ON MAPS DETERMINED BY ZERO PRODUCTS

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Abstract. The main result describes a bijective additive map h between prime rings with nontrivial idempotents that satisfies $h(x)h(y)h(z) = 0$ whenever $xy = yz = 0$. The proof is based on the consideration of a multiadditive map satisfying a related condition.

1. Introduction

The study of maps on rings and algebras that are determined by the action on pairs of elements whose product is zero has a long history. We refer the reader to [1] and [9] for references and historic details. In this paper we shall continue the study of such maps in prime rings with nontrivial idempotents, the topic initiated in the influental paper [9] by M. A. Chebotar, W.-F. Ke and P.-H. Lee.

In [9] the authors considered a bijective additive map h between prime rings $\mathcal{A}$ and $\mathcal{B}$ with the property that $xy = 0$ implies $h(x)h(y) = 0$. Their main result in particular shows that if $\mathcal{A}$ has a nontrivial idempotent (and satisfies a technical condition that was later removed in [5]) then there exists λ in the extended centroid of $\mathcal{B}$ such that $h(xy) = \lambda h(x)h(y)$ for all $x, y \in \mathcal{A}$. In the case where $\mathcal{A}$ is unital the proof is much simpler and in fact λ then lies in the center of $\mathcal{B}$. In the non-unital case the proof is based on functional identities (see [6]).

In [5] the second author considered a more general condition where a map $h : \mathcal{A} \to \mathcal{B}$ satisfies

$$xy = yz = 0 \Longrightarrow h(x)h(y)h(z) = 0 \text{ for all } x, y, z \in \mathcal{A}. \tag{1}$$

However, only the case where $\mathcal{A}$ is unital and $h(1) = 1$ was treated. In the present paper we will deal with a considerably more involved non-unital case. We shall first consider a more general problem concerning a triadditive map B from $\mathcal{A}^3 = \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ to an additive group $\mathcal{X}$ with the property that $B(x, y, z) = 0$ whenever $xy = yz = 0$. In the main result of Section 2 we will actually consider a multiadditive map in an arbitrary number of variables satisfying a condition of this sort (Theorem 2.1). Here we were motivated by several recent works [1, 2, 7, 8, 10, 11, 14] dealing with biadditive maps satisfying some related conditions. An application of Theorem 2.1 to maps satisfying (1) will lead to a functional identity which is not covered by the general theory [6]. Still, by using some of the methods of functional identities we will be able to obtain the desired conclusion (Theorem 3.2).

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The study of maps determined by the action on elements satisfying $xy = yz = 0$ was motivated by applications to local derivations [5]. More precisely, the following condition on a map $d : \mathcal{A} \to \mathcal{A}$ is relevant in this context:

$$xy = yz = 0 \Longrightarrow xd(y)z = 0 \text{ for all } x, y, z \in \mathcal{A}. \tag{2}$$

We will consider (2) in Section 4. The results that we will obtain are not really new. However, it may be of some interest to show that the approach based on Theorem 2.1 makes it possible for one to treat (1) and (2) in a unified manner.

2. Multiadditive maps and zero products

In the next theorem we do not assume the existence of unity in our ring $\mathcal{A}$, but still formally use the symbol 1 whose meaning is determined by $1x = x1 = x$ for every $x \in \mathcal{A}$. This simplifies our notation.

Theorem 2.1. *Let $\mathcal{A}$ be a ring and let $\mathcal{R}$ be its subring generated by all idempotents in $\mathcal{A}$. Let $\mathcal{X}$ be an abelian group and let $B : \mathcal{A}^n \to \mathcal{X}$, $n \geq 2$, be a multiadditive map such that for all $x_i \in \mathcal{A}$, $1 \leq i \leq n$,*

$$x_1x_2 = x_2x_3 = \ldots = x_{n-1}x_n = 0 \quad \Longrightarrow \quad B(x_1, x_2, \ldots, x_n) = 0.$$

Then for all $r_i \in \mathcal{R}$, $1 \leq i \leq n-1$, and all $x_i \in \mathcal{A}$, $1 \leq i \leq n$, we have

$$\sum B(x_1s_1, t_1x_2s_2, t_2x_3s_3, \ldots, t_{n-2}x_{n-1}s_{n-1}, t_{n-1}x_n) = 0,$$

where the sum is taken over all $(s_i, t_i) \in \{(r_i, 1), (1, -r_i)\}$.

Proof. We first consider the case where $n = 2$. Our goal is to show that for every $r \in \mathcal{R}$ we have $B(x_1r, x_2) = B(x_1, rx_2)$ for all $x_1, x_2 \in \mathcal{A}$. Note that the set of all $r \in \mathcal{A}$ satisfying this condition is a subring of $\mathcal{A}$. Therefore it suffices to show that $B(x_1e, x_2) = B(x_1, ex_2)$ holds for all $x_1, x_2 \in \mathcal{A}$ and every $e = e^2 \in \mathcal{A}$. This follows easily from the identities $x_1e(x_2 - ex_2) = 0$ and $(x_1 - x_1e)ex_2 = 0$. Namely, according to our assumption they imply that $B(x_1e, x_2 - ex_2) = 0$ and $B(x_1 - x_1e, ex_2) = 0$, and the desired formula follows.

We may now assume that $n \geq 3$ and that the theorem holds for all integers smaller than n. Let $x_2, x_3, \ldots, x_n$ be such that $x_2x_3 = x_3x_4 = \ldots = x_{n-1}x_n = 0$. Then for every idempotent $e \in \mathcal{A}$ and every $x_1 \in \mathcal{A}$ we have

$$x_1e(x_2 - ex_2) = (x_2 - ex_2)x_3 = \ldots = x_{n-1}x_n = 0,$$
$$(x_1 - x_1e)ex_2 = ex_2\, x_3 = x_3x_4 = \ldots = x_{n-1}x_n = 0.$$

Our assumption yields

$$B(x_1e, x_2 - ex_2, x_3, \ldots, x_{n-1}, x_n) = 0,$$
$$B(x_1 - x_1e, ex_2, x_3, \ldots, x_{n-1}, x_n) = 0,$$

which further implies

$$B(x_1e, x_2, x_3, \ldots, x_{n-1}, x_n) - B(x_1, ex_2, x_3, \ldots, x_{n-1}, x_n) = 0. \tag{3}$$

Thus, (3) holds for all $x_1 \in \mathcal{A}$, all idempotents $e \in \mathcal{A}$, and all $x_2, \dots, x_n \in \mathcal{A}$ satisfying $x_2x_3 = x_3x_4 = \dots = x_{n-1}x_n = 0$. Fixing $x_1 \in \mathcal{A}$ and $e = e^2 \in \mathcal{A}$ this means that the map $H : \mathcal{A}^{n-1} \to \mathcal{X}$ defined by

$$H(x_2, x_3, \dots, x_n) = B(x_1e, x_2, x_3, \dots, x_n) - B(x_1, ex_2, x_3, \dots, x_n)$$

satisfies the condition of the induction assumption. Consequently, for all $x_2, x_3, \dots, x_n \in \mathcal{A}$ and all $r_i \in \mathcal{R}$ we have

$$\sum H(x_2s_2, t_2x_3s_3, \dots, t_{n-1}x_n) = 0,$$

where the sum is taken over all $(s_i, t_i) \in \{(r_i, 1), (1, -r_i)\}$. That is,

$$\sum \Big(B(x_1e, x_2s_2, t_2x_3s_3, \dots, t_{n-1}x_n) - B(x_1, ex_2s_2, t_2x_3s_3, \dots, t_{n-1}x_n) \Big) = 0. \tag{4}$$

Since the set of all $r \in \mathcal{A}$ such that

$$\sum \Big(B(x_1r, x_2s_2, t_2x_3s_3, \dots, t_{n-1}x_n) - B(x_1, rx_2s_2, t_2x_3s_3, \dots, t_{n-1}x_n) \Big) = 0$$

for all $r_i \in \mathcal{R}$, $2 \le i \le n-1$, and all $x_i \in \mathcal{A}$, $1 \le i \le n$, where the sum is taken over all $(s_i, t_i) \in \{(r_i, 1), (1, -r_i)\}$, is readily seen to be a subring of $\mathcal{A}$, and since, according to (4), it contains all idempotents, the conclusion of the theorem follows. □

Let us restate Theorem 2.1 for $n = 3$, i.e., the case we shall consider in applications.

Corollary 2.2. *Let $\mathcal{A}$, $\mathcal{R}$ and $\mathcal{X}$ be as in Theorem 2.1. If $B : \mathcal{A}^3 \to \mathcal{X}$ is a triadditive map such that $B(x, y, z) = 0$ whenever $x, y, z \in \mathcal{A}$ satisfy $xy = yz = 0$, then*

$$B(x, rys, z) + B(xr, y, sz) = B(xr, ys, z) + B(x, ry, sz) \tag{5}$$

holds for all $x, y, z \in \mathcal{A}$ and $r, s \in \mathcal{R}$.

Remark 2.3. For applications of these results it is important to note that the subring $\mathcal{R}$ generated by all idempotents of a ring $\mathcal{A}$ is often large. In particular, it contains the ideal I generated by all elements of the form $ex - xe$ where $e, x \in \mathcal{A}$ and e is an idempotent [5, Lemma 2.1]. Thus, if $\mathcal{A}$ is a prime ring containing a nontrivial idempotent (i.e., an idempotent different from 0 and 1), then $\mathcal{R}$ contains a nonzero ideal of $\mathcal{A}$. Namely, nontrivial idempotents in a prime ring cannot be central.

Remark 2.4. If $\mathcal{A}$ is a simple ring with a nontrivial idempotent, then $\mathcal{R} = \mathcal{A}$ in view of the preceding remark. There are other important examples of rings for which $\mathcal{R} = \mathcal{A}$ holds, cf. [5]. If $\mathcal{A}$ is such a ring and furthermore $\mathcal{A}$ is unital, then Corollary 2.2 gives the optimal conclusion about B. Namely, setting $x = z = 1$ we get

$$B(r, y, s) = B(r, ys, 1) + B(1, ry, s) - B(1, rys, 1)$$

for all $r, y, s \in \mathcal{A}$. This implies that B can be expressed through biadditive maps $F, G : \mathcal{A}^2 \to \mathcal{A}$ as follows

$$B(r, y, s) = F(ry, s) + G(r, ys), \quad r, y, s \in \mathcal{A}.$$

Conversely, if B is of such a form for some biadditive maps F and G, then the condition of Corollary 2.2 clearly holds.

A similar remark can be stated with respect to Theorem 2.1.

3. Maps satisfying (1)

In this section we shall use the extended centroid and related notions in our arguments. We refer the reader to the book [3] for a full account on this topic. Let us mention here just a few facts that will be needed. Let $\mathcal{B}$ be a prime ring. Then its *symmetric Martindale ring of quotients* $Q_s(\mathcal{B})$ is a prime ring containing $\mathcal{B}$ as its subring, and has, in particular, the following property: If $q_1, q_2 \in Q_s(\mathcal{B})$ and $q_1\mathcal{B}q_2 = 0$, then $q_1 = 0$ or $q_2 = 0$. The center C of $Q_s(\mathcal{B})$ is a field, called the *extended centroid of* $\mathcal{B}$. It coincides with the centralizer of $\mathcal{B}$ in $Q_s(\mathcal{B})$. If $a, b, c, d \in \mathcal{B}$ are such that $a \neq 0$, $c \neq 0$, and $aub = cud$ for all $u \in \mathcal{B}$, then b and d are linearly dependent over C.

Slightly different versions of the next lemma appear, sometimes just implicitly, in a number of publications on functional identities. The proof of the version that we need is given in [4, Example 2.5].

Lemma 3.1. *Let $\mathcal{B}$ be a prime ring. Suppose there exist functions $E, F : \mathcal{B} \to \mathcal{B}$ and nonzero elements $c, d \in \mathcal{B}$ such that $E(u)vd = cuF(v)$ for all $u, v \in \mathcal{B}$. Then there exists $q \in Q_s(\mathcal{B})$ such that $E(u) = cuq$ and $F(v) = qvd$ for all $u, v \in \mathcal{B}$.*

We are now in a position to handle maps satisfying (1).

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be prime rings. Assume that $\mathcal{A}$ contains a nontrivial idempotent. If a bijective additive map $h : \mathcal{A} \to \mathcal{B}$ satisfies* (1)*, then there exists $\lambda \in \mathcal{C}$, the extended centroid of $\mathcal{B}$, such that $h(xy) = \lambda h(x)h(y)$ for all $x, y \in \mathcal{A}$.*

Proof. We shall write

$$f(x, y, z) = h(x)h(yz) - h(xy)h(z)$$

for all $x, y, z \in \mathcal{A}$. Our first goal is to show that there exists a nonzero ideal J of $\mathcal{A}$ such that

$$f(\mathcal{A}, J, J) = 0 \quad \text{or} \quad f(J, J, \mathcal{A}) = 0. \tag{6}$$

After establishing this it will be easy to complete the proof.

Note that Corollary 2.2 can be applied to $B : \mathcal{A}^3 \to \mathcal{B}$ defined by

$$B(x, y, z) = h(x)h(y)h(z).$$

Hence it follows that

$$h(x)h(rys)h(z) + h(xr)h(y)h(sz) = h(xr)h(ys)h(z) + h(x)h(ry)h(sz) \tag{7}$$

for all $x, y, z \in \mathcal{A}, r, s \in \mathcal{R}$. In particular, (7) holds for all r, s from a nonzero ideal I of $\mathcal{R}$, cf. Remark 2.3.

We can rewrite (7) as

$$f(x, r, ys)h(z) = f(x, r, y)h(sz), \tag{8}$$

$$h(x)f(ry, s, z) = h(xr)f(y, s, z) \tag{9}$$

for all $x, y, z \in \mathcal{A}, r, s \in I$. Substituting zz' for z in (8) we obtain

$$f(x, r, ys)h(zz') = f(x, r, y)h(szz') = f(x, r, ysz)h(z').$$

A similar computation can be made after substituting xx' for x in (9). Thus, we have

$$f(x, r, ysz)h(z') = f(x, r, ys)h(zz'), \tag{10}$$

$$h(x)f(x'ry, s, z) = h(xx')f(ry, s, z) \tag{11}$$

for all $x, y, z, x', z' \in \mathcal{A}, r, s \in I$. Note that (10) and (11) yield

$$f(x,r,ysz)h(z')f(s'y',r',x') = f(x,r,ys)h(z)f(z's'y',r',x') \tag{12}$$

for all $x, y, z, x', y', z' \in \mathcal{A}, r, s, r', s' \in I$.

If $f(x,r,ys) = 0$ for all $x, y \in \mathcal{A}$, $r, s \in I$, then the first identity in (6) holds for $J = \mathcal{A}I$. We may therefore assume that $c = f(x_1, r_1, y_1 s_1) \neq 0$ for some $x_1, y_1 \in \mathcal{A}$, $r_1, s_1 \in I$. Similarly, we may assume that $d = f(s_2 y_2, r_2, x_2) \neq 0$ for some $y_2, x_2 \in \mathcal{A}$, $s_2, r_2 \in I$. Defining $E, F : \mathcal{B} \to \mathcal{B}$ by

$$E(u) = f(x_1, r_1, y_1 s_1 h^{-1}(u)) \quad \text{and} \quad F(v) = f(h^{-1}(v) s_2 y_2, r_2, x_2)$$

we see from (12) that $E(u)vd = cuF(v)$ for all $u, v \in \mathcal{B}$. Lemma 3.1 implies, in particular, that $F(v) = qvd$ for all $v \in \mathcal{B}$ and some fixed $q \in Q_s(\mathcal{B})$. Writing $h(z')$ for v we thus have

$$f(z' s_2 y_2, r_2, x_2) = qh(z')d \quad \text{for all } z' \in \mathcal{A}.$$

Using this in (12) we get

$$\Big(f(x,r,ysz) - f(x,r,ys)h(z)q\Big)h(z')d = 0 \quad \text{for all } x, y, z, z' \in \mathcal{A},\ r, s \in I.$$

The primeness of $\mathcal{B}$ implies

$$f(x,r,ysz) = f(x,r,ys)h(z)q \quad \text{for all } x, y, z \in \mathcal{A},\ r, s \in I. \tag{13}$$

Again using (12) it now follows that

$$f(zsy,r,x) = qh(z)f(sy,r,x) \quad \text{for all } x, y, z \in \mathcal{A},\ r, s \in I. \tag{14}$$

We may assume that $q \neq 0$ since otherwise (6) holds with $J = \mathcal{A}I\mathcal{A}$.

Using (10) together with (13) we get

$$f(x,r,ys)h(zz') = f(x,r,ys)h(z)qh(z').$$

In particular, $ch(zz') = ch(z)qh(z')$ holds for all $z, z' \in \mathcal{A}$. Consequently, for all $x, y, z \in \mathcal{A}$ we have

$$ch(x)qh(yz) = ch(xyz) = ch(xy)qh(z) = ch(x)qh(y)qh(z).$$

Since $c \neq 0$ and $\mathcal{B}$ is prime it follows that

$$qh(yz) = qh(y)qh(z) \quad \text{for all } y, z \in \mathcal{A}. \tag{15}$$

Similarly, from (11) and (14) one can derive

$$h(yz)q = h(y)qh(z)q \quad \text{for all } y, z \in \mathcal{A}. \tag{16}$$

The last two identities will make it possible for us to simplify our fundamental formula (7). Multiplying this formula from the right by q and using (16) we get

$$\begin{aligned} 0 &= h(x)h(rys)h(z)q + h(xr)h(y)h(sz)q - h(x)h(ry)h(sz)q - h(xr)h(ys)h(z)q \\ &= h(x)h(rys)h(z)q + h(xr)h(y)h(s)qh(z)q - h(x)h(ry)h(s)qh(z)q - h(xr)h(ys)h(z)q \\ &= \Big(h(x)h(rys) + h(xr)h(y)h(s)q - h(x)h(ry)h(s)q - h(xr)h(ys)\Big)h(z)q. \end{aligned}$$

From the primeness of $\mathcal{B}$ it follows that

$$h(x)h(rys) + h(xr)h(y)h(s)q = h(x)h(ry)h(s)q + h(xr)h(ys) \quad \text{for all } x, y \in \mathcal{A},\ r, s \in I. \tag{17}$$

Now multiply (17) by q from the left, use (15), and argue similarly as above. Then we get

$$h(rys) + qh(r)h(y)h(s)q = h(ry)h(s)q + qh(r)h(ys) \quad \text{for all } y \in \mathcal{A},\ r, s \in I. \tag{18}$$

If we now multiply (18) by q on both sides, and again use (15) and (16), we easily get

$$\Big(q^2h(r) - qh(r)q\Big)h(y)\Big(h(s)q^2 - qh(s)q\Big) = 0$$

for all $y \in \mathcal{A}, r, s \in I$. Therefore either $q^2h(r) = qh(r)q$ for all $r \in I$ or $h(s)q^2 = qh(s)q$ for all $s \in I$. Let us consider only the first possibility. The second one can be considered similarly. We have $q^2h(rx) = qh(rx)q$ for all $r \in I$ and $x \in \mathcal{A}$. Applying (15) it follows that $q^2h(r)qh(x) = qh(r)qh(x)q$, i.e., $qh(r)q[q, u] = 0$ for every $u(= h(x))$ in $\mathcal{B}$; here, $[q, u]$ denotes the commutator $qu - uq$. Replacing u by uv and using $[q, uv] = [q, u]v + u[q, v]$ we obtain $qh(I)q\mathcal{B}[q, \mathcal{B}] = 0$. Thus either $qh(I)q = 0$ or $[q, \mathcal{B}] = 0$. If the latter is true, then $q \in C$ and hence the conclusion of the theorem follows immediately from (15) (since $\lambda = q$ is an invertible element). Therefore we may assume that $qh(I)q = 0$. Then (16) yields $h(\mathcal{A}\mathcal{I})q = 0$. Substituting s by zs in (17) it then follows that $h(x)h(ryzs) = h(xr)h(yzs)$ for all $x, y, z \in \mathcal{A}$, $r, s \in I$. This implies that the first identity of (6) holds for $J = \mathcal{A}^2I$. We have thereby finally proved that (6) holds in any case.

Among two possibilities in (6) we choose to consider the first one. The second one can be treated similarly. Thus, we have

$$h(x)h(rs) = h(xr)h(s) \quad \text{for all } x \in \mathcal{A},\ r, s \in J. \tag{19}$$

Accordingly,

$$h(x)h(yrs) = h(xyr)h(s) = h(xy)h(rs)$$

for all $x, y \in \mathcal{A}, r, s \in J$. That is, by setting $K = J^2$ we have

$$h(x)h(yt) = h(xy)h(t) \quad \text{for all } x, y \in \mathcal{A},\ t \in K.$$

Of course, K is also a nonzero ideal of $\mathcal{A}$. Consequently, for all $x, y, z \in \mathcal{A}$ and $t, v, w \in K$ we have

$$\begin{aligned} h(xy)h(z)h(w)h(tv) &= h(xy)h(z)h(wt)h(v) = h(xy)h(zw)h(t)h(v) \\ &= h(x)h(yzw)h(t)h(v) = h(x)h(y)h(zwt)h(v) \\ &= h(x)h(y)h(z)h(wtv). \end{aligned}$$

That is,

$$h(xy)uh(w)h(tv) = h(x)h(y)uh(wtv) \quad \text{for all } x, y \in \mathcal{A},\ t, v, w \in K,\ u \in \mathcal{B}. \tag{20}$$

Since $K^3 \neq 0$, $h(wtv) \neq 0$ for some $t, v, w \in K$. From (20) it obviuosly follows that $h(w)h(tv)$ is also nonzero. Similarly we see that there exist $x_0, y_0 \in \mathcal{A}$ such that $h(x_0y_0) \neq 0$ and $h(x_0)h(y_0) \neq 0$. By a well-known property of the extended centroid (mentioned above) we deduce from (20) that there exists $\lambda \in C$ such that $h(wtv) = \lambda h(w)h(tv)$. Accordingly,

$$\Big(h(xy) - \lambda h(x)h(y)\Big)uh(w)h(tv) = 0 \quad \text{for all } x, y \in \mathcal{A},\ u \in \mathcal{B}.$$

This clearly implies $h(xy) = \lambda h(x)h(y)$ for all $x, y \in \mathcal{A}$. $\square$

Remark 3.3. The condition $h(xy) = \lambda h(x)h(y)$ can be expressed as that $\varphi(x) = \lambda h(x)$ is a homomorphism. Thus, h is a homomorphism multiplied by a nonzero element in C, i.e., $h(x) = \lambda^{-1}\varphi(x)$.

Theorem 3.2 extends [5, Theorem 3.1] which treats the same condition under the additional assumption that $\mathcal{A}$ is unital and $h(1) = 1$ (which yields $\lambda = 1$, i.e., h is a homomorphism). The arguments are considerably easier in this setting. On the other hand, Theorem 3.2 also extends [5, Corollary 4.3] and [9, Theorem 1].

4. Maps satisfying (2)

The results that will be given in this section are not new. Neglecting some minor differences they were basically obtained by Wang [15] who appropriately modified the arguments from [5] in which only the unital case was treated. Still, it is perhaps of some interest to see how a more conceptual approach, based on Corollary 2.2, easily yields the same conclusions.

There are several alternative definitions of the notion of a generalized derivation which are "almost" equivalent, that is, they coincide on various nice classes of rings but may differ in some special rings. Here we will choose the following definition: an additive map d from a ring $\mathcal{A}$ into itself is called a *generalized derivation* if

$$d(xyz) + xd(y)z = d(xy)z + xd(yz) \quad \text{for all } x, y, z \in \mathcal{A}. \tag{21}$$

If $\mathcal{A}$ is unital, then this is easily seen to be equivalent to the following condition: there are $a \in \mathcal{A}$ and a derivation δ of $\mathcal{A}$ such that $d(x) = \delta(x) + ax$ for every $x \in \mathcal{A}$.

Theorem 4.1. *Let $\mathcal{A}$ be a prime ring with a nontrivial idempotent. If an additive map $d : \mathcal{A} \to \mathcal{A}$ satisfies* (2)*, then d is a generalized derivation.*

Proof. Setting $B(x, y, z) = xd(y)z$ and applying Corollary 2.2 it follows that

$$xd(rys)z + xrd(y)sz = xrd(ys)z + xd(ry)sz$$

for all $x, y, z \in \mathcal{A}$, $r, s, \in \mathcal{R}$. Since $\mathcal{A}$ is prime, this gives

$$d(rys) + rd(y)s = rd(ys) + d(ry)s \quad \text{for all } y \in \mathcal{A},\ r \in \mathcal{R}. \tag{22}$$

Recall that $\mathcal{R}$ contains a nonzero ideal I (Remark 2.3). Pick $z \in \mathcal{A}$ and $s \in I$, and let us now make two substitutions in (22): the first one is replacing y by $yz \in \mathcal{A}$, and the second one is replacing s by $zs \in I \subseteq \mathcal{R}$. Comparing the two identities, so obtained, we get

$$d(ryz)s + rd(y)zs = d(ry)zs + rd(yz)s.$$

Since s is an arbitrary element from a nonzero ideal I it follows that

$$d(ryz) + rd(y)z = d(ry)z + rd(yz) \quad \text{for all } y \in \mathcal{A},\ r \in \mathcal{R}. \tag{23}$$

In a similar fashion, by letting $r \in I$ and $x \in \mathcal{A}$, first replacing y by xy in (23) and then replacing r by rx in (23), it follows that (21) holds. $\square$

The original motivation for studying the condition (2) was its connection to local derivations and related maps, cf. [5, Section 3]. The concept of a local derivation was introduced independently in [12] and [13] at the beginning of the 1990's, and since then studied in numerous papers.

Recall that a *local generalized derivation* of a ring $\mathcal{A}$ is an additive map $d : \mathcal{A} \to \mathcal{A}$ such that for every $y \in \mathcal{A}$ there exists a generalized derivation d_y of $\mathcal{A}$ satisfying $d(y) = d_y(y)$. If $x, y, z \in \mathcal{A}$ are such that $xy = yz = 0$, then

$$xd(y)z = xd_y(y)z = d_y(xy)z + xd_y(yz) - d_y(xyz) = 0.$$

Thus d satisfies the conditions of Theorem 4.1. Accordingly, we have

Corollary 4.2. *Every local generalized derivation of a prime ring with a nontrivial idempotent is a generalized derivation.*

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