Varieties of mutual-visibility and general position on Sierpiński graphs

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Abstract

The variety of mutual-visibility problems contains four members, as does the variety of general position problems. The basic problem is to determine the cardinality of the largest such sets. In this paper, these eight invariants are investigated on Sierpiński graphs S_p^n. They are determined for the Sierpiński graphs S_p^2, $p \geq 3$. All, but the outer mutual-visibility number and the outer general position number, are also determined for S_3^n, $n \geq 3$. In many of the cases the corresponding extremal sets are enumerated.

Keywords: mutual-visibility set, general position set, Sierpiński graph

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1 Introduction

General position and mutual-visibility are two fresh areas in metric and algorithmic graph theory. These concepts are complementary to each other, and together they represent a flourishing field of research.

After general position sets were independently introduced (in a general setting) to graph theory in [5] and in [22], research in this area has expanded rapidly, a recent review article [4] lists 115 references. These investigations include several interesting variations including edge general position sets [23], monophonic position sets [26], Steiner position sets [15], vertex position sets [25], mobile position sets [14], and lower general position sets [10, 20]. See also recent studies [1, 13, 27, 31–33].

Given a set X of vertices in a graph G, two vertices u and v are *X-positionable*, if for every shortest u, v-path P we have $V(P) \cap X \subseteq \{u, v\}$. (Note that if $uv \in E(G)$, then u and v are X-positionable.) Then X is a *general position set*, if every $u, v \in X$ are X-positionable. A largest general position set is a *gp-set* and its size is the *general position number* $\mathrm{gp}(G)$ of G.

Based on the motivation of robotic visibility, the graph mutual-visibility problem was introduced in 2022 by Di Stefano [9]. Given a set X of vertices in a graph G, two vertices u and v are *mutually-visible* with respect to X, shortly *X-visible*, if there exists a shortest u, v-path P such that $V(P) \cap X \subseteq \{u, v\}$. The set X is a *mutual-visibility set* if any two vertices from X are X-visible. A largest mutual-visibility set of G is a *μ-set* and its size is the *mutual-visibility number* $\mu(G)$ of G. Although only recently introduced, the mutual-visibility sets has already received a lot of attention, here we would like to point in particular to [2, 8, 17, 21, 24, 28].

In [7], the total mutual-visibility number was introduced, while the variety of mutual-visibility invariants was rounded off in [6] by adding to the list the outer mutual-visibility number and the dual mutual-visibility number. A set $X \subseteq V(G)$ is an *outer mutual-visibility set* in G if X is a mutual-visibility set and every pair of vertices $u \in X$, $v \in V(G) \setminus X$ are X-visible. X is a *dual mutual-visibility set* if X is a mutual-visibility set and every pair of vertices $u, v \in V(G) \setminus X$ are X-visible. Finally, X is a *total mutual-visibility set* if every pair of vertices in G are X-visible. The cardinality of a largest outer/dual/total mutual-visibility sets are respectively denoted by $\mu_{\mathrm{o}}(G)$, $\mu_{\mathrm{d}}(G)$, $\mu_{\mathrm{t}}(G)$.

Following the pattern of mutual-visibility, the variety of general position invariants was presented in [30]. The definition of the *outer/dual/total general position set* in G is analogous, we just need to replace everywhere "X-visible" by "X-positionable." Largest corresponding sets are called gp_{o}*-sets*, gp_{d}*-sets*, gp_{t}*-sets* and their sizes are the *outer/dual/total general position number* of G, respectively denoted by $\mathrm{gp}_{\mathrm{o}}(G)$, $\mathrm{gp}_{\mathrm{d}}(G)$, $\mathrm{gp}_{\mathrm{t}}(G)$.

Recently, Korže and Vesel [18] investigated Sierpiński triangle graphs ST_3^n and determined $\tau(ST_3^n)$ for $\tau \in \{\mu, \mu_{\rm t}, \mu_{\rm o}, \mu_{\rm d}, {\rm gp}\}$. Sierpiński triangle graphs ST_3^n are obtained from the classical Sierpiński graphs S_3^n by contracting all the edges which do not lie in triangles. Continuing the above investigation, in this paper we determine $\tau(S_3^n)$ for $\tau \in \{\mu, \mu_{\rm t}, \mu_{\rm d}, {\rm gp}, {\rm gp}_{\rm t}, {\rm gp}_{\rm d}\}$ and bound $\mu_{\rm o}(S_3^n)$ and ${\rm gp}_{\rm o}(S_3^n)$. We also determine all the eight invariants for the Sierpiński graphs S_p^2 for any $p \geq 3$. In many of the cases we also enumerate the corresponding extremal sets.

2 Preliminaries

For any positive integer k we set $[k] = \{1, 2, \ldots, k\}$ and $[k]_0 = \{0, 1, \ldots, k-1\}$.

Let $G = (V(G), E(G))$ be a graph. The *degree* of a vertex u of G is the number of its adjacent vertices in G. For the vertices u and v of G, the length of a shortest u,v-path is called the *distance* between u and v, and is denoted by $d_G(u, v)$.

If $X \subseteq V(G)$, then the subgraph of G induced by X is denoted by $G[X]$. A vertex of a graph is *simplicial* if its neighborhood induces a complete graph. The set of simplicial vertices of G is denoted by $S(G)$ and we set $s(G) = |S(G)|$. A subgraph H of G is *convex*, if for every two vertices u and v of H, every shortest u,v-path in G is contained in H.

Let $\tau \in \{\mu, \mu_{\rm t}, \mu_{\rm o}, \mu_{\rm d}, {\rm gp}, {\rm gp}_{\rm t}, {\rm gp}_{\rm o}, {\rm gp}_{\rm d}\}$. By a τ-set of G we mean a set with the property τ of cardinality $\tau(G)$, and by $\#\tau(G)$ we denote the number of τ-sets of G. The following fact is often very useful, parts of it are already known from the literature.

Lemma 2.1 *If G is a connected graph and $\tau \in \{\mu, \mu_{\rm t}, \mu_{\rm o}, \mu_{\rm d}, {\rm gp}, {\rm gp}_{\rm o}, {\rm gp}_{\rm d}\}$, then $\tau(G) \geq s(G)$.*

Proof. Since any two vertices of G are $S(G)$-positionable, ${\rm gp}_{\rm t}(G) \geq s(G)$. The assertion follows because ${\rm gp}_{\rm t}(G) \leq \tau(G)$ for $\tau \in \{\mu, \mu_{\rm t}, \mu_{\rm o}, \mu_{\rm d}, {\rm gp}, {\rm gp}_{\rm t}, {\rm gp}_{\rm o}, {\rm gp}_{\rm d}\}$. □

We next collect several known results that will be needed later.

Lemma 2.2 [9, Lemma 2.1] *If H is a convex subgraph of G, and X a mutual-visibility set of G, then $X \cap V(H)$ is a mutual-visibility set of H.*

Theorem 2.3 [3, Theorem 5.2] *If G is a connected graph and $X \subseteq V(G)$, then X is a total mutual-visibility set of G if and only if any two vertices u and v of G with $d_G(u, v) = 2$ are X-visible.*

Theorem 2.4 [30, Theorems 2.1, 3.1] *If G is a connected graph and $X \subseteq V(G)$, then the following hold.*

(i) X is a total general position set of G if and only if $X \subseteq S(G)$. Moreover, $\mathrm{gp_t}(G) = s(G)$.

(ii) If X is a general position set of G, then X is a dual general position set if and only if $G - X$ is convex.

In the rest of the preliminaries we introduced Sierpiński graphs S_p^n and related notation required. These graphs were introduced in [16] as graphs of a particular variant of the well-known Tower of Hanoi problem [12].

If $p \geq 3$ and $n \geq 1$, then S_p^n is defined as follows. The vertex set is $V(S_p^n) = [p]_0^n$, we will simplify the notation of a vertex $(i_1, \ldots, i_n)$ of S_p^n to $i_1 \cdots i_n$. Vertices $i_1 \cdots i_n$ and $j_1 \cdots j_n$ being adjacent if there exists an index $h \in [n]$, such that

(i) $\forall\, t,\, t < h \implies i_t = j_t$,

(ii) $i_h \neq j_h$,

(iii) $\forall\, t,\, t > h \implies i_t = j_h$ and $j_t = i_h$.

In the case $p = 3$, these graphs are isomorphic to the graphs of the classical Tower of Hanoi problem.

If $s \in [p]_0^{n-k}$, where $k \in [n-1]$, then the subgraph of S_p^n induced by the vertices of the form $\{st:\ t \in [p]_0^k\}$, is isomorphic to S_p^k, it will be denoted by $\underline{s}S_p^k$. If $i \in [p]_0$, then the notation $\underline{i}S_p^1$ will be simplified to iS_p^1. Note that iS_p^1 is isomorphic to K_p. The subgraphs $\underline{s}S_p^k$ indicate the fractal nature of Sierpiński graphs by which we mean that $V(S_p^n)$ can be partitioned into p^{n-k} sets each of which induces a subgraph isomorphic to S_p^k.

Now, consider $p = 3$. Let k be a positive integer, where $1 \leq k \leq n-2$, and let $s \in [3]_0^k$. In the subgraph $\underline{s}S_3^{n-k}$ of S_3^n, each of the three vertices si^{n-k}, $i \in [3]_0$, is the degree 2 vertex of an induced bull, which we denote by $\underline{s}B_i^n$. (Recall that the *bull graph* is a graph of order five obtained from a triangle by attaching two pendant vertices to its two different vertices.) Note that some of these bulls can be isomorphic. In particular, for a fixed $i \in [3]_0$, the bulls $\underline{i^k}B_i^n$, $1 \leq k \leq n-2$, are one and the same bull with the degree two vertex being the vertex i^n. See Fig. 1 where S_3^4 is presented and some of its bulls emphasized. We can infer that

$$V(\underline{s}B_i^n) = \{si^{n-k-2}ji, si^{n-k-1}j:\ j \in [3]_0\}\,.$$

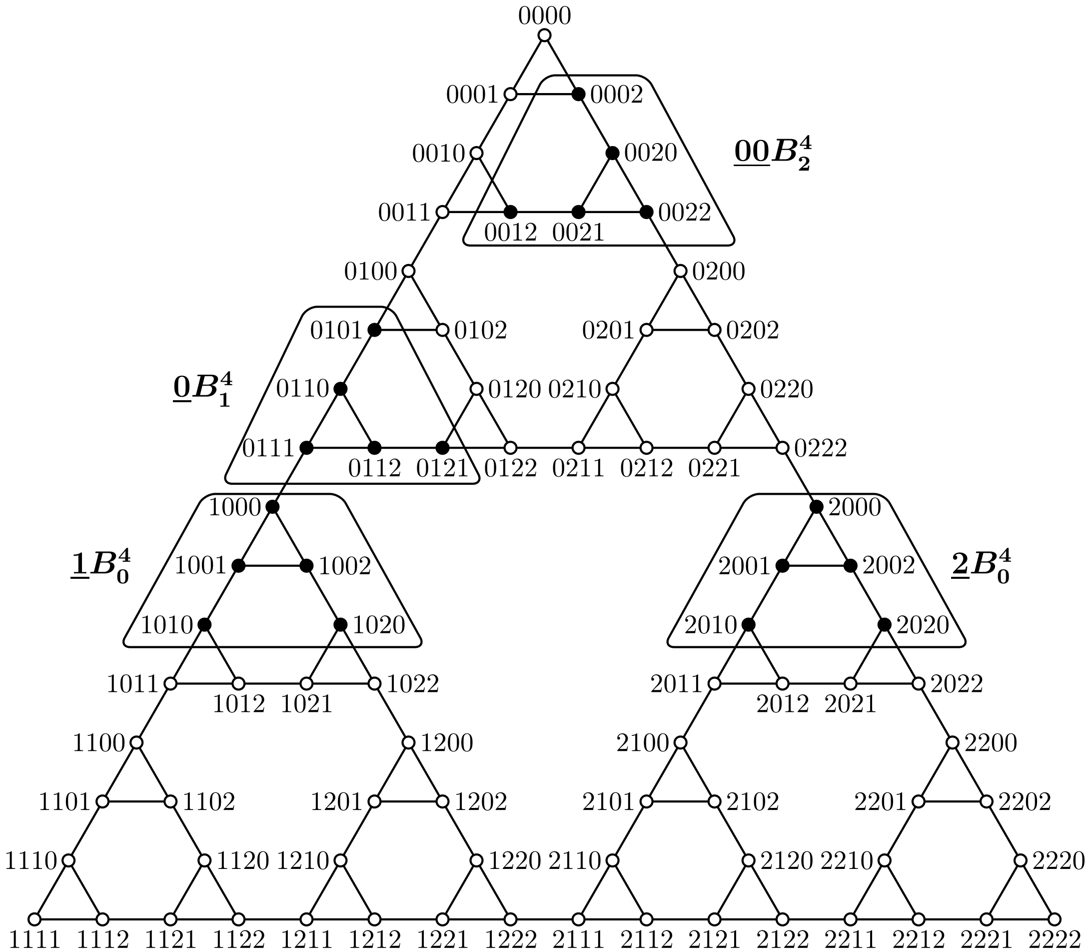

Figure 1: S_3^4 and some of its bulls

3 Sierpiński graphs S_p^2

In the seminal paper on Sierpiński graphs [16] it was proved that there are at most two shortest paths between any two vertices of S_p^n. It was also described when one of the two cases happens. In particular, in S_p^2 there exist two shortest paths between any pair of vertices of the form ik and jk, these are the paths ik, ij, ji, jk and ik, ki, kj, jk. For all the remaining pair of vertices there exists a unique shortest path between them.

Theorem 3.1 *If $p \geq 3$ then,*

$$\mu(S_p^2) = \begin{cases} \frac{(p+1)^2}{4}; & p \text{ odd}, \\ \frac{p(p+2)}{4}; & p \text{ even}, \end{cases} \quad \text{and} \quad \#\mu(S_p^2) = \begin{cases} \binom{p}{\frac{p+1}{2}}; & p \text{ odd}, \\ \binom{p+1}{\frac{p+2}{2}}; & p \text{ even}. \end{cases}$$

Proof. If $p \geq 3$ is odd, then let

$$X_1 = \Big\{ii, ij : \ i \in [(p+1)/2]_0, j \in [p]_0 \setminus [(p+1)/2]_0\Big\},$$

and if $p \geq 4$ is even, then let

$$X_2 = \Big\{ii, ij : \ i \in [p/2]_0, j \in [p]_0 \setminus [p/2]_0\Big\}.$$

It is straightforward to check that X_1 is a mutual-visibility set of S_p^2 if p is odd, whilst X_2 is a mutual-visibility set of S_p^2 if p is even. Since $|X_1| = (p+1)^2/4$ and $|X_2| = p(p+2)/4$, we have thus shown that

$$\mu(S_p^2) \geq \begin{cases} \frac{(p+1)^2}{4}; & p \text{ odd}, \\ \frac{p(p+2)}{4}; & p \text{ even}. \end{cases}$$

To prove that this lower bound is also an upper bound, consider an arbitrary μ-set X of S_p^2. We may without loss of generality assume that

$$|X \cap V(0S_p^1)| = \max\{|X \cap V(iS_p^1)| : \ i \in [p]_0\},$$

and let $|X \cap V(0S_p^1)| = k$. Since we have assumed that X is a μ-set of S_p^2, the already proved lower bound implies that $k \geq 2$. We consider the following two cases.

Case 1: $00 \in X \cap V(0S_p^1)$.
Let $0j \in X \cap V(0S_p^1)$, where $j \in [p-1]$. Since $00, 0j, j0, ji$, where $i \in [p]_0$, is the unique shortest path between 00 and ji, it follows that $X \cap V(jS_p^1) = \emptyset$. As $X \cap V(0S_p^1)$ contains $k-1$ vertices different from 00, this in turn implies that $k-1$ subgraphs of the form iS_p^1 contain no vertex from X. By the definition of k we get that $|X| \leq k \cdot (p-k+1)$.

Case 2: $00 \notin X \cap V(0S_p^1)$.
Let $0j, 0j' \in X \cap V(0S_p^1)$, where $j \neq j'$ and $j, j' \in [p-1]$. We claim that either $X \cap V(jS_p^1) = \emptyset$ or $X \cap V(j'S_p^1) = \emptyset$. Suppose not. Since $0j', 0j, j0, j\ell$ is the unique shortest path between $0j'$ and $j\ell$, where $\ell \in [p]_0 \setminus \{j'\}$, it follows that $X \cap V(jS_p^1) = \{jj'\}$. Analogously, since $0j, 0j', j'0, j'\ell'$ is the unique shortest between $0j$ and $j'\ell'$,

where $\ell' \in [p]_0 \setminus \{j\}$, we get $X \cap V(j'S_p^1) = \{j'j\}$. Hence $\{0j, 0j', jj', j'j\} \subseteq X$. But the vertices $0j$, $0j'$, $j0$, jj', $j'j$, and $j'0$ induce a cycle C_6, it contradicts with the fact that $\mu(C_6) = 3$. Since $X \cap V(0S_p^1)$ contains $k-1$ vertices different from $0j$, it implies that $k-1$ subgraphs of the form iS_p^1 contain no vertex from X. By the definition of k we thus have $|X| \le k \cdot (p-k+1)$.

From the above, we have

$$|X| \le k \cdot (p-k+1) \le \max\{k \cdot (p-k+1) :\; k \in [p]\}\,.$$

Note that

$$\max\{k \cdot (p-k+1) :\; k \in [p]\} = \begin{cases} \frac{(p+1)^2}{4}; & p \text{ odd}\,, \\ \frac{p(p+2)}{4}; & p \text{ even}\,. \end{cases}$$

As a consequence, we conclude that

$$\mu(S_p^2) = \begin{cases} \frac{(p+1)^2}{4}; & p \text{ odd}\,, \\ \frac{p(p+2)}{4}; & p \text{ even}\,. \end{cases}$$

In remains to determine the number of μ-sets X. Assume first that p is odd. In this case $|X| = \frac{(p+1)^2}{4}$, which is if and only if $k = \frac{p+1}{2}$. That is, X intersects exactly $\frac{p+1}{2}$ subgraphs iS_p^1 in exactly $\frac{p+1}{2}$ vertices each. The selection of these subgraphs can be made in $\binom{p}{\frac{p+1}{2}}$ ways. We now claim that as soon as such a selection is made, X is uniquely determined. To prove it, assume without loss of generality that X has vertices in iS_p^2 for $i \in [(p+1)/2]_0$. Hence, if $j, k \in [(p+1)/2]_0$, then $jk \notin X$. The remaining vertices in each of iS_p^2 for $i \in [(p+1)/2]_0$ must thus lie in X, that is, X is uniquely determined. This proves that $\#\mu(S_p^2) = \binom{p}{\frac{p+1}{2}}$ when p is odd.

The argument for p is even is similar, except that now a μ-set either intersects $\frac{p}{2}$ copies iS_p^1 in exactly $\frac{p+2}{2}$ vertices each, or intersects $\frac{p+2}{2}$ copies of iS_p^1 in exactly $\frac{p}{2}$ vertices each. In each of these cases we then proceed as above to see that a μ-set is unique as soon as we select the subgraphs iS_p^1 which contain vertices from the μ-set. Therefore if p is even,

$$\#\mu(S_p^2) = \binom{p}{\frac{p}{2}} + \binom{p}{\frac{p+2}{2}} = \binom{p+1}{\frac{p+2}{2}}\,,$$

and we are done. □

Theorem 3.1 is illustrated in Fig. 2 for $p \in \{3,4\}$. For S_3^2 all three μ-sets are shown, while for S_4^2 the left figure shows one of six μ-sets that interest two subgraphs iS_4^1, and the right figure shows one of four μ-sets that interest three subgraphs iS_4^1.

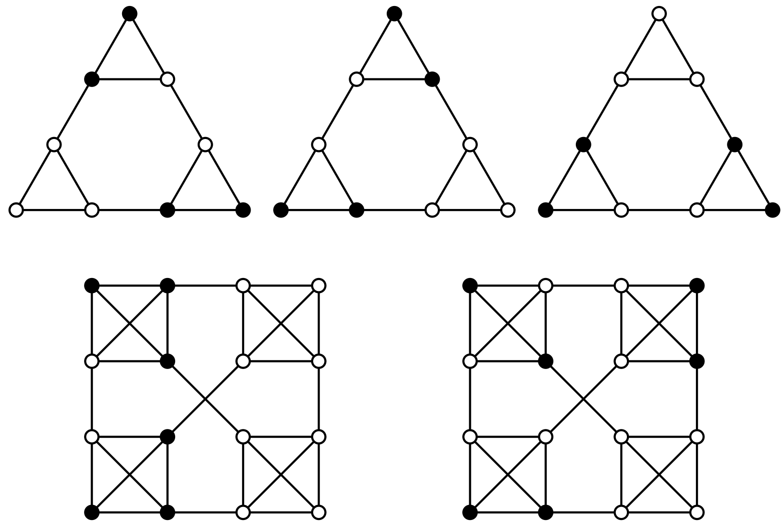

Figure 2: μ-sets in S_3^2 and in S_4^2

Corollary 3.2 *If $p \geq 3$ then,*

$$\mathrm{gp}(S_p^2) = \begin{cases} \frac{(p+1)^2}{4}; & p \text{ odd}, \\ \frac{p(p+2)}{4}; & p \text{ even}, \end{cases} \quad \text{and} \quad \#\mathrm{gp}(S_p^2) = \begin{cases} \binom{p}{\frac{p+1}{2}}; & p \text{ odd}, \\ \binom{p+1}{\frac{p+2}{2}}; & p \text{ even}. \end{cases}$$

Proof. The μ-sets X_1 and X_2 mentioned in the proof of Theorem 3.1 are general position sets of S_p^2. Since $\mathrm{gp}(S_p^2) \leq \mu(S_p^2)$, the result follows. □

Theorem 3.3 *If $p \geq 3$, then $\mu_{\mathrm{d}}(S_p^2) = p$ and $\#\mu_{\mathrm{d}}(S_p^2) = p + 1$.*

Proof. By Lemma 2.1, $\mu_{\mathrm{d}}(S_p^2) \geq p$. To prove the upper bound, we consider an arbitrary μ_{d}-set X of S_p^2. Let $X_i = X \cap V(iS_p^1)$, and let $x_i = |X_i|$ for $i \in [p]_0$. We distinguish three cases.

If for each $i \in [p]_0$ we have $k_i \leq 1$, then $|X| \leq p$.

Assume second that there exists an index $i \in [p]_0$ such that $k_i = p$, then $k_{i'} = 0$, where $i' \in [p]_0 \setminus \{i\}$. Indeed, for otherwise $00, 0i', i'0, i'j$ is the unique shortest path between 00 and $i'j$, where $j \in [p]_0$, but this implies that the vertices 00 and $i'j$ are not X-visible as $0i' \in X$. Then $|X| \leq p$.

In the remaining case we may assume without loss of generality that $2 \leq x_0 \leq p-1$ and that $x_0 = \max\{x_i :\ i \in [p]_0\}$. Then there exist vertices $0i \in X$ and $0j \notin X$. In iS_p^1 there exists a vertex $ik \notin X$. Since $0j, 0i, i0, ik$ is the unique shortest path, the vertices $0j$ and jk are not X-visible, hence this last case is not possible. In consequence, we have $|X| \leq p$, and we can conclude that $\mu_{\rm d}(S_p^2) = p$.

From the above, the only possibilities that X is a $\mu_{\rm d}$-set is that X contains all vertices of some iS_p^1, or that X has exactly one vertex from each of them. In the first case, we find $\mu_{\rm d}$-sets $V(iS_p^1)$, $i \in [p]_0$, while in the second case the only $\mu_{\rm d}$-set is $\{ii : i \in [p]_0\}$. Hence $\#\mu_{\rm d}(S_p^2) = p + 1$. □

Theorem 3.3 is illustrated in Fig. 3 on the case of S_4^2. The left figure shows one of four $\mu_{\rm d}$-sets which respectively contain sets $V(iS_4^1)$, the right figure shows the unique μ-set which interests each subgraph iS_4^1.

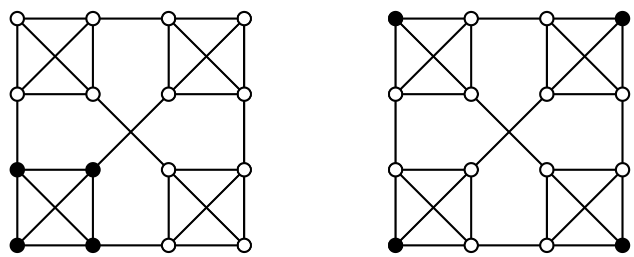

Figure 3: $\mu_{\rm d}$-sets in S_4^2

Corollary 3.4 *If $p \geq 3$, then $\mathrm{gp}_{\rm d}(S_p^2) = p$ and $\#\mathrm{gp}_{\rm d}(S_p^2) = 1$.*

Proof. It is straightforward to check that the set $\{ii : i \in [p]_0\}$ is a dual general position set of S_p^2, hence $\mathrm{gp}_{\rm d}(S_p^2) \geq p$. By the definitions of mutual-visibility and general position we have $\mathrm{gp}_{\rm d}(G) \leq \mu_{\rm d}(G)$, in view of Theorem 3.3, hence $\mathrm{gp}_{\rm d}(S_p^2) = p$. Moreover, the set $\{ii : i \in [p]_0\}$ is the only largest dual general position set of S_p^2, hence we are done. □

Theorem 3.5 *If $p \geq 3$ and $\tau \in \{\mu_{\rm t}, \mu_{\rm o}, \mathrm{gp}_{\rm t}, \mathrm{gp}_{\rm o}\}$, then $\tau(S_p^2) = p$ and $\#\tau(S_p^2) = 1$.*

Proof. By Lemma 2.1 we have $\tau(S_p^2) \geq p$ for any $\tau \in \{\mu_{\rm t}, \mu_{\rm o}, \mathrm{gp}_{\rm t}, \mathrm{gp}_{\rm o}\}$.

We first consider the total mutual-visibility. Since $\mu_{\rm t}(S_p^2) \leq \mu_{\rm d}(S_p^2)$, Theorem 3.3 implies $\mu_{\rm t}(S_p^2) = p$. Let X be an arbitrary $\mu_{\rm t}$-set of S_p^2. We will show that $X = \{ii : i \in [p]_0\}$. If $i \neq j$, then considering the vertices ii, ij, and ji, Theorem 2.3 implies that $ij \notin X$ and $ji \notin X$. It follows that $X \subseteq \{ii : i \in [p]_0\}$. Moreover, the set $\{ii : i \in [p]_0\}$ is a total mutual-visibility set, hence this set is the unique $\mu_{\rm t}$-set of S_p^2.

Consider next the outer mutual-visibility. To prove that $\mu_{\rm o}(S_p^2) \le p$, let Y be an arbitrary $\mu_{\rm o}$-set of S_p^2 and let $Y_i = Y \cap V(iS_p^1)$ for $i \in [p]_0$. If $|Y_i| \le 1$ for each $i \in [p]_0$, then there is nothing to prove. In the rest we may hence assume that $|Y_0| \ge 2$ and that $|Y_0| = \max\{|Y_i| : \ i \in [p]_0\}$. Then there exists a vertex $0j \in Y_0$, where $j \in [p-1]$. Since $0i, 0j, j0$ is the unique shortest path between $0i$ and $j0$, we get that $0i \notin Y$ and $j0 \notin Y$. But this implies that $0j$ is the unique vertex in Y_0, which contradicts our assumption that $|Y_0| \ge 2$. Hence $|Y| \le p$, and we have $\mu_{\rm o}(S_p^2) = p$. This argument also implies that Y is a $\mu_{\rm o}$-set if and only if $|Y_i| = 1$ and $Y \cap V(iS_p^1) = \{ii\}$ for $i \in [p]_0$.

Similar as to the above arguments, the set $\{ii : 1 \le i \le p\}$ is the only ${\rm gp}_{\rm t}$-set as well as the only ${\rm gp}_{\rm o}$-set of S_p^2. Hence ${\rm gp}_{\rm t}(S_p^2) = {\rm gp}_{\rm o}(S_p^2) = p$ and $\#{\rm gp}_{\rm t}(S_p^2) = \#{\rm gp}_{\rm o}(S_p^2) = 1$. □

4 Sierpiński graphs S_3^n

In this section, we consider varities of mutual-visibility problems and general position problems on the Sierpiński graphs S_3^n.

Theorem 4.1 *If $n \ge 1$, then $\mu_{\rm t}(S_3^n) = \mu_{\rm d}(S_3^n) = 3$. Moreover, $\#\mu_{\rm t}(S_3^n) = 1$ and $\#\mu_{\rm d}(S_3^n) = 4$.*

Proof. Clearly, $\mu_{\rm t}(S_3^1) = \mu_{\rm d}(S_3^1) = 3$ and by Theorems 3.5 and 3.3 also $\mu_{\rm t}(S_3^2) = \mu_{\rm d}(S_3^2) = 3$. Hence in the remaining proof we may assume that $n \ge 3$.

By Lemma 2.1 we have $\mu_{\rm t}(S_3^n) \ge 3$ so that $3 \le \mu_{\rm t}(S_3^n) \le \mu_{\rm d}(S_3^n)$. To prove that $\mu_{\rm t}(S_3^n) = \mu_{\rm d}(S_3^n) = 3$ it thus suffices to show that $\mu_{\rm d}(S_3^n) \le 3$. Let X be an arbitrary $\mu_{\rm d}$-set of S_3^n. We claim that

$$X \subseteq W = V(0^{n-1}S_3^1) \cup V(1^{n-1}S_3^1) \cup V(2^{n-1}S_3^1)\,.$$

Suppose on the contrary that there exists a vertex $x \in X \setminus W$. Then the degree of x is 3, let x_1, x_2, x_3 be the neighbors of x, where $x_2x_3 \in E(S_3^n)$. Since X is a dual mutual-visibility set, either $x_1 \in X$ and $x_2, x_3 \notin X$, or $x_1 \notin X$ and $x_2, x_3 \in X$. In the first case consider a convex P_4 in which the edge xx_1 is the middle edge to get a contradiction that X is a dual mutual-visibility set. In the second case we proceed similarly, expect that now the middle edge of a considered convex P_4 is xx_2. This contradiction proves the claim.

Assume now that $i^{n-1}j \in X$, where $i, j \in [3]_0$, $i \ne j$. Then as above, considering the neighbors of $i^{n-1}j$ we infer that then $X = \{i^n, i^{n-1}j, i^{n-1}k\}$, where $\{i, j, k\} = [3]_0$. In this way we get the following $\mu_{\rm d}$-sets: $\{0^n, 0^{n-1}1, 0^{n-1}2\}$, $\{1^n, 1^{n-1}0, 1^{n-1}2\}$, and

$\{2^n, 2^{n-1}0, 2^{n-1}1\}$. So, if some vertex of the form $i^{n-1}j$ lies in X, then X is one of these three sets. The last possibility for X is then $\{0^n, 1^n, 2^n\}$ which is also a dual mutual-visibility set. We have thus proved that $\mu_{\rm d}(S_3^n) \le 3$ and that $\#\mu_{\rm d}(S_3^n) = 4$.

Finally, note that any vertex of S_p^n of degree p is the middle vertex of a convex P_3. By [28, Lemma 5] we get that such a vertex lies in no total mutual-visibility set. We can conclude that $\{0^n, 1^n, 2^n\}$ is the unique total mutual-visibility set. □

Corollary 4.2 *If $n \ge 1$, then ${\rm gp}_{\rm t}(S_3^n) = {\rm gp}_{\rm d}(S_3^n) = 3$. Moreover, $\#{\rm gp}_{\rm t}(S_3^n) = 1$ and $\#{\rm gp}_{\rm d}(S_3^n) = 1$.*

Proof. Using Theorem 2.4(i) and Theorem 4.1 we have

$$3 = {\rm gp}_{\rm t}(S_3^n) \le {\rm gp}_{\rm d}(S_3^n) \le \mu_{\rm d}(S_3^n) = 3\,.$$

This in turn also implies that the maximum sets from Theorem 4.1 remain also for the total/dual general position case. □

Next we focus on the mutual-visibility set. We first settle small cases which will serve as our basis for the later induction.

Proposition 4.3 *The following holds.*

(i) $\mu(S_3^2) = 4$. *Moreover, the sets $\{ii, ij, kj, kk\}$, where $\{i, j, k\} = [3]_0$, are the unique μ-sets of S_3^2.*

(ii) $\mu(S_3^3) = 6$. *Moreover, if X is a μ-set of S_3^3, then either $|X \cap V(iS_3^3)| = 2$ for every $i \in [3]_0$, or $|X \cap V(iS_3^3)| = 3$ for exactly two $i \in [3]_0$.*

(iii) $\mu(S_3^4) = 12$. *Moreover, if X is a μ-set of S_3^4, then $|X \cap V(iS_3^3)| = 4$ for $i \in [3]_0$. In addition, if X is a mutual-visibility set of S_3^4, then $|X \cap (V(iS_3^3) \cup V(jS_3^3))| \le 10$ for $i, j \in [3]_0$.*

Proof. (i) This follows from Theorem 3.1 and its proof, see also the top part of Fig. 2.

(ii) The set $X = \{iji : i \ne j \text{ and } i, j \in [3]_0\} \subseteq V(S_3^3)$ is a mutual-visibility set of S_3^3, hence $\mu(S_3^3) \ge 6$.

To prove that $\mu(S_3^3) \le 6$, consider an arbitrary μ-set T of S_3^3. Since iS_3^2, $i \in [3]_0$, is a convex subgraph of S_3^3, we have $|T \cap V(iS_3^2)| \le 4$ by (i). There is nothing to prove that $|T| \le 6$ if $|T \cap V(iS_3^2)| \le 2$ for $i \in [3]_0$. In this case we also have that $|T \cap V(iS_3^3)| = 2$, for $i \in [3]_0$. In the rest we may hence without loss of generality assume that $|T \cap V(0S_3^2)| \ge 3$.

Assume first that $|T \cap V(0S_3^2)| = 4$. Then these vertices must be (up to symmetry) $000, 011, 021, 020$ or $011, 010, 020, 022$. In both cases we infer that no vertex from $V(1S_3^2) \cup V(2S_3^2)$ lies in T, hence $|T| = 4$.

We are left with the case when $|T \cap V(0S_3^2)| = 3$. Setting $Y_1 = \{000, 001, 010, 011\}$ we see that $|T \cap Y_1| \leq 2$. Moreover, if $|T \cap Y_1| = 2$, then $T \cap V(1S_3^2) = \emptyset$. As in the case where $|T \cap V(2S_3^2)| \geq 4$ was previously ruled out, we can conclude that $|T| \leq 6$. The same conclusion can be derived by considering the set $Y_2 = \{000, 002, 020, 022\}$. Hence assume in the rest that $|T \cap Y_1| \leq 1$ and $|T \cap Y_2| \leq 1$ in the rest of this proof and consider the following subcases.

Consider the case where $|T \cap Y_1| = 1$ and $|T \cap Y_2| = 1$. Assume first that $000 \in T \cap Y_1$. Then since $|T \cap V(0S_3^2)| = 3$, we have $T \cap V(0S_3^3) = \{000, 012, 021\}$. Since $021, 012, 1ij$ is the unique shortest path between 021 and each vertex $1ij \in V(1S_3^2) \setminus \{122\}$, we have $|T \cap V(1S_3^2)| \leq 1$. Analogously, $012, 021, 2ij$ is the unique shortest path between 012 and each vertex $2ij \in V(1S_3^2) \setminus \{211\}$, we have $|T \cap V(2S_3^2)| \leq 1$. Hence $|T| \leq 5$ in this case.

Assume second that $000 \notin T \cap Y_1$. Since we have assumed that $|T \cap V(0S_3^3)| = 3$, we get $|\{001, 010, 011\} \cap T| = 1$, $|\{002, 020, 022\} \cap T| = 1$, and $|\{012, 021\} \cap T| = 1$. Assume first that $011 \in T$. Since $|\{012, 021\} \cap T| = 1$, we infer that T can have at most one vertex in $1S_3^2$, and if so, this vertex is 122. Now, if $122 \notin T$, then T has vertices only in $0S_3^2$ and in $2S_3^2$, hence by our case assumption $|T| \leq 6$. And if $122 \in T$, then $T \cap V(2S_3^2) = \emptyset$, and we have $|T| \leq 4$. Analogously we get the conclusion if $022 \in T$. Hence we are left with the cases when $|\{001, 010\} \cap T| = 1$, $|\{002, 020\} \cap T| = 1$, and $|\{012, 021\} \cap T| = 1$. Then a case analysis similar to the above leads to the required conclusion. Since we have assumed $|T \cap V(0S_3^3)| = 3$, we are now left with the case where exactly one among $T \cap Y_1$ and $T \cap Y_2$ is of cardinality 1 and the other is empty. As in the proof of the previous cases, none other than 122 and 211 can be in T from $V(1S_3^2) \cup V(2S_3^2)$. Hence, $|T| \leq 5$ in this case. We can conclude that in each case $|T| \leq 6$, and therefore $\mu(S_3^3) = 6$. In each of the cases, we also see that if $|T \cap V(0S_3^2)| = 3$, then $|T \cap V(iS_3^2)| = 3$ and $|T \cap V(jS_3^2)| = 0$, where $\{i, j\} = \{1, 2\}$ (or otherwise $|T| < 6$).

(iii) (To help the reader follow this part of the proof, we invite the reader to use Fig. 1.) Consider S_3^4 and let $X = \{iiii, i012, i120, i201 : i \in [3]_0\}$. Since X is a mutual-visibility set of cardinality 12, we get $\mu(S_3^4) \geq 12$.

To prove the reverse inequality, let T be a mutual-visibility set of S_3^4. Since each iS_3^3 is a convex subgraph of S_3^4, combining Lemma 2.2 with (i) we have $|T \cap V(iS_3^3)| \leq 6$. We can also observe that $V(iS_3^3) \cup V(jiS_3^2) \cup V(jjiS_3^1) \cup \{jjji, jjjj\}$ induces a convex subgraph of S_3^4 for $i, j \in [3]_0$.

We claim that $|T \cap (V(iS_3^3) \cup V(jS_3^3))| \leq 10$ for $i, j \in [3]_0$. Let $|T \cap V(2S_3^3)| = \max\{|T \cap V(iS_3^3)| : i \in [3]_0\}$. In view of (ii), we have $|T \cap V(2S_3^3)| \leq 6$. If

$|T \cap V(2S_3^3)| \leq 5$, there is nothing to prove and the inequality holds. If $|T \cap V(2S_3^3)| = 6$, we will show that $|T \cap V(iS_3^3)| \leq 4$ for each $i \in [2]_0$. Suppose not and assume, without loss of generality, that $|T \cap V(1S_3^3)| \geq 5$. Since $|T \cap V(2S_3^3)| = 6$, by (ii) we have $|T \cap V(2iS_3^2)| \leq 3$ for each $i \in [3]_0$. It follows that $|T \cap (V(20S_3^2) \cup V(220S_3^1) \cup \{2220\})| \leq 5$ and $|T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})| \geq 1$.

Next, we show that $|T \cap (V(12S_3^2) \cup V(112S_3^1) \cup \{1112, 1111\})| \geq 1$. It is straightforward to check that $|T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})| \leq 2$ if $|T \cap V(10S_3^2)| \neq 0$. By (i) we known that $|T \cap V(1iS_3^2)| \leq 4$ for each $i \in [3]_0$. But if $|T \cap V(1iS_3^2)| = 4$, then $|T \cap V(1jS_3^2)| = 0$ for $j \in [3]_0 \setminus \{i\}$. Hence $|T \cap V(1S_3^3)| = 4$ contradicts what we have assumed $|T \cap V(1S_3^3)| \geq 5$, so $|T \cap V(1iS_3^2)| \leq 3$ for each $i \in [3]_0$. Assume first that $|T \cap V(10S_3^2)| \leq 2$. Since $|T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})| \leq 2$, we have $|T \cap (V(10S_3^2) \cup V(110S_3^1) \cup \{1110\})| \leq 4$, then our assumption implies that $|T \cap (V(12S_3^2) \cup V(112S_3^1) \cup \{1112, 1111\})| \geq 1$.

Assume second that $|T \cap V(10S_3^2)| = 3$. If $|T \cap (V(110S_3^1) \cup \{1110\})| \leq 1$, then $|T \cap (V(12S_3^2) \cup V(112S_3^1) \cup \{1112, 1111\})| \geq 1$ as the assumption $|T \cap V(1S_3^3)| \geq 5$. If $|T \cap (V(110S_3^1) \cup \{1110\})| = 2$, since the vertex 1100 lies on the unique shortest path between a vertex of $T \cap \{1101, 1102, 1110\}$ and each vertex of $T \cap V(10S_3^2)$, we see that $1100 \notin T$. Moreover, the vertices 1110, 1101, and 1102 lie on a convex P_3 in S_3^4, then $1110 \in T$ or $1101 \in T$. Furthermore, if $\{1101, 1102\} \subseteq T \cap V(1S_3^3)$ is the case, since $|T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})| \geq 1$, the vertex 1102 lies on the unique shortest path between 1101 and a vertex of $T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})$. This is a contradiction. If $\{1110, 1102\} \subseteq T$, it is easy to verify that only $\{1000, 1002, 1012\} \subseteq T \cap V(10S_3^2)$ is the case. But the vertex 1102 lies on the unique shortest path between 1000 and a vertex of $T \cap (V(21S_3^2) \cup V(221S_3^1) \cup \{2221, 2222\})$, a contradiction. Therefore, $|T \cap (V(10S_3^2) \cup V(110S_3^1) \cup \{1110\})| \leq 4$, we obtain $|T \cap (V(12S_3^2) \cup V(112S_3^1) \cup \{1112, 1111\})| \geq 1$.

Let $x \in T \cap (V(12S_3^2) \cup V(112S_3^1) \cup \{1112, 1111\})$. Since each vertex in $T \cap V(2S_3^3)$ is T-visible with x, we obtain $2111 \notin T \cap V(2S_3^3)$ and each vertex in $T \cap V(2S_3^3)$ is $T \cap V(2S_3^3)$-visible with 2111. This implies that $(T \cap V(2S_3^3)) \cup \{2111\}$ is a mutual-visibility set of S_3^3 of cardinality seven, which is a contradiction. As a consequence, we conclude that $|T \cap (V(iS_3^3) \cup V(jS_3^3))| \leq 10$ for $i, j \in [3]_0$.

Now assume that T is an arbitrary μ-set of S_3^4. We show that $|T \cap V(iS_3^3)| \leq 4$ for each $i \in [3]_0$. Suppose not and we may without loss of generality let $|T \cap V(0S_3^3)| \geq 5$. Since we have proved that $|T| \geq 12$ and $|T \cap (V(iS_3^3) \cup V(jS_3^3))| \leq 10$ for each $i, j \in [3]_0$, it follows that $|T \cap V(iS_3^3)| \geq 2$ for each $i \in [2]$. In fact, there are only two cases. Either $|T \cap V(iS_3^3)| \geq 4$ for some $i \in [2]$ or $|T \cap V(iS_3^3)| \geq 3$ for each $i \in [2]$. For each $i \in [2]$, since $|T \cap V(iS_3^3)| \geq 2$, we get

$$|T \cap \{0000, 000i, 00i0, 00ii, 0i00, 0i0i, 0ii0, 0iii\}| \leq 1\,.$$

Let $T' = T \cap \{0111, 0112, 0121, 0122, 0211, 0212, 0221, 0222\}$. It is also observed that $|T'| \leq 2$. Assume that $|T'| = 2$, let x_i be the vertex in T' closer to iS_3^3 for each $i \in [2]$. Since $|T \cap V(iS_3^3)| \geq 2$, it is straightforward to check that $x_i \in \{0iii, 0iij\}$, where $i \in [2]$. If for some $i \in [2]$ we have $|T \cap V(iS_3^3)| \geq 4$, there is no choice for x_j, where $x_j \in T'$ and $j \in [2] \setminus \{i\}$, which is a contradiction. If $|T \cap V(iS_3^3)| \geq 3$ for each $i \in [2]$, then $x_i = 0iii$, which is again not possible. These two contradictions imply that $|T'| \leq 1$.

Recall the definition of bull graph $\underline{s}B_i^n$, where $i \in [3]_0$. Since assumption $|T \cap V(0S_3^3)| \geq 5$, we have $|T \cap V(0B_i^4)| \leq 2$ for $i \in [3]_0$. If $|T \cap V(0B_i^4)| = 2$ for some $i \in [2]$, then all the remaining vertices in $T \cap V(0S_3^3)$ must be from $V(0jjS_3^1) \cup \{0j0j\}$ as $|T \cap V(iS_3^3)| \geq 2$. But $|T \cap (V(0jjS_3^1) \cup \{0j0j\})| \leq 2$, which is a contradiction to $|T \cap V(0S_3^3)| \geq 5$. (Notice that $\{0010, 0020\} \subseteq T \cap V(0B_0^4)$.) This contradiction implies that $|T \cap V(0B_i^4)| \leq 1$ for each $i \in [2]$.

Now, let C be the set of twelve vertices in $0S_3^3$ whose induced subgraph is a cycle C_{12}. Then $2 \leq |T \cap C| \leq 3$.

Consider first the case case $|T \cap C| = 2$. Since we have proved that $|T \cap \{0000, 000i, 00i0, 00ii, 0i00, 0i0i, 0ii0, 0iii\}| \leq 1$ and $|T'| \leq 1$, where $i \in [2]$, the assumption $|T \cap V(0S_3^3)| \geq 5$ implies that T does not intersect $\{00ii, 0i00, 0ijj : i, j \in [2]\}$. It follows that T intersects each of the three sets $\{0iii, 0iij, 0iji : i, j \in [2]\}$ and $\{0000, 000i, 00i0, 0i0i, 0ii0, 0iii\}$ for $\{i, j\} = [2]$. There are then two subcases. If T intersects $\{0000, 000i, 00i0\}$, $\{0iii, 0iij, 0iji\}$, and $\{0jjj, 0jj0, 0j0j\}$ for some $i \in [2]$, we see that $T \cap (V(0S_3^3) \setminus C) \subseteq (V(0iiS_3^1) \cup \{0i0i\})$ since $|T \cap V(jS_3^3)| \geq 2$, where $j \in [3]_0$, which is a contradiction. In the other subcase, $0010, 0020 \in T$, and T intersects $\{0iii, 0iij, 0iji\}$ for some $i \in [2]$. Then T does not intersect $\{0012, 0021\}$. Hence T intersects $\{0102, 0120\}$ and $\{0201, 0210\}$. It follows that $T \cap (V(0S_3^3) \setminus C) \subseteq V(0jjS_3^1) \cup \{0j0j\}$ since $|T \cap V(iS_3^3)| \geq 2$, which is again a contradiction.

Consider next the case $|T \cap C| = 3$. Since $|T \cap V(0S_3^3)| \geq 5$, the set T intersects at least two of the sets $\{0iii, 0iij, 0iji : i, j \in [2]\}$ and $\{0000, 000i, 00i0, 0i0i, 0ii0, 0iii\}$ for $i \in [2]$. Thus T intersects each of the three sets $\{0011, 0012, 0021, 0022\}$ and $\{0i00, 0i0j, 0ij0, 0ijj\}$ for $i, j \in [2]$. Since $|T \cap V(iS_3^3)| \geq 2$ for $i \in [2]$, this implies that T does not intersect $\{00i0, 0i00, 0i0i, 0ii0, 0iii, 0iij, 0iji, 0ijj : i, j \in [2]\}$. Consequently, we can conclude that 0001 and 0002 are in T, which is a contradiction, since T already intersects $\{0i0j, 0ij0 : i, j \in [2]\}$.

Hence $\mu(S_3^4) = 12$. Moreover, if X is a μ-set of S_3^4, then $|X \cap V(iS_3^3)| = 4$ for $i \in [3]_0$. □

Theorem 4.4 *If $n \geq 2$, then $\mu(S_3^n) = 3^{n-2} + 3$. Moreover, if $n \geq 4$ and X is a μ-set of S_3^n, then $|X \cap V(iS_3^{n-1})| = 3^{n-3} + 1$ for $i \in [3]_0$. In addition, if $n \geq 4$ and*

X is a mutual-visibility set of S_3^n, then $|X \cap (V(iS_3^{n-1}) \cup V(jS_3^{n-1}))| \leq 2.3^{n-3} + 4$ for $i, j \in [3]_0$.

Proof. Proposition 4.3 yields the correctness of the formula for $n \leq 4$. Also, note that when $n = 4$, $2.3^{n-3} + 4 = 10$, so that the remaining part of the statement also follows from Proposition 4.3. Hence assume in the rest that $n \geq 5$. The set

$$X = \{s012, s120, s201 : \ s \in [[3]_0]^{n-3}\} \cup \{i^n : \ i \in [3]_0\}$$

is a mutual-visibility set of cardinality $3^{n-2} + 3$. Therefore, $\mu(S_3^n) \geq 3^{n-2} + 3$. Also, note that $|X \cap V(iS_3^{n-1})| = 3^{n-3} + 1$, for $i \in [3]_0$.

We first claim that if T is a mutual-visibility set of S_3^n, then $|T \cap (V(iS_3^{n-1}) \cup V(jS_3^{n-1}))| \leq 2.3^{n-3} + 4$ for $i, j \in [3]_0$. For this it is enough to show that if $|T \cap V(iS_3^{n-1})| = 3^{n-3} + 3$ for some $i \in [3]_0$ then $|T \cap V(jS_3^{n-1})| \leq 3^{n-3} + 1$ for each $j \in [3]_0 \setminus \{i\}$. Assume the contrary. Without loss of generality let $|T \cap V(2S_3^{n-1})| = 3^{n-3} + 3$ and $|T \cap V(1S_3^{n-1})| \geq 3^{n-3} + 2$. Considering $1S_3^{n-1}$, by induction hypothesis, we know that $|T \cap (V(10S_3^{n-2}) \cup V(11S_3^{n-2}))| \leq 2.3^{n-4} + 4$. Since $2.3^{n-4} + 4 < 3^{n-3} + 2$, we get that T intersects $12S_3^{n-2}$. This implies that $21^{n-1} \notin T$ and each vertex in $T \cap V(2S_3^{n-1})$ is $T \cap V(2S_3^{n-1})$-visible with 21^{n-1}. Hence by adding 21^{n-1} to the set $T \cap V(2S_3^{n-1})$ we obtain a mutual-visibility set of S_3^{n-1} with cardinality $3^{n-3} + 4$, which is a contradiction to our induction hypothesis. This contradiction proves the claim.

If $i, j, k \in [3]_0$, then by the fact that $\mu(S_3^n) \geq 3^{n-2} + 3$ and by the above claim we obtain that

$$\begin{aligned} |T \cap V(iS_3^{n-1})| &= |T| - |T \cap (V(jS_3^{n-1}) \cup V(kS_3^{n-1}))| \\ &\geq (3^{n-2} + 3) - (2 \cdot 3^{n-3} + 4) \\ &= 3^{n-3} - 1\,. \end{aligned} \tag{1}$$

Now suppose T is a μ-set of S_3^n. Then $|T| \geq 3^{n-2} + 3$. If $|T \cap V(iS_3^{n-1})| \leq 3^{n-3} + 1$ for each $i \in [3]_0$, then we are done. Suppose now that, without loss of generality, $|T \cap V(0S_3^{n-1})| \geq 3^{n-3} + 2$. Then considering $0S_3^{n-1}$, by induction hypothesis, we know that $|T \cap (V(00S_3^{n-2}) \cup V(0iS_3^{n-2}))| \leq 2 \cdot 3^{n-4} + 4$. Since $2 \cdot 3^{n-4} + 4 < 3^{n-3} + 2$, we get that T intersects $0iS_3^{n-2}$ for each $i \in [2]$.

Considering iS_3^{n-1}, by induction hypothesis, we know that $|T \cap (V(iiS_3^{n-2}) \cup V(ijS_3^{n-2}))| \leq 2 \cdot 3^{n-4} + 4$. For $n \geq 6$, since $2 \cdot 3^{n-4} + 4 < 3^{n-3} - 1$, we get that T intersects $i0S_3^{n-2}$ for each $i \in [3]_0$. Hence for $n \geq 6$, we obtain that the vertices on the shortest $01^{n-1}, 02^{n-1}$-path are not in T. In addition, we obtain that every vertex in $T \cap V(0S_3^{n-1})$ is $(T \cap V(0S_3^{n-1}))$-visible with 01^{n-1} and 02^{n-1}. Hence by adding 01^{n-1} and 02^{n-1} to the set $T \cap V(0S_3^{n-1})$, we get a mutual-visibility set of

S_3^{n-1} with cardinality at least $3^{n-3}+4$, which is a contradiction to our induction hypothesis.

Now we are left with the case $n=5$. In this case, we claim that T intersects $V(i0S_3^3) \cup V(ii0S_3^2) \cup V(iii0S_3^1) \cup V(iiiiS_3^1)$ for each $i \in [2]$. Assume the contrary. Then by (1) we have $|T \cap V(iS_3^5)| \geq 8$. But now we have, $|T \cap (V(ijS_3^3) \cup V(iijS_3^2) \cup V(iiijS_3^1))| \geq 8$. Then $|T \cap V(ijS_3^3)| \leq 5$ since otherwise, we can form a mutual-visibility set of S_3^3 with cardinality seven, which is a contradiction. Also, $|T \cap (V(iijS_3^2) \cup V(iiijS_3^1))| \leq 3$. Hence the only possibility is that $|T \cap V(ijS_3^3)| = 5$ and $|T \cap (V(iijS_3^2) \cup V(iiijS_3^1))| = 3$. But $|T \cap V(ijS_3^3)| = 5$ happens only if $ij00i$ and $ij00j$ are T in which case, $ij00j$ is not mutually-visible with the vertices in $V(iijS_3^2) \cup V(iiijS_3^1)$. This is a contradiction. Thus T intersects $V(i0S_3^3) \cup V(ii0S_3^2) \cup V(iii0S_3^1) \cup V(iiiiS_3^1)$ for both $i \in [2]$. Then again as mentioned for $n \geq 6$, we can add 01^4 and 02^4 to the set $T \cap V(0S_3^4)$ to get a mutual-visibility set of S_3^4 with cardinality at least 13, which is a contradiction. Thus if $n \geq 5$, then $\mu(S_3^n) = 3^{n-2}+3$. Moreover, if X is a μ-set of S_3^n, then $|X \cap V(iS_3^{n-1})| = 3^{n-3}+1$ for $i \in [3]_0$. □

Corollary 4.5 *If* $n \geq 2$*, then* $\mathrm{gp}(S_3^n) = 3^{n-2}+3$.

Proof. Since $\mathrm{gp}(S_3^n) \leq \mu(S_3^n)$, Theorem 4.4 implies that $\mathrm{gp}(S_3^n) \leq \mu(S_3^n) = 3^{n-2}+3$. On the other hand, the set $\{s012, s120, s201 : s \in [[3]_0]^{n-3}\} \cup \{i^n : i \in [3]_0\}$ is a general position set of S_3^n of cardinality $3^{n-2}+3$ for $n \geq 3$, and we are done. □

It remains to consider the outer mutual-visibility number and the outer general position number of S_3^n. The sets

$$X = \{0^k 1 2^{n-k-1} : k \in [n]_0\} \quad \text{and}$$
$$Y = \{0^n, i000^k i j^{n-k-4} : i,j \in [2], k \in [n-4]_0\}$$

are outer general position sets with cardinality n and $2n-7$, respectively. Thus, if $n \geq 3$, then

$$\mu_{\mathrm{o}}(S_3^n) \geq \mathrm{gp}_{\mathrm{o}}(S_3^n) \geq \max\{n, 2n-7\}\,.$$

5 Concluding remarks

We have finished the previous section by bounding from below $\mu_{\mathrm{o}}(S_3^n)$ and $\mathrm{gp}_{\mathrm{o}}(S_3^n)$. It remains open to determine the exact values of $\mu_{\mathrm{o}}(S_3^n)$ and of $\mathrm{gp}_{\mathrm{o}}(S_3^n)$ for $n \geq 3$.

The Sierpiński graph S_p^2 is isomorphic to the Sierpiński product graph $K_p \otimes_f K_p$, where f is the identity function. (For the definition of Sierpiński product graphs see the seminal paper [19], cf. also [11].) In [29] it was proved that $\min_f \mathrm{gp}(K_p \otimes_f K_p) = p$ and that $\max_f \mathrm{gp}(K_p \otimes_f K_p) = p(p-1)$. Hence Corollary 3.2 asserts that S_p^2 is somewhere in between the minimum and the maximum over all functions f.

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