On the local metric dimension of K_4-free graphs

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June 3, 2025

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Abstract

Let G be a graph of order $n(G)$, local metric dimension $\dim_l(G)$, and clique number $\omega(G)$. It has been conjectured that if $n(G) \geq \omega(G) + 1 \geq 4$, then $\dim_l(G) \leq \left(\frac{\omega(G)-2}{\omega(G)-1}\right) n(G)$. In this paper the conjecture is confirmed for the case $\omega(G) = 3$. Consequently, a problem regarding the local metric dimension of planar graphs is also resolved.

Keywords: metric dimension; local metric dimension; clique number; planar graph

AMS Math. Subj. Class. (2020): 05C12, 05C69

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote the clique number of G by $\omega(G)$ and the order of G by $n(G)$. The open neighborhood $N_G(u)$ of $u \in V(G)$ is the set of vertices adjacent to u. The degree $d_G(u)$ of u is the cardinality of $N_G(u)$. The maximum degree of G is denoted by $\Delta(G)$. If V' is a subset of the vertex set $V(G)$, then the notation $G[V']$ refers to the subgraph of G that is induced by V'. Additionally, if H and H' are subgraphs of G, then $E_G(H, H')$ denotes the set of edges connecting vertices from H to vertices in H'. The union $G \cup G'$ of graphs G and G' is the graph with vertex set $V(G) \cup V(G')$

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and edge set $E(G) \cup E(G')$. For a positive integer $l \geq 1$, the notation $[l]$ represents the set $\{1, \ldots, l\}$. We also set $[0] = \emptyset$.

The distance $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of shortest u, v-path in G. The vertices u and v are *distinguished* by $w \in V(G)$, or equivalently, w *distinguishes* u and v, if $d_G(u, w) \neq d_G(v, w)$. A subset $W \subseteq V(G)$ is a *resolving set* for G if, for any two vertices u and v in $V(G) - W$, there exists at least one vertex in W that distinguishes u and v. Similarly, W is referred to as a *local resolving set* if, for any adjacent vertices u and v from $V(G) - W$, there is a vertex in W that distinguishes u and v. The *metric dimension* $\dim(G)$ and the *local metric dimension* $\dim_l(G)$ of G are defined as the sizes of the smallest resolving sets and the smallest local resolving sets for G, respectively. Clearly, $\dim_l(G) \leq \dim(G)$.

The metric dimension of graphs has a rich history, initially defined by Harary and Melter [10] and by Slater [19]. Determining the metric dimension is known to be NP-complete for general graphs [12] and also for restricted cases involving planar graphs with a maximum degree six [4]. Research in this area is extensive, partly because metric dimension has numerous real-world applications, including robot navigation, image processing, privacy in social networks, and tracking intruders in networks. The 2023 overview [20] of the essential results and applications of metric dimension contains well over 200 references.

Several variations of metric dimension gained a wider attention. The survey [15] focusing on these variants also cites over 200 papers. One particularly interesting variation is the local metric dimension, introduced in 2010 by Okamoto, Phinezy, and Zhang [17]. Like the standard metric dimension, the local metric dimension is computationally challenging [5, 6] and has been explored in several studies [1–3, 7, 8, 14, 16, 18]. We should also mention closely related research on the fractional local metric dimension [11] and the nonlocal metric dimension [13]. Okamoto et al. [17] proved several important relationships between local metric dimension and clique number: $\dim_l(G) = n(G) - 1$ if and only if $G \cong K_{n(G)}$; $\dim_l(G) = n(G) - 2$ if and only if $\omega(G) = n(G) - 1$; $\dim_l(G) = 1$ if and only if G is bipartite; and $\dim_l(G) \geq \max\left\{\lceil \log_2 \omega(G) \rceil, n(G) - 2^{n(G)-\omega(G)}\right\}$. Furthermore, Abrishami et al. [1] established that $\dim_l(G) \leq \frac{2}{5}n(G)$ when $\omega(G) = 2$ and $n(G) \geq 3$. They also posed:

Problem 1. [1, Problem 1] *If G is a planar graph with $n(G) \geq 2$, is it then true that*

$$\dim_l(G) \leq \left\lceil \frac{n(G) + 1}{2} \right\rceil ?$$

In [1] it was confirmed that Problem 1 has a positive answer for triangle-free planar graphs. On the other hand, it was demonstrated in [9] that the problem has a negative answer when $\omega(G) = 4$. Additionally, in [8] it was proved that

$\dim_l(G) \le \left(\frac{\omega(G)-1}{\omega(G)}\right) n(G)$, with equality holding if and only if $G \cong K_{n(G)}$, thus verifying a conjecture from [1].

Our second main motivation is:

Conjecture 2. [8, Conjecture 2] *If G is a graph with $n(G) \ge \omega(G)+1 \ge 4$, then*

$$\dim_l(G) \le \left(\frac{\omega(G)-2}{\omega(G)-1}\right) n(G)\,.$$

It is demonstrated in [8] that if Conjecture 2 is true, then the bound is asymptotically best possible, and that Conjecture 2 holds for all graphs G with $\omega(G) \in \{n(G)-1, n(G)-2, n(G)-3\}$. It is also established that when $\omega(G) = n(G)-2$, then $n(G)-4 \le \dim_l(G) \le n(G)-3$, and that when $\omega(G) = n(G)-3$, then $n(G)-8 \le \dim_l(G) \le n(G)-3$.

The main result of this paper reads as follows.

Theorem 3. *If G is a graph with $n(G) \ge 4 > \omega(G)$, then*

$$\dim_l(G) \le \left\lfloor \frac{n(G)}{2} \right\rfloor .$$

Theorem 3 implies that Problem 1 and Conjecture 2 have positive answers when $\omega(G) = 3$. Additionally, there exist infinitely many planar graphs that achieve the equality stated in the theorem. For any positive odd number n, let $G = \frac{n-1}{2}K_2 + K_1$, that is, a graph constructed from $\frac{n-1}{2}$ disjoint complete graphs K_2 by adding a new vertex and connecting it to all the vertices of the $\frac{n-1}{2}K_2$. It is straightforward to observe that the local metric dimension of G equals $\lfloor \frac{n}{2} \rfloor$. Therefore, there are infinitely many planar graphs G such that $\dim_l(G) = \left\lfloor \frac{n(G)}{2} \right\rfloor$.

So, Problem 1 has a positive answer for planar graph G with $\omega(G) \le 3$, while we know from before that there are planar graph G containing K_4 for which the bound of Problem 1 does not apply. It is important to highlight that Conjecture 2 remains unresolved for graphs G where $4 \le \omega(G) \le n(G)-4$.

2 Proof of Theorem 3

We begin the proof by describing a key approach to it. First, let F_i, $i \in [9]$, be the graphs of order at most 4 and with no isolated vertices, except K_4, as illustrated in Fig. 1.

Let now G be a graph with $n(G) \ge 4 > \omega(G)$. In the following, we will sequentially select in G and in its induced subgraphs maximum sets of vertex disjoint

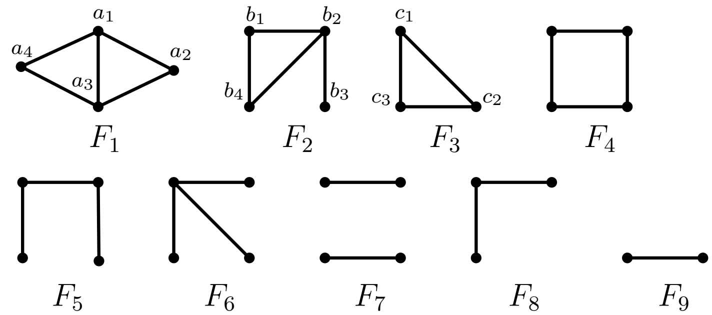

Figure 1: All non-complete graphs with at most 4 vertices and no isolated vertex.

induced subgraphs isomorphic to some F_i. Such a selection is not necessarily unique, but we will select one and fix it, so the following notation is well-defined for the purposes of the proof.

- Let $\mathcal{F}_1(G)$ be a maximum set of vertex disjoint induced subgraphs of G isomorphic to F_1.
- Set $G_1 = G$. For $i = 2, 3, \ldots, 9$, let $\mathcal{F}_i(G)$ be a maximum set of vertex disjoint induced subgraphs of $G_i = G_{i-1} - \bigcup_{H \in \mathcal{F}_{i-1}(G)} V(H)$ isomorphic to F_i.
- Note that $G_9 - \bigcup_{H \in \mathcal{F}_9(G)} V(H)$ is a set of isolated vertices, let $\mathcal{F}_{10}(G)$ be the set of these induced subgraphs isomorphic to K_1.
- Let $\mathcal{F}(G) = \{\mathcal{F}_i(G) :\ i \in [10]\}$, and call it a *local vertex division* of G.
- For $i \in [10]$, let $V_i = \cup_{H \in \mathcal{F}_i(G)} V(H)$.

The sets V_i, $i \in [10]$, form a partition of $V(G)$. We further emphasize that some of these sets may be empty, that is, some of the sets $\mathcal{F}_i(G)$ may be empty. The definition of the local vertex division and the assumption $\omega(G) \leq 3$ imply the following facts, where we use vertex labels from Fig. 1.

1. Let $H \in \mathcal{F}_1(G)$ and $v \in V(G - H)$. If for some $i \in \{2, 4\}$, $va_i \in E(G)$, then there exists $j \in \{1, 3\}$ such that $va_j \notin E(G)$.
2. Let $H \in \mathcal{F}_2(G)$ and $v \in V(G_2 - H)$. Then at most one vertex in the set $\{b_1, b_2, b_4\}$ is adjacent to v.
3. Let $H \in \mathcal{F}_4(G)$ and $v \in V(G_4 - H)$. Then at most two vertices of H are adjacent to v, and $G[V(H) \cup \{v\}]$ does not contain cycles of length 3.

4. Let $H \in \mathcal{F}_5(G)$ and $v \in V(G_5 - H)$. Then at most two vertices of H can be adjacent to v, and $G[V(H) \cup \{v\}]$ does not contain cycles of length 3 or 4.

5. For $i \in \{6, 7, 8, 9\}$, let $H \in \mathcal{F}_i(G)$ and $v \in V(G_i - H)$. Then at most one vertex of H is adjacent to v.

Based on the above five statements we claim that there exists a subset S of $V(G - \cup_{H \in \mathcal{F}_3(G)} H)$ such that

- for any $H \in \mathcal{F}_i(G)$, $i \in [10] \setminus \{3\}$, we have $|S \cap V(H)| \geq \frac{1}{2} n(H)$, and
- $V(G) - S$ is a local resolving set for G.

We construct S as follows, where we initially set $S = \emptyset$. For any graph $H_1 \in \mathcal{F}_1(G)$, we add to S its vertices a_2 and a_4 (see Fig. 1). Similarly, for any $H_2 \in \mathcal{F}_2(G)$, add to S either b_1 and b_2, or b_2 and b_4. Next, for any graph $H \in \mathcal{F}_4(G) \cup \mathcal{F}_5(G) \cup \mathcal{F}_6(G) \cup \mathcal{F}_7(G)$, add to S two arbitrary non-adjacent vertices of H. Further, for any $H_8 \in \mathcal{F}_8(G)$, add to S two arbitrary vertices of H_8, while for any element $H_9 \in \mathcal{F}_9(G)$, add to S an arbitrary vertex of H_9. Finally, set $S = S \cup \mathcal{F}_{10}(G)$. Since it is evident that S satisfies the required two conditions, the claim is proved.

If $\mathcal{F}_3(G)$ is empty, then the above constructed local resolving set $V(G) - S$ yields the theorem's assertion. Hence assume in the rest that $\mathcal{F}_3(G) \neq \emptyset$. Then the following statements hold, where in each case the argument is based on the fact that otherwise we could increase the cardinality of $\mathcal{F}_1(G)$ or $\mathcal{F}_2(G)$.

I. Let H and H' be two disjoint elements in the set $\mathcal{F}_3(G)$. Then, for any $h \in V(H)$ and any $h' \in V(H')$, it holds that $hh' \notin E(G)$.

II. There is no edge between the vertices that lie on the elements of $\mathcal{F}_3(G)$ and those that lie on the elements of $\cup_{i=4}^{10} \mathcal{F}_i(G)$. In other words, for each $u \in \big(\cup_{H \in \mathcal{F}_3(G)} V(H)\big)$ and $v \in \big(\cup_{i=4}^{10} \cup_{H \in \mathcal{F}_i(G)} V(H)\big)$, it holds that $uv \notin E(G)$.

III. Let $F \in \mathcal{F}_3(G)$ and $V(F) = \{c_1, c_2, c_3\}$. If for $v \in \big(\cup_{i=1}^{2} \cup_{H \in \mathcal{F}_i(G)} V(H)\big)$ and $i \in [3]$, we have $vc_i \in E(G)$, then v distinguishes either (c_i and c_j) or (c_i and c_k), where $[3] - \{i\} = \{j, k\}$.

IV. If for $H \in \mathcal{F}_2(G)$ and $F, F' \in \mathcal{F}_3(G)$, we have $V(H) \cap \big(\cup_{v \in V(F)} N_G(v)\big) \neq \emptyset$ and $V(H) \cap \big(\cup_{u \in V(F')} N_G(u)\big) \neq \emptyset$, then

$$|\big(V(H) \cap \big(\cup_{v \in V(F)} N_G(v)\big)\big) \cup \big(V(H) \cap \big(\cup_{u \in V(F')} N_G(u)\big)\big)| = 1.$$

In the rest of the proof we are going to add to the above constructed set S some of the vertices from the triangles from $\mathcal{F}_3(G)$, such that $V(G) - S$ remains a local resolving set for G. For this sake, some additional notation is needed.

- Let $\mathcal{F}_3(G) \neq \emptyset$, and let $\mathcal{A}$ be a subset of $\mathcal{F}_3(G)$. For $H \in \mathcal{F}_1(G)$, let $\eta_1(H, \mathcal{F}_3(G), \mathcal{A})$ represent the set of elements in $\mathcal{F}_3(G) - \mathcal{A}$ such that at least one vertex from each of these elements is adjacent to some vertices of H, that is,
$$\eta_1(H, \mathcal{F}_3(G), \mathcal{A}) = \{F : F \in \mathcal{F}_3(G) - \mathcal{A}, |E_G(H, F)| \geq 1\}.$$
- Analogously, for $H' \in \mathcal{F}_2(G)$, let $\eta_2(H', \mathcal{F}_3(G), \mathcal{A})$ represent the set of elements in $\mathcal{F}_3(G) - \mathcal{A}$ such that at least one vertex from each of these elements is adjacent to some vertices of H', that is,
$$\eta_2(H', \mathcal{F}_3(G), \mathcal{A}) = \{F : F \in \mathcal{F}_3(G) - \mathcal{A}, |E_G(H', F)| \geq 1\}.$$
- Also, let $H'' \in \mathcal{F}_1(G) \cup \mathcal{F}_2(G)$. For a subset U of $V(H'')$ and a subset Y of $\mathcal{F}_3(G)$, the notation $D(U, Y)$ represents the largest set of two-subsets $\{x, y\}$ such that the following conditions hold: (i) xy is an edge from a triangle in Y; (ii) x and y are distinguished by a vertex from U; (iii) any triangle from Y has at most one two-subset in $D(U, Y)$.

1^{st} process:

(1.1) Set $\mathcal{A} = \emptyset$ and consider the above-defined set S.

(1.2) Select $H \in \mathcal{F}_1(G)$ such that $\eta_1(H, \mathcal{F}_3(G), \mathcal{A})$ has maximum cardinality.

(1.3) If $|\eta_1(H, \mathcal{F}_3(G), \mathcal{A})| \leq 1$, then return S and $\mathcal{A}$, and end the process, otherwise go to (1.4).

(1.4) If $|\eta_1(H, \mathcal{F}_3(G), \mathcal{A})| \geq 4$, then set
$$\begin{aligned} S &= (S - V(H)) \cup D(V(H), \eta_1(H, \mathcal{F}_3(G), \mathcal{A})), \\ \mathcal{A} &= \mathcal{A} \cup \eta_1(H, \mathcal{F}_3(G), \mathcal{A}), \end{aligned}$$
and proceed to (1.2), otherwise go to (1.5).

(1.5) If $|\eta_1(H, \mathcal{F}_3(G), \mathcal{A})| \in \{2, 3\}$ and $h \in V(H)$ has the property $\eta_1(H, \mathcal{F}_3(G), \mathcal{A}) = \eta_1(H - h, \mathcal{F}_3(G), \mathcal{A})$, then set
$$\begin{aligned} S &= (S - V(H)) \cup D(V(H) - \{h\}, \eta_1(H, \mathcal{F}_3(G), \mathcal{A})) \cup \{h\}, \\ \mathcal{A} &= \mathcal{A} \cup \eta_1(H, \mathcal{F}_3(G), \mathcal{A}), \end{aligned}$$
and proceed to (1.2).

2nd process:

(2.1) Consider the sets $\mathcal{A}$ and S that are returned in the 1st process.

(2.2) Select $H \in \mathcal{F}_2(G)$ such that $\eta_2(H, \mathcal{F}_3(G), \mathcal{A})$ has maximum cardinality.

(2.3) If $|\eta_2(H, \mathcal{F}_3(G), \mathcal{A})| \leq 1$, then return S and $\mathcal{A}$, and end the process, otherwise go to (2.4).

(2.4) For $h \in V(H)$ such that $d_H(h) \geq 2$ and h is adjacent to no vertex from $\eta_2(H, \mathcal{F}_3(G), \mathcal{A})$, set

$$S = (S - V(H)) \cup \{h\} \cup D(V(H) - \{h\}, \eta_2(H, \mathcal{F}_3(G), \mathcal{A})),$$
$$\mathcal{A} = \mathcal{A} \cup \eta_2(H, \mathcal{F}_3(G), \mathcal{A}),$$

and proceed to (2.2).

3rd process:

(3.1) Consider the sets $\mathcal{A}$ and S that are returned in the 2nd process.

(3.2) If $\mathcal{F}_3(G) - \mathcal{A} \neq \emptyset$, then go to (3.3), otherwise return S and $\mathcal{A}$, and end the process.

(3.3) Take an element F of $\mathcal{F}_3(G)$. If there exists an element H in $\mathcal{F}_1(G)$ such that for a vertex h of it, $d_H(h) = 3$ and h is adjacent to a vertex of F, then set

$$S = S \cup D(\{h\}, \{F\}),$$
$$\mathcal{A} = \mathcal{A} \cup \{F\},$$

and proceed to (3.2), otherwise go to (3.4).

(3.4) If there exists an element H in $\mathcal{F}_2(G)$ such that for a vertex h of it, $d_H(h) \leq 2$ and h is adjacent to a vertex of F, then for $h_1, h_2 \in V(H)$ such that $d_H(h_1) = 2$, $d_H(h_2) = 3$, and $h_1 \neq h$, set

$$S = (S - V(H)) \cup \{h_1, h_2\} \cup D(\{h\}, \{F\}),$$
$$\mathcal{A} = \mathcal{A} \cup \{F\},$$

and proceed to (3.2).

Before starting the 4th process, let's consider the sets $\mathcal{A}$ and S that are returned in the 3rd process. Also, if z_1 and z_2 are integers such that $z_1 + z_2 = |\mathcal{F}_3(G) - \mathcal{A}|$, then let $\mathcal{F}_3(G) - \mathcal{A} = \{F_i^1 : i \in [z_1]\} \cup \{F_j^2 : j \in [z_2]\}$, where for $i \in [z_1]$, F_i^1 has adjacency with an element in $\mathcal{F}_1(G)$, and for $j \in [z_2]$, F_j^2 has adjacency with an element in $\mathcal{F}_2(G)$. Now, assume for $i \in [z_1]$, $H_i^1 \in \mathcal{F}_1(G)$ has adjacency with F_i^1, and for $j \in [z_2]$, $H_j^2 \in \mathcal{F}_2(G)$ has adjacency with F_j^2. Also, for $i \in [z_1]$, let's consider $V(F_i^1) = \{f_{i_1}^1, f_{i_2}^1, f_{i_3}^1\}$, $V(H_i^1) = \{h_{i_1}^1, h_{i_2}^1, h_{i_3}^1, h_{i_4}^1\}$, $d_{H_i^1}(h_{i_2}^1) = d_{H_i^1}(h_{i_4}^1) = 2$, $d_{H_i^1}(h_{i_1}^1) = d_{H_i^1}(h_{i_3}^1) = 3$, $f_{i_1}^1 h_{i_2}^1 \notin E(G)$, and $f_{i_2}^1 h_{i_2}^1 \in E(G)$. Plus, for $j \in [z_2]$, let's consider $V(F_j^2) = \{f_{j_1}^2, f_{j_2}^2, f_{j_3}^2\}$, $V(H_j^2) = \{h_{j_1}^2, h_{j_2}^2, h_{j_3}^2, h_{j_4}^2\}$, $d_{H_j^2}(h_{j_1}^2) = d_{H_j^2}(h_{j_4}^2) = 2$, $d_{H_j^2}(h_{j_2}^2) = 3$, $d_{H_j^2}(h_{j_3}^2) = 1$, and $f_{j_2}^2 h_{j_2}^2 \in E(G)$.

4th process:

(4.1) Consider the set S that is returned in the 3rd process and set $Z_1 = [z_1]$.

(4.2) If there is $i \in Z_1$ and $x \in \mathcal{F}_{10}(G)$ such that $\{xh_{i_1}^1, xh_{i_4}^1\} \subseteq E(G)$, then go to (4.3), otherwise return S, Z_1, $\mathcal{F}_{10}(G)$ and end the process.

(4.3) Set

$$\begin{aligned} S &= (S - \{x, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ \mathcal{F}_{10}(G) &= \mathcal{F}_{10}(G) - \{x\}, \end{aligned}$$

and proceed to (4.2).

Before starting the 5th process, let's set up $\mathcal{F}_9(G) = \{F_1^9, \ldots, F_{|\mathcal{F}_9(G)|}^9\}$.

5th process:

(5.1) Consider the sets S and Z_1 that are returned in the 4th process. Also, for $i \in [|\mathcal{F}_9(G)|]$, set $V_i^9 = V(F_i^9)$.

(5.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_9(G)|]$, such that for a vertex x in V_j^9 we have $\{xh_{i_1}^1, xh_{i_4}^1\} \subseteq E(G)$, then go to (5.3), otherwise return S, Z_1, and end the process.

(5.3) Set

$$\begin{aligned} S &= ((S \cup V_j^9) - \{x, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ V_j^9 &= V_j^9 - \{x\}, \end{aligned}$$

and proceed to (5.2).

Before starting the 6th process, let's set up $\mathcal{F}_8(G) = \{F^8_1, \ldots, F^8_{|\mathcal{F}_8(G)|}\}$.

<u>6th process:</u>

(6.1) Consider the sets S and Z_1 that are returned in the 5th process. Also, for $i \in [|\mathcal{F}_8(G)|]$, set $V^8_i = V(F^8_i)$.

(6.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_8(G)|]$, such that for a vertex x in V^8_j we have $\{xh^1_{i_1}, xh^1_{i_4}\} \subseteq E(G)$, then go to (6.3), otherwise return S, Z_1, and end the process.

(6.3) Set

$$\begin{aligned}
S &= ((S \cup V^8_j) - \{x, h^1_{i_2}\}) \cup \{h^1_{i_3}, f^1_{i_1}, f^1_{i_2}\}, \\
Z_1 &= Z_1 - \{i\}, \\
V^8_j &= V^8_j - \{x\},
\end{aligned}$$

and proceed to (6.2).

We set up next $\mathcal{F}_7(G) = \{\{x_{i_1}x_{i_2}, y_{i_1}y_{i_2}\} : i \in [|\mathcal{F}_8(G)|]\}$.

<u>7th process:</u>

(7.1) Consider the sets S and Z_1 that are returned in the 6th process. Also, for $i \in [|\mathcal{F}_7(G)|]$ and $a \in \{x, y\}$, set $V^7_{a_i} = \{a_{i_1}, a_{i_2}\}$.

(7.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_7(G)|]$, such that for $a \in \{x, y\}$ and $v \in V^7_{a_j}$ we have $\{vh^1_{i_1}, vh^1_{i_4}\} \subseteq E(G)$, then go to (7.3), otherwise return S, Z_1, and end the process.

(7.3) Set

$$\begin{aligned}
S &= ((S \cup V^7_{a_j}) - \{v, h^1_{i_2}\}) \cup \{h^1_{i_3}, f^1_{i_1}, f^1_{i_2}\}, \\
Z_1 &= Z_1 - \{i\}, \\
V^7_{a_j} &= V^7_{a_j} - \{v\},
\end{aligned}$$

and proceed to (7.2).

Let's set up $\mathcal{F}_6(G) = \{F_i^6 : i \in [|\mathcal{F}_6(G)|]\}$. Also, for $i \in [|\mathcal{F}_6(G)|]$, set up $V(F_i^6) = \{f_i^6 : i \in [4]\}$ and $d_{F_i^6}(f_4^6) = 3$.

8th process:

(8.1) Consider the sets S and Z_1 that are returned in the 7th process. Also, for $i \in [|\mathcal{F}_6(G)|]$, set $V_i^6 = V(F_i^6)$.

(8.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_6(G)|]$, such that for a vertex v in V_j^6 we have $\{vh_{i_1}^1, vh_{i_4}^1\} \subseteq E(G)$, then go to (8.3), otherwise return S, Z_1, and end the process.

(8.3) If $|V_j^6| \leq 2$, then set

$$\begin{aligned} S &= ((S \cup V_j^6) - \{v, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ V_j^6 &= V_j^6 - \{v\}, \end{aligned}$$

and proceed to (8.2), otherwise set

$$\begin{aligned} S &= ((S \cup (V_j^6 \cap \{f_i^6 : i \in [3]\}) - \{v, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ V_j^6 &= V_j^6 - \{v\}, \end{aligned}$$

and proceed to (8.2).

Set up now $\mathcal{F}_5(G) = \{F_i^5 : i \in [|\mathcal{F}_5(G)|]\}$, and for $X \subseteq V(G)$, let $\overline{X}$ be a maximum subset of X such that $E(G[X]) = \emptyset$.

9th process:

(9.1) Consider the sets S and Z_1 that are returned in the 8th process. Also, for $i \in [|\mathcal{F}_5(G)|]$, set $V_i^5 = V(F_i^5)$.

(9.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_5(G)|]$, such that for a vertex v in V_j^5 we have $\{vh_{i_1}^1, vh_{i_4}^1\} \subseteq E(G)$, then go to (9.3), otherwise return S, Z_1, and end the process.

(9.3) Set

$$\begin{aligned} X &= V_j^5 - \{v\}, \\ S &= ((S \cup \overline{X}) - \{v, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ V_j^5 &= V_j^5 - \{v\}, \end{aligned}$$

and proceed to (9.2).

Next, set up $\mathcal{F}_4(G) = \{F_i^4 : i \in [|\mathcal{F}_4(G)|]\}$.

<u>10^{th} process:</u>

(10.1) Consider the sets S and Z_1 that are returned in the 9^{th} process. Also, for $i \in [|\mathcal{F}_4(G)|]$, set $V_i^4 = V(F_i^4)$.

(10.2) If there is $i \in Z_1$ and $j \in [|\mathcal{F}_4(G)|]$, such that for a vertex v in V_j^4 we have $\{vh_{i_1}^1, vh_{i_4}^1\} \subseteq E(G)$, then go to (10.3), otherwise return S, Z_1, and end the process.

(10.3) Set

$$\begin{aligned} X &= V_j^4 - \{v\}, \\ S &= ((S \cup \overline{X}) - \{v, h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ V_j^4 &= V_j^4 - \{v\}, \end{aligned}$$

and proceed to (10.2).

<u>11^{th} process:</u>

(11.1) Consider the sets S and Z_1 that are returned in the 10^{th} process. Also, set $Z_2 = [z_2]$.

(11.2) If there are $i \in Z_1$ and $j \in Z_2$ such that $\{h_{i_1}^1 h_{j_3}^2, h_{i_4}^1 h_{j_3}^2\} \subseteq E(G)$, then go to (11.3), otherwise return S, Z_1, Z_2, and end the process.

(11.3) Set

$$\begin{aligned} S &= (S - \{h_{i_2}^1\}) \cup \{h_{i_3}^1, f_{i_1}^1, f_{i_2}^1, f_{j_1}^1\}, \\ Z_1 &= Z_1 - \{i\}, \\ Z_2 &= Z_2 - \{j\}, \end{aligned}$$

and proceed to (11.2).

<u>12^{th} process:</u>

(12.1) Consider the sets S and Z_1 that are returned in the 11^{th} process.

(12.2) If there are $i, j \in Z_1$ such that $\{h_{i_1}^1 h_{j_4}^2, h_{i_4}^1 h_{j_4}^2\} \subseteq E(G)$, then go to (12.3), otherwise return S, Z_1, and end the process.

(12.3) Set

$$S = (S - \{h^1_{i_2}, h^2_{j_2}, h^2_{j_4}\}) \cup \{h^1_{i_3}, f^1_{i_1}, f^1_{i_2}, h^2_{j_3}, f^2_{j_1}, f^2_{j_2}\},$$
$$Z_1 = Z_1 - \{i, j\},$$

and proceed to (12.2).

13$^{\text{th}}$ process:

(13.1) Consider the sets S and Z_1 that are returned in the 12$^{\text{th}}$ process.

(13.2) If $Z_1 \neq \emptyset$, then take an element i of Z_1 and go to (13.3), otherwise return S and end the process.

(13.3) Set

$$S = (S - \{h^1_{i_2}\}) \cup \{h^1_{i_3}, f^1_{i_1}, f^1_{i_2}\},$$
$$Z_1 = Z_1 - \{i\},$$

and proceed to (13.2).

14$^{\text{th}}$ process:

(14.1) Consider the set $\mathcal{F}_{10}(G)$ that is returned in the 4$^{\text{th}}$ process, the set Z_2 that is returned in the 11$^{\text{th}}$ process, and the set S that is returned in the 13$^{\text{th}}$ process.

(14.2) If there is $i \in Z_2$ and $x \in \mathcal{F}_{10}(G)$ such that $\{xh^2_{i_2}, xh^2_{i_3}\} \subseteq E(G)$, then go to (14.3), otherwise return S, Z_2, and end the process.

(14.3) Set

$$S = (S - \{x, h^2_{i_2}\}) \cup \{h^2_{i_3}, f^2_{i_1}, f^2_{i_2}\},$$
$$Z_2 = Z_2 - \{i\},$$
$$\mathcal{F}_{10}(G) = \mathcal{F}_{10}(G) - \{x\},$$

and proceed to (14.2).

15$^{\text{th}}$ process:

(15.1) Consider the sets S and Z_2 that are returned in the 14$^{\text{th}}$ process.

(15.2) If $Z_2 \neq \emptyset$, then take an element i of Z_2 and go to (15.3), otherwise return S and end the process.

(15.3) Set

$$
\begin{aligned}
S &= (S - \{h_{i_2}^2\}) \cup \{h_{i_3}^2, f_{i_1}^2, f_{i_2}^2\}, \\
Z_2 &= Z_2 - \{i\},
\end{aligned}
$$

and proceed to (15.2).

Now, let's examine the set S that is produced in the 15^{th} processes. It is clear that $|S| \geq \frac{n(G)}{2}$. Furthermore, by utilizing $\omega(G) \leq 3$ and the maximality of $\mathcal{F}_i(G)$ for $i \in [9]$, we can observe that $V(G) - S$ serves as a local resolving set for G. Since $\dim_l(G)$ is an integer, we have proved Theorem 3.

Acknowledgments

The research of Ali Ghalavand and Xueliang Li was supported the by NSFC No. 12131013 and 12161141006. Sandi Klavžar was supported by the Slovenian Research Agency (ARIS) under the grants P1-0297, N1-0355, and N1-0285.

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