$ from (u_1, v_1) to (u_3, v_3) that passes through (u_2, v_2) . Let Q and R be monophonic paths in G from u_1 to u_2 and from u_2 to u_3 respectively. Also, recall that \tilde{Q} is the reverse path of Q , i.e. the path formed by traversing Q in the opposite direction from u_2 to u_1 . Without loss of generality there are four possibilities to consider: (i) u_1, u_2 and u_3 are pairwise distinct, (ii) $u_1 = u_2 \neq u_3$, (iii) $u_1 = u_3 \neq u_2$, and (iv) $u_1 = u_2 = u_3$. Note that the path u_2P passes through (u_2, v_2) .

Case	Section 1	Section 2	Section 3
u_1, u_2, u_3 distinct	Q_{v_1}	u_2P	R_{v_3}
$u_1 = u_2 \neq u_3$	u_1P	R_{v_3}	
$u_1 = u_3 \neq u_2$	Q_{v_1}	u_2P	\tilde{Q}_{v_3}
$u_1 = u_2 = u_3$	u_1P		

Table 1:

The desired monophonic paths in $G \square H$ are constructed by concatenating the paths in Table 1 in order. Each section of the paths is monophonic by construction and it is easily verified that there are no chords between different sections of the

displayed paths. Observe that $d_H(v_1, v_3) \geq 2$, so that there is no edge in $G \square H$ between a vertex (u, v_1) and (u, v_3) . \square

Note that the converse of Lemma 3.2 is not true. As an example, consider $K_3 \square P_3$. Let $V(P_3) = \{1, 2, 3\} = V(K_3)$. Then $S = \{(1, 1), (2, 2), (1, 3)\}$ is not a monophonic position set of G , whereas $\pi_{K_3}(S) = \{1, 2\}$ is a monophonic position set of K_3 .

Lemma 3.3. *Let G and H be connected graphs and let S be a monophonic position set of $G \square H$. If $(u, v) \in S$, then $V({}^uH) \cap S = \{(u, v)\}$ or $V(G^v) \cap S = \{(u, v)\}$.*

Proof. Suppose that the result is not true, i.e. that there exist $u' \neq u$ in G and $v' \neq v$ in H such that (u, v) , (u', v) and (u, v') all belong to S . Let P be a monophonic u', u -path in G and Q be a monophonic v, v' -path in H . Then the concatenation of P_v and ${}_uQ$ would be a monophonic (u', v) , (u, v') -path in $G \square H$ through (u, v) , a contradiction. \square

Lemma 3.4. *If $S = \{(u_i, v_i) : i \in [|S|]\}$ is a monophonic position set of $G \square H$, then one of the following holds.*

- (a) S lies in a single G -layer, or S lies in a single H -layer.
- (b) $u_1, \dots, u_{|S|}$ are distinct vertices of G and $v_1, \dots, v_{|S|}$ are distinct vertices of H .
- (c) $\pi_G(S)$ is a clique of G with order at least 2 and $v_1, \dots, v_{|S|}$ are distinct vertices of H , or $\pi_H(S)$ is a clique of H with order at least 2 and $u_1, \dots, u_{|S|}$ are distinct vertices of G .

Proof. Suppose that S satisfies neither of the first two statements; without loss of generality, we can assume that $u_1 = u_2$ and there is a $u_i \in \pi_G(S)$ with $u_i \neq u_1$. We wish to show that $\pi_G(S)$ induces a clique in G and that the vertices $v_1, v_2, \dots, v_{|S|}$ of H are all distinct.

Let P be a monophonic u_1, u_i -path in G and Q and R be monophonic v_1, v_2 - and v_2, v_i -paths in H . As the concatenated path ${}_uQ, P_{v_2}, {}_{u_i}R$ cannot be monophonic there must be a chord between the paths ${}_uQ$ and ${}_{u_i}R$, which is the case if and only if $u_1 \sim u_i$ in G and the paths Q and R intersect in H . Therefore, $u_1 \sim u_i$ in G for every i such that $u_1 \neq u_i$ which implies that $\pi_G(S)$ induces a connected subgraph of G . Lemmas 2.1 and 3.2 then yields that the set $\pi_G(S)$ induces a clique in G .

Suppose now that not all vertices $v_1, \dots, v_{|S|}$ are distinct; say $v_i = v_j$. As we are assuming that S does not lie in a single layer, there must be a $v_k \in \pi_H(S)$ with $v_k \neq v_i$. Let P' be a monophonic v_i, v_k -path in H . As $\pi_G(S)$ is a clique, the path formed by following the edge $(u_i, v_i) \sim (u_j, v_i)$, the path ${}_{u_j}P'$ and then the edge

$(u_j, v_k) \sim (u_k, v_k)$ (if $u_k = u_j$ we omit the final edge) will be monophonic unless $u_i = u_k$ and there is an edge $v_i \sim v_k$; however, in this case we can interchange u_i and u_j in the argument to produce a contradiction. \square

Corollary 3.5. *If G and H are connected graphs, then*

$$\text{mp}(G \square H) \leq \max\{\text{mp}(G), \text{mp}(H)\}.$$

Proof. By Lemma 3.4, $|\pi_G(S)| = |S|$ or $|\pi_H(S)| = |S|$ (or both), thus the conclusion follows from Lemma 3.2. \square

As the bounds in Observation 3.1 and Corollary 3.5 coincide for products of paths and cycles, the following corollary follows immediately.

Corollary 3.6. *For paths P_m, P_n of order at least two and cycles C_r, C_s of length at least four,*

$$\text{mp}(P_m \square P_n) = \text{mp}(P_n \square C_r) = \text{mp}(C_r \square C_s) = 2.$$

Our bounds also make it easy to evaluate $\text{mp}(K_n \square H)$ for graphs H with small mp-number.

Corollary 3.7. *If H is a connected graph and $n \geq \text{mp}(H)$, then $\text{mp}(K_n \square H) = n$.*

Corollary 3.7 in particular yields $\text{mp}(K_n \square P_m) = n$ for $n \geq 2$, $\text{mp}(K_n \square C_m) = n$ for $n \geq 3$, and $\text{mp}(K_n \square K_m) = \max\{n, m\}$.

We will call a monophonic position set of Type (a) *layered*, of Type (b) *varied*, and of Type (c) *cliquey*. These three types of monophonic position sets are shown schematically in Fig. 1.

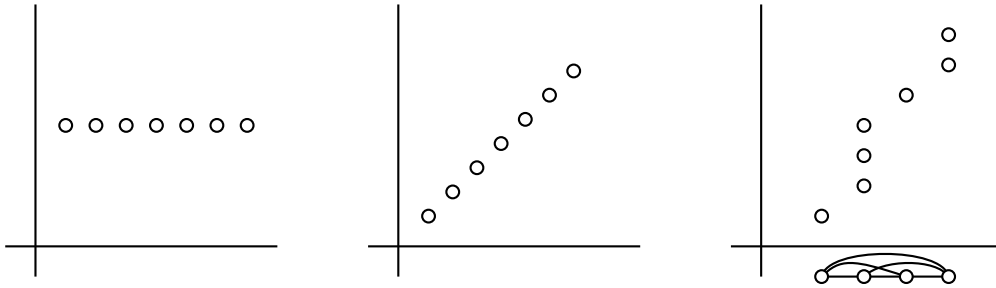


Figure 1: Layered (left), varied (middle), and cliquey (right) monophonic position set

Lemma 3.8. *If $u, u' \in V(G)$ and $v, v' \in V(H)$ are such that $u' \notin N_G[u]$ and $v' \notin N_H[v]$, then the set $\{(u, v), (u', v')\}$ is a maximal monophonic position set of $G \square H$.*

Proof. Suppose that (u, v) and (u', v') satisfy the stated conditions, but that there exists a third vertex (x, y) of $G \square H$ such that $\{(u, v), (u', v'), (x, y)\}$ is a monophonic position set. It follows from Lemmas 3.2 and 3.4 that $\{u, u', x\}$ is a set of distinct vertices of G in monophonic position, and likewise $\{v, v', y\}$ in an monophonic position set of H of order three.

Let P and P' be monophonic u, x - and x, u' -paths in G respectively and Q and Q' be monophonic v, y - and y, v' -paths in H respectively. Consider the path formed by the concatenation of $P_v, {}_xQ, P'_y$ and ${}_uQ'$. Trivially there are no chords between consecutive sections of the concatenated path, e.g. between P_v and ${}_xQ$. There is no chord between P_v and ${}_uQ'$; otherwise there is an edge $(w, v) \sim (u', z)$ in $G \square H$, where w lies on P and z lies on Q' and, since either $w = u'$ or $z = v$ we would have a violation of Lemma 3.2.

Finally if we assume that $x \not\sim u'$ and $v \not\sim y$, then there are no chords between P_v and P'_y or between ${}_xQ$ or ${}_uQ'$. This contradicts our assumption that $\{(u, v), (u', v'), (x, y)\}$ is an monophonic position set, so we conclude that either $x \sim u'$ or $y \sim v$. Similarly we must have $x \sim u$ or $y \sim v'$. By Lemma 3.2 we cannot have both $x \sim u$ and $x \sim u'$, or both $y \sim v$ and $y \sim v'$, so it follows that either (i) $x \sim u$ and $y \sim v$ or (ii) $x \sim u'$ and $y \sim v'$. Suppose that $x \sim u$ and $y \sim v$ (the other case is similar). Let P'' be a monophonic u, u' -path in G and Q'' be a monophonic v, v' -path in H . Then the path formed by the edges $(x, y) \sim (x, v)$ and $(x, v) \sim (u, v)$ followed by the path ${}_uQ''$ and P''_v is a monophonic path passing through (u, v) , a contradiction. We conclude that $\{(u, v), (u', v')\}$ is a maximal monophonic position set. \square

Corollary 3.9. *If $G \square H$ has a maximum monophonic position set that is varied, then $\text{mp}(G \square H) = \max\{\omega(G), \omega(H)\}$.*

Proof. We have $\text{mp}(G \square H) \geq \max\{\omega(G), \omega(H)\}$ by Observation 3.1. Assume that S is a largest monophonic position set of $G \square H$ that is varied. Suppose for a contradiction that $\text{mp}(G \square H) > \max\{\omega(G), \omega(H)\}$. By Lemmas 3.2 and 2.1, the sets $\{u_1, \dots, u_{\text{mp}(G \square H)}\}$ and $\{v_1, \dots, v_{\text{mp}(G \square H)}\}$ are independent unions of cliques in G and H respectively, so we can assume by Lemma 3.8 that there exist (u_1, v_1) , (u_2, v_2) and (u_3, v_3) in S such that in G we have $u_1 \sim u_2$ and $u_1 \not\sim u_3$, whilst in H we have $v_1 \not\sim v_2$ and $v_1 \sim v_3$. Let P be a monophonic u_2, u_3 -path in G and Q and Q' be monophonic v_1, v_2 - and v_2, v_3 -paths in H respectively. As u_1 and u_2 are at distance at least two from u_3 , the path formed by following the edge $(u_1, v_1) \sim (u_2, v_1)$ followed

by the paths $_{u_2}Q$, P_{v_2} and $_{u_3}Q'$ in that order is bad, a contradiction. It follows that either $\pi_G(S)$ or $\pi_H(S)$ induces a clique, so that $|S| \leq \max\{\omega(G), \omega(H)\}$. \square

We can now confine our attention to layered and cliquy monophonic position sets.

Lemma 3.10. *If $\{u\} \times S' \subseteq S$, where $S' \subseteq V(H)$, then if S is either cliquy or S is layered with $|S| > \max\{\omega(G), \omega(H)\}$, then S' induces an independent set in H , with a similar result for subsets $S'' \times \{v\} \subseteq S$.*

Proof. Let $\{u\} \times S' \subseteq S$. We show firstly that S' is either a clique or an independent set. If $|S'| \leq 2$ then there is nothing to prove, so we assume that $|S'| \geq 3$. By Lemmas 3.2 and 2.1, the set $\pi_H(S)$ is an independent union of cliques, so if S' is not a clique or an independent set, then there are vertices $v_1, v_2, v_3 \in S'$ such that $v_1 \sim v_2$, $v_1 \not\sim v_3$ and $v_2 \not\sim v_3$ in H . Let u' be any neighbour of u in G and P be a monophonic v_2, v_3 -path in H . Then the path formed by concatenating the path $(u, v_1) \sim (u, v_2) \sim (u', v_2)$ with the path $_{u'}P$ and then the edge $(u', v_3) \sim (u, v_3)$ in that order would be $\{u\} \times S'$ -bad. Thus S' is either a clique or an independent set.

Suppose that there are $v_1, v'_1 \in S'$ with $v_1 \sim v'_1$ in H and let $u_2 \in \pi_G(S) \setminus \{u\}$, where $(u_2, v_2) \in S$. Let P be a monophonic v'_1, v_2 -path. Then the path $(u, v_1), (u, v'_1), (u_2, v'_1)$ followed by $_{u_2}P$ is a bad path, so there is no such edge in S' . If S is layered and $|S| > \max\{\omega(G), \omega(H)\}$, then S' cannot be a clique and hence must be independent. \square

We will use the following labelling convention for a cliquy monophonic position set S of $G \square H$ (we assume that $\pi_G(S)$ is a clique without loss of generality). With each vertex $u_i \in \pi_G(S)$ we associate the set $S'_i \subseteq V(H)$ such that $\{u_i\} \times S'_i = S \cap (\{u_i\} \times V(H))$. Recall that by Lemma 3.4 the sets S_i are pairwise disjoint, so that these sets partition $\pi_H(S)$. The sets S'_i are schematically presented in Fig. 2.

Lemma 3.11. *Let S be a cliquy monophonic position set of $G \square H$, where $\pi_G(S)$ is a clique. If $|\pi_G(S)| \geq 3$ or $|S| \geq 3$, then $\pi_H(S)$ is an independent set.*

Proof. By Lemma 3.10 each set S'_i is an independent set, so we need only prove that there are no edges between S'_i and S'_j for $i \neq j$.

We show firstly that if $|S'_i| \geq 2$ there are no edges from S'_i to S'_j in H for $i \neq j$. We assume that $|S'_i| \geq 2$, with $(u_i, v_1), (u_i, v_2) \in S$. Suppose that $v_2 \sim v_3$ in H for some $v_3 \in S'_j$, $i \neq j$. By Lemma 3.10 $v_1 \not\sim v_2$. Let P be a monophonic v_1, v_2 -path in H ; in this case the path formed by concatenating $_{u_i}P$ with the path $(u_i, v_2), (u_j, v_2), (u_j, v_3)$ would again be bad. Hence there can be no edge between S'_i and S'_j .

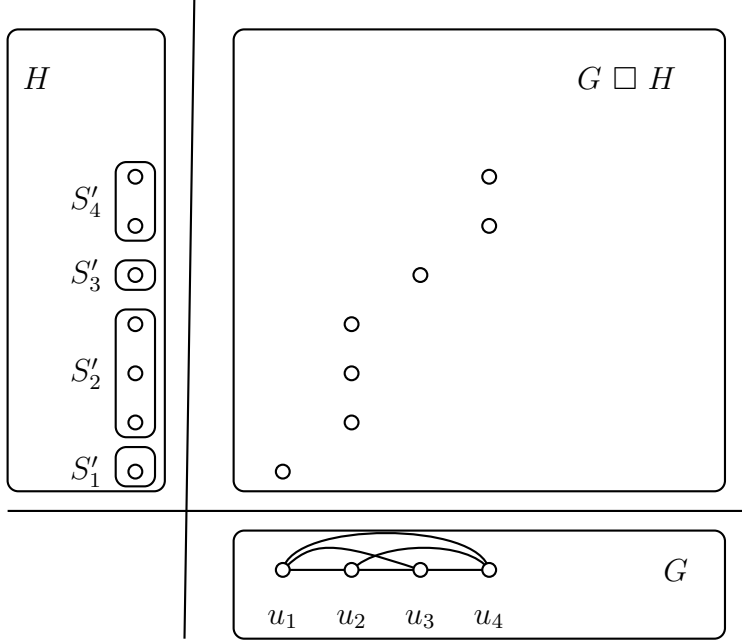


Figure 2: Sets S'_i

Now we show that if $|\pi_G(S)| \geq 3$ then there are no edges between S'_i and S'_j for $i \neq j$. Suppose that $|\pi_G(S)| \geq 3$ and that there is an edge between $v_1 \in S'_i$ and $v_2 \in S'_j$, where $i \neq j$. Let $u_k \in \pi_G(S) \setminus \{u_i, u_j\}$. If there is an edge from $v_3 \in S'_k$ to $\{v_1, v_2\}$, then $\{v_1, v_2, v_3\}$ is a clique and

$$(u_i, v_1), (u_j, v_1), (u_j, v_2), (u_k, v_2), (u_k, v_3)$$

would be bad. Otherwise, there is no edge between v_3 and $\{v_1, v_2\}$ and, letting P be a monophonic v_2, v_3 -path in H , the path formed by concatenating (u_i, v_1) , (u_j, v_1) , (u_j, v_2) , (u_k, v_2) and $u_k P$ would again be bad.

Finally, suppose that $|\pi_G(S)| = 2$ and $|S| > 2$. Write $\pi_G(S) = \{u_1, u_2\}$. If there are $v_1 \in S'_1$, $v_2 \in S'_2$ and an edge $v_1 \sim v_2$, then we can assume that there is a $v_3 \in S'_2 \setminus \{v_2\}$ and, taking P to be a monophonic v_2, v_3 path in H , the path formed from (u_1, v_1) , (u_1, v_2) , (u_2, v_2) followed by $u_2 P$ would again be bad. \square

Lemma 3.12. *If S is a layered or cliquey monophonic position set of $G \square H$ (where $\pi_G(S)$ is a clique) and $|S| > \max\{\omega(G), \omega(H)\}$, then every vertex from $\pi_G(S)$ is simplicial.*

Proof. Suppose that $u_i \in \pi_G(S)$ is not simplicial, with $w_1, w_2 \in N(u_i)$ and $w_1 \not\sim w_2$ in G . Firstly assume that $|S'_i| \geq 3$, with $\{v_1, v_2, v_3\} \subseteq S'_i$. Let P and P' be

monophonic v_1, v_2 - and v_2, v_3 -paths respectively. Having Lemma 3.10 in mind, the path formed by concatenating the edge $(u_i, v_1) \sim (w_1, v_1)$, the path ${}_{w_1}P$, the path (w_1, v_2) , (u_i, v_2) , (w_2, v_2) , the path ${}_{w_2}P'$ and finally the edge $(w_2, v_3) \sim (u_i, v_3)$ in that order would be bad.

Now suppose that $|S'_i| = 2$. As $|S| > \max\{\omega(G), \omega(H)\}$, S must be cliquey, so there is a $u_j \in \pi_G(S) \setminus \{u_i\}$. Let $v_1, v_2 \in S'_i$, $v_3 \in S'_j$, P and P' be monophonic v_1, v_2 - and v_2, v_3 -path respectively and Q be a shortest path from $\{w_1, w_2\}$ to u_j , which we assume without loss of generality to be a w_2, u_j -path (notice that Q has length at most two and that $w_2 = u_j$ is possible). Then the following concatenated path is bad: the edge $(u_i, v_1) \sim (w_1, v_1)$, the path ${}_{w_1}P$, the path (w_1, v_2) , (u_i, v_2) , (w_2, v_2) , the path ${}_{w_2}P'$, and finally the path Q_{v_3} . Thus $|S'_i| = 1$.

Finally suppose that $|S'_i| = 1$. As $|S| > \max\{\omega(G), \omega(H)\}$, S is cliquey and there is a $u_j \in \pi_G(S) \setminus \{u_i\}$ with $|S'_j| \geq 2$, say $S'_i = \{v_1\}$ and $\{v_2, v_2\} \subseteq S'_j$. Now u_i has a neighbour w such that $w \not\sim u_j$. Let P and P' be monophonic v_1, v_2 - and v_2, v_3 -paths respectively and Q be a monophonic w, u_j -path. The path formed as the concatenation of the edge $(u_i, v_1) \sim (w, v_1)$ and the paths ${}_wP$, Q_{v_2} , and ${}_{u_j}P'$ in that order would be bad. \square

A vertex subset S of a graph that is simultaneously an independent set and in monophonic position is called *independent monophonic position set*. The largest order of an independent monophonic position set is the *independent monophonic position number* of G , denoted by $\text{mp}_i(G)$. With this notation in hand, we prove the following tight bounds.

Theorem 3.13. *If G and H are connected graphs, then*

$$\max\{\omega(G), \omega(H)\} \leq \text{mp}(G \square H) \leq \max\{\omega(G), \omega(H), \text{mp}_i(G), \text{mp}_i(H)\}.$$

Furthermore, if neither G nor H has simplicial vertices, then

$$\text{mp}(G \square H) = \max\{\omega(G), \omega(H)\}.$$

Proof. Suppose that $G \square H$ contains a monophonic position set S with $|S| > \max\{\omega(G), \omega(H)\}$. Then it follows from Corollary 3.9 that S is either cliquey or layered. If S is cliquey, then using Lemma 3.4, we may assume that $\pi_G(S)$ is a clique and $|\pi_H(S)| = |S|$. Now, Lemmas 3.2 and 3.11 show that $\pi_H(S)$ is an independent monophonic position set of H and hence $\text{mp}(G \square H) = |S| = |\pi_H(S)| \leq \text{mp}_i(H)$. On the other hand, if S is a layered monophonic position set of G with $\pi_G(S) = \{u\}$, then it follows from Lemmas 3.2 and 3.10 that $\pi_H(S)$ is an independent monophonic position set of H . This again shows that $\text{mp}(G \square H) = |S| = |\pi_H(S)| \leq \text{mp}_i(H)$ and thus the bounds follow.

The second assertion of the theorem follows from Lemma 3.12. \square

The bounds in Theorem 3.13 are sharp. As an example, consider $K_{1,4} \square K_3$. Let $V(K_{1,4}) = \{0, 1, 2, 3, 4\}$, where 0 is the center vertex; and let $V(K_3) = \{1, 2, 3\}$. Then $S = (\{1, 2\} \times \{1\}) \cup (\{3, 4\} \times \{2\})$ is a cliquey monophonic position set of $K_{1,4} \square K_3$ and $T = \{1, 2, 3, 4\} \times \{1\}$ is a layered monophonic position set of $K_{1,4} \square K_3$. Hence $\text{mp}(K_{1,4} \square K_3) = 4 = \text{mp}_i(K_{1,4})$.

The structural properties established in the earlier lemmas about the monophonic position sets in the Cartesian product of graphs can be used to determine the monophonic position number for numerous classes of graphs. For instance, the following theorem for bipartite graphs shows that for integers $m, n \geq 2$, $\text{mp}(K_{1,m} \square K_{1,n}) = \max\{m, n\}$. To this end, we set $\sigma(G) = 1$ if $\delta(G) = 1$, and $\sigma(G) = 0$ otherwise.

Theorem 3.14. *If G and H are connected bipartite graphs of order at least 3, then*

$$\text{mp}(G \square H) \leq \max\{2, \sigma(G)\Delta(G), \sigma(H)\Delta(H)\}.$$

Moreover, the bound is tight when both G and H are star graphs.

Proof. Let S be a largest monophonic position set of $G \square H$. If $|S| \geq 3$, then S is either cliquey or layered by Corollary 3.9. Suppose that S is cliquey and $\pi_G(S) = \{u_1, u_2\}$ induces a clique, that is, $u_1 \sim u_2$ in G . But the order of G is at least three, which is in contradiction with Lemma 3.12. Hence S must be layered; and again by Lemma 3.12, we may assume that $\pi_G(S) = \{u\}$, where u is a vertex of G of degree 1. Let $u' \in V(G)$ be such that $d_G(u, u') = 2$ and let $P : u = u_0, u_1, u_2 = u'$ be a shortest u, u' -path in G . As in the proof of Theorem 3.13, we get that $\pi_H(S)$ is an independent monophonic position set of H . Let $v_1, v_2 \in \pi_H(S)$. In the following, we first prove that any v_1, v_2 -monophonic path in H has length exactly 2.

Assume on the contrary that there exists a v_1, v_2 -monophonic path, say $Q : v_1 = x_0, x_1, \dots, x_k = v_2$ with $k \geq 3$. Choose $v_3 \in \pi_H(S)$ such that v_3 is distinct from both v_1 and v_2 . This is possible since $|\pi_H(S)| = |S| \geq 3$. Let i be the least suffix such that $i \in [k]$ and $d_H(v_3, x_i)$ is minimum. Let Q_1 be a v_3, x_i -shortest path in H . Then Q_1 together with the v_1, x_i -subpath of Q is a v_3, v_1 -monophonic path in H , say Q_2 . Then $i \in [k-1]$, since $\pi_H(S)$ is independent monophonic position set of H . Now, if $i \in [k-2]$, then the concatenation of the paths ${}_u Q_2, P_{v_1, u_2} Q$, and \tilde{P}_{v_2} forms a monophonic $(u, v_3), (u, v_2)$ -path in $G \square H$ containing the vertex (u, v_1) . This is impossible and hence $x_i = x_{k-1}$. Now, suppose that Q_1 together with the edge $x_{k-1}x_k$ is a monophonic path in H , say Q_3 . Then as above the concatenation of the paths ${}_u Q_3, P_{v_2, u_2} \tilde{Q}$, and \tilde{P}_{v_1} forms a monophonic $(u, v_3), (u, v_1)$ -path in $G \square H$ containing the vertex (u, v_2) , which is not possible. Therefore, we can assume that the v_3, v_2 -path Q_3 is not monophonic. Now, fix $Q_3 : v_3 = z_1, z_2, \dots, z_l = x_i, v_2$. Let

j be the least suffix such that $j \in [l - 1]$ and z_j is adjacent to v_2 . This shows that the v_3, z_j -subpath of Q_3 together with the edge $z_j v_2$ is a monophonic v_3, v_2 -path in H , say Q_4 . Then the concatenation of the paths ${}_u Q_4, P_{v_2}, {}_{u_2} \tilde{Q}$, and \tilde{P}_{v_1} forms a monophonic $(u, v_3), (u, v_1)$ -path in $G \square H$ containing the vertex (u, v_2) , another contradiction. Thus we can conclude that each monophonic path in H joining any pair of vertices of $\pi_H(S)$ has length exactly 2.

Now let v_1, v_2 and v_3 be three distinct vertices in $\pi_H(S)$ and let $Q : v_1, y, v_2$ be a monophonic v_1, v_2 -path of length 2 in H . Suppose that the vertices v_3 and y are non-adjacent in H . Let Q' be a monophonic v_2, v_3 -path in H . Then the concatenation of the paths ${}_u Q, P_{v_2}, {}_{u_2} \tilde{Q}'$, and \tilde{P}_{v_3} is a monophonic $(u, v_1), (u, v_3)$ -path in $G \square H$ containing the vertex (u, v_1) , which is not possible. Hence v_2 must be adjacent to the vertex y . This proves that $|\pi_H(S)| \leq |N_H(y)| = \deg_H(y)$ and so $\text{mp}(G \square H) \leq \Delta(H)$. Thus the bound follows. Furthermore, if both G and H are star graphs, then for any degree 1 vertex u of G and for any $v \in V(H)$, one can easily check that the set $u \times N_H(v)$ is a monophonic position set of $G \square H$. This proves the sharpness of the bound for star graphs. \square

4 Lexicographic products

The variety of general position sets was introduced in [23] and has already been investigated on lexicographic products in [6]. In this section, we determine the monophonic position number of arbitrary lexicographic products.

Before stating the main result, we recall the distance function of lexicographic products (see [8, Proposition 5.12]) and state two lemmas.

Proposition 4.1. *If (g, h) and (g', h') are distinct vertices of $G \circ H$, then*

$$d_{G \circ H}((g, h), (g', h')) = \begin{cases} d_G(g, g'); & g \neq g', \\ d_H(h, h'); & g = g', \deg_G(g) = 0, \\ \min\{d_H(h, h'), 2\}; & g = g', \deg_G(g) \neq 0. \end{cases}$$

Lemma 4.2. *If S is a monophonic position set of $G \circ H$, then $\pi_G(S)$ is a monophonic position set of G .*

Proof. Suppose for a contradiction that $\pi_G(S)$ is not a monophonic position set in G . Then there exist vertices $u_1, u_2, u_3 \in \pi_G(S)$ and a monophonic u_1, u_3 -path P in G which passes through u_2 . As $u_1, u_2, u_3 \in \pi_G(S)$, there exist vertices $v_1, v_2, v_3 \in V(H)$ such that $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in S$. But now it is straightforward to lift the path P to a monophonic $(u_1, v_1), (u_3, v_3)$ -path in $G \circ H$ which passes (u_2, v_2) , a contradiction. \square

Let $u \in V(G)$ and let P be a monophonic path in H . Then the isomorphic copy of P in the layer uH of $G \circ H$ is a monophonic path of $G \circ H$. This fact implies the second announced lemma.

Lemma 4.3. *If S is a monophonic position set of $G \circ H$, then for any $u \in \pi_G(S)$, $\pi_H({}^uH \cap S)$ is a monophonic position set of H .*

Note that $\pi_H(S)$ need not be a monophonic position set of H . As an example consider $P_2 \circ P_3$ (See Fig. 3). Let $V(P_2) = \{u_1, u_2\}$, $V(P_3) = \{v_1, v_2, v_3\}$ and let $S = \{(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3)\}$. As S is a clique, it is a monophonic position set of $P_2 \circ P_3$. But $\pi_{P_3}(S) = \{v_1, v_2, v_3\}$ is not a monophonic set in P_3 .

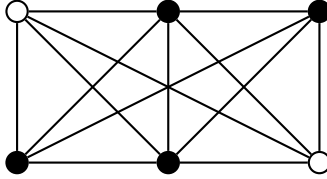


Figure 3: Example demonstrating that a monophonic position set of $G \circ H$ need not project to a monophonic position set in H

In view of Lemma 2.1, in the rest of this section, for a monophonic position set M of G we denote the components of $G[M]$ by: $A_1, A_2, \dots, A_k, B_1, \dots, B_r$, where $|A_i| \geq 2$ for each $i \in [k]$ and $|B_j| = 1$ for each $j \in [r]$. Also we fix $n_M = \sum_{i=1}^k |A_i|$. Then for any monophonic position set M of G we have $|M| = n_M + r$. Now we are ready for our main result of this section.

Theorem 4.4. *Let G be a connected graph of order at least 2 and let \mathcal{M} be the collection of all monophonic position sets of G . Then*

$$\text{mp}(G \circ H) = \max_{M \in \mathcal{M}} \{n_M \cdot \omega(H) + r \cdot \text{mp}(H)\}.$$

Proof. Let S be a monophonic position set of $G \circ H$. If $|\pi_G(S)| = |S|$, then by Lemma 4.2 we have $|S| = |\pi_G(S)| \leq \text{mp}(G)$. Now, for any monophonic position set M of G , we have that $|M| = n_M + r \leq n_M \cdot \omega(H) + r \cdot \text{mp}(H)$. This shows that $|S| \leq \max_{M \in \mathcal{M}} \{n_M \cdot \omega(H) + r \cdot \text{mp}(H)\}$.

Next consider the case $|\pi_G(S)| < |S|$. If $|\pi_G(S)| = 1$, then $\pi_H(S) = \pi_H({}^uH \cap S)$, for some $u \in \pi_G(S)$. Lemma 4.3 yields that $\pi_H({}^uH \cap S)$ is a monophonic position set of H . Thus $|S| = |\pi_H(S)| \leq \text{mp}(H)$. So in the following, we assume that $1 < |\pi_G(S)| < |S|$. Let $\{A_i\}_{i \in [k]}$ be the non-trivial components of $G[\pi_G(S)]$.

Claim: If $u \in A_i$, $i \in [k]$, then $\pi_H({}^u H \cap S)$ is a clique in H .

Let $u \in A_i$. If $|\pi_H({}^u H \cap S)| = 1$, then there is nothing to prove. So assume that $|\pi_H({}^u H \cap S)| > 1$. Choose distinct vertices v_1, v_2 in $\pi_H({}^u H \cap S)$. Then $(u, v_1), (u, v_2) \in S$. Since $|A_i| \geq 2$, we can choose $u_1 \in A_i$ distinct from u ; and $v_3 \in V(H)$ such that $(u_1, v_3) \in S$. By Lemma 2.1, A_i induces a clique in G . If $v_1 \not\sim v_2$, then by Proposition 4.1, the path $P : (u, v_1), (u_1, v_3), (u, v_2)$ is a monophonic path in $G \circ H$, a contradiction to the fact that S is a monophonic position set of $G \circ H$. Hence $v_1 \sim v_2$ in H which proves the claim.

By the above claim and the fact that for each $u \in \pi_G(S)$, the set $\pi_H({}^u H \cap S)$ is a monophonic position set of H , we can conclude that $\text{mp}(G) = |S| \leq n_M \cdot \omega(H) + r \cdot \text{mp}(H)$. To complete the proof it suffices to show that for any $M \in \mathcal{M}$, we are able to construct a monophonic position set S in $G \circ H$ with $|S| = n_M \cdot \omega(H) + r \cdot \text{mp}(H)$.

Consider an arbitrary monophonic position set M of G . Let $G[M] = (\cup_{i=1}^k A_i) \cup (\cup_{j=1}^r B_j)$, where $|A_i| \geq 2$ for each $i \in [k]$ and $|B_j| = 1$ for each $j \in [r]$. Let C be a largest clique and let D be a largest monophonic position set of H , respectively. Now for each $a_l \in \cup_{i=1}^k A_i$, $l \in [n_M]$, fix $C_l = a_l \times C$, a clique in ${}^{a_l} H$; and for $b_j \in B_j$, fix $D_j = b_j \times D$, a monophonic position set of ${}^{b_j} H$. We claim that $S = (\cup_{l=1}^{n_M} C_l) \cup (\cup_{j=1}^r D_j)$ is a monophonic position set of $G \circ H$. If possible, suppose that there exist vertices $x = (x_1, y_1)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ in S such that $y \in J_{G \circ H}[x, z]$. Fix $F = \{x, y, z\}$. Let $A = \cup_{l=1}^{n_M} C_l$, $B = \cup_{j=1}^r D_j$ and P be a monophonic path connecting x and z containing y . We distinguish the following four cases.

Case 1: $x_1 = y_1 = z_1$.

In this case, since $y \in J_{G \circ H}[x, z]$, $\pi_H(F)$ is not a clique in H . Thus $F \subseteq B$. Since $\pi_H(F) \subseteq D$ and D is a monophonic position set in H , we have that P is not contained in ${}^{x_1} H$. This shows that P cannot be a monophonic path in $G \circ H$, a contradiction.

Case 2: $x_1 = y_1 \neq z_1$.

Then there will be a chord from x to any path connecting y and z , a contradiction to the fact that $y \in J_{G \circ H}[x, z]$.

Case 3: $x_1 = z_1 \neq y_1$.

In this case, since P is a monophonic x, z -path, the length of P must be 2. Hence $x_1 \sim y_1$ and $z_1 \sim y_1$ in G which implies $F \subseteq A$. But then by our construction, $x_2 \sim z_2$ in H , a contradiction to the fact that $y \in J_{G \circ H}[x, z]$.

Case 4: x_1, y_1 and z_1 are distinct vertices of G .

By Proposition 4.1, $y \in J_{G \circ H}[x, z]$ if and only if $y_1 \in J_G[x_1, z_1]$. This contradicts the fact that M is a monophonic position set in G .

Since in all cases we have arrived at a contradiction, we can conclude that S is a monophonic position set of $G \circ H$ and so $\text{mp}(G \circ H) \geq |S| = n_M \cdot \omega(H) + r \cdot \text{mp}(H)$. \square

Corollary 4.5. *If G be a connected bipartite graph and H is a connected graph, then $\text{mp}(G \circ H) = \text{mp}(G) \cdot \text{mp}(H)$.*

Corollary 4.6. *If H is a connected graph and $n \geq 2$, then $\text{mp}(K_n \circ H) = n \cdot \omega(H)$.*

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