S-packing chromatic critical graphs

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Abstract

For a non-decreasing sequence of positive integers $S = (s_1, s_2, \ldots)$, the S-packing chromatic number of a graph G is denoted by $\chi_S(G)$. In this paper, χ_S-critical graphs are introduced as the graphs G such that $\chi_S(H) < \chi_S(G)$ for each proper subgraph H of G. Several families of χ_S-critical graphs are constructed, and 2- and 3-colorable χ_S-critical graphs are presented for all packing sequences S, while 4-colorable χ_S-critical graphs are found for most of S. Cycles which are χ_S-critical are characterized under different conditions. It is proved that for any graph G and any edge $e \in E(G)$, the inequality $\chi_S(G-e) \geq \chi_S(G)/2$ holds. Moreover, in several important cases, this bound can be improved to $\chi_S(G-e) \geq (\chi_S(G)+1)/2$. The sharpness of the bounds is also discussed. Along the way an earlier result on χ_S-vertex-critical graphs is supplemented.

Keywords: packing coloring; S-packing coloring; S-packing critical graph; independence number; cycle graph

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1 Introduction

Let $S = (s_1, s_2, \ldots)$ be a non-decreasing sequence of positive integers and let $G = (V(G), E(G))$ be a graph. A mapping $c : V(G) \to [k] = \{1, \ldots, k\}$ is an *S-packing k-coloring* of G if the equality $c(u) = c(v) = i$ for $u \neq v \in V(G)$ implies $d_G(u, v) > s_i$. The *S-packing chromatic number* $\chi_S(G)$ of G is the smallest integer k such that G admits an S-packing k-coloring [16].

In the special case when $S = (1, 2, 3, \ldots)$, the S-packing chromatic number is the standard packing chromatic number χ_ρ, which was first explored under the name broadcast chromatic number [15] and given the present name in [7]. The 2020 review article [6] on packing colorings (including S-packing colorings) contains 68 references, but research continues, see [3, 14, 17–19]. The greatest emphasis in recent years has been on S-packing colorings, especially on subcubic graphs, see [4, 8, 12, 21, 27, 33, 36, 37, 40].

It should be stressed that the concept of S-packing coloring is very general. As said, it contains the packing coloring as a particular instance. In addition, the special case $S = (k, k, k, \ldots)$, $k \geq 1$, is studied in the literature as *k-distance colorings*, the corresponding $(k, k, k, \ldots)$-packing chromatic number is denoted by χ_k. Note that $\chi_1 = \chi$. Up to 2008, these investigations were surveyed in [30], while for some recent related papers see [20, 26, 32]. In the last years, however, the main focus was on 2-distance colorings of planar graphs, cf. [2, 9, 10, 31, 41].

Independently, and almost simultaneously, two different packing criticality concepts were introduced. In [25], a graph G was defined to be *χ_ρ-vertex-critical* if $\chi_\rho(G - u) < \chi_\rho(G)$ for each $u \in V(G)$. Moreover, if G is a χ_ρ-vertex-critical graph with $\chi_\rho(G) = k$, then G is called *k-χ_ρ-vertex-critical*. On the other hand, according to [5], G is *χ_ρ-critical* if $\chi_\rho(H) < \chi_\rho(G)$ for each proper subgraph H of G. If G has no isolated vertices, this is equivalent to the requirement that $\chi_\rho(G - e) < \chi_\rho(G)$ holds for each $e \in E(G)$. If G is a χ_ρ-critical graph with $\chi_\rho(G) = k$, then G is called *k-χ_ρ-critical*. The paper [13] further investigated χ_ρ-vertex-critical graphs and provided a characterization of 4-χ_ρ-vertex-critical graphs.

In the same way as packing colorings extend to S-packing colorings, one can extend χ_ρ-vertex-critical graphs and χ_ρ-critical graphs to χ_S-vertex-critical graphs and χ_S-critical graphs. The first of these generalizations has been done in [22], while in the follow-up paper [24] a characterization of 4-χ_S-vertex-critical graphs for packing sequences with $s_1 = 1$ and $s_2 \geq 3$ is given. The second of these generalizations, that is, χ_S-critical graphs, has not yet been studied, we fill this gap in this paper. We say that G is *χ_S-critical* if $\chi_S(H) < \chi_S(G)$ for each proper subgraph H of G. If G is a χ_S-critical graph with $\chi_S(G) = k$, then G is called *k-χ_S-critical*. Note that we do not consider the empty graph as a proper subgraph. Then, by our definition, the isolated vertex K_1 has no proper subgraph, and it is 1-χ_S-critical for every packing sequence S.

The paper is organized as follows. In the next section, we give some definitions, introduce useful notation, and present basic observations about S-packing critical graphs. In Section 3, some families of χ_S-critical graphs are discussed. We determine 2-χ_S-critical and 3-χ_S-critical graphs for all packing sequences S, and determine 4-χ_S-critical graphs for most of S. For the case of 4-χ_S-critical graphs, we supplement an earlier result from the literature on χ_S-vertex-critical graphs. In Section 4 we characterize cycles which are χ_S-critical under different conditions. In Section 5 we consider the impact of edge removal on the S-packing chromatic number. We prove that $\chi_S(G-e) \geq \chi_S(G)/2$ for any graph G and any edge $e \in E(G)$, and that in several important cases the bound can be improved to $\chi_S(G-e) \geq (\chi_S(G)+1)/2$. For many S, infinitely many sharp examples are constructed. In the last section we identify several open problems for further research.

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph. The open neighborhood $N_G(u)$ of u in G is the set of the neighbors of u. A support vertex of G is a vertex adjacent to a leaf. The girth of G is denoted by $g(G)$. If G has no cycles, we set $g(G) = \infty$. As usual, $\alpha(G)$ is the independence number of G. The distance $d_G(u, v)$ between $u, v \in V(G)$ is the shortest-path distance. The diameter of G is denoted by $\mathrm{diam}(G)$. A subgraph H of G is isometric, if for every two vertices $u, v \in V(H)$ we have $d_H(u, v) = d_G(u, v)$. The path, the cycle, and the complete graph of order n are respectively denoted by P_n, C_n, and K_n, while the order of a graph G will be denoted by $n(G)$.

The set of all *packing sequences* will be denoted by $\mathcal{S}$, that is,

$$\mathcal{S} = \{(s_1, s_2, \dots) :\ 1 \leq s_1 \leq s_2 \leq \cdots\}.$$

For a given $S \in \mathcal{S}$ we will always assume that $S = (s_1, s_2, \dots)$. Unless stated otherwise, the packing sequences are considered to be infinite in this paper.

We will consider sets of packing sequences such that some of their first coordinates are fixed or bounded from below. Instead of introducing the notation in general, consider the following example. Assume we wish to consider the set of packing sequences $S = (s_1, s_2, s_3, \dots)$ with $s_1 = 1$, $s_2 = 3$, and $s_3 \geq 4$. Then we set

$$\mathcal{S}_{1,3,\overline{4}} = \{(s_1, s_2, s_3, \dots) :\ s_1 = 1, s_2 = 3, s_3 \geq 4\}\,.$$

The general notation should be clear from this example. For instance, $\mathcal{S}_{1,\overline{3},5}$ is the set of packing sequences with $s_1 = 1$, $3 \leq s_2 \leq 5$, and $s_3 = 5$. Note also that $\mathcal{S} = \mathcal{S}_{\overline{1}}$.

It was stated in [16, Observation 2] that every graph G and any edge e of it satisfy the inequality

$$\chi_S(G-e) \le \chi_S(G).$$

As the removal of isolated vertices does not change $\chi_S(G)$, this inequality also implies $\chi_S(H) \le \chi_S(G)$ for every subgraph H of G. We may also infer that if $\chi_S(G) = k$, then G contains a subgraph that is k-χ_S-critical.

Observation 2.1 *Let $S \in \mathcal{S}$ and let G be a graph.*

(i) If G contains no isolated vertex, then G is χ_S-critical if and only if $\chi_S(G-e) < \chi_S(G)$ holds for every edge $e \in E(G)$.

(ii) K_1 is the unique 1-χ_S-critical graph.

3 Families of S-packing critical graphs

In this section we present several families of S-packing critical graphs. We first show that graphs of diameter k and girth at least $k+2$ are χ_S-critical for each packing sequence $S \in \mathcal{S}_k$. This result is then extended to specific generalized lexicographic products. We end the section by classifying k-χ_S-critical graphs for almost all S and $k \in \{2, 3, 4\}$. But first we give two general, simple properties of S-packing critical graphs.

Lemma 3.1 *If $S \in \mathcal{S}$ and G is a χ_S-critical graph, then G is connected.*

Proof. Suppose to the contrary that G is not a connected graph such that $H_1, \ldots, H_r$ are the components of G, where $r \ge 2$. Since $\chi_S(G) = \max_{i\in[r]} \chi_S(H_i)$, there exists a component H_j such that $\chi_S(H_j) = \chi_S(G)$. Now consider a component H_k for some $k \ne j$. The removal of H_k from G yields a proper subgraph H with $\chi_S(H) = \chi_S(H_j) = \chi_S(G)$, which contradicts the assumption that G is χ_S-critical. Therefore, G must be connected. $\square$

Lemma 3.2 *If G is a χ_S-critical graph, then G is a χ_S-vertex-critical graph.*

Proof. If $G \cong K_1$, then it is χ_S-critical and χ_S-vertex-critical. Otherwise, consider an arbitrary vertex $x \in V(G)$. Since $G - x$ is a proper subgraph of G, χ_S-criticality implies $\chi_S(G-x) < \chi_S(G)$ and proves that G is a χ_S-vertex-critical graph. $\square$

A graph G is called *diameter k-critical* if $\mathrm{diam}(G) = k$ and $\mathrm{diam}(G-e) > \mathrm{diam}(G)$ holds for every $e \in E(G)$ (see [11,34,39]).

Proposition 3.3 *Let $k \geq 1$ and $S \in \mathcal{S}_k$. Then every diameter k-critical graph is χ_S-critical.*

Proof. Let G be a diameter k-critical graph. Since $s_1 = k$, no two vertices of G can receive the same color, that is, $\chi_S(G) = n(G)$. Let now e be an arbitrary edge of G. Since $\mathrm{diam}(G - e) \geq k + 1$, there are two vertices u and v with $d_{G-e}(u, v) \geq k + 1$. Therefore, in an S-coloring of $G - e$, we can color u and v with color 1, and assign a unique color to every other vertex. Hence $\chi_S(G - e) \leq n(G) - 1 < \chi_S(G) = n(G)$ which yields the conclusion. □

Since every graph G with $\mathrm{diam}(G) = k$ and girth $g(G) \geq k + 2$ is diameter k-critical, we deduce the following statement from Proposition 3.3.

Corollary 3.4 *Let $k \geq 1$ and $S \in \mathcal{S}_k$. If G is a graph with $\mathrm{diam}(G) = k$ and $g(G) \geq k + 2$, then G is χ_S-critical.*

As Proposition 3.3 and Corollary 3.4 are true for trees, we may infer that every tree is χ_S-critical for infinitely many packing sequences S. We also prove the following property for trees.

Proposition 3.5 *For every tree T and every $S \in \mathcal{S}$, the tree T is χ_S-critical if and only if it is χ_S-vertex-critical.*

Proof. If T is an isolated vertex, the equivalence holds. From now on, we may assume $n(T) \geq 2$. The first direction of the statement follows from Lemma 3.2. To prove the other direction, consider a χ_S-vertex-critical tree T and remove an arbitrary edge $e = u_i u_j$. Let T_i and T_j be the two components of $T - e$ such that $u_i \in V(T_i)$ and $u_j \in V(T_j)$. We know that $\chi_S(T - e) = \max\{\chi_S(T_i), \chi_S(T_j)\}$ and may assume $\chi_S(T - e) = \chi_S(T_i)$. As T_i is also a component in $T - u_j$, the χ_S-vertex-criticality of T implies

$$\chi_S(T - e) = \chi_S(T_i) \leq \chi_S(T - u_j) < \chi_S(T).$$

Since $\chi_S(T - e) < \chi_S(T)$ holds for every edge and T is isolate-free, we may conclude that T is χ_S-critical. This finishes the proof of the equivalence. □

We conclude the section by considering k-χ_S-critical graphs, where $k \in \{2, 3, 4\}$.

Proposition 3.6 *If $S \in \mathcal{S}$, then a graph G is 2-χ_S-critical if and only if $G \cong K_2$.*

Proof. It is straightforward that $G \cong K_2$ is 2-χ_S-critical: its packing chromatic number is 2, and removing any vertex or edge reduces the packing chromatic number to 1. Conversely, suppose G is 2-χ_S-critical. Then, by Lemma 3.2, G is also 2-χ_S-vertex-critical. It is shown in [22] that the only graph with this property is K_2. Thus, the result follows. $\square$

In the first item of [22, Theorem 5.1] it is claimed that if $S \in \mathcal{S}_{2,2,2}$, then a graph G is 4-χ_S-vertex-critical if and only if G is one of the graphs $K_{1,3}$, C_4, Z_1, $K_4 - e$, K_4, where Z_1 denotes the graph obtained by adding a pendant edge to a C_3. However, there is one example missing from the proof, which we explain below.

Consider a 4-χ_S-vertex-critical graph G and let $u \in V(G)$, so that $\chi_S(G-u) \leq 3$. In the proof of [22, Theorem 5.1], when the case $\deg_G(u) = 2$ is considered, it is correctly stated that if $G - u$ is connected, then, for $G - u \cong P_3$ we get $G \cong C_4$ or $G \cong Z_1$, for $G - u \cong C_3$ we get $G \cong K_4 - e$. Afterwards, it is stated that in any other case no χ_S-critical graph is obtained. But the vertex u can be adjacent to two degree 1 vertices of $G - u$, that is, to the end-vertices of a path in which case G is a cycle. Note that $\chi_S(C_n) = 3$ if and only if n is divisible by 3. Hence, C_n is a 4-χ_S-vertex-critical graph if and only if n is not divisible by 3.

According to the above, the first item of [22, Theorem 5.1] must be supplemented as follows.

Proposition 3.7 *If $S \in \mathcal{S}_{2,2,2}$, then a graph G is 4-χ_S-critical if and only if*

$$G \in \{K_{1,3}, Z_1, K_4 - e, K_4\} \cup \{C_n : \ n \geq 4, \ n \not\equiv 0 \pmod 3\}.$$

With Proposition 3.7 in hand, we can state the following result.

Theorem 3.8 *Let $S \in \mathcal{S}$ and let G be a graph.*

(i) If $S \in \mathcal{S}_{1,1}$, then G is 3-χ_S-critical if and only if $G \in \{C_{2k+1} : k \geq 1\}$.

(ii) If $S \in \mathcal{S}_{1,\overline{2}}$, then G is 3-χ_S-critical if and only if $G \in \{C_3, P_4\}$

(iii) If $S \in \mathcal{S}_{\overline{2}}$, then G is 3-χ_S-critical if and only if $G \cong P_3$.

(iv) If $S \in \mathcal{S}_{2,2,2}$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}\} \cup \{C_n : \ n \geq 4, \ n \not\equiv 0 \pmod 3\}$.

(v) If $S \in \mathcal{S}_{2,2,\overline{3}}$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, C_4, P_6\}$.

(vi) If $S \in \mathcal{S}_{2,\overline{3}}$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, C_4, P_5\}$.

(vii) If $S \in \mathcal{S}_{\overline{3}}$, then G is 4-χ_S-critical if and only if $G \in \{K_{1,3}, P_4\}$.

Proof. Let $S \in \mathcal{S}$. Then it was proved in [22, Theorem 4.1] that (i) if $S \in \mathcal{S}_{1,1}$, then G is 3-χ_S-vertex-critical if and only if $G \in \{C_{2k+1} : k \geq 1\}$; (ii) if $S \in \mathcal{S}_{1,\overline{2}}$, then G is 3-χ_S-vertex-critical if and only if $G \in \{C_3, C_4, P_4\}$; and (iii) if $S \in \mathcal{S}_{\overline{2}}$, then G is 3-χ_S-vertex-critical if and only if $G \in \{C_3, P_3\}$. By Lemma 3.2, it remains to verify which of the above-listed graphs is 3-χ_S-critical. Doing it one by one, the first three assertions of the theorem follow.

By Proposition 3.7 we know the list of 4-χ_S-vertex-critical where $S \in \mathcal{S}_{2,2,2}$. We next recall that it was further proved in Theorem [22, Theorem 5.1] that (a) if $S \in \mathcal{S}_{2,2,\overline{3}}$, then G is 4-χ_S-vertex-critical if and only if $G \in \{K_{1,3}, C_4, Z_1, K_4 - e, K_4, P_6, C_6\}$; (b) if $S \in \mathcal{S}_{2,\overline{3}}$, then G is 4-χ_S-vertex-critical if and only if $G \in \{K_{1,3}, C_4, Z_1, K_4 - e, K_4, P_5\}$; and (c) if $S \in \mathcal{S}_{\overline{3}}$, then G is 4-χ_S-vertex-critical if and only if $G \in \{K_{1,3}, P_4, C_4, Z_1, K_4 - e, K_4\}$. Applying Lemma 3.2 again, we need to verify which of the above-listed graphs are 4-χ_S-critical. Carefully checking all of them, the last four assertions of the theorem follow. $\square$

To state the following theorem, we first need to introduce two families of graphs and some specific graphs. If $S \in \mathcal{S}$, then let

$$\mathcal{C}_{s_4} = \{C_n, n \geq 5 : \ (n \equiv 1, 2 \pmod 4) \text{ or } (n \equiv 3 \pmod 4 \text{ and } s_4 < \lfloor n/2 \rfloor)\}.$$

So $\mathcal{C}_{s_4}$ is a subclass of cycles that depends on the fourth term of S. Next, let X_{2k}, $k \geq 3$, be the graph obtained from P_{2k} by attaching a pendant vertex to each of the support vertices of P_{2k}. Finally, we need the graphs G_i, $i \in \{1, \ldots, 8\}$, which are shown in Fig. 1. Note that $G_6 \cong X_6$.

Theorem 3.9 *If G is a graph, then the following assertions hold.*

(i) If $S \in \mathcal{S}_{1,3,3}$, then G is 4-χ_S-critical if and only if

$$G \in \{K_4, G_1, G_2\} \cup \mathcal{C}_{s_4} \cup \{X_{2k} : k \geq 3\}.$$

(ii) If $S \in \mathcal{S}_{1,3,\overline{4}}$, then G is 4-χ_S-critical if and only if

$$G \in \{K_4, C_5, C_6, P_8\} \cup \{G_i : \ i \in \{1, \ldots, 7\}\}.$$

(iii) If $S \in \mathcal{S}_{1,\overline{4}}$, then G is 4-χ_S-critical if and only if

$$G \in \{K_4, C_5, P_6, G_8\}.$$

Proof. Let $S \in \mathcal{S}$. In each of the three cases listed above, the set of all 4-χ_S-vertex-critical graphs has been completely classified in [24]. This classification includes

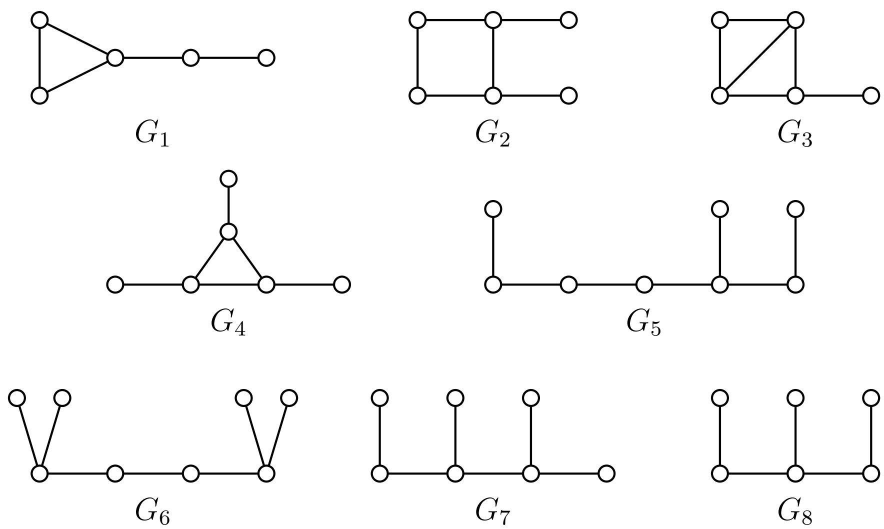

Figure 1: The graphs $G_1, \ldots, G_8$

sporadic examples, as well as several graph families. By Lemma 3.2, any 4-χ_S-critical graph must also be 4-χ_S-vertex-critical. Therefore, it suffices to determine which of the graphs identified in [24] are also χ_S-critical.

When $S \in \mathcal{S}_{1,3,3}$, the classification from [24] includes nine individual graphs and four graph families, two of which are infinite. Only those listed in item (i) satisfy the χ_S-criticality condition.

For $S \in \mathcal{S}_{1,3,\overline{4}}$, the classification from [24] consists of thirteen graphs and two graph families. Only the graphs listed in item (ii) are χ_S-critical.

For $S \in \mathcal{S}_{1,\overline{4}}$, the classification from [24] includes twelve graphs and two graph families. Among them, only the graphs in item (iii) were found to be χ_S-critical through direct verification. This completes the proof. $\square$

4 On S-packing critical cycles

Recall that $(k, k, \ldots)$-packing colorings are known as k-distance colorings and that the $(k, k, \ldots)$-packing chromatic number is denoted by χ_k. For cycles, it is proved in [38] that $\chi_k(C_n) = k + 1 + \lceil \frac{r}{\ell} \rceil$, where $n = \ell(k + 1) + r$, $0 \le r \le k$. (The special case when $r = 0$ was earlier observed in [29].) Since $n \ge k+1$ implies $\chi_k(P_n) = k+1$, we can conclude that C_n, where $n \ge k + 1$, is $(k, k, , \ldots)$-packing critical if and only if $n \not\equiv 0 \pmod{k + 1}$.

Proposition 4.1 *If $S \in \mathcal{S}$ and $n \geq 3$, then the following hold.*

(i) If $n \leq s_1 + 1$, then C_n is not S-packing critical.

(ii) If $s_1 + 2 \leq n \leq 2s_1 + 1$, then C_n is S-packing critical.

Proof. (i) Since $n \leq s_1 + 1$, we have $\chi_S(C_n) = n$. Moreover, $C_n - e = P_n$ and $\mathrm{diam}(P_n) = n - 1 \leq s_1$, hence $\chi_S(C_n - e) = n$. So C_n is not S-packing critical.

(ii) Let $s_1 + 2 \leq n \leq 2s_1 + 1$. Since $\mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{2s_1+1}{2} \rfloor = s_1$, we have $\chi_S(C_n) = n$. On the other hand, $\mathrm{diam}(C_n - e) = \mathrm{diam}(P_n) = n - 1 \geq (s_1 + 2) - 1 = s_1 + 1$, so we can color the two leaves of P_n with the same color and henceforth, $\chi_S(C_n - e) \leq n - 1$. So C_n is S-packing critical in this case. $\square$

Theorem 4.2 *If $S \in \mathcal{S}_1$ and $n \geq 3$, then the following hold.*

(i) If $n \leq s_2 + 2$, then C_n is S-packing critical if and only if n is odd.

(ii) If $s_2 + 3 \leq n \leq 2s_2 + 1$, then C_n is S-packing critical.

Proof. (i) If $n \leq s_2 + 1$, then $\mathrm{diam}(C_n) < \mathrm{diam}(P_n) \leq s_2$. Therefore, no color t with $t \geq 2$ can be repeated in an S-packing coloring of P_n, and we get an optimal coloring by assigning color 1 to as many vertices as possible. Then $\chi_S(P_n) = n - \alpha(P_n) + 1 = n - \lceil \frac{n}{2} \rceil + 1$. For cycles, the situation is similar and we obtain $\chi_S(C_n) = n - \alpha(C_n) + 1 = n - \lfloor \frac{n}{2} \rfloor + 1$. Since $\chi_S(C_n - e) = \chi_S(P_n)$ for every $e \in E(C_n)$, the above formulas show that $\chi_S(C_n - e) < \chi_S(C_n)$ if and only if n is odd.

Now let $n = s_2 + 2$. We observe that $\mathrm{diam}(C_n) = \lfloor \frac{n}{2} \rfloor \leq s_2$ always holds and then no color t with $t \geq 2$ can be repeated in an S-packing coloring of C_n. Thus, $\chi_S(C_n) = n - \alpha(C_n) + 1 = n - \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil + 1$. After deleting an edge from C_n, we obtain P_n, whose diameter is $n - 1 = s_2 + 1$. We may have two types of S-coloring c of the n-path $v_1 \ldots v_n$. The first possibility is that no color t with $t \geq 2$ is repeated, and then c uses at least $n - \alpha(P_n) + 1 = \lfloor \frac{n}{2} \rfloor + 1$ colors. The second possibility is to assign color 2 to v_1 and v_n, and use color 1 on an independent set in $v_2 \ldots v_{n-1}$. This needs at least $n - 2 - \alpha(P_{n-2}) + 2 = n - \lceil \frac{n-2}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1$ colors. We may therefore infer that $\chi_S(C_n) = \lfloor \frac{n}{2} \rfloor + 1$. Comparing $\chi_S(C_n)$ and $\chi_S(C_n - e) = \chi_S(P_n)$, we conclude that C_n is χ_S-critical if and only if n is odd, as stated.

(ii) Suppose first that n is even. Since $\mathrm{diam}(C_n) = \frac{n}{2} \leq s_2$, no color t with $t \geq 2$ can be repeated in an S-packing coloring of C_n. Thus, $\chi_S(C_n) = n - \alpha(C_n) + 1 = \frac{n}{2} + 1$. For the n-path $v_1 v_2 \ldots v_n$, consider the coloring c that assigns color 1 to the vertices $v_1, v_3, \ldots, v_{n-1}$, color 2 to v_2 and v_n, while the remaining vertices are colored pairwise differently with colors $3, \ldots, \frac{n}{2}$. As $d_{P_n}(v_2, v_n) = n - 2 \geq s_2 + 1$, c is an

S-packing-coloring. Then, we conclude that $\chi_S(P_n) \le \frac{n}{2} < \chi_S(C_n)$, proving the χ_S-criticality of C_n.

If n is odd, $\mathrm{diam}(C_n) = \frac{n-1}{2} \le s_2$ implies that no color different from 1 can be repeated in an S-packing coloring. We infer again that $\chi_S(C_n) = n - \alpha(C_n) + 1 = \frac{n+1}{2} + 1$. The path P_n can be colored such that only color 1 is repeated. Hence,

$$\chi_S(P_n) \le n - \alpha(P_n) + 1 = \frac{n-1}{2} + 1 < \chi_S(C_n)$$

that proves the χ_S-criticality of C_n. $\square$

Theorem 4.3 *If $n \ge 3$, then the following hold.*

(i) If $S \in \mathcal{S}_{1,1}$, then C_n is S-packing critical if and only if n is odd.

(ii) If $S \in \mathcal{S}_{1,2,2}$, then C_n is S-packing critical if and only if it is C_3 or C_5.

(iii) If $S \in \mathcal{S}_{1,\overline{2},3}$, then C_n is S-packing critical if and only if $n \not\equiv 0 \pmod 4$.

Proof. Throughout the proof, let $v_1, \dots, v_n$ be consecutive vertices of C_n.

(i) Let $S \in \mathcal{S}_{1,1}$. An even cycle C_n can be colored alternately with colors 1 and 2. Hence, $\chi_S(C_n) = 2$. If n is odd, a 2-packing-coloring is not possible, but three colors are clearly enough. On the other hand, $\chi_S(P_n) = 2$ for every $n \ge 2$. It follows that $\chi_S(C_n - e) < \chi_S(C_n)$ holds if and only if n is odd.

(ii) Let $S \in \mathcal{S}_{1,2,2}$. Consider a path P_n, for $n \ge 4$, and an S-packing-coloring c of P_n. On every four consecutive vertices of the path, the coloring uses at least three colors. Let $(123)^*$ denote the sequence of colors in which 123 repeats an arbitrary number of times. Using the color pattern $(123)^*$, starting with the first vertex of the path, and where from the last block 123 the required number of elements is used (possibly zero), we obtain an S-packing coloring. Consequently, $\chi_S(P_n) = 3$ if $n \ge 4$. As follows, $\chi_S(C_n) \ge 3$ holds for every $n \ge 4$.

If $n \ge 6$, we consider the following colorings of C_n. The referred patterns start from vertex v_1, and after a specified initial sequence, the coloring repeats pattern 123 so that the color of v_n will be 3. If $n \equiv 0 \pmod 3$, we color C_n with $(123)^*$. If $n \equiv 1 \pmod 3$, we color C_n as $1213\,(123)^*$. If $n \equiv 2 \pmod 3$, then $n \ge 8$, and we can color C_n as $1213\,1213\,(123)^*$. It shows $\chi_S(C_n) \le 3$ and in turn, $\chi_S(C_n) = 3$, for every $n \ge 6$. We conclude that in this case there is no S-packing critical cycle on more than 5 vertices.

For the small cases, we observe $\chi_S(P_3) = 2 < \chi_S(C_3) = 3$; $\chi_S(P_4) = 3 = \chi_S(C_4)$; and $\chi_S(P_5) = 3 < \chi_S(C_5) = 4$. Now, we may conclude that C_3 and C_5 are the only S-packing critical cycles if $S \in \mathcal{S}_{1,2,2}$.

(iii) Under the conditions $n \geq 4$ and $S \in \mathcal{S}_{1,\overline{2},3}$, any S-packing-coloring of P_n or C_n requires at least 3 colors. A path P_n with $n \geq 4$ can be colored by $(1213)^*$ no matter whether $n \equiv 0 \pmod 4$ is valid or not. Naturally, if $n \not\equiv 0 \pmod 4$, then from the last block 1213 the required number of elements is used. Therefore, $\chi_S(P_n) = 3$ when $n \geq 4$. If $n \equiv 0 \pmod 4$, we can take the same type of coloring for C_n and get $\chi_S(C_n) = 3$. It also shows that no n-cycle with $n \equiv 0 \pmod 4$ is χ_S-critical.

Suppose now that c is an S-packing coloring of C_n, for $n \geq 5$ that uses only colors 1, 2, 3. We claim that there are no two neighbors colored with 2 and 3. Assume, without loss of generality, that $c(v_i) = 3$ and $c(v_{i+1}) = 2$. Then, v_{i+2} cannot get a color different from 1. But then, as $s_2 \geq 2$ and $s_3 = 3$, neither of colors 1, 2, and 3 can be assigned to v_{i+3}. This contradiction proves that every second vertex of the cycle is colored with 1. As neither of the patterns 1212 and 1313 may occur in the coloring, we obtain that the pattern 1213 must be repeated along the cycle. If $n \not\equiv 0 \pmod 4$, it is impossible to have 3 colors and we conclude $\chi_S(C_n) \geq 4$ for these cases. Therefore, C_n is χ_S-critical for every $n \geq 5$ if $n \not\equiv 0 \pmod 4$. Observing also that $\chi_S(P_3) = 2 < \chi_S(C_3) = 3$ we obtain that C_3 is χ_S-critical. This completes the proof for (iii). $\square$

5 Impact of edge removal on χ_S

In view of Observation 2.1 (i), the question naturally arises as to what extent removing an edge of G can affect $\chi_S(G)$. Before we answer this question, recall the following well-known sets (see [1, 23, 35]) which are defined for an arbitrary edge $e = uv$ of a graph G:

$$\begin{aligned} W_{uv}^G &= \{w \in V(G):\ d_G(u,w) < d_G(v,w)\}, \\ W_{vu}^G &= \{w \in V(G):\ d_G(v,w) < d_G(u,w)\}, \\ {}_vW_u^G &= \{w \in V(G):\ d_G(u,w) = d_G(v,w)\}\,. \end{aligned}$$

Clearly, $V(G) = W_{uv} \cup W_{vu} \cup {}_vW_u$. We will use the next lemma throughout the rest of the section mostly without explicitly mentioning it.

Lemma 5.1 *If $e = uv \in E(G)$, then $W_{uv}^G = W_{uv}^{G-e}$ and $W_{vu}^G = W_{vu}^{G-e}$.*

Proof. Assume first that $w \in W_{uv}^G$. Then $e = uv$ does not lie on any shortest w,u-path, thus we have

$$d_{G-e}(w,u) = d_G(w,u) < d_G(w,v) \leq d_{G-e}(w,v)\,,$$

hence $w \in W_{uv}^{G-e}$.

Assume second that $w \in W_{uv}^{G-e}$, that is, $d_{G-e}(w,u) < d_{G-e}(w,v)$. Then no matter whether there exists a shortest w,v-path in G which passes e, we have

$$d_G(w,v) \geq d_G(w,u) + 1\,,$$

that is, $w \in W_{uv}^G$. We can conclude that $W_{uv}^G = W_{uv}^{G-e}$. The argument for the equality $W_{vu}^G = W_{vu}^{G-e}$ is parallel. $\square$

In the proof of the next result, we use some ideas similar to those in the proof of [5, Theorem 1].

Theorem 5.2 *Let S be a packing sequence and let $e = uv$ be an edge of a graph G. Then the following statements hold.*

(i) $\chi_S(G-e) \geq \dfrac{\chi_S(G)}{2}$*. Moreover, there are infinitely many sharp examples for every packing sequence* $S \in \mathcal{S}_{1,\overline{3}} \cup \mathcal{S}_{2,\overline{5}} \cup \mathcal{S}_{\overline{3}}$.

(ii) If G contains a component on at least three vertices and $S \in \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2}$, then $\chi_S(G-e) \geq \dfrac{\chi_S(G)+1}{2}$ *holds. Moreover, there are infinitely many sharp examples for every* $S \in \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2}$.

(iii) If $S \in \mathcal{S}_{2,2,2}$ and $\chi_S(G-e) \geq 3$, then $\chi_S(G-e) \geq \dfrac{\chi_S(G)+1}{2}$ *holds.*

Proof. (i) Let $c'\colon V(G) \to [\chi_S(G-e)]$ be a χ_S-packing-coloring of $G-e$. For a color $t \in [\chi_S(G-e)]$, we say that a vertex pair $\{x,y\}$ is *t-problematic* if $c'(x) = c'(y) = t$ but $d_G(x,y) \leq s_t$. Since c' is an S-packing coloring of $G-e$, we have $d_{G-e}(x,y) \geq s_t + 1$. Then $d_{G-e}(x,y) > d_G(x,y)$ and therefore, in G, every shortest (x,y)-path goes through the edge e. It also follows that, for every problematic pair $\{x,y\}$, one vertex is in W_{uv}^G and the other is in W_{vu}^G. Note that $W_{uv}^G = W_{uv}^{G-e}$ and $W_{vu}^G = W_{vu}^{G-e}$ hold by Lemma 5.1.

We say that a vertex z *covers* a problematic pair $\{x,y\}$ if $z = x$ or $z = y$ and state the following key property of problematic pairs.

Claim: For every $t \in [\chi_S(G-e)]$, either there is no t-problematic pair or there exists a vertex that covers all t-problematic pairs.

Proof. Consider the bipartite graph F_t with partite classes W_{uv}^G, W_{vu}^G, where xy is an edge if $\{x,y\}$ is a t-problematic pair in G. Suppose for a contradiction that the claim is not true, that is, $E(F_t) \neq \emptyset$ and that one vertex cannot cover all edges of

F_t. König's theorem [28] implies that the matching number of F_t is at least 2. So we may suppose that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are two vertex-disjoint t-problematic pairs in G.

Without loss of generality, let $x_i \in W_{uv}^G$ and $y_i \in W_{vu}^G$ for $i \in \{1, 2\}$. Let us set $d_G(x_i, u) = a_i$ and $d_G(y_i, v) = b_i$ for $i \in \{1, 2\}$. Note that these distances remain the same in $G - e$. Consider first x_1 and x_2. As both vertices belong to W_{uv}, we have $d_G(x_1, x_2) = d_{G-e}(x_1, x_2)$. Since c' is a χ_S-packing-coloring of $G - e$, it holds that $d_{G-e}(x_1, x_2) \geq s_t + 1$. Further, the length $a_1 + a_2$ of the (x_1, x_2)-path through u gives an upper bound on the distance between x_1 and x_2. We obtain

$$a_1 + a_2 \geq d_G(x_1, x_2) \geq s_t + 1. \tag{1}$$

A similar reasoning gives

$$b_1 + b_2 \geq d_G(y_1, y_2) \geq s_t + 1. \tag{2}$$

By our assumption, both $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are t-problematic pairs and so

$$a_1 + 1 + b_1 = d_G(x_1, y_1) \leq s_t \tag{3}$$

and

$$a_2 + 1 + b_2 = d_G(x_2, y_2) \leq s_t. \tag{4}$$

Inequalities (1)-(4) imply

$$2s_t + 2 \leq a_1 + a_2 + b_1 + b_2 \leq 2s_t - 2.$$

This contradiction finishes the proof of the claim. ($\square$)

By the claim, for every color t with a t-problematic pair, we can specify a vertex z_t that covers all t-problematic pairs. If we remove z_t from the corresponding color class, then no t-problematic pair remains, and hence, any two remaining vertices have a distance of at least $s_t + 1$ in G. Let Z contain all specified vertices z_t. Then $|Z| \leq \chi_S(G-e)$. Define now a new coloring c which keeps the color $c'(x)$ if $x \notin Z$ and assigns a unique color to every vertex $x \in Z$ from $\{\chi_S(G-e)+1, \ldots, \chi_S(G-e)+|Z|\}$.

It is clear that c uses at most $2\chi_S(G - e)$ colors. We now prove that c is an S-packing coloring of G. Suppose that $c(x) = c(y) = p$, where $x \neq y$. Since every color q with $q > \chi_S(G-e)$ is assigned to only one vertex, we infer that $p \in [\chi_S(G-e)]$. As all p-problematic pairs were destroyed by recoloring one vertex from the pair, $\{x, y\}$ is not a problematic pair and hence, $d_G(x, y) \geq s_p + 1$. Thus, c is an S-packing coloring of G, which implies $\chi_S(G) \leq 2\chi_S(G - e)$ as stated.

We now prove the sharpness of the inequality. If $S \in \mathcal{S}_{1,\overline{3}}$, let G be constructed by taking two copies of the star $K_{1,k}$ with $k \geq 3$ and connecting them by an edge

e between two leaves. It is clear that $\chi_S(G-e) = \chi_S(K_{1,k}) = 2$. We show that $\chi_S(G) = 4$. In G, the path P between the centers of the stars is an isometric subgraph of diameter 3. Hence, either all four vertices of P get different colors, or color 1 is assigned to two vertices. In the latter case, at least one center receives color 1, and then the k neighbors get pairwise different colors. In either case, the number of colors is at least 4. On the other hand, a 4-packing-coloring can be obtained by assigning color 1 to all leaves and one vertex of degree 2. Thus, $\chi_S(G) = 4$ and G is a sharp example for the bound in (i).

If $S \in \mathcal{S}_{2,\overline{5}}$, let G_k, $k \geq 2$, be the graph obtained from the disjoint union of K_k and K_{k+1} by adding a path of length 3 between a vertex of K_k and a vertex of K_{k+1}. Let e be the edge of this path attached to K_{k+1}. As $\mathrm{diam}(G_k) = 5$, no color except 1 can be repeated in an S-packing coloring of G_k and it is easy to check that $\chi_S(G_k) = 2k + 2 = 2\chi_S(G_k - e)$.

Assume now that $S \in S_{\overline{3}}$ and consider the following example. Let H be a graph with a universal vertex and let G be the graph obtained from the disjoint union of two copies of H by adding an edge e between a universal vertex of one copy of H and a universal vertex of the other copy of H. Then $\mathrm{diam}(H) = 3$ which implies that $\chi_S(G) = 2n(H)$. On the other hand, $\chi_S(G-e) = n(H)$. This demonstrates the sharpness of (i) for every $S \in S_{\overline{3}}$.

(ii) Let $S \in \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2}$. If the largest component of G contains at least three vertices, $\chi_S(G-e) \geq 2$ holds for every $e \in E(G)$. We prove that there is a color $t \in \{1,2\}$ without a t-problematic pair in G. Assume that $\{x,y\}$ is a 1-problematic pair. Then $d_G(x,y) \leq s_1 = 1$ and all shortest (x,y)-paths contain $e = uv$. It implies $\{x,y\} = \{u,v\}$ and $c'(u) = c'(v) = 1$. Consequently, for every two vertices x' and y' with $c'(x') = c'(y') = 2$, either $d_G(x',y') = d_{G-e}(x',y') \geq s_2 + 1$ or, in G, every shortest (x',y')-path contains e and $d_G(x',y') \geq 3 \geq s_2 + 1$. It follows that one of the colors 1 and 2 has no problematic pair, and then, the proof of part (i) can be improved by claiming $|Z| \leq \chi_S(G-e) - 1$. We conclude $\chi_S(G) \leq 2\chi_S(G-e) - 1$ as stated.

For a packing sequence $S \in \mathcal{S}_{1,1}$, we take the odd cycles which are 3-χ_S-critical graphs according to Theorem 3.8 (i). Thus, $\chi_S(C_{2k+1}) = 3$ and $\chi_S(C_{2k+1} - e) = 2$, and the odd cycles are sharp examples for the inequality in (ii).

When $S \in \mathcal{S}_{1,2}$, we consider two vertex-disjoint stars $K_{1,k}$, for $k \geq 3$, and add an edge e between the centers to obtain the graph G. It is easy to check that $\chi_S(G-e) = 2$ and $\chi_S(G) = 3$. It provides then a sharp example for (ii). Remark that C_3 and P_4 are also sharp examples for $S \in \mathcal{S}_{1,2}$, according to Theorem 3.8 (ii).

(iii) Assume that $S \in \mathcal{S}_{2,2,2}$ and $\chi_S(G-e) \geq 3$. We prove that for at least one color $t \in \{1,2,3\}$, G contains no t-problematic pair. Let us choose t from $\{1,2,3\}$ such that $t \neq c'(u)$ and $t \neq c'(v)$. Then, for every two vertices x and y with

$c'(x) = c'(y) = t$, all shortest (x, y)-paths contain $e = uv$ and the distance $d_G(x, y)$ is at least $3 = s_t + 1$. Therefore, we have $|Z| \le \chi_S(G-e) - 1$ again and may conclude $\chi_S(G) \le 2\chi_S(G-e) - 1$. $\square$

We note that the inequalities in Theorem 5.2 (i) and (ii) remain valid if the packing sequence S is finite and we suppose that G is S-packing colorable. Indeed, if $2\chi_S(G-e) \le |S|$, the proof given above remains valid. If $2\chi_S(G-e) > |S|$, then the S-packing colorability of G immediately implies $\chi_S(G) \le |S| \le 2\chi_S(G-e) - 1$ and the two inequalities follow.

Setting $S = (1, 2, 3, \ldots)$ in Theorem 5.2 (ii), we get the following:

Corollary 5.3 [5, Theorem 1] *If $e \in E(G)$, then $\chi_\rho(G-e) \ge \frac{\chi_\rho(G)+1}{2}$.*

To see that the bound in Theorem 5.2 (i) is asymptotically sharp also when e is not a cut-edge, consider the following example for the constant packing sequence $S = (3, 3, \ldots)$. Let H be a graph with two universal vertices x and y, and let H' be an isomorphic copy of H with respective universal vertices x' and y'. Let G be the graph obtained from the disjoint union of H and H' by adding the edge $e = xx'$, and by connecting y and y' with a path of length 3.

Note that $n(G) = 2n(H) + 2$ and that $\mathrm{diam}(G) = 3$. Therefore, $\chi_S(G) = 2n(H) + 2$. Consider now $G - e$. Then we can assign color 1 to x and y', color 2 to y and x', whilst assigning each color from $\{3, \ldots, n(H)\}$ to the remaining pairs of vertices respectively, one from each of H and H'. Two further colors, $n(H) + 1$ and $n(H) + 2$ are used to color the two vertices outside $V(H) \cup V(H')$. In this way, we infer that $\chi_S(G-e) = n(H) + 2$. So $\lim_{n\to\infty} \frac{\chi_S(G-e)}{\chi_S(G)} = \frac{1}{2}$.

If the removed edge is a cut-edge, we can slightly improve Theorem 5.2.

Proposition 5.4 *Let $S \in \mathcal{S}$ and $s_2 \le 2$. If e is a cut-edge in a graph G and $\chi_S(G-e) \ge 2$, then $\chi_S(G-e) \ge \dfrac{\chi_S(G)+1}{2}$.*

Proof. Theorem 5.2 (ii) establishes the lower bound if $s_1 = 1$ and $s_2 \le 2$. Hence, it suffices to prove the lower bound for $s_1 = s_2 = 2$. Let $e = uv$ be a cut-edge in G, and G_1, G_2 be the two components in $G - e$. We may suppose that $u \in V(G_1)$ and $v \in V(G_2)$. We use the notations from the proof of Theorem 5.2. Assume first that some color $t \in \{1, 2\}$ is not in $\{c'(u), c'(v)\}$ and $c'(x) = c'(y) = t$. If x and y belong to the same component G_i, then $d_G(x, y) = d_{G-e}(x, y) \ge s_t + 1$ as c' is an S-packing coloring in $G - e$. If $x \in V(G_1)$ and $y \in V(G_2)$, then the distance $d_G(x, y) \ge 3 = s_t + 1$. We conclude that there is no t-problematic pair in G and $\chi_S(G) \le 2\chi_S(G-e) - 1$ holds for this case.

If both colors 1 and 2 are used on vertices u, v by c', we define a coloring c'' of $G - e$ by switching colors 1 and 2 in G_2. Since $s_1 = s_2$, coloring c'' remains an S-packing coloring. Moreover, as $c''(u) = c''(v)$ holds, $\chi_S(G) \leq 2\chi_S(G - e) - 1$ follows by the same reasoning as above. $\square$

In the sharp examples with $\chi_S(G) = 2\chi_S(G - e)$ given in the proof of Theorem 5.2 (i), the edge e is always a cut-edge. Therefore, the inequality in Proposition 5.4 does not hold for all graphs when $S \in \mathcal{S}_{1,\overline{3}} \cup \mathcal{S}_{2,\overline{5}} \cup \mathcal{S}_{\overline{3}}$.

6 Concluding remarks

- In Theorems 3.8 and 3.9 we have characterized 4-χ_S-critical graphs for most of the packing sequences S. The missing cases which remain to be considered are $S \in \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2}$. In fact, these are also the missing cases of 4-χ_S-vertex-critical graphs.

- In Theorem 4.3 we have characterized cycles which are S-packing critical for $S \in \mathcal{S}_{1,1}$, $S \in \mathcal{S}_{1,2,2}$, and $S \in \mathcal{S}_{1,\overline{2},3}$. The remaining cases are still to be explored.

- In Theorem 5.2 we have demonstrated that there are infinitely many sharp examples for the inequality $\chi_S(G - e) \geq \frac{\chi_S(G)}{2}$, for each $S \in \mathcal{S}_{1,\overline{3}} \cup \mathcal{S}_{2,\overline{5}} \cup \mathcal{S}_{\overline{3}}$, where e is a cut-edge. We next provide another sporadic example for the sharpness when $S \in \mathcal{S}_{2,3,\overline{11}}$. For this purpose, consider P_{14} and its middle edge e. Using case analysis, it can be checked that $\chi_S(P_{14}) = 8$. On the other hand, $P_{14} - e$ contains two components both of which are isomorphic to P_7 and we obtain $\chi_S(P_{14} - e) = 4 = \frac{\chi_S(P_{14})}{2}$. Proposition 5.4 shows that if $s_2 \leq 2$ and G contains a component with at least two edges, then the stronger inequality $\chi_S(G - e) \geq \frac{\chi_S(G)+1}{2}$ holds for every cut-edge e of G. The remaining cases are packing sequences with

  - $s_1 = 2$, $s_2 = 3$, and $3 \leq s_3 \leq 10$;
  - $s_1 = 2$, $s_2 = 4$.

  For these cases, it remains an open question whether $\chi_S(G - e) \geq \frac{\chi_S(G)+1}{2}$ holds whenever e is a cut-edge of G.

- In the above example when $S \in \mathcal{S}_{2,3,\overline{11}}$, we have stated that $\chi_S(P_{14}) = 8$. Establishing this result is not completely straightforward. In general, it would be of interest to determine $\chi_S(P_n)$ for any $S \in \mathcal{S}$ and any n.

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References

[1] A. Ali, T. Došlić, Mostar index: results and perspectives, Appl. Math. Comput. 404 (2021) 126245.

[2] K. Aoki, Improved 2-distance coloring of planar graphs with maximum degree 5, Discrete Math. 348 (2025) 114225.

[3] E. Bidine, T. Gadi, M. Kchikech, The exponential growth of the packing chromatic number of iterated Mycielskians, Discrete Appl. Math. 341 (2023) 232–241.

[4] B. Brešar, J. Ferme, P. Holub, M. Jakovac, P. Melicharová, S-packing colorings of distance graphs with distance sets of cardinality 2, Appl. Math. Comput. 490 (2025) 129200.

[5] B. Brešar, J. Ferme, Graphs that are critical for the packing chromatic number, Discuss. Math. Graph Theory 42 (2022) 569–589.

[6] B. Brešar, J. Ferme, S. Klavžar, D.F. Rall, A survey on packing colorings, Discuss. Math. Graph Theory 40 (2020) 923–970.

[7] B. Brešar, S. Klavžar, D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, Discrete Appl. Math. 155 (2007) 2303–2311.

[8] B. Brešar, K. Kuenzel, D.F. Rall, Claw-free cubic graphs are $(1, 1, 2, 2)$-colorable, Discrete Math. 348 (2025) 114477.

[9] M. Chen, L. Jiang, M Wang, S. Zhou, 2-distance coloring of planar graphs with maximum degree 5, Discrete Math. Algorithms Appl. 17 (2025) 2430002.

[10] Z. Deniz, On 2-distance 16-coloring of planar graphs with maximum degree at most five, Discrete Math. 348 (2025) 114379.

[11] A. Devillers, N. Kamčev, B. McKay, P. Ó Catháin, G. Royle, G. Van de Voorde, I. Wanless, D.R. Wood, Triangle-free graphs with diameter 2, `arXiv:2406.00246` (2024).

[12] A. El Zein, M. Mortada, Every non-regular subcubic graph is $(1,1,2,2)$-packing colorable, `arXiv:2503.20239` (2025).

[13] J. Ferme, A characterization of 4-χ_ρ-(vertex-)critical graphs, Filomat 36 (2022) 6481–6501.

[14] J. Ferme, D. Štesl, On distance dominator packing coloring in graphs, Filomat 35 (2021) 4005–4016.

[15] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris, D.F. Rall, Broadcast chromatic numbers of graphs, Ars Combin. 86 (2008) 33–49.

[16] W. Goddard, H. Xu, The S-packing chromatic number of a graph, Discuss. Math. Graph Theory 32 (2012) 795–806.

[17] D. Gözüpek, I. Peterin, Grundy packing coloring of graphs, Discrete Appl. Math. 371 (2025) 17–30.

[18] P. Gregor, J. Kranjc, B. Lužar, K. Štorgel, Packing coloring of hypercubes with extended Hamming codes, Discrete Appl. Math. 359 (2024) 269–277.

[19] H. Grochowski, K. Junosza-Szaniawski, Partial packing coloring and quasi-packing coloring of the triangular grid, Discrete Math. 348 (2025) 114308.

[20] M. Hasanvand, K. Ozeki, A note on 3-distance coloring of planar graphs, Bull. Iranian Math. Soc. 50(2) (2024) 25.

[21] P. Holub, J. Hofman, On S-packing colourings of distance graphs $D(1,t)$ and $D(1,2,t)$, Appl. Math. Comput. 447 (2023) 127855.

[22] P. Holub, M. Jakovac, S. Klavžar, S-packing chromatic vertex-critical graphs, Discrete Appl. Math. 285 (2020) 119–127.

[23] J. Jerebic, S. Klavžar, D.F. Rall, Distance-balanced graphs, Ann. Combin. 12 (2008) 71–79.

[24] S. Klavžar, H. Lei, X. Lian, Y. Shi, A characterization of 4-χ_S-vertex-critical graphs for packing sequences with $s_1 = 1$ and $s_2 \geq 3$, Discrete Appl. Math. 338 (2023) 46–55.

[25] S. Klavžar, D.F. Rall, Packing chromatic vertex-critical graphs, Discrete Math. Theor. Comput. Sci. 21(3) (2019) paper #8, 18 pp.

[26] S. Koley, S. Ghosh, On the span of ℓ distance coloring of infinite hexagonal grid, Internat. J. Found. Comput. Sci. 35 (2024) 791–813.

[27] A. Kostochka, X. Liu, Packing $(1, 1, 2, 4)$-coloring of subcubic outerplanar graphs, Discrete Appl. Math. 302 (2021) 8–15.

[28] D. König, Gráfok és mátrixok, Mat. Fiz. Lapok, 38 (1931) 116–119.

[29] F. Kramer, H. Kramer, Un probleme de coloration des sommets d'un graphe, CR Acad. Sci. Paris A 268(7) (1969) 46–48.

[30] F. Kramer, H. Kramer, A survey on the distance-coloring of graphs, Discrete Math. 308 (2008) 422–426.

[31] H. La, M. Montassier, 2-distance 4-coloring of planar subcubic graphs with girth at least 21, Discrete Math. Theor. Comput. Sci. 26 (2024) 19.

[32] H. La, M. Montassier, 2-distance $(\Delta + 1)$-coloring of sparse graphs using the potential method. Discrete Math. 348 (2025) 114292.

[33] R. Liu, X. Liu, M. Rolek, G. Yu, Packing $(1, 1, 2, 2)$-coloring of some subcubic graphs, Discrete Appl. Math. 283 (2020) 626–630.

[34] P.-S. Loh, J. Ma, Diameter critical graphs, J. Combin. Theory Ser. B 117 (2016) 34–58.

[35] Š. Miklavič, P. Šparl, ℓ-distance-balanced graphs, Discrete Appl. Math. 244 (2018) 143–154.

[36] M. Mortada, About S-packing coloring of 3-irregular subcubic graphs, Discrete Appl. Math. 359 (2024) 16–18.

[37] M. Mortada, O. Togni, Further results and questions on S-packing coloring of subcubic graphs, Discrete Math. 348 (2025) 114376.

[38] P.K. Naranjan, S.R. Kola, The distance chromatic number of trees and cycles, AKCE Int. J. Graphs Combin. 16 (2019) 230–235.

[39] G.F. Royle, Regular vertex diameter critical graphs, Australas. J. Combin. 26 (2002), 209–217.

[40] W. Yang, B. Wu, On packing S-colorings of subcubic graphs, Discrete Appl. Math. 334 (2023) 1–14.

[41] J. Zhu, Y. Bu, H. Zhu, Wegner's conjecture on 2-distance coloring for planar graphs, Theoret. Comput. Sci. 926 (2022) 71–74.