

Minimum mitigating matchings and the chromatic edge stability index

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Abstract

A set of edges whose deletion decreases the chromatic index of a graph G is called a mitigating set. Let $es_{\chi'}(G)$ be the minimum cardinality of such a set. In [European J. Combin. 111 (2023) Article 103690] it was conjectured that every nonempty graph G has a mitigating set of cardinality $es_{\chi'}(G)$ which is a matching. In this note the conjecture is verified for graphs G with $es_{\chi'}(G) = 3$. On the other hand, the conjecture is generally refuted, since for every integer $k \geq 4$ there exists a connected graph G with $es_{\chi'}(G) = k$ and $\min_{\chi'}(G) = k+1$.

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1 Introduction

Over the past decade, the chromatic edge stability number and the chromatic edge stability index have been the subject of considerable interest. The *chromatic edge stability number* of a graph is the minimum number of edges whose deletion results in a graph with smaller chromatic number. Similarly, the *chromatic edge stability index* of G , denoted by $es_{\chi'}(G)$, is the minimum number of edges whose deletion

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results in a graph H with $\chi'(H) = \chi'(G) - 1$. In this note, we focus on the latter invariant; for the chromatic edge stability number, which was initiated back in 1980 by Staton [10], we refer the reader to [2, 5–8]. (Note that in [6], the chromatic edge stability number was called the chromatic stability index!)

The chromatic edge stability index was proposed and studied for the first time in [3]. The invariant was further investigated in [4], where, among other results, $\text{es}_{\chi'}(G)$ was determined for bipartite graphs in terms of a maximum matching in the subgraph induced by the vertices of maximum degree. Very recently, Abelyan and Petrosyan [1] studied the chromatic edge stability index in planar and regular graphs. Now, calling a set of edges whose deletion decreases the chromatic index of a given graph a *mitigating set*, the following conjecture from the seminal paper was our main motivation for this note.

Conjecture 1.1. [3, Conjecture 1] *Every nonempty graph has a minimum mitigating set which is a matching.*

On the positive side, in Section 3 we prove that Conjecture 1.1 holds true for graphs G with $\text{es}_{\chi'}(G) = 3$. On the negative side, in Section 4 we demonstrate that the conjecture is false in general. Before we move on to the results listed above, we'll take care of all the necessary preparations in the next section.

2 Preliminaries

All graphs considered in this paper are simple. For a graph G , write $n = |V(G)|$ and $\Delta = \Delta(G)$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S . A matching with k edges is called a k -*matching*. The chromatic index $\chi'(G)$ is the least number of colors in a proper edge coloring of G . By Vizing's theorem [11],

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

Graphs with $\chi'(G) = \Delta$ are called *Class 1 graphs*, and those with $\chi'(G) = \Delta + 1$ are called *Class 2 graphs*.

If G is a nonempty graph, then the *chromatic edge stability index* of G is

$$\text{es}_{\chi'}(G) = \min\{|F| : F \subseteq E(G), \chi'(G - F) < \chi'(G)\},$$

where $G - F$ is the graph obtained from G by deleting the edges in F . Recall that a set F whose deletion decreases $\chi'(G)$ is called a mitigating set. We further say a mitigating set is *minimum* if $|F| = \text{es}_{\chi'}(G)$. Setting

$$\min_{\chi'}(G) = \min\{|M| : M \subseteq E(G) \text{ is a matching and } \chi'(G - M) < \chi'(G)\},$$

we can give an equivalent, short form of Conjecture 1.1: $\text{es}_{\chi'}(G) = \min_{\chi'}(G)$ holds for every G . Since every color class of an optimal edge coloring is a mitigating matching, we have the following chain of inequalities:

$$1 \leq \text{es}_{\chi'}(G) \leq \min_{\chi'}(G) \leq n/2. \quad (1)$$

Conjecture 1.1 was proved in [3] for bipartite graphs and for graphs with $\text{es}_{\chi'}(G) = 2$. Together with the other cases established there, it is known that

$$\text{es}_{\chi'}(G) = \min_{\chi'}(G) \quad \text{if } G \text{ is bipartite or } \text{es}_{\chi'}(G) \in \left\{1, 2, \left\lfloor \frac{n}{2} \right\rfloor - 1, \left\lfloor \frac{n}{2} \right\rfloor\right\}. \quad (2)$$

An edge set *covers* a vertex if it contains an incident edge. Clearly, every mitigating set of a Class 1 graph covers all vertices of maximum degree.

A *maximum Δ -edge-colorable spanning subgraph* of G is a spanning subgraph with the largest size among those admitting a proper Δ -edge coloring. Mkrtchyan and Steffen proved the following:

Theorem 2.1. [9, Theorem 7] *Every simple graph G has a maximum Δ -edge-colorable spanning subgraph whose omitted edges form a matching.*

Using this result, we can now quickly prove that Conjecture 1.1 holds for all Class 2 graphs.

Corollary 2.2. *If G is Class 2, then $\text{es}_{\chi'}(G) = \min_{\chi'}(G)$.*

Proof. By Theorem 2.1, the complement of some maximum Δ -edge-colorable spanning subgraph is a matching. For a Class 2 graph, it is a minimum mitigating set. $\square$

Moreover, for regular graphs we have the following specified confirmation of the conjecture.

Corollary 2.3. *Every nonempty regular graph G has a minimum mitigating matching. If G is Class 1, then $\text{es}_{\chi'}(G) = n/2$ and every minimum mitigating set is a perfect matching.*

Proof. The Class 2 case follows from Corollary 2.2. For the Class 1 case, the assertion is obvious. $\square$

3 Conjecture 1.1 holds true for $\text{es}_{\chi'}(G) = 3$

As we know from (2), the case $\text{es}_{\chi'}(G) = 2$ of Conjecture 1.1 was already settled, see [3, Theorem 4.4]. In the main result of this section we prove the case $\text{es}_{\chi'}(G) = 3$.

For a proper edge coloring φ , let $\overline{\varphi}(x)$ denote the set of colors missing at x . The edges of two colors α, β form some paths and even cycles. Interchanging α and β on a component preserves the proper coloring.

Lemma 3.1. *If G satisfies $\text{es}_{\chi'}(G) = 3$ and $\min_{\chi'}(G) > 3$, then some minimum mitigating set has the form*

$$F = \{xy, xz, uv\}, \quad (3)$$

where x, y, z, u, v are distinct.

Proof. Choose an edge e in a minimum mitigating set. Then $\chi'(G - e) = \chi'(G)$ and $\text{es}_{\chi'}(G - e) = 2$. By (2), $G - e$ has a mitigating 2-matching M . Thus $F = M \cup \{e\}$ is a minimum mitigating set in G . Since F is not a matching, the only case not covered by (3) is a path $F = \{ab, bc, cd\}$.

Let φ be a proper $(\chi'(G) - 1)$ -edge coloring of $G - F$, and choose $\alpha \in \overline{\varphi}(b)$. By minimality, α occurs at both a and c . Let aa' have color α . If $aa' \notin \{ac, ad\}$, uncolor aa' and color ab with α , obtaining (3). If $aa' = ad$, uncolor the edge of color α at c and color bc with α , obtaining (3) as before. We may therefore assume $\varphi(ac) = \alpha$. Applying the same argument at the other end of the path, we may also assume $\varphi(bd) = \beta$ for some $\beta \in \overline{\varphi}(c)$. Since α occurs at c , we have $\beta \neq \alpha$.

The vertices b and c lie on the same alternating (α, β) -path P . Otherwise, interchanging the colors on the component containing b would allow bc to be colored with β , contrary to $\text{es}_{\chi'}(G) = 3$. The path P starts with bd and ends with ac . Let dw follow bd on P . Then $w \notin \{a, b, c, d\}$. Uncolor dw , interchange α, β on the segment of P from w to c , and color bc with α . The uncolored set is now $\{ab, cd, dw\}$, as required. $\square$

We next consider a 5-cycle together with disjoint paths attached to its vertices.

Lemma 3.2. *Let C be a 5-cycle and let R be a graph whose components are paths or even cycles. Assume that $E(C) \cap E(R) = \emptyset$, each vertex of C is incident with at most one edge of R , and that every component of R meeting C in two vertices has odd length. If R contains at most one chord of C , then $Q = C \cup R$ has a matching N with $|N| \leq 3$ such that $Q - N$ is 2-edge-colorable.*

Proof. Components of R disjoint from C are already 2-edge-colorable. Every other non-trivial component is a path meeting C only at one or both endpoints. We may add a pendant edge at each vertex of C incident with no edge of R . If fewer than two

paths join pairs of vertices of C , join the outside endpoints of two paths meeting C only once. Use an edge or a 2-edge path so that the resulting path has odd length. Repeat until two paths join pairs of vertices of C , leaving one path attached to the remaining vertex. All new vertices are chosen distinct. Every new edge has an endpoint outside C , so no chord of C is added. A matching N obtained in the enlarged graph gives the required matching $N \cap E(Q)$ in the original graph. Write $C = v_0v_1v_2v_3v_4v_0$, with v_4 incident with the path having only one endpoint on C . The other four vertices are paired by two odd paths, denoted by P_{ij} according to their endpoints. There are three possible pairings.

Case 1: the paths are P_{01} and P_{23} .

Since $v_0v_1 \in E(C)$, the path P_{01} has length at least three. Let v_0w be its first edge and take $N = \{v_1v_2, v_3v_4, v_0w\}$. Every vertex of C loses one incident edge. The only cycle remaining in the part meeting C is $P_{23} \cup \{v_2v_3\}$, which has even length.

Case 2: the paths are P_{02} and P_{13} .

At most one of them is a chord. So, by reversing the order on C if necessary, assume that P_{02} has length at least three. Let v_0w be its first edge and again take $N = \{v_1v_2, v_3v_4, v_0w\}$. After these deletions, the part meeting C is a path.

Case 3: the paths are P_{03} and P_{12} .

The path P_{12} has length at least three. Let v_1w be its first edge and take $N = \{v_2v_3, v_4v_0, v_1w\}$. Again, the part meeting C is a path. In each case, $w \notin V(C)$, so N is a 3-matching. Its deletion leaves maximum degree at most two and no odd cycle. Thus the remaining graph is 2-edge-colorable. $\square$

Theorem 3.3. *If $\text{es}_{\chi'}(G) = 3$, then $\min_{\chi'}(G) = 3$.*

Proof. Suppose that $\min_{\chi'}(G) > 3$. By Corollary 2.2, G is Class 1. If $\Delta \leq 2$, then G is bipartite, contrary to (2). Thus $\Delta \geq 3$. Let $F = \{xy, xz, uv\}$ be as in (3), put $k = \Delta - 1$, and fix a proper k -edge coloring of $G - F$. Some color α is missing at x . By minimality, it occurs at both y and z . If the edges of color α at y, z are distinct and one of them, say yy' , avoids $\{u, v\}$, uncolor yy' and color xy with α . This gives the mitigating 3-matching $\{yy', xz, uv\}$, a contradiction. Hence, after possibly interchanging u, v , either

$$\varphi(yz) = \alpha \tag{4}$$

or

$$\varphi(yu) = \varphi(zv) = \alpha. \tag{5}$$

We first exclude (5) for every choice of F and its coloring. The edges in $F \cup \{yu, zv\}$ form the 5-cycle $C = xyuvzx$. Fix $\beta \neq \alpha$, and let R consist of the edges

colored α or β , except yu, zv . Its components are paths or even cycles. Each vertex of C is incident in R only with an edge of color β , if any. Hence every path in R joining two vertices of C begins and ends with color β , and has odd length. If R contained at most one chord of C , Lemma 3.2 would give a matching of at most three edges whose deletion makes $C \cup R$, 2-edge-colorable. Keeping the other $k - 2$ colors would then give a mitigating matching of G , a contradiction. Thus each $\beta \neq \alpha$ supplies two independent chords of C . Since $\chi'(G) = k + 1$ and each color class contains at most two edges in $G[V(C)]$,

$$2k + 3 = 5 + 2(k - 1) \leq |E(G[V(C)])| \leq 2(k + 1),$$

a contradiction.

Suppose (4), and choose $\beta \neq \alpha$. Let P be the component containing x in the subgraph formed by colors α, β . Since α is missing at x , this component is a path with endpoint x , or the single vertex x . The edge yz has color α , so P contains either both y, z or neither of them.

If P contains neither y nor z , interchange the colors on P . Now, β is missing at x , while yz still has color α . The argument preceding (4), with β instead of α , gives a mitigating 3-matching or (5) with the colors interchanged. Both have been excluded. Otherwise, suppose P meets y, z in that order from x . Uncolor yz , color xy with α , and interchange α, β on the segment of P from z to its other endpoint. We obtain a proper k -edge coloring of $G - F'$, where $F' = \{xz, yz, uv\}$, with β missing at z and $\varphi(xy) = \alpha$. The same argument, applied at z with β as the missing color, gives the final contradiction. $\square$

4 A counterexample to Conjecture 1.1

We now give a counterexample to Conjecture 1.1.

Let H be obtained from the path $v_0v_1 \cdots v_{11}$ by adding the edges

$$D = \{v_0v_4, v_0v_{10}, v_1v_8, v_3v_9\}.$$

The graph H is shown in Figure 1, where the edges in D are dashed.

Theorem 4.1. *The graph H satisfies*

$$\text{es}_{\chi'}(H) = 4 < 5 = \min_{\chi'}(H).$$

In particular, H is a counterexample to Conjecture 1.1.

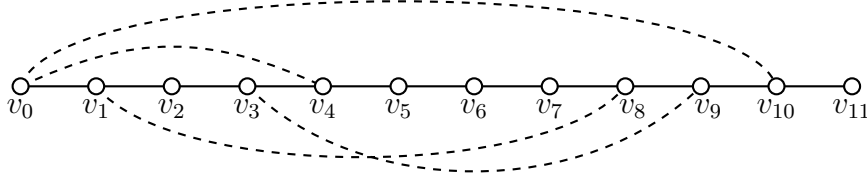


Figure 1: The graph H

Proof. The color classes

$$\begin{aligned} A &= \{v_0v_{10}, v_1v_2, v_3v_4, v_5v_6, v_8v_9\}, \\ B &= \{v_0v_4, v_1v_8, v_3v_9, v_6v_7, v_{10}v_{11}\}, \\ C &= \{v_0v_1, v_2v_3, v_4v_5, v_7v_8, v_9v_{10}\} \end{aligned}$$

give a proper 3-edge coloring of H . Hence $\chi'(H) = \Delta(H) = 3$ and $\min_{\chi'}(H) \leq 5$.

Every mitigating set covers the seven vertices

$$T = \{v_0, v_1, v_3, v_4, v_8, v_9, v_{10}\}$$

of degree 3, while $H - D$ is a path. Thus $\text{es}_{\chi'}(H) = 4$.

A 4-matching covering T has exactly one endpoint outside T . Its edge incident with that endpoint is one of

$$v_1v_2, \quad v_2v_3, \quad v_4v_5, \quad v_7v_8, \quad v_{10}v_{11}.$$

The remaining edges form a perfect matching on the six uncovered vertices of T . The graph $H[T]$ consists of the three internally disjoint paths $v_0v_{10}v_9$, $v_0v_4v_3v_9$, and $v_0v_1v_8v_9$. Each of the first four choices forces this matching, while the last leaves a 6-cycle and gives two choices. The six resulting matchings are listed in Table 1.

Each deletion leaves an odd cycle, so none is mitigating. We can conclude that $\min_{\chi'}(H) = 5$. $\square$

Having the graph H in hand, it is easy to produce infinite families of counterexamples to Conjecture 1.1. For instance, one can attach a path of arbitrary length to the vertex v_{11} .

We conclude the note by showing that Conjecture 1.1 fails for every value of the chromatic edge stability index larger than 3.

Theorem 4.2. *For every integer $k \geq 4$ there exists a connected graph G with $\text{es}_{\chi'}(G) = k$ and $\min_{\chi'}(G) = k + 1$.*

	Deleted matching	Surviving odd cycle
1	$\{v_1v_2, v_0v_{10}, v_3v_4, v_8v_9\}$	$(v_0, v_4, v_5, v_6, v_7, v_8, v_1)$
2	$\{v_2v_3, v_0v_4, v_1v_8, v_9v_{10}\}$	$(v_3, v_4, v_5, v_6, v_7, v_8, v_9)$
3	$\{v_4v_5, v_0v_{10}, v_1v_8, v_3v_9\}$	$(v_0, v_4, v_3, v_2, v_1)$
4	$\{v_7v_8, v_0v_1, v_3v_4, v_9v_{10}\}$	$(v_3, v_2, v_1, v_8, v_9)$
5	$\{v_{10}v_{11}, v_0v_1, v_3v_4, v_8v_9\}$	$(v_3, v_2, v_1, v_8, v_7, v_6, v_5, v_4, v_0, v_{10}, v_9)$
6	$\{v_{10}v_{11}, v_0v_4, v_1v_8, v_3v_9\}$	$(v_8, v_7, v_6, v_5, v_4, v_3, v_2, v_1, v_0, v_{10}, v_9)$

Table 1: All 4-matchings covering the vertices of degree 3 in H .

Proof. Let $t = k - 4$, and let H_t be the graph obtained from H by adding the path $v_{11}v_{12} \cdots v_{11+2t}$ and, for each $i \in \{1, \dots, t\}$, a new vertex w_i adjacent to v_{10+2i} . Thus $H_0 = H$, and in general H_t arises from H by attaching a tree at v_{11} . Hence H_t is connected, $\Delta(H_t) = 3$, every cycle of H_t is contained in H , and a proper 3-edge coloring of H extends greedily to H_t , so that $\chi'(H_t) = 3$.

Let $U = \{v_{10+2i} : 1 \leq i \leq t\}$ and $W = \{v_{10+2i}w_i : 1 \leq i \leq t\}$. The vertices of U have degree 3 in H_t , they are pairwise nonadjacent, and they are incident with no edge of H , while v_{11} and the remaining new vertices have degree at most two. Recall that a graph is 2-edge-colorable if and only if it has maximum degree at most two and no odd cycle. Since every vertex of H other than v_{11} is incident with the same edges in H_t as in H , it follows that a set $F \subseteq E(H_t)$ is mitigating in H_t if and only if $F \cap E(H)$ is mitigating in H and $F \setminus E(H)$ covers U . No edge covers two vertices of U , so at least t edges are needed to cover U . On the other hand, W is a matching that covers U , and no edge of W is incident with a vertex of H . Therefore $\text{es}_{\chi'}(H_t) = \text{es}_{\chi'}(H) + t$ and $\min_{\chi'}(H_t) = \min_{\chi'}(H) + t$, and Theorem 4.1 yields $\text{es}_{\chi'}(H_t) = k$ and $\min_{\chi'}(H_t) = k + 1$. $\square$

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Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of Interest Statement

The authors declare no conflict of interest.

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