Complex Analysis

Rafael Andrist

University of Ljubljana, Winter 2023/24
17 January 2024

Contents

1 Cauchy Integral Formula

By $\mathbb{C}$ we denote the field of complex numbers. By $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ we denote the unit disk. For an open subset $\Omega \subseteq \mathbb{C}$ we denote the set of holomorphic function on Ω by $\mathcal{O}(\Omega)$.

Definition 1.1. *Let $\mathbb{C} \ni z = x + iy$ with $x, y \in \mathbb{R}$. The* Wirtinger derivatives *are defined as follows:*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Remark 1.2. *Let f be a complex valued function that is real differentiable in a point a. Then f is complex differentiable in a if and only if $\frac{\partial f}{\partial \overline{z}}(a) = 0$.*

Theorem 1.3. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathcal{C}^1$-smooth boundary, and let f be a function in $\mathcal{C}^1(\overline{\Omega})$. Then*

$$f(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\iint_{\Omega} \frac{\partial f}{\partial \overline{\zeta}} \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z}$$

holds for every $z \in \Omega$.

Theorem 1.4. *Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $f \in \mathcal{O}(\Omega)$. For $a \in \Omega$ let $r := \operatorname{dist}(a, \partial\Omega)$. Then f can be developed in a complex power series about a that converges absolutely and uniformly to f on every disk $D_\rho(a)$ for any $\rho \in (0, r)$. Moreover, $f(z) = \sum_{k=0}^{\infty} c_k(z-a)^k$ has coefficients determined by*

$$c_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i}\oint_{\partial D_\rho(a)} \frac{f(\zeta)}{(\zeta - a)^{k+1}} d\zeta$$

Theorem 1.5 (Identity Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$. Assume that there exists $A \subset \Omega$ with an accumulation point inside Ω. If $f|A = 0$ then $f = 0$.*

Lemma 1.6. *Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $f \in \mathcal{O}(\Omega)$. Let $a \in \Omega$ and $r > 0$ such that $\overline{D_r(a)} \subset \Omega$. Assume that*

$$f(a) < \min_{z \in \partial D_r(a)} |f(z)|$$

Then f has a zero in $D_r(a)$.

Theorem 1.7 (Open Mapping Theorem). *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$. If f is not constant, then f is an open mapping.*

Theorem 1.8 (Maximum Principle).

1. *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in \mathcal{O}(\Omega)$. If $|f|$ attains a local maximum in Ω then f is constant.*

2. *Let $\Omega \subset\subset \mathbb{C}$ be a bounded domain and let $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then $|f|$ attains its maximum on the boundary $\partial\Omega$.*

Theorem 1.9 (Riemann Removable Singularity Theorem). *Let $\Omega \subseteq \mathbb{C}$ be an open subset, $a \in \Omega$ and let $f \in \mathcal{O}(\Omega \setminus \{a\})$. If f is locally bounded near a, then there exists a uniquely determined $F \in \mathcal{O}(\Omega)$ such that $F|\Omega \setminus \{a\} = f$.*

Theorem 1.10 (Lemma of Schwarz). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then the following holds:*

1. *$|f(z)| \leq |z|$ for all $z \in \mathbb{D}$*

2. *$f'(0)| \leq 1$*

If there exists a point $z_0 \in \mathbb{D}, z_0 \neq 0$, with $|f(z_0)| = |z_0|$ or if $f'(0)| = 1$, then

$$f(z) = \lambda \cdot z$$

for some $\lambda \in \partial\mathbb{D}$.

Definition 1.11. *We call an $f \in \mathcal{O}(\mathbb{C})$ an* entire *function.*

Theorem 1.12 (Liouville). *A bounded entire function is constant.*

Theorem 1.13. *Let $\Omega \subset \mathbb{C}$ be an open subset. Let $f_j \in \mathcal{O}(\Omega), j \in \mathbb{N}$, be a sequence of holomorphic functions that converges uniformly on compacts of Ω. Then the limit function $f := \lim_{j\to\infty} f_j$ exists and is also holomorphic on Ω. Moreover, then also $\lim_{j\to\infty} f_j^{(k)} = f^{(k)}$ uniformly on compacts of Ω for any derivative of order $k \in \mathbb{N}$.*

2 The $\overline{\partial}$-equation

Lemma 2.1. *Let $g \in \mathcal{C}^\infty(\mathbb{C})$ with compact support. Then there exist $f \in \mathcal{C}^\infty(\mathbb{C})$ such that*

$$\frac{\partial f}{\partial \overline{z}} = g$$

Theorem 2.2 (Dolbeault Lemma)**.** *Let $D_R := \{z \in \mathbb{C} : |z| < R\}$ be the disk of radius R where $0 < R \leq \infty$. For any given $g \in \mathcal{C}^\infty(D_R)$ there exist $f \in \mathcal{C}^\infty(D_R)$ such that*

$$\frac{\partial f}{\partial \overline{z}} = g$$

Definition 2.3. *Let $\Omega \subseteq \mathbb{C}$ be an open subset. A function is called meromorphic on Ω if there exists a subset $A \subset \Omega$ such that*

1. *A does not have any accumulation points in Ω*
2. *f is holomorphic on $\Omega \setminus A$*
3. *for each $a \in A$ there exists some $k \in \mathbb{N}$ with $\lim_{z \to a}(z-a)^k f(z) \neq 0$*

We call $a \in A$ a pole *of f.*

An *additive Cousin problem* for an open subset $\Omega \subseteq \mathbb{C}$ is a pair $(\{U_j\}_{j\in J}, \{f_j\}_{j \in J})$ of an open cover $\{U_j\}_{j\in J}$ of Ω and meromorphic functions $f_j \in \mathcal{M}(U_j)$ such that

$$f_j|U_j \cap U_k - f_k|U_j \cap U_k \in \mathcal{O}(U_j \cap U_k) \quad \text{for all } j, k \in J$$

A *solution of the additive Cousin problem* is a meromorphic $f \in \mathcal{M}(X)$ with $f|U_j - f_j \in \mathcal{O}(U_j)$ for all $j \in J$.

A *generalized additive Cousin problem* for an open subset $\Omega \subseteq \mathbb{C}$ is a pair $(\{U_j\}_{j\in J}, \{f_{jk}\}_{(j,k)\in J^2})$ of an open cover $\{U_j\}_{j\in J}$ of Ω and holomorphic functions $f_{jk} \in \mathcal{O}(U_j \cap U_k)$ such that:

- $f_{jk} = -f_{kj}$ on $U_j \cap U_k$
- $f_{jk} + f_{k\ell} + f_{\ell j} = 0$ on $U_j \cap U_k \cap U_\ell$

for all $j, k, \ell \in J$. A *solution of the generalized additive Cousin problem* are holomorphic functions $\widetilde{f}_j \in \mathcal{O}(U_j), j \in J$, with $\widetilde{f}_j - \widetilde{f}_k = f_{jk}$ for all $(j,k) \in J^2$.

Remark 2.4. *For each additive Cousin problem we can write down a corresponding generalized additive Cousin problem. A solution of the generalized additive Cousin problem then induces a solution of the original additive Cousin problem.*

Lemma 2.5. *Let* $(\{U_j\}_{j\in J}, \{f_{jk}\}_{j,k\in J})$ *be a generalized additive Cousin problem on an open set* $\Omega \subseteq \mathbb{C}$. *Then there exist smooth functions* $g_j \in \mathcal{C}^\infty(U_j)$ *with the property that* $g_j - g_k = f_{jk}$ *on* $U_j \cap U_k$.

Theorem 2.6. *Let* $D_R := \{z \in \mathbb{C} : |z| < R\}$ *be the disk of radius* R *where* $0 < R \le \infty$. *Then every generalized additive Cousin problem on* D_R *has a solution.*

For $m \in \mathbb{N}$ we call $\sum_{k=-m}^{-1} c_k(z-a)^k$ a *finite principal part* in the point a.

Theorem 2.7 (Theorem of Mittag-Leffler)**.** *Let* $(a_n)_n \subset \mathbb{C}$ *be a sequence without repetition and without accumulation point in* $\mathbb{C}$, *and let*

$$f_n = \sum_{k=-m_n}^{-1} c_{kn}(z-a_n)^k$$

be a finite principal part in each a_n *for* $n \in \mathbb{N}$. *Then there exists a meromorphic function* f *on* $\mathbb{C}$ *such that* f *has poles precisely in* $(a_n)_n$ *and with the principal part* f_n *in each* a_n.

3 Normal Families and Montel's Theorem

Definition 3.1. *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset. A family of functions* $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ *is called* locally bounded, *if for every* $p \in \Omega$ *there exists* $\delta > 0$ *and* $M > 0$ *such that*

$$\sup_{f\in\mathcal{F}} \sup_{z\in D_\delta(p)} |f(z)| < M$$

Theorem 3.2 (Montel's Theorem for sequences)**.** *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset. Let* $f_n\colon \Omega \to \mathbb{C}, n \in \mathbb{N}$, *be a sequence of holomorphic functions that is locally bounded. Then* $(f_n)_n$ *admits a subsequence that convergence uniformly on compacts.*

Theorem 3.3 (Arzelà–Ascoli)**.** *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset. Let* $\mathcal{F}$ *be an infinite family of functions* $\Omega \to \mathbb{C}$. *The family* $\mathcal{F}$ *contains a sequence that converges uniformly on compacts if*

1. $\mathcal{F}$ *is point-wise bounded, i.e.*

$$\forall x \in \Omega\, \exists M > 0\, \forall f \in \mathcal{F} \;:\; |f(x)| < M$$

2. $\mathcal{F}$ *is locally equi-continuous, i.e.*

$$\forall x \in \Omega\, \exists \rho > 0\, \forall \epsilon > 0\, \exists \delta > 0\, \forall f \in \mathcal{F}\, \forall z, w \in \Omega \cap D_\rho(x) \;: \\ |z-w| < \delta \Longrightarrow |f(z) - f(w)| < \varepsilon$$

Lemma 3.4. *Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $f_n \colon \Omega \to \mathbb{C}, n \in \mathbb{N}$, be a sequence of holomorphic functions that is locally bounded. Then $(f_n)_n$ is locally equi-continuous.*

Definition 3.5. *Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is called a* normal family *if every sequence in $\mathcal{F}$ contains a convergent subsequence.*

Definition 3.6 (Montel's Theorem for families)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open subset. A family $\mathcal{F} \subseteq \mathcal{O}(\Omega)$ is a normal family if and only if it is locally bounded.*

Theorem 3.7 (Vitali's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f_n \colon \Omega \to \mathbb{C}, n \in \mathbb{N}$, be a sequence of locally bounded holomorphic functions. Then the following are equivalent:*

1. *$(f_n)_n$ converges uniformly on compacts of Ω*
2. *$\exists p \in \Omega \, : \, (f_n^{(k)}(p))_n$ converges for all $k \in \mathbb{N}_0$*
3. *The set $A := \{w \in \Omega \, : \, \lim_{n\to\infty} f_n(w)$ exists$\}$ has an accumulation point in Ω*

4 Riemann Mapping Theorem

Theorem 4.1 (Riemann mapping theorem)**.** *For a domain $\Omega \subsetneq \mathbb{C}$ the following are equivalent:*

1. *Ω is simply connected.*
2. *There exist holomorphic logarithms on Ω for any $f \in \mathcal{O}^*(\Omega)$.*
3. *There exist holomorphic square-roots on Ω for any $f \in \mathcal{O}^*(\Omega)$.*
4. *There exists a biholomorphic mapping $f \colon \Omega \to \mathbb{D}$ onto the unit disk.*

Remark 4.2. *While $\mathbb{C}$ obviously can't be mapped biholomorphically to $\mathbb{D}$, it is easy to find a homeomorphism from $\mathbb{C}$ to $\mathbb{D}$. The Riemann mapping theorem hence implies the purely topological statement that every simply connected domain in $\mathbb{C}$ is homeomorphic to the unit disk.*

Lemma 4.3 (Existence of injections)**.** *Let $\Omega \subsetneq \mathbb{C}$ be a domain that admits holomorphic square-roots for nowhere-vanishing functions. Let $a \in \Omega$. Then there exists a holomorphic injection $f \colon \Omega \to \mathbb{D}$ with $f(a) = 0$.*

Definition 4.4. *Let $\Omega \subset \mathbb{D}$ be a domain, $0 \in \Omega$. A holomorphic injection $\kappa \colon \Omega \to \mathbb{D}$ with $\kappa(0) = 0$ and $|\kappa(z)| > |z|$ for all $z \in G \setminus \{0\}$ is called an* expansion.

By $g_c \colon \mathbb{D} \to \mathbb{D}$ we denote the automorphisms

$$g_(z) = \frac{z - c}{\bar{c}z - 1}, \quad c \in \mathbb{D}$$

Lemma 4.5 (Existence of expansions)**.** *Let $\Omega \subset \mathbb{D}$ be a domain that admits holomorphic square-roots for nowhere-vanishing functions. Let $c \in \Omega$ with $c^2 \notin \Omega$. Let v be the square-root of $g_{c^2}|\Omega \in \mathcal{O}(\Omega)$ with $v(0) = c$. Then the map $\kappa(z) = g_c(v(z))$ is an expansion of Ω, and $\mathrm{id}_\Omega = \psi_c \circ \kappa$ where $\psi_c = g_{c^2} \circ j \circ g_c$ and $j(z) = z^2$.*

Lemma 4.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f_n \colon \Omega \to \mathbb{C}, n \in \mathbb{N}$, be a sequence of holomorphic functions that converges uniformly on compacts of Ω to a non-constant function $f \in \mathcal{O}(\Omega)$. Then for any $p \in \Omega$ there exists a sequence $(p_n)_n \subseteq \Omega$ with $\lim_{n\to\infty} p_n = p$ and $f_n(p_n) = f(p)$ for every $n \in \mathbb{N}$.*

Theorem 4.7 (Hurwitz injection theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ and $\Omega' \subseteq \mathbb{C}$ be domains, and let $f_n \colon \Omega \to \Omega', n \in \mathbb{N}$, be a sequence of holomorphic injections that converges uniformly on compacts of Ω to a non-constant function $f \in \mathcal{O}(\Omega)$. Then $f(\Omega) \subseteq \Omega'$ and $f \colon \Omega \to \Omega'$ is injective.*

5 Theorems of Bloch, Schottky and Picard

Lemma 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and let $f \colon \overline{\Omega} \to \mathbb{C}$ be a continuous and $f|\Omega \colon \Omega \to \mathbb{C}$ an open mapping. Let $a \in \Omega$ be a point such that $s := \min_{z\in\partial\Omega} |f(z) - f(a)| > 0$. Then Ω contains the disk $D_s(f(a))$.*

Lemma 5.2. *Let $V \subseteq \mathbb{C}$ be an open subset. Let f be holomorphic in a neighborhood of $\overline{V}$ and non-constant. Assume that $\|f'\|_V \leq 2 \cdot |f'(a)|$. Then we have*

$$D_R(f(a)) \subseteq f(V)$$

where $R := (3 - 2\sqrt{2}) \cdot r \cdot |f'(a)|$

Theorem 5.3 (Bloch's Theorem)**.** *Let f be holomorphic in a neighborhood of $\overline{\mathbb{D}}$ with $f'(0) = 1$. Then $f(\mathbb{D})$ contains disks of radius $\frac{3}{2} - \sqrt{2} > \frac{1}{12}$.*

Lemma 5.4. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, and let $f \in \mathcal{O}(\Omega)$ with $1 \notin f(\Omega)$ and $-1 \notin \Omega$. Then there exists $F \in \mathcal{O}(\Omega)$ such that $f = \cos F$.*

Lemma 5.5. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain, and let $f \in \mathcal{O}(\Omega)$ with $0 \notin f(\Omega)$ and $1 \notin \Omega$. Then there exists $g \in \mathcal{O}(\Omega)$ such that*

$$f = \frac{1}{2}(1 + \cos(\pi(\cos(\pi g)))) \tag{1}$$

If $g \in \mathcal{O}(\Omega)$ is a function that satisfies (1), then $g(\Omega)$ does not contain a disk of radius 1.

Theorem 5.6 (Picard's Little Theorem)**.** *Every non-constant entire function omits at most one complex number as value.*

Let $S(r)$ denote the set of all functions $f \in \mathcal{O}(\overline{\mathbb{D}})$ with $|f(0)| \leq r$ that do not assume the values 0 and 1.
We choose any constant $\gamma > 0$ for which Bloch's Theorem holds, e.g. $\gamma = \frac{3}{2} - \sqrt{2} > \frac{1}{12}$.
We define $L\colon (0,1) \times (0,+\infty) \to \mathbb{R}^+$ by

$$L(\Theta, r) := \exp\left(\pi \exp\left(\pi\left(3 + 2r + \frac{\Theta}{\gamma(1-\Theta)}\right)\right)\right)$$

Theorem 5.7 (Schottky's Theorem)**.** *For any function $f \in S(r)$ we have that*

$$|f(z)| \leq L(\Theta, r)$$

for all $z \in \mathbb{D}$ with $|z| \leq \Theta$ where $\Theta \in (0,1)$.

Theorem 5.8 (Sharpened form of Montel's Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain in $\mathbb{C}$. Then the family $\mathcal{F} := \{f \in \mathcal{O}(\Omega) : f$ omits the values 0 and 1$\}$ is normal in Ω where we also allow convergence to ∞.*

Definition 5.9. *Let $\Omega \subseteq \mathbb{C}$ be a domain in $\mathbb{C}$, and $f_n \in \mathcal{O}(\Omega)$, $n \in \mathbb{N}$. We say that $(f_n)_n$ converges* to infinity *uniformly on compacts of Ω, if* $\lim_{n\to\infty} (\inf\{|f(z)| : z \in K\}) = \infty$

Theorem 5.10 (Picard's Great Theorem)**.** *Let $c \in \mathbb{C}$ be an isolated essential singularity of f. Then, in every neighborhood of c, f assumes every complex number as a value infinitely many times, with at most one exception.*

6 Infinite Products

Definition 6.1. *Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers. Then*

$$\left(\prod_{k=1}^{n} a_k\right)_{n \in NN}$$

is called the sequence of partial products *with* factors a_k.

Definition 6.2. *We call an* infinite product $\prod_{k=1}^{\infty} a_k$ convergent*, if there exists an index m such that*

$$\lim_{n\to\infty} \prod_{k=m}^{n} a_k =: \widehat{a}_m \neq 0$$

We define $a_1 \cdots a_{m-1} \cdot \widehat{a}_m$ to be the limit *of* $\prod_{k=1}^{\infty} a_k$.

Definition 6.3. *Let $X \subseteq \mathbb{C}$ be a subset and let $f_k \in \mathcal{C}(X), k \in \mathbb{N}$, be a sequence of functions. We call an* infinite product $\prod_{k=1}^{\infty} f_k$ convergent uniformly on compacts*, if for every compact $L \subset X$ there exists an index m such that*

$$\lim_{n\to\infty} \prod_{k=m}^{n} f_k =: \widehat{f}_m$$

exists as a uniform limit on L and such that $\widehat{f}_m$ is nowhere vanishing on L We define $f_1 \cdots f_{m-1} \cdot \widehat{f}_m$ to be the limit *of* $\prod_{k=1}^{\infty} f_k$ *on L.*

Definition 6.4. *Let $X \subseteq \mathbb{C}$ be a subset and let $f_k = 1 + g_k \in \mathcal{C}(X), k \in \mathbb{N}$, be a sequence of functions. We call an* infinite product $\prod_{k=1}^{\infty} f_k$ normally convergent*, if the series $\sum_{k=1}^{\infty} g_k$ is normally convergent, i.e. for every compact $L \subset X$ we have that*

$$\sum_{k=1}^{\infty} \|g_k\|_L < +\infty$$

Theorem 6.5 (Reordering Theorem)**.** *Let $X \subseteq \mathbb{C}$ be a subset and let $\prod_{k=1}^{\infty} f_k$ with $f_k \in \mathcal{C}(X)$ be normally convergent. Then there exists a function $f\colon X \to \mathbb{C}$ such that for all bijections $\tau\colon \mathbb{N} \to \mathbb{N}$ the product $\prod_{k=1}^{\infty} f_k$ converges to f uniformly on compacts of X.*

Lemma 6.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f = \prod_{k=1}^{\infty} f_k$ be a in Ω normally convergent product of holomorphic functions $f_k \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we have that*

1. $f \neq 0$
2. $Z(f) = \bigcup_{k=1}^{\infty} Z(f_k)$
3. $o_c(f) = \sum_{k=1}^{\infty} o_c(f_k)$ *for any* $c \in \Omega$

Lemma 6.7. *Let $\Omega \subseteq \mathbb{C}$ be a domain. If $f = \prod_{k=1}^{\infty} f_k$ is a in Ω normally convergent product of holomorphic functions $f_k \in \mathcal{O}(\Omega)$, then $\prod_{k=n}^{\infty} f_k =: \widehat{f}_n \to 1$ uniformly on compacts of Ω for $n \to \infty$.*

Definition 6.8. *Let $\Omega \subseteq \mathbb{C}$ be a domain. For functions meromorphic in Ω we call the series $\sum_{k=1}^{\infty} g_k$ normally convergent, if for every compact $L \subset X$ there exists an index m such that*

$$\sum_{k=m}^{\infty} \|g_k\|_L < +\infty$$

Theorem 6.9 (Logarithmic differentiation)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f = \prod_{k=1}^{\infty} f_k$ be a in Ω normally convergent product of holomorphic functions $f_k \in \mathcal{O}(\Omega) \setminus \{0\}$. Then*

$$\sum_{k=1}^{\infty} \frac{f_k'}{f_k}$$

is a normally convergent series of functions meromorphic in Ω and

$$\sum_{k=1}^{\infty} \frac{f_k'}{f_k} = \frac{f'}{f}$$

Lemma 6.10. *Let g be meromorphic in $\mathbb{C}$ with poles in $\mathbb{Z}$ where the principal part in $m \in \mathbb{Z}$ is given by $\frac{1}{z-m}$. Moreover, assume that g is an odd function that satisfies*

$$2g(2z) = g(z) + g(z + \frac{1}{2})$$

Then $g(z) = \pi \cot(\pi z)$ for $z \in \mathbb{C} \setminus \mathbb{Z}$.

Corollary 6.11.

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}$$

Theorem 6.12.

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

Lemma 6.13.

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$$

converges normally in $\mathbb{C}$.

We set $H(z) := z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$

Lemma 6.14.
$$H(1) = e^{-\gamma}$$
where
$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)\right)$$
is Euler's constant.

We set $\Delta(z) := e^{\gamma z} \cdot H(z)$.

Lemma 6.15. *1.* $\Delta(z) = z \cdot \Delta(z+1)$ *and* $\Delta(1) = 1$

2. $\pi \cdot \Delta(z) \cdot \Delta(1-z) = \sin(\pi z)$

Definition 6.16. *We define the* Gamma function $\Gamma(z) := \frac{1}{\Delta(z)}$.

Theorem 6.17. *Let F be holomorphic in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and assume the identity $F(z+1) = z \cdot F(z)$. If F is bounded on $\{z \in \mathbb{C} : 1 \leq \operatorname{Re} z < 2\}$ and $F(1) = 1$, then $F = \Gamma$.*

7 Weierstraß Products

Definition 7.1. *We define the following* Weierstraß factors*:*

$$E_0(z) = z - 1$$
$$E_n(z) = (z-1) \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^\ell}{\ell}\right), \quad n \geq 1$$

Lemma 7.2. *1.* $E_n'(z) = -z^n \cdot \exp\left(\sum_{\ell=1}^{n} \frac{z^\ell}{\ell}\right)$ *for* $n \geq 1$

2. $E_n(z) = 1 + \sum_{k=n+1}^{\infty} a_k z^k$ *where* $\sum_{k=n+1}^{\infty} |a_k| = 1$ *for* $n \geq 0$.

3. $|z| \leq 1 \implies |E_n(z) - 1| \leq |z|^{n+1}$ *for* $n \geq 0$

Theorem 7.3 (Weierstraß factorization theorem)**.** *For any sequence $(a_k)_k \subset \mathbb{C}$ without accumulation point there exists a* Weierstraß product

$$z^q \cdot \prod_{a_k \neq 0} E_{k-1}\left(\frac{z}{a_k}\right), \quad q = \#\{k \in \mathbb{N} : a_k = 0\}$$

that converges normally in $\mathbb{C}$.

Theorem 7.4 (Weierstraß product theorem)**.** *Every $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ can be written as*

$$f(z) = e^{g(z)} \cdot z^q \cdot \prod_{k=1}^{\infty} E_{k-1}\left(\frac{z}{a_k}\right)$$

where $g \in \mathcal{O}(\mathbb{C})$. The product over k may be empty or finite.

Lemma 7.5. *Let $A \subset \mathbb{C}$ be a discrete set such that $A' = \overline{A} \setminus A \neq \emptyset$.*

$$\begin{aligned} A_1 &:= \{z \in A \, : \, |z| \cdot \operatorname{dist}(z, A') \geq 1\} \\ A_2 &:= \{z \in A \, : \, |z| \cdot \operatorname{dist}(z, A') < 1\} \\ A_2(\varepsilon) &:= \{z \in A_2 \, : \, \operatorname{dist}(z, A') \geq \varepsilon\} \end{aligned}$$

Then A_1 has no accumulation point in $\mathbb{C}$, and $A_2(\varepsilon)$ is a finite set for every $\varepsilon > 0$.

Theorem 7.6 (Weierstraß product theorem for open subsets)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open subset. Let $(a_k)_k$ be a sequence without accumulation points in Ω. Set $A = \bigcup_{k=1}^{\infty}\{a_k\}$ and $A' := \overline{A} \setminus A$. Then there exists a Weierstraß product for $(a_k)_k$ that converges normally in $\mathbb{C} \setminus A'$.*

Example 7.7 (Blaschke products)**.** *Let $(a_k)_k \subset \mathbb{D}\setminus\{0\}$ be a sequence without accumulation points in $\mathbb{D}$. If*

$$\sum_{k=1}^{\infty} (1 - |a_k|) < +\infty$$

then

$$\prod_{k=1}^{\infty} E_0\left(\frac{a_k - b_k}{z - b_k}\right)$$

with $b_k = 1/\overline{a_k}$ converges normally in $\mathbb{D}$ and has zeros precisely in $(a_k)_k$, with multiplicity according to repetition.

Theorem 7.8. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{O}(\Omega) \setminus \{0\}$. Then we can write*

$$f = g \cdot \prod_{k=1}^{\infty} f_k$$

as a product that converges normally in Ω, where $g \in \mathcal{O}^*(\Omega)$ is a unit and $f_k \in \mathcal{O}(\Omega)$ are Weierstraß factors.

Theorem 7.9. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $f \in \mathcal{M}(\Omega) \setminus \{0\}$. Then $f = g/h$ for $g, h \in \mathcal{O}(\Omega)$ with no common zeros.*

8 Mittag-Leffler Series

Definition 8.1. *Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $(a_k)_k \subset \Omega$ be a sequence without repetition and without accumulation points in Ω. For each k, let $q_k = \sum_{\ell=1}^{\infty} c_{k,\ell}(z-a_k)^{-\ell}$ be a principal part in a_k. If there exist $g_k \in \mathcal{O}(\Omega)$ s.t.*

$$f(z) := \sum_{k=1}^{\infty}(q_k - g_k)$$

converges normally in Ω, then we call it a Mittag-Leffler series *in Ω for the* distribution of principal parts $(a_k, q_k)_k$.

Theorem 8.2 (Mittag-Leffler theorem for $\mathbb{C}$)**.** *For every distribution of principal parts in $\mathbb{C}$ there exists a Mittag-Leffler series.*

Theorem 8.3 (Mittag-Leffler theorem for open subsets)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open subset. For every distribution of principal parts $(a_k, q_k)_k$ in Ω there exists a Mittag-Leffler series that converges normally in $\mathbb{C} \setminus A'$ where $A' = \overline{A} \setminus A$ and $A = \bigcup_{k=1}^{\infty}\{a_k\}$.*

Theorem 8.4 (Mittag-Leffler osculation theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open subset and let $(a_k)_k \subset \Omega$ be a sequence without repetition and without accumulation points in Ω. Set $A = \bigcup_{k=1}^{\infty}\{a_k\}$. For each k, let $f_k = \sum_{\ell=-\infty}^{n(k)} c_{k,\ell}(z-a_k)^{\ell}$ with $n(k) \in \mathbb{N}$ be normally convergent on $\mathbb{C} \setminus A$. Then there exists $f \in \mathcal{O}(\Omega \setminus A)$ such that*

$$o_{a_k}(f - f_k) > n(k)$$

for each $k \in \mathbb{N}$.

Theorem 8.5 (Wedderburn's Lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $f, g \in \mathcal{O}(\mathbb{C})$ be relatively prime. Then they satisfy an equation*

$$af + bg = 1$$

with functions $a, b \in \mathcal{O}(\mathbb{C})$ and such that a has no zeros in Ω.

Theorem 8.6. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Every finitely generated ideal in $\mathcal{O}(\Omega)$ is a principal ideal.*

Definition 8.7. *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $I \subseteq \mathcal{O}(\Omega)$ be an ideal.*

1. *We say that $p \in \Omega$ is* zero *of I if $f(p) = 0$ for every $f \in I$.*
2. *We say that I is a* closed *if for every sequence $(f_k)_k \subseteq I$ that converges uniformly on compacts of Ω to $f \in \mathcal{O}(\Omega)$ we have $f \in I$.*

Theorem 8.8. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $I \subseteq \mathcal{O}(\Omega)$ be an ideal that has no zeros and is closed. Then $I = \mathcal{O}(\Omega)$.*

9 Runge Approximation

Lemma 9.1. *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset and let* $K \neq \emptyset$ *be a compact subset of* Ω*. Then there exist finitely many oriented horizontal or vertical line segments* $\sigma_1, \ldots, \sigma_n$ *of equal length in* $\Omega \setminus K$ *such that for every* $f \in \mathcal{O}(\Omega)$ *we have that:*

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\sigma_k} \frac{f(\zeta)}{z - \zeta} d\zeta \qquad \text{for } z \in K$$

Lemma 9.2. *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset and let* $K \neq \emptyset$ *be a compact subset of* Ω*. Then there exist finitely many oriented horizontal or vertical line segments* $\sigma_1, \ldots, \sigma_n$ *of equal length in* $\Omega \setminus K$ *such that for every* $f \in \mathcal{O}(\Omega)$ *and every* $\varepsilon > 0$ *there exists a rational function*

$$q(z) := \sum_{k=1}^{m} \frac{c_k}{z - w_k}, \quad c_k \in \mathbb{C}, w_k \in \bigcup_{k=1}^{n} \sigma_k$$

with

$$\|f - q\|_K < \varepsilon$$

Lemma 9.3. *Let* $K \subset \mathbb{C}$ *be a compact, and let* $a, b \in P$ *where* P *is a connected component of* $\mathbb{C} \setminus K$*. For any* $\varepsilon > 0$ *the function* $(z-a)^{-1}$ *can be approximated by a polynomial* q *in the variable* $(z-b)^{-1}$ *such that*

$$\|q(z) - \frac{1}{z-a}\|_K < \varepsilon$$

If P *is the unbounded component of* $K \subset \mathbb{C}$*, then* q *can be chosen to be a polynomial in* z*.*

For any set $P \subset \mathbb{C}$ we denote by $\mathbb{C}_P[z]$ the family of rational functions on $\mathbb{C}$ whose pole sets are contained in P. We note that $\mathbb{C}[z] \subset \mathbb{C}_P[z] \subset \mathcal{O}(O \setminus \overline{P})$.

Theorem 9.4. *Let* $K \subset \mathbb{C}$ *be a compact. If* P *has non-empty intersection with every connected component of* $\mathbb{C} \setminus K$*, then every function* f *that is holomorphic in a neighborhood of* K *can be approximated uniformly on* K *by functions from* $\mathbb{C}_P[z]$*, i.e. for every such* f *and for every* $\varepsilon > 0$ *there exist a* $q \in \mathbb{C}_P[z]$ *such that*

$$\|q - f\|_K < \varepsilon$$

Theorem 9.5. *Let* $\Omega \subseteq \mathbb{C}$ *be an open subset and let* $K \subset \Omega$ *be a compact. If every connected component of* $\mathbb{C} \setminus K$ *intersects* $\mathbb{C} \setminus \Omega$*, then every function* f *that is holomorphic in a neighborhood of* K *can be approximated uniformly on* K *by a function that is holomorphic on* Ω*, i.e. for every* f *and for every* $\varepsilon > 0$ *there exist a* $q \in \mathcal{O}(\Omega)$ *such that*

$$\|q - f\|_K < \varepsilon$$

Theorem 9.6 (Runge's Little Theorem). *Let $K \subset \mathbb{C}$ be a compact. If $\mathbb{C} \setminus K$ is connected, then every function f that is holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials, i.e. for every such f and for every $\varepsilon > 0$ there exist a $q \in \mathbb{C}[z]$ such that*

$$\|q - f\|_K < \varepsilon$$

Definition 9.7. *Let $\Omega \subseteq \mathbb{C}$ be an open subset, and let $\mathcal{F}$ be a family of certain functions $\Omega \to \mathbb{C}$. For every compact $K \subset \Omega$ we define the $\mathcal{F}$-*convex hull *of K by*

$$\widehat{K}_{\mathcal{F}} := \{z \in \Omega \,:\, |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{F}\} = \bigcap_{f \in \mathcal{F}} \{z \in \Omega \,:\, |f(z)| \leq \|f\|_K\}$$

Remark 9.8. *We call $\widehat{K}_{\mathcal{O}(\Omega)}$ the* holomorphically convex hull *of K, and we call $\widehat{K}_{\mathbb{C}[z]}$ the* polynomially convex hull *of K.*

Theorem 9.9. *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $K \subset \Omega$ be a compact. Then the following are equivalent:*

1. *$\Omega \setminus K$ has no relatively compact component in Ω.*
2. *Every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus \Omega$.*
3. *Every function holomorphic in a neighborhood of K can be uniformly approximated by rational functions with poles outside Ω.*
4. *Every function holomorphic in a neighborhood of K can be uniformly approximated by holomorphic functions on Ω.*
5. *For every $p \in \Omega \setminus K$ there exists $f \in \mathcal{O}(\Omega)$ such that $|f(p)| > \|f\|_K$.*
6. *$K = \widehat{K}_{\mathcal{O}(\Omega)}$*

Definition 9.10. *Let $\Omega \subseteq \mathbb{C}$ be an open subset. We call a compact component of $\mathbb{C} \setminus \Omega$ a* hole *of Ω.*

Lemma 9.11. *Let $\Omega \subseteq \mathbb{C}$ be an open subset. For every compact $K \subset \Omega$ there exists a compact K' such that $K \subseteq K' \subset \Omega$ and such that every bounded component of $\mathbb{C} \setminus K'$ contains a hole of Ω.*

Theorem 9.12. *Let $\Omega \subseteq \Omega' \subseteq \mathbb{C}$ be open subsets. Then the following are equivalent:*

1. *$\Omega' \setminus \Omega$ has no compact components.*

2. *The algebra of all rational functions without poles in* Ω' *is dense in* $\mathcal{O}(\Omega)$.

3. (Ω, Ω') *is a Runge pair.*

4. $\Omega' \setminus \Omega$ *contains no relatively open compact* $\neq \emptyset$.

10 Applications of Runge Approximation

Definition 10.1. *Let* V *be a topological vector space and let* $T\colon V \to V$ *be a linear operator. We call* T hypercyclic *if there exists a* (hyper-)cyclic *vector* $f \in V$ *such that the set of iterates*

$$\{f, T(f), T^2(f), \dots\}$$

is dense in V.

Theorem 10.2 (G.D. Birkhoff). *Let* $\tau\colon \mathbb{C} \to \mathbb{C}, \tau(z) = z+a, a \neq 0$, *be a non-trivial translation. Then the translation operator* $T\colon \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C}), T(f) = f \circ \tau$ *is hypercyclic.*

Definition 10.3. *Let* X *and* Y *be topological space. A map* $f\colon X \to Y$ *is called* proper *if for every compact* $K \subseteq Y$ *the pre-image* $f^{-1}(K)$ *is also compact.*

Definition 10.4. *Let* $\Omega \subseteq \mathbb{C}$ *be a domain. We call* $f\colon \Omega \to \mathbb{C}^n$ *an* embedding *if* f *is continuously differentiable, injective and immersive, i.e.* $f'(z) \neq 0$ *for all* $z \in \Omega$.

Theorem 10.5. *There exists a proper holomorphic embedding* $f\colon \mathbb{D} \to \mathbb{C}^3$ *of the unit disk.*

Theorem 10.6. *There exists a sequence of polynomials* $p_n \in \mathbb{C}[z]$ *such that*

1. $\lim_{n\to\infty} p_n(0) = 1$ *and* $\lim_{n\to\infty} p_n(z) = z$ *for* $z \in \mathbb{C}^*$

2. $\lim_{n\to\infty} p_n^{(k)}(z) = 0$ *for* $z \in \mathbb{C}$, $k \geq 1$

3. *For every* k, *the sequence* $p_n^{(k)}$ *converges uniformly on compacts of* $\mathbb{C} \setminus \mathbb{R}_0^+$, *but never converges uniformly in a neighborhood of any point in* $\mathbb{R}_0^+$.