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Tracial Nullstellensätze

Matej Brešar[1] and Igor Klep[2]

Abstract. We survey some of the latest developments on the geometry of polynomials in noncommuting variables, focusing on various Nullstellensätze both in the dimension-free and the dimension-dependent setting. After a brief review of Amitsur's and Bergman's Nullstellensatz, we focus on the trace. For instance, we show that a polynomial all of whose evaluations at $d \times d$ matrices have trace zero, is a sum of commutators and a polynomial identity of $d \times d$ matrices. The main new contribution is a dimension-free tracial Nullstellensatz with multilinear constraints.

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1. Introduction

Hilbert's Nullstellensatz is a classical result in algebraic geometry. Over an algebraically closed field it characterizes polynomials vanishing on an algebraic set (i.e., zero set of a set of polynomials):

Theorem 1.1 (Hilbert's Nullstellensatz). *Let $f, g_1, \dots, g_s \in \mathbb{C}[\underline{X}]$ and*

$$Z := \{a \in \mathbb{C}^n \mid g_1(a) = \cdots = g_s(a) = 0\}.$$

If $f|_Z = 0$, then for some $r \in \mathbb{N}$, f^r belongs to the ideal $(g_1, \dots, g_s)$.

Due to its importance it has been generalized and extended in many different directions. In this expository article we will focus on *free noncommutative Nullstellensätze* describing vanishing in free algebras.

In Section 2 we briefly introduce the central notions used in the paper. Then our starting point is Amitsur's Nullstellensatz [Ami1] which is a direct generalization of Hilbert's Nullstellensatz. It describes noncommutative polynomials vanishing on the vanishing set of a given finite set of polynomials in a full matrix algebra. We then move to directional zeros of noncommutative polynomials and the Nullstellensatz of Bergman [HM]. Finally, the trace is thoroughly analyzed in Section 5. For instance, our tracial Nullstellensatz shows that a polynomial all of whose evaluations at $d \times d$ matrices have trace zero, is a sum of commutators and a polynomial identity of $d \times d$ matrices. Most of this material is taken from [BK1]. The main new contribution in this paper is a dimension-free tracial Nullstellensatz with multilinear constraints, see Section 5.5.

It is our hope this note will be of interest to a wider audience with various backgrounds, so we have included several proofs exhibiting different circles of ideas and the main techniques currently used in the area, to serve as a gentle introduction.

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2. Notation and set-up

In this section we fix the basic notation we will be using throughout the paper. By $\mathbb{F}$ we denote a field, which we shall assume, for the sake of convenience, to be of characteristic 0.

2.1. The free algebra

By $\mathbb{F}\langle \underline{X}\rangle$ we denote the free algebra generated by $\underline{X} = \{X_1, X_2, \ldots\}$, i.e., the algebra of all polynomials in noncommuting variables X_i. We write $\langle \underline{X}\rangle$ for the monoid freely generated by $\underline{X}$, i.e., $\langle \underline{X}\rangle$ consists of *words* in the letters $X_1, X_2, \ldots$ (including the empty word denoted by 1). An element of the form aw where $0 \neq a \in \mathbb{F}$ and $w \in \langle \underline{X}\rangle$ is called a *monomial* and a its *coefficient*. Hence words are monomials whose coefficient is 1. Write $\mathbb{F}\langle \underline{X}\rangle_k$ for the vector space consisting of the polynomials of degree at most k and $\langle \underline{X}\rangle_k$ for the set of words $w \in \langle \underline{X}\rangle$ of length at most k.

2.2. The free $*$-algebra

For dealing with matrices and their transposes, we introduce the analogue of a free algebra in the category of algebras with involution. Let $\mathbb{F}$ be a field with an involution $*$. By $\mathbb{F}\langle \underline{X}, \underline{X}^*\rangle$ we denote the free $*$-algebra over $\mathbb{F}$ generated by $\underline{X} = \{X_1, X_2, \ldots\}$, i.e., the $\mathbb{F}$-algebra of all polynomials in noncommuting variables X_i, X_j^*. Further, by $\operatorname{Sym}\mathbb{F}\langle \underline{X}, \underline{X}^*\rangle$ we denote the set of all symmetric, and by $\operatorname{Skew}\mathbb{F}\langle \underline{X}, \underline{X}^*\rangle$ we denote the set of all skew-symmetric polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^*\rangle$ (with respect to the canonical involution, of course).

2.3. Evaluations and representations

In contrast to classical representation theory, we are interested in the image of a *fixed* element of a free algebra under all representations in a suitably chosen class. Our focus will be mainly on *finite dimensional representations* of a free algebra with an occasional foray into bounded operators on infinite dimensional Hilbert spaces. This forces the theory into two branches. The dimension-free (in the sense that the we are considering evaluations at tuples of matrices of *all* sizes or even bounded operator on Hilbert spaces) setting is developed much better due to the works of Helton with coauthors [Hel, HP] and the authors [KS, BK1, BK2]. This branch of the theory has a certain operator-algebraic flavor. On the other hand, a mixture of central simple algebras, and the theory of polynomial identities forms the dimension dependent branch of the theory, initiated by the seminal paper [PS] of Procesi and Schacher. They studied the Albert-Weil notion of positive involutions and orderings on central simple algebras. How this relates to dimension-dependent positivity in free algebras is explored in some detail in [KU, K].

3. Amitsur's Nullstellensatz

In this section we fix the number n of variables $\underline{X}$. An n-tuple of matrices $\underline{A} \in M_d(\mathbb{F})^n$ gives rise to the evaluation representation

$$\operatorname{ev}_{\underline{A}} : \mathbb{F}\langle \underline{X}\rangle \to M_d(\mathbb{F}), \quad p \mapsto p(\underline{A}).$$

Amitsur's Nullstellensatz [Ami1, Theorem 1] is our first noncommutative Nullstellensatz in a dimension-dependent setting as it works over a fixed matrix size. It is a generalization of Hilbert's Nullstellensatz which can be recovered from Theorem 3.1 by setting $d = 1$.

Theorem 3.1 (Amitsur's Nullstellensatz). *Fix $d \in \mathbb{N}$ and let $f, g_1, \ldots, g_s \in \mathbb{C}\langle \underline{X}\rangle$,*

$$Z(d) := \{\underline{A} \in M_d(\mathbb{C})^n \mid g_1(\underline{A}) = \cdots = g_s(\underline{A}) = 0\}.$$

If $f|_{Z(d)} = 0$*, then for some* $r \in \mathbb{N}$*,*

$$f^r \in T_d + (g_1, \dots, g_s). \tag{3.1}$$

Here $(g_1, \dots, g_s)$ is the two sided ideal generated by the g_j, and T_d denotes the ideal of all polynomial identities $h \in \mathbb{C}\langle \underline{X} \rangle$ for $d \times d$ matrices. That is, $h \in T_d$ if and only if for all tuples $\underline{A} \in M_d(\mathbb{C})^n$, $h(\underline{A}) = 0$.

Proof. We give only a sketch of the proof. Let

$$Q_d := T_d + (g_1, \dots, g_s), \quad J_d := \bigcap \{P \mid P \supseteq Q_d \text{ primitive ideal}\}.$$

In the first step we prove $f \in J_d$. Assume otherwise and let $P \supseteq Q_d$ be a primitive ideal avoiding f. Then $D := \mathbb{C}\langle \underline{X} \rangle / P$ is a primitive ring satisfying all identities of $d \times d$ matrices, so is by Kaplansky's theorem ([Row, §1.5] or [Pro, Theorem II.1.1]) a central simple algebra. Furthermore, its degree is $\leq d$.

Let Z be the center of D and construct $D \otimes_Z \bar{Z} \cong M_k(\bar{Z})$, where $k \leq d$ and $\bar{Z}$ denotes the algebraic closure of Z. Consider the following first order sentence:

$$\varphi: \quad \exists d \times d \text{ matrices } A_1, \dots, A_n : g_1(\underline{A}) = \cdots = g_s(\underline{A}) = 0 \neq f(\underline{A}).$$

By assumption, this statement is true in $\bar{Z}$, i.e., $\bar{Z} \models \varphi$. By the model completeness of the theory of algebraically closed fields [Hod, Theorem A.5.1], this implies $\mathbb{C} \models \varphi$. Hence there are matrices $\underline{A} = (A_1, \dots, A_n) \in M_d(\mathbb{C})^n$ satisfying $g_1(\underline{A}) = \cdots = g_s(\underline{A}) = 0 \neq f(\underline{A})$. But this obviously contradicts $f|_{Z(d)} = 0$.

The second and final step of the proof now invokes Amitsur's result [Lam, Theorem 4.20] stating that the Jacobson radical of a finitely generated algebra over an uncountable field is nil. In particular, this yields $f^r \in Q_d$ for some $r \in \mathbb{N}$. ∎

It is tempting to guess an adaptation of Theorem 3.1 to hold in a dimension-free setting (e.g. no T_d in (3.1)). However, this fails due to scarceness of finite dimensional representations.

Example 3.2. Let $n = 2$ and

$$g := X_1 X_2 - X_2 X_1 - 1 \in \mathbb{C}\langle \underline{X} \rangle.$$

Then $Z(d) = \varnothing$ for every d, since $\mathbb{C}\langle \underline{X} \rangle / (g)$ is the first Weyl algebra $\mathcal{A}_1(\mathbb{C})$, well-known not to have any finite dimensional or bounded infinite dimensional representations.

Consider $f = 1$. Then $f|_{Z(d)} = 0$, but $1 = f^r \notin (g) \subsetneq \mathbb{C}\langle \underline{X} \rangle$ for all $r \in \mathbb{N}$.

For a suitable non-finite dimensional version of Theorem 3.1, we need to work with primitive rings [Lam, Chapter 4]. This is [Ami1, Theorem 2]:

Theorem 3.3 (Amitsur). *Let* $f, g_1, \dots, g_s \in \mathbb{C}\langle \underline{X} \rangle$ *and*

$$Z(\infty) := \{\underline{A} \in R^n \mid R \text{ primitive, } g_1(\underline{A}) = \cdots = g_s(\underline{A}) = 0\}.$$

If $f|_{Z(\infty)} = 0$*, then for some* $r \in \mathbb{N}$*,*

$$f^r \in (g_1, \dots, g_s). \tag{3.2}$$

The proof of Theorem 3.3 is similar to the proof of the previous theorem, so is omitted. For details we refer to [Ami1].

4. Directional Nullstellensatz

A relaxation of the notion of vanishing in the free algebra is given by *directional zeros.* A directional zero of $p \in \mathbb{F}\langle \underline{X} \rangle$ is a pair $(\underline{A}, v)$, where $\underline{A}$ is a tuple of linear operators on an $\mathbb{F}$-vector space $\mathcal{H}$, and $v \in \mathcal{H}$, satisfying

$$p(\underline{A})v = 0. \tag{4.1}$$

Similarly, one introduces directional zeros in a free $*$-algebra. Directional zeros are important in understanding boundaries of noncommutative sets, cf. [HP, HKMS, HKM].

4.1. Bergman's Nullstellensatz

The first result in this setting is the dimension-free Nullstellensatz due to Bergman-Helton-McCullough [HM, Theorem 6.3]:

Theorem 4.1 (Bergman-Helton-McCullough). *Let $f, g_1, \dots, g_s \in \mathbb{F}\langle \underline{X} \rangle$ and*

$$d := \max\{\deg g_i,\ \deg f\}.$$

Let V be an $\mathbb{F}$-vector space of dimension $\sum_{j=0}^{d} n^j$, and

$$Z := \{(\underline{A}, v) \in \operatorname{End}(V)^n \times V \mid g_1(\underline{A})v = \cdots = g_s(\underline{A})v = 0\}.$$

If for all $(\underline{A}, v) \in Z$ we have $f(\underline{A})v = 0$, then f is in the left ideal

$$Q := \mathbb{F}\langle \underline{X} \rangle g_1 + \cdots + \mathbb{F}\langle \underline{X} \rangle g_s$$

generated by the g_i.

Proof. Consider the vector space $V := \mathbb{F}\langle \underline{X} \rangle_d / (Q \cap \mathbb{F}\langle \underline{X} \rangle_d)$, where $\mathbb{F}\langle \underline{X} \rangle_d$ denotes the set of all polynomials of degree $\leq d$. We use $p \mapsto \bar{p}$ to denote the quotient mapping. Let W denote the subspace

$$\{\bar{p} \mid p \in \mathbb{F}\langle \underline{X} \rangle_{d-1}\},$$

and choose a basis $\{\bar{f}_1, \dots, \bar{f}_m\}$ for W. Extend it to a basis

$$\{\bar{f}_1, \dots, \bar{f}_m, \bar{f}_{m+1}, \dots, \bar{f}_{m+\ell}\}$$

of V. Without loss of generality, $\deg f_j < d$ for $j = 1, \dots, m$.

Define

$$\hat{X}_i : V \to V, \quad \bar{f}_k \mapsto \begin{cases} \overline{X_i f_k} & 1 \leq k \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Then for every polynomial p of degree $\leq d$,

$$p(\underline{\hat{X}})\bar{1} = \bar{p}.$$

Clearly, $g_j(\underline{\hat{X}})\bar{1} = \bar{g}_j = 0$ since $g_j \in Q$. Hence by assumption,

$$\bar{f} = f(\underline{\hat{X}})\bar{1} = 0,$$

so $f \in Q$. ∎

The above theorem and its proof readily generalize to noncommutative polynomials with matrix coefficients [HKMS, Theorem 6.8].

4.2. The Helton-McCullough-Putinar directional Nullstellensatz

Let us now consider directional Nullstellensätze in a free $*$-algebra. Here the situation is somewhat more complicated, as observed by Helton and McCullough [HM, Example 6.1].

Example 4.2. Let $q = (X^*X + XX^*)^2$ and $p = X + X^*$ where X is a single variable. Then, for every matrix A and vector v (belonging to the space where A acts), $q(A)v = 0$ implies $p(A)v = 0$. However, there does not exist a positive integer m and $r, r_j \in \mathbb{R}\langle \underline{X}, \underline{X}^* \rangle$, so that

$$p^{2m} + \sum r_j^* r_j = qr + r^* q. \tag{4.2}$$

Nevertheless, a clear result can be derived for a special kind of polynomials. Polynomials in $\mathbb{F}\langle \underline{X} \rangle \subseteq \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ are called *analytic polynomials* (they contain no variables X_j^*).

Theorem 4.3 (Helton-McCullough-Putinar). *Let $g_1, \dots, g_s \in \mathbb{R}\langle \underline{X} \rangle$ be analytic polynomials, and let $p \in \mathbb{R}\langle \underline{X}, \underline{X}^* \rangle$. Assume that for every n-tuple $\underline{A}$ of linear operators acting on a finite dimensional Hilbert space $\mathcal{H}$, and every vector $v \in \mathcal{H}$, we have:*

$$(q_j(\underline{A})v = 0,\ 1 \le j \le s) \ \Rightarrow\ p(\underline{A}, \underline{A}^*)v = 0. \tag{4.3}$$

Then p belongs to the left ideal $\mathbb{R}\langle \underline{X}, \underline{X}^* \rangle g_1 + \cdots + \mathbb{R}\langle \underline{X}, \underline{X}^* \rangle g_s$.

The proof of Theorem 4.3 is similar to the proof of the Bergman-Helton-McCullough Nullstellensatz in that it uses well chosen, separating, $*$-representations of the free $*$-algebra. However, this proof is more involved, as it depends on a different "dilation type" argument. We will not give the full proof here, for that we refer the reader to [HMP, Theorem 2]. Let us instead say a few words about the intuition behind it. Assume (4.3) holds. On a very large vector space if $\underline{A}$ is determined on a small number of vectors, then $\underline{A}^*$ is not heavily constrained; it is almost like being able to take $\underline{A}^*$ to be a completely independent tuple $\underline{B}$. If it were independent, we would have

$$(q_j(\underline{A})v = 0,\ 1 \le j \le s) \ \Rightarrow\ p(\underline{A}, \underline{B})v = 0.$$

In this case Theorem 4.1 would yield the desired conclusion. Since $\underline{A}^*$ is dependent on $\underline{A}$, an operator extension with certain properties is needed to make the above argument work. For details see [HMP].

We finish this section by referring the reader to the preprint [CHMN] for a more detailed study of ideals on which such kind of Nullstellensätze hold.

5. Tracial Nullstellensätze

Let us now turn to our last type of vanishing in the free algebra. That is, to the trace. We shall give a global Nullstellensatz in both the dimension-dependent, and as a consequence, also in the dimension-free case. In the last subsection we give a new result, a Nullstellensatz for multilinear polynomials with constraints.

Fix $d \in \mathbb{N}$. One of the results we shall describe is the following: A polynomial has zero trace when evaluated at $d \times d$ matrices if and only if it is a sum of commutators and a polynomial identity of $d \times d$ matrices (see Corollaries 5.8 and 5.20 below).

The zero trace problem motivates one to consider the following more general topic: *What is the linear span of all the values of a polynomial on a given algebra $\mathcal{A}$?* Studying this question has turned out to be quite fruitful. Its answer yields the tracial Nullstellensatz described in the previous paragraph, and, on the other hand, it is interesting in its own right because of its connections to certain Lie structure topics and also to polynomial identities.

Our crucial observation is that the linear span of values of a polynomial is a Lie ideal of the algebra $\mathcal{A}$ in question (Theorem 5.2). This paves the way for the precise description of the linear span of all the values of a polynomial on certain algebras. A glance at Theorem 5.7 below shows a type of results that can be obtained.

Another line of results deals with algebras with involution. The consideration in this context is similar, but more involved. We consider noncommutative polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^*\rangle$ and observe that the linear span of values of such a polynomial need not be a Lie ideal, but it is always closed under Lie products with *skew-symmetric* elements (Theorem 5.11). We call subspaces having this property *Lie skew-ideals* and classify them for full matrix algebras (Theorems 5.15 and 5.16). This enables us to categorize polynomials into classes depending on their evaluations on a full matrix algebra (Theorems 5.18 and 5.19).

After a brief notational section, we survey the results from our paper [BK1] in the subsequent two sections. We will present them in a simpler context than in the original paper. Still, some of the proofs are almost identical. We have selected these proofs from [BK1] in hope that they will be of interest to a wider audience, because of their connections to other mathematical areas (such as Lie theory and polynomial identities).

5.1. More notation

Let us fix the notation that will be used in this section. By $\mathbb{F}$ we denote a field of characteristic 0, and all our algebras will be algebras over $\mathbb{F}$. Let $\mathcal{A}$ be an (associative) algebra. By $\mathcal{Z}$ we denote its center. If $\mathcal{A}$ is a $*$-algebra, i.e., an algebra with involution $*$, then by $\mathcal{S}$ (resp. $\mathcal{K}$) we denote the set of all symmetric (resp. skew-symmetric) elements in $\mathcal{A}$:

$$\mathcal{S} = \{a \in \mathcal{A} \mid a^* = a\}, \quad \mathcal{K} = \{a \in \mathcal{A} \mid a^* = -a\}.$$

The advantage of this notation is brevity, but the reader should be warned against possible confusion. Let us point out that $\mathcal{S}$ and $\mathcal{K}$ depend on the involution.

5.2. Involution-free case

Let $\mathcal{A}$ be an algebra over $\mathbb{F}$, and let $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \underline{X}\rangle$. If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are subsets of $\mathcal{A}$, then by $f(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ we denote the set of all values $f(a_1, \ldots, a_n)$ with $a_i \in \mathcal{L}_i$, $i = 1, \ldots, n$. If all $\mathcal{L}_i$ are equal to $\mathcal{A}$, then we simplify the notation and write $f(\mathcal{A})$ instead of $f(\mathcal{A}, \ldots, \mathcal{A})$. If $\mathcal{U}$ is a subset of $\mathcal{A}$, then by $\operatorname{span}\mathcal{U}$ we denote the linear span of $\mathcal{U}$. One of the goals of this section is to describe $\operatorname{span} f(\mathcal{A})$ for all polynomials f and certain algebras $\mathcal{A}$. Of course it can happen that $\operatorname{span} f(\mathcal{A}) = 0$ even when $f \neq 0$; such a polynomial f is called a (*polynomial*) *identity* of $\mathcal{A}$. Algebras satisfying (nontrivial) polynomial identities are called *PI algebras*. This class of algebras includes all finite dimensional algebras.

We say that a polynomial $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \underline{X}\rangle$ is *homogeneous in* X_i if each monomial of f has the same degree with respect to X_i; if this degree is 1, then we say that f is *linear in* X_i. Further, we say that f is *multihomogeneous* if it is homogeneous in every X_i, $i = 1, \ldots, n$. Every polynomial is a sum of multihomogeneous polynomials. A polynomial is said to be *multilinear* if it is linear in every X_i, $i = 1, \ldots, n$. Thus, a multilinear polynomial in $X_1, \ldots, X_n$ is a linear combination of monomials of the form $X_{\sigma(1)} \ldots X_{\sigma(n)}$ where σ is a permutation of $\{1, \ldots, n\}$.

5.2.1. Image of a polynomial and Lie theory. We recall that an associative algebra $\mathcal{A}$ becomes a Lie algebra when replacing the ordinary product in $\mathcal{A}$ by the *Lie product*

$$[x, y] := xy - yx \quad \text{for } x, y \in \mathcal{A}.$$

Ideals of $\mathcal{A}$ with respect to this product are called *Lie ideals* of $\mathcal{A}$. Thus, a Lie ideal of $\mathcal{A}$ is a linear subspace $\mathcal{L}$ of $\mathcal{A}$ such that $[\mathcal{L},\mathcal{A}]\subseteq\mathcal{L}$.

Let us now indicate the connection of polynomial values to Lie theory. From the identity

$$[X_{\sigma(1)}\dots X_{\sigma(n)},X_{n+1}] = [X_{\sigma(1)},X_{n+1}]X_{\sigma(2)}\dots X_{\sigma(n)}$$
$$+X_{\sigma(1)}[X_{\sigma(2)},X_{n+1}]X_{\sigma(3)}\dots X_{\sigma(n)}+\dots+X_{\sigma(1)}\dots X_{\sigma(n-1)}[X_{\sigma(n)},X_{n+1}]$$

it follows easily that every multilinear polynomial h satisfies (cf. [BCM, p. 170])

$$\begin{aligned}&[h(X_1,\dots,X_n),X_{n+1}] = h([X_1,X_{n+1}],X_2,\dots,X_n)\\ &+h(X_1,[X_2,X_{n+1}],X_3,\dots,X_n)+\dots+h(X_1,\dots,X_{n-1},[X_n,X_{n+1}]).\end{aligned}\tag{5.1}$$

This clearly implies that $\operatorname{span} h(\mathcal{A})$ is a Lie ideal of $\mathcal{A}$. As we will now show, a considerably more general result holds. In its proof we shall need the following simple lemma. It can be proved by a standard Vandermonde-type argument. We omit details; a proof is given in [BK1, Lemma 2.2].

Lemma 5.1. *Let $\mathcal{V}$ be a linear space over $\mathbb{F}$, and let $\mathcal{U}$ be a subspace. Suppose that $c_0,c_1,\dots,c_n\in\mathcal{V}$ are such that*

$$\sum_{i=0}^{n}\lambda^i c_i\in\mathcal{U}\tag{5.2}$$

holds for at least $n+1$ different scalars λ. Then each $c_i\in\mathcal{U}$.

Theorem 5.2. *Let $\mathcal{A}$ be an $\mathbb{F}$-algebra, and let $\mathcal{L}_1,\dots,\mathcal{L}_n$ be Lie ideals of $\mathcal{A}$. Then for every $f=f(X_1,\dots,X_n)\in\mathbb{F}\langle\underline{X}\rangle$, $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$ is again a Lie ideal of $\mathcal{A}$.*

Proof. We can write $f=f_0+f_1+\dots+f_m$ where f_i is the sum of all monomials of f that have degree i in X_1. Note that

$$f(\lambda a_1,a_2,\dots,a_n)=\sum_{i=0}^{m}\lambda^i f_i(a_1,\dots,a_n)\in\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$$

for all $\lambda\in\mathbb{F}$ and all $a_i\in\mathcal{L}_i$, and so $f_i(a_1,\dots,a_n)\in\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$ by Lemma 5.1. Repeating the same argument with respect to other variables we see that values of each of the multihomogeneous components of f lie in $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$. But then there is no loss of generality in assuming that f itself is multihomogeneous. Accordingly, we can write

$$f=h(X_1,\dots,X_1,X_2,\dots,X_2,\dots,X_n,\dots,X_n)$$

where $h\in\mathbb{F}\langle\underline{X}\rangle$ is multilinear, X_1 appears k_1 times, X_2 appears k_2 times, etc. Considering $f(a_1+\lambda a_1',a_2,\dots,a_n)$ we thus arrive at the relation

$$\sum_{i=0}^{k_1}\lambda^i c_i\in\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n),$$

where, in particular,

$$\begin{aligned}c_1&=h(a_1',a_1,\dots,a_1,a_2,\dots,a_2,\dots,a_n,\dots,a_n)\\ &+h(a_1,a_1',a_1,\dots,a_1,a_2,\dots,a_2,\dots,a_n,\dots,a_n)\\ &+\dots+h(a_1,\dots,a_1,a_1',a_2,\dots,a_2,\dots,a_n,\dots,a_n).\end{aligned}$$

By Lemma 5.1, each c_i, including of course c_1, belongs to $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$; here, $a_1,a_1'\in\mathcal{L}_1$ $a_2\in\mathcal{L}_2,\dots,a_n\in\mathcal{L}_n$ are arbitrary elements. Similar statements can be established with respect to other variables.

Now, using (5.1) we see that for all $a_i \in \mathcal{L}_i$ and $b \in \mathcal{A}$ we have

$$\begin{aligned}[f(a_1,\dots,a_n),b] = h([a_1,b],a_1,\dots,a_1,a_2,\dots,a_2,\dots,a_n,\dots,a_n)\\ +\dots+h(a_1,\dots,a_1,[a_1,b],a_2,\dots,a_2,\dots,a_n,\dots,a_n)\\ +\dots+h(a_1,\dots,a_1,[a_2,b],a_2,\dots,a_2,\dots,a_n,\dots,a_n)\\ +\dots+h(a_1,\dots,a_1,a_2,\dots,a_2,[a_2,b],\dots,a_n,\dots,a_n)\\ +\dots+h(a_1,\dots,a_1,a_2,\dots,a_2,\dots,[a_n,b],a_n\dots,a_n)\\ +\dots+h(a_1,\dots,a_1,a_2,\dots,a_2,\dots,a_n\dots,a_n,[a_n,b]).\end{aligned}$$

Let us point out that $[a_i,b] \in \mathcal{L}_i$ since $\mathcal{L}_i$ is a Lie ideal of $\mathcal{A}$. In view of the above observation $c_1 \in \operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$ it follows that the sum of the first k_1 summands that involve $[a_1,b]$ lies in $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$. Similarly we see that the sum of summands involving $[a_2,b]$ lies in $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$, etc. Accordingly, $[f(a_1,\dots,a_n),b] \in \operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$, proving that $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$ is a Lie ideal of $\mathcal{A}$. ■

The following result is folklore.

Lemma 5.3. *Let $\mathcal{A} = M_d(\mathbb{F})$, $d \ge 2$. Then $\mathcal{A}$ contains exactly four Lie ideals: 0, $\mathcal{Z}$, $[\mathcal{A},\mathcal{A}]$ and $\mathcal{A}$.*

Here, the center $\mathcal{Z}$ is equal to $\mathbb{F}$, the set of all scalar matrices, and $[\mathcal{A},\mathcal{A}]$ is the set of all commutators $[A,B]$, $A,B \in \mathcal{A}$, or equivalently, the set of all matrices with zero trace.

A general remark about notation: if $\mathcal{U}$ and $\mathcal{V}$ are subspaces of an algebra $\mathcal{A}$, then by $[\mathcal{U},\mathcal{V}]$ we denote the linear span of all commutators $[u,v]$, $u \in \mathcal{U}$, $v \in \mathcal{V}$. By chance in the case of $\mathcal{A} = M_d(\mathbb{F})$ the linear space $[\mathcal{A},\mathcal{A}]$ coincides with the set of all commutators $[A,B]$, but in general this is not true.

One can prove Lemma 5.3 by a direct computation. On the other hand, the lemma follows immediately from a substantially more general result by Herstein [Her, Theorem 1.5] stating that under very mild assumptions a Lie ideal of a simple algebra $\mathcal{A}$ either contains $[\mathcal{A},\mathcal{A}]$ or is contained in $\mathcal{Z}$.

Our next goal is to classify the polynomials in $\mathbb{F}\langle \underline{X}\rangle$ according to their values on full matrix algebras, and then as corollaries of these classification results derive what we call "tracial Nullstellensätze".

The following notion was introduced in [KS].

Definition 5.4. We say that polynomials f,g in $\mathbb{F}\langle \underline{X}\rangle$ are *cyclically equivalent* (notation $f \overset{\text{cyc}}{\sim} g$) if $f-g$ is a sum of commutators in $\mathbb{F}\langle \underline{X}\rangle$.

The next remark shows that cyclic equivalence can be checked easily and that it is "stable" under scalar extensions in the following sense: Given a field extension $\mathbb{F} \subseteq \mathbb{K}$ and $f,g \in \mathbb{F}\langle \underline{X}\rangle$, then $f \overset{\text{cyc}}{\sim} g$ in $\mathbb{F}\langle \underline{X}\rangle$ if and only if $f \overset{\text{cyc}}{\sim} g$ in $\mathbb{K}\langle \underline{X}\rangle$.

Remark 5.5.

(a) Two words $v,w \in \langle \underline{X}\rangle$ are cyclically equivalent if and only if there are words $v_1,v_2 \in \langle \underline{X}\rangle$ such that $v = v_1v_2$ and $w = v_2v_1$.

(b) Two polynomials $f = \sum_{w\in\langle \underline{X}\rangle} a_w w$ and $g = \sum_{w\in\langle \underline{X}\rangle} b_w w$ ($a_w,b_w \in \mathbb{F}$) are cyclically equivalent if and only if for each $v \in \langle \underline{X}\rangle$,

$$\sum_{w \overset{\text{cyc}}{\sim} v} a_w = \sum_{w \overset{\text{cyc}}{\sim} v} b_w.$$

The next lemma is simple, but will be of fundamental importance in the sequel.

Lemma 5.6. *Let $f = f(X_1, \dots, X_n) \in \mathbb{F}\langle \underline{X} \rangle$. If f is linear in X_n, then there exists $g = g(X_1, \dots, X_{n-1}) \in \mathbb{F}\langle \underline{X} \rangle$ such that $f \overset{\text{cyc}}{\sim} gX_n$.*

Proof. It suffices to treat the case when f is a monomial, that is $f = mX_nm'$ where m and m' are monomials in $X_1, \dots, X_{n-1}$. But then the result follows immediately from the identity $mX_nm' - m'mX_n = [mX_n, m']$. ∎

Consider now $\mathcal{A} = M_d(\mathbb{F})$. Let $f \in \mathbb{F}\langle \underline{X} \rangle$. Theorem 5.2 and Lemma 5.3 imply that $\operatorname{span} f(\mathcal{A})$ can be either 0, $\mathcal{Z}$, $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$. Each of the four possibilities indeed occurs. Finding polynomials f such that $\operatorname{span} f(\mathcal{A})$ is either $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$ is trivial (say, take $X_1X_2 - X_2X_1$ and X_1). Since $\mathcal{A}$ is a PI algebra, we can find (nonzero) polynomials f such that $\operatorname{span} f(\mathcal{A}) = 0$. The existence of polynomials f such that $\operatorname{span} f(\mathcal{A}) = \mathcal{Z}$ is nontrivial; cf. [Row, Appendix A] or [Pro]. These are the so-called *central polynomials*, i.e., polynomials which are not identities on $\mathcal{A}$ but all their values lie in $\mathcal{Z}$.

Theorem 5.7. *Let $\mathcal{A} = M_d(\mathbb{F})$, let $f \in \mathbb{F}\langle \underline{X} \rangle$, and let us write $\mathcal{L} := \operatorname{span} f(\mathcal{A})$. Then exactly one of the following four possibilities holds:*

(i) *f is an identity of $\mathcal{A}$; in this case $\mathcal{L} = 0$;*
(ii) *f is a central polynomial of $\mathcal{A}$; in this case $\mathcal{L} = \mathcal{Z}$;*
(iii) *f is not an identity of $\mathcal{A}$, but is cyclically equivalent to an identity of $\mathcal{A}$; in this case $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$;*
(iv) *f is not a central polynomial of $\mathcal{A}$ and is not cyclically equivalent to an identity of $\mathcal{A}$; in this case $\mathcal{L} = \mathcal{A}$.*

Proof. As just mentioned, Theorem 5.2 and Lemma 5.3 tell us that $\mathcal{L}$ is either 0, $\mathcal{Z}$, $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$. It is clear that $\mathcal{Z} \cap [\mathcal{A}, \mathcal{A}] = 0$, since $\mathcal{Z}$ is the set of scalar matrices, and $[\mathcal{A}, \mathcal{A}]$ is the set of all trace-zero matrices.

Suppose first that f is cyclically equivalent to an identity. Then $f(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}]$ and hence $\mathcal{L} \subseteq [\mathcal{A}, \mathcal{A}]$. Since $\mathcal{Z} \cap [\mathcal{A}, \mathcal{A}] = 0$, there are only two possibilities: either $\mathcal{L} = 0$ or $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$. If f itself is an identity, then of course (i) holds. If f is not an identity, then $\mathcal{L} \neq 0$ and so (iii) must hold.

Assume now that f is not cyclically equivalent to an identity. If f is a central polynomial, then (ii) holds. Assume therefore that f is not a central polynomial. We must show that $\mathcal{L} = \mathcal{A}$. Obviously, $\mathcal{L} \neq 0$ and $\mathcal{L} \neq \mathcal{Z}$. We still have to eliminate the possibility that $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$. Assume that this possibility actually occurs, so in particular $f(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}]$. Writing f as a sum of multihomogeneous polynomials, and then arguing as at the beginning of the proof of Theorem 5.2 we see that each of these homogeneous components has the same property that its values lie in $[\mathcal{A}, \mathcal{A}]$. It is obvious that at least one of these summands is not cyclically equivalent to an identity. Thus, there exists a multihomogeneous polynomial, let us call it $h = h(X_1, \dots, X_n)$, which is not cyclically equivalent to an identity and has the property $h(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}]$. We will show that this is impossible by induction on the degree of h with respect to X_n. Let us denote this degree by k. If $k = 1$, then we can use Lemma 5.6 to find a polynomial $g = g(X_1, \dots, X_{n-1})$ such that $h \overset{\text{cyc}}{\sim} gX_n$. Consequently, $(gX_n)(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}]$. Pick $a_1, \dots, a_{n-1} \in \mathcal{A}$ and write $w = g(a_1, \dots, a_{n-1})$. Then $wx \in [\mathcal{A}, \mathcal{A}]$ for every $x \in \mathcal{A}$, which clearly implies that the same is true for every $x \in \mathcal{A}$. If $w \neq 0$, then because of the simplicity of $\mathcal{A}$ there exist $u_i, v_i \in \mathcal{A}$ such that $1 = \sum_i u_iwv_i$. But then

$$1 = \sum_i [u_i, wv_i] + w \sum_i v_iu_i \in [\mathcal{A}, \mathcal{A}],$$

contradicting $\mathcal{Z} \cap [\mathcal{A}, \mathcal{A}] = 0$. Thus $w = 0$, i.e., $g(a_1, \ldots, a_{n-1}) = 0$ for all $a_i \in \mathcal{A}$. That is, g, and hence also gX_n, is an identity of $\mathcal{A}$. This contradicts our assumption that h is not cyclically equivalent to an identity. Now let $k > 1$ and consider the polynomial

$$\begin{aligned} h'(X_1, \ldots, X_n, X_{n+1}) &= h(X_1, \ldots, X_{n-1}, X_n + X_{n+1}) \\ &\quad - h(X_1, \ldots, X_{n-1}, X_n) - h(X_1, \ldots, X_{n-1}, X_{n+1}). \end{aligned}$$

Obviously the values of h' also lie in $[\mathcal{A}, \mathcal{A}]$, and so the same is true for each of multihomogeneous components of h'. Since the degree in X_n of each of these components is smaller than k, the induction assumption implies that each of them is cyclically equivalent to an identity. But then h' itself is cyclically equivalent to an identity. However, since

$$h(X_1, \ldots, X_n) = \frac{1}{2^k - 2} h(X_1, \ldots, X_n, X_n)$$

it follows that h is also cyclically equivalent to an identity - a contradiction. ■

Theorem 5.7 works at a greater level of generality - it can be proved for finite dimensional central simple algebras (and a version even holds for prime PI algebras), see [BK1].

5.2.2. Tracial Nullstellensätze. We record the following two easily obtained corollaries related to [KS, Theorem 2.1]. We call them tracial Nullstellensätze; the first one deals with the dimension-dependent setting and the second one is dimension-free.

Corollary 5.8. *Let $d \geq 2$, and let $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \underline{X} \rangle$. Then $\operatorname{tr}(f(\underline{A})) = 0$ for all $\underline{A} \in M_d(\mathbb{F})^n$ if and only if f is cyclically equivalent to an identity of $M_d(\mathbb{F})$.*

Proof. Note $\operatorname{tr}(f(\underline{A})) = 0$ for all $\underline{A} \in M_d(\mathbb{F})^n$ if and only if $\operatorname{span} f(M_d(\mathbb{F}))$ equals 0 or $[M_d(\mathbb{F}), M_d(\mathbb{F})]$. Hence the conclusion follows easily from Theorem 5.7. ■

Corollary 5.9. *Let $f = f(X_1, \ldots, X_n) \in \mathbb{F}\langle \underline{X} \rangle$. Then $\operatorname{tr}(f(\underline{A})) = 0$ for all $\underline{A} \in M_d(\mathbb{F})^n$ and all $d \geq 2$ if and only if $f \overset{\text{cyc}}{\sim} 0$.*

5.3. Involution case

In this section we present the results of the previous section in the setting of algebras with involution.

Let $\mathbb{F}$ be a field of characteristic 0 with an involution $*$. Recall by $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ we denote the free $*$-algebra over $\mathbb{F}$ generated by $\underline{X} = \{X_1, X_2, \ldots\}$. By the *degree* of X_i in a monomial $M \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ we shall mean the number of appearances of X_i *or* X_i^* in M. For example, both X_1^2 and $X_1 X_1^*$ have degree 2 in X_1. The concepts of (multi)homogeneity and (multi)linearity of polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ are defined accordingly. For example, $X_1 X_2 X_1^* + X_2^* X_1^2$ is multihomogeneous and linear in X_2.

Let $\mathcal{A}$ be an algebra with involution $*$ and let $f = f(X_1, \ldots, X_n, X_1^*, \ldots, X_n^*) \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are subsets of $\mathcal{A}$, then by $f(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ we denote the set of all values $f(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*)$ with $a_i \in \mathcal{L}_i$, $i = 1, \ldots, n$. Again, if $\mathcal{L}_i = \mathcal{A}$ for every i, then we simply write $f(\mathcal{A})$ instead of $f(\mathcal{A}, \ldots, \mathcal{A})$.

5.3.1. $*$-images of polynomials and Lie theory. Theorem 5.2 does not hold for polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. For example, if $f = X_1 + X_1^*$, then $f(\mathcal{A}) = \mathcal{S}$ and so $\operatorname{span} f(\mathcal{A})$ is only exceptionally a Lie ideal of $\mathcal{A}$. However, it does satisfy a weaker version of the definition of a Lie ideal: while it is, in general, not closed under commutation with elements from $\mathcal{S}$, it is certainly closed under commutation with elements from $\mathcal{K}$ since $[\mathcal{S}, \mathcal{K}] \subseteq \mathcal{S}$. Subspaces satisfying this property will be one of the central topics of this section.

Definition 5.10. A linear subspace $\mathcal{L}$ of an algebra $\mathcal{A}$ with involution will be called a *Lie skew-ideal* of $\mathcal{A}$ if $[\mathcal{L},\mathcal{K}] \subseteq \mathcal{L}$.

Theorem 5.11. *Let $\mathcal{A}$ be an $\mathbb{F}$-algebra with involution, and let $\mathcal{L}_1,\dots,\mathcal{L}_n$ be Lie skew-ideals of $\mathcal{A}$. Then for every $f = f(X_1,\dots,X_n,X_1^*,\dots,X_n^*) \in \mathbb{F}\langle \underline{X},\underline{X}^*\rangle$, $\operatorname{span} f(\mathcal{L}_1,\dots,\mathcal{L}_n)$ is again a Lie skew-ideal of $\mathcal{A}$.*

Proof. The proof is almost the same as the proof of Theorem 5.2, so is omitted. ■

Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{F}$. Every Lie ideal of $\mathcal{A}$ is also a Lie skew-ideal of A, while the converse is not true in general. For example, $\mathcal{S}$ and $\mathcal{K}$ are Lie skew-ideals, which are only rarely Lie ideals. Obviously, Lie skew-ideals are closed under sums and intersections. Further, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lie skew-ideals, then $[\mathcal{L}_1,\mathcal{L}_2]$ is also a Lie skew-ideal. This can be easily checked by using the Jacobi identity.

Let us mention eight examples of Lie skew-ideals: 0, $\mathcal{Z}$, $\mathcal{K}$, $[\mathcal{S},\mathcal{K}]$, $\mathcal{S}$, $\mathcal{Z}+\mathcal{K}$, $[\mathcal{A},\mathcal{A}]$, and $\mathcal{A}$. As indicated above, there are other natural examples. The reasons for pointing out these eight examples will become clear shortly.

Let $\mathcal{A} = M_d(\mathbb{F})$ be a full matrix $*$-algebra. Then $*$ is called *orthogonal* if $\dim_{\mathbb{F}} \mathcal{S} = \frac{d(d+1)}{2}$ and *symplectic* if $\dim_{\mathbb{F}} \mathcal{S} = \frac{d(d-1)}{2}$. Symplectic involutions only exist for even d. For a full account on algebras with involutions we refer the reader to [KMRT].

The basic example of an orthogonal involution on the algebra $\mathcal{A} = M_d(\mathbb{F})$ is the *transpose involution*, $A \mapsto A^t$. The *usual symplectic involution* on $\mathcal{A} = M_d(\mathbb{F})$ is defined when d is even, $d = 2d_0$, as follows:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix} \quad \text{where } A,B,C,D \in M_{d_0}(\mathbb{F}).$$

An involution on an algebra $\mathcal{A}$ is said to be *of the first kind* if it fixes its center $\mathcal{Z}$ pointwise and *of the second kind* otherwise. Involutions of the second kind are also called *unitary involutions.* Both the transpose and the usual symplectic involution are of course involutions of the first kind.

Lemma 5.12. *Let $\mathcal{A} = M_d(\mathbb{F})$ be endowed with the transpose involution. If $d \neq 2,4$, then 0, $\mathcal{Z}$, $\mathcal{K}$, $[\mathcal{S},\mathcal{K}]$, $\mathcal{S}$, $\mathcal{Z}+\mathcal{K}$, $[\mathcal{A},\mathcal{A}]$, and $\mathcal{A}$ are the only Lie skew-ideals of $\mathcal{A}$.*

Proof. Let us begin by noting that $\mathcal{Z}$ consists of all scalar matrices, $[\mathcal{S},\mathcal{K}]$ consists of all symmetric matrices with trace 0, and $[\mathcal{A},\mathcal{A}]$ consists of all matrices with trace 0.

Since $d \neq 2,4$, $\mathcal{K}$ is a simple Lie algebra. This is well-known and easy to see (see for example [BMM, p. 443]). Given a Lie skew-ideal $\mathcal{L}$ of $\mathcal{A}$, we have that $\mathcal{L}\cap\mathcal{K}$ is a Lie ideal of $\mathcal{K}$, and hence either $\mathcal{L}\cap\mathcal{K} = 0$ or $\mathcal{L}\cap\mathcal{K} = \mathcal{K}$. That is,

$$\mathcal{L}\cap\mathcal{K} = 0 \quad \text{or} \quad \mathcal{K} \subseteq \mathcal{L}. \tag{5.3}$$

Let us first consider the case where $\mathcal{L} \subseteq \mathcal{Z}+\mathcal{K}$. If $\mathcal{L} \subseteq \mathcal{Z}$, then of course either $\mathcal{L} = 0$ or $\mathcal{L} = \mathcal{Z}$. If $\mathcal{L} \nsubseteq \mathcal{Z}$, then $\mathcal{L}$ contains a matrix $\lambda I + K_0$ where $\lambda \in \mathbb{F}$ and $0 \neq K_0 \in \mathcal{K}$. Picking $K_1 \in \mathcal{K}$ which does not commute with K_0 it follows that $0 \neq [K_0,K_1] = [\lambda I + K_0, K_1] \in \mathcal{L}\cap\mathcal{K}$. Therefore $\mathcal{K} \subseteq \mathcal{L}$ by (5.3). But then either $\mathcal{L} = \mathcal{K}$ or $\mathcal{L} = \mathcal{Z}+\mathcal{K}$.

Assume from now on that $\mathcal{L} \nsubseteq \mathcal{Z}+\mathcal{K}$. Therefore there exists $A = (a_{ij}) \in \mathcal{L}$ such that for some $i \neq j$, either $\alpha = a_{jj} - a_{ii} \neq 0$ or $\beta = a_{ij} + a_{ji} \neq 0$. Since for every $K \in \mathcal{K}$ also $K^3 \in \mathcal{K}$, we have

$$K^2AK - KAK^2 = \frac{1}{3}\Big([[[A,K],K],K] - [A,K^3]\Big) \in \mathcal{L}.$$

For $K = E_{ij} - E_{ji}$ we get

$$\alpha(E_{ij} + E_{ji}) + \beta(E_{ii} - E_{jj}) \in \mathcal{L}. \tag{5.4}$$

Pick k different from i and j (recall that $d \neq 2!$). Since $E_{jk} - E_{kj} \in \mathcal{L}$, it follows that $\mathcal{L}$ contains

$$[[\alpha(E_{ij}+E_{ji})+\beta(E_{ii}-E_{jj}), E_{jk}-E_{kj}], E_{jk}-E_{kj}] = -\alpha(E_{ij}+E_{ji})+2\beta(E_{jj}-E_{kk}).$$

Using this together with (5.4) it follows that $\beta(E_{ii} + E_{jj} - 2E_{kk}) \in \mathcal{L}$, and hence also

$$\beta(E_{ik} + E_{ki}) = \frac{1}{3}[\beta(E_{ii} + E_{jj} - 2E_{kk}), E_{ik} - E_{ki}] \in \mathcal{L}.$$

If $\beta \neq 0$, then this yields $E_{ik} + E_{ki} \in \mathcal{L}$. If, however, $\beta = 0$, then $\alpha \neq 0$ and hence $E_{ij} + E_{ji} \in \mathcal{L}$ by (5.4). Thus, in any case $\mathcal{L}$ contains a matrix of the form $E_{uv} + E_{vu}$ with $u \neq v$. We claim that this implies that $\mathcal{L}$ contains all matrices of the form $E_{pq} + E_{qp}$ with $p \neq q$. Indeed, if $\{p, q\} \cap \{u, v\} = \varnothing$, then this follows from $E_{pq} + E_{qp} = [[E_{uv} + E_{vu}, E_{vp} - E_{pv}], E_{uq} - E_{qu}]$, and if $\{p, q\} \cap \{u, v\} \neq \varnothing$, then the proof is even easier. Consequently, $E_{qq} - E_{pp} = \frac{1}{2}[E_{pq} + E_{qp}, E_{pq} - E_{qp}] \in \mathcal{L}$. Note that all these relations can be summarized as

$$[\mathcal{S}, \mathcal{K}] \subseteq \mathcal{L}. \tag{5.5}$$

Suppose that $\mathcal{L} \cap \mathcal{K} = 0$. We claim that in this case $\mathcal{L} \subseteq \mathcal{S}$. Indeed, if this was not true, then $\mathcal{L}$ would contain a matrix $K_0 + S_0$ with $0 \neq K_0 \in \mathcal{K}$ and $S_0 \in \mathcal{S}$. Picking $K_1 \in \mathcal{K}$ that does not commute with K_0 it then follows from (5.5) that $0 \neq [K_0, K_1] = [K_0 + S_0, K_1] - [S_0, K_1] \in \mathcal{L} \cap \mathcal{K}$, a contradiction. Thus $[\mathcal{S}, \mathcal{K}] \subseteq \mathcal{L} \subseteq \mathcal{S}$ and so either $\mathcal{L} = [\mathcal{S}, \mathcal{K}]$ or $\mathcal{L} = \mathcal{S}$.

It remains to consider the case where $\mathcal{L} \cap \mathcal{K} \neq 0$. In this case $\mathcal{K} \subseteq \mathcal{L}$ by (5.3). Since $\mathcal{L}$ also contains $[\mathcal{S}, \mathcal{K}]$ and since $[\mathcal{S}, \mathcal{K}] + \mathcal{K} = [\mathcal{A}, \mathcal{A}]$, it follows that $[\mathcal{A}, \mathcal{A}] \subseteq \mathcal{L} \subseteq \mathcal{A}$. But then either $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$ or $\mathcal{L} = \mathcal{A}$. ■

The cases where $d = 2$ or $d = 4$ are indeed exceptional (see [BK1] for details).

Our next aim is to prove a version of Lemma 5.12 for the usual symplectic involution. For this we need the following lemma which describes the structure of certain subspaces of $M_d(\mathbb{F})$ that are in particular Lie skew-ideals of $M_d(\mathbb{F})$ with respect to the transpose involution. Since the restriction $d \neq 2, 4$ is unnecessary in this situation, we cannot apply Lemma 5.12. In any case a direct computational proof could be easily given. However, a result by Montgomery [Mon, Corollary 1] describing additive subgroups $\mathcal{M}$ of simple rings $\mathcal{A}$ with involution satisfying $a\mathcal{M}a^* \subseteq \mathcal{M}$ for all $a \in \mathcal{A}$ will make it possible for us to use a shortcut. This result implies that if $\mathcal{A}$ is a simple algebra over $\mathbb{F}$, the involution $*$ is of the first kind, and $\mathcal{M}$ is such a linear subspace of $\mathcal{A}$, then $\mathcal{M}$ must be either 0, $\mathcal{K}$, $\mathcal{S}$, or $\mathcal{A}$.

Lemma 5.13. *Let $\mathcal{A} = M_d(\mathbb{F})$ be endowed with the transpose involution. If $\mathcal{M}$ is a linear subspace of $\mathcal{A}$ such that $MA^t + AM \in \mathcal{M}$ for all $M \in \mathcal{M}$ and $A \in \mathcal{A}$, then $\mathcal{M}$ is either* 0, *$\mathcal{K}$, $\mathcal{S}$, or $\mathcal{A}$.*

Proof. From the identity

$$AMA^t = \frac{1}{2}\Big(\big((MA^t + AM)A^t + A(MA^t + AM)\big) - \big(M(A^2)^t + A^2M\big)\Big)$$

it follows that $AMA^t \in \mathcal{M}$ for all $A \in \mathcal{A}$ and $M \in \mathcal{M}$. Therefore the result follows immediately from [Mon, Corollary 1]. ■

Lemma 5.14. *Let $\mathcal{A} = M_{2d_0}(\mathbb{F})$, let $*$ be the usual symplectic involution on $\mathcal{A}$. Then* 0, *$\mathcal{Z}$, $\mathcal{K}$, $[\mathcal{S}, \mathcal{K}]$, $\mathcal{S}$, $\mathcal{Z} + \mathcal{K}$, $[\mathcal{A}, \mathcal{A}]$, and $\mathcal{A}$ are the only Lie skew-ideals of $\mathcal{A}$.*

Proof. Set $\mathcal{A}_0 = M_{d_0}(\mathbb{F})$ and let $\mathcal{K}_0$ and $\mathcal{S}_0$ denote the sets of symmetric and skew-symmetric matrices in $\mathcal{A}_0$ with respect to the transpose involution. Note that $\mathcal{K}$ consists of all matrices of the form

$$\begin{bmatrix} A & S \\ T & -A^t \end{bmatrix} \quad \text{where } A \in \mathcal{A}_0,\ S, T \in \mathcal{S}_0,$$

and $\mathcal{S}$ consists of all matrices of the form

$$\begin{bmatrix} A & K \\ L & A^t \end{bmatrix} \quad \text{where } A \in \mathcal{A}_0,\ K, L \in \mathcal{K}_0.$$

Let $\mathcal{L}$ be a Lie skew-ideal of $\mathcal{A}$, and let $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}$. Commuting this matrix with $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{K}$ it follows that $\begin{bmatrix} 0 & -B \\ C & 0 \end{bmatrix} \in \mathcal{L}$. Furthermore, commuting the latter matrix with $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ one easily shows that actually both $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$ belong to $\mathcal{L}$. Thus, we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L} \Rightarrow \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathcal{L}. \tag{5.6}$$

Let $\mathcal{M}_0$ be the set of all $M \in \mathcal{A}_0$ such that $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \in \mathcal{L}$. Commuting this matrix with $\begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix} \in \mathcal{K}$ it follows that $\mathcal{M}_0$, considered as a subspace of $\mathcal{A}_0$, satisfies the condition of Lemma 5.13. Therefore $\mathcal{M}_0$ is 0, $\mathcal{K}_0$, $\mathcal{S}_0$, or $\mathcal{A}_0$. Each of these four cases shall be considered separately.

Assume that $\mathcal{M}_0 = 0$. From (5.6) we see that then any matrix in $\mathcal{L}$ is of the form $\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$. Commuting such a matrix with $\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} \in \mathcal{K}$ it follows that $AS = SD$ for all $S \in \mathcal{S}_0$. It is easy to see that this is possible only if $A = D$ is a scalar matrix. Consequently, commuting $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$ with $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ it follows that $C = -C$, i.e., $C = 0$. Therefore $\mathcal{L}$ consists only of scalar matrices. There are just two possibilities: either $\mathcal{L} = 0$ or $\mathcal{L} = \mathcal{Z}$.

Next we consider the case where $\mathcal{M}_0 = \mathcal{K}_0$. Pick $K \in \mathcal{K}_0$ and $S \in \mathcal{S}_0$. Commuting $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \in \mathcal{L}$ with $\begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} \in \mathcal{K}$ it follows that $\begin{bmatrix} KS & 0 \\ 0 & -SK \end{bmatrix} \in \mathcal{L}$. It is easy to see that every matrix in $\mathcal{A}_0$ of the form KS has trace 0, and conversely, every matrix in $\mathcal{A}_0$ with trace 0 is a linear span of matrices of the form KS. Therefore $\mathcal{L}$ contains all matrices $\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix}$ with $A \in [\mathcal{A}_0, A_0]$. Now take any matrix in $\mathcal{L}$ of the form $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Its commutator with $\begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} \in \mathcal{K}$ is $\begin{bmatrix} 0 & AS - SD \\ 0 & 0 \end{bmatrix}$. Since this matrix must be in $\mathcal{L}$ it follows that $AS - SD \in \mathcal{K}_0$ for every $S \in \mathcal{S}_0$. This condition can be rewritten as $S(A^t - D) + (A^t - D)^t S = 0$ for every $S \in \mathcal{S}_0$. It is easy to see that this forces $A^t = D$. Therefore the "diagonal part" of $\mathcal{L}$ consists only of matrices of the form $\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix}$, and there are two possibilities: either all such matrices with an arbitrary $A \in \mathcal{A}_0$ are in $\mathcal{L}$, or only all such matrices with the restriction that A has trace 0, i.e., $A \in [\mathcal{A}_0, \mathcal{A}_0]$. It remains to examine the "lower corner" part. Pick $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathcal{L}$. Commuting it with $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathcal{K}$ we get $\begin{bmatrix} -C & 0 \\ 0 & C \end{bmatrix} \in \mathcal{L}$.

But then C must lie in $\mathcal{K}_0$. Conversely, as the commutator of $\begin{bmatrix} A & 0 \\ 0 & A^t \end{bmatrix} \in \mathcal{L}$ with $\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \in \mathcal{K}$ is $\begin{bmatrix} 0 & 0 \\ A^t - A & 0 \end{bmatrix}$, and since every $K \in \mathcal{K}_0$ can be written as $K = A^t - A$ with $A \in [\mathcal{A}_0, \mathcal{A}_0]$, it follows that $\mathcal{L}$ contains all matrices $\begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}$ with $K \in \mathcal{K}_0$. We can now gather all the information derived in the following conclusion: $\mathcal{L}$ either consists of all matrices $\begin{bmatrix} A & K \\ L & A^t \end{bmatrix}$ with $A \in \mathcal{A}_0$, $S, T \in \mathcal{K}_0$ or of all such matrices with $A \in [\mathcal{A}_0, \mathcal{A}_0]$, $S, T \in \mathcal{K}_0$. In the first case $\mathcal{L} = \mathcal{S}$ and in the second case $\mathcal{L} = [\mathcal{S}, \mathcal{K}]$.

The cases where $\mathcal{M}_0 = \mathcal{S}_0$ or $\mathcal{M}_0 = \mathcal{A}_0$ can be treated similarly as the $\mathcal{M}_0 = \mathcal{K}_0$ case. One can show that $\mathcal{M}_0 = \mathcal{S}_0$ implies that $\mathcal{L} = \mathcal{K}$ or $\mathcal{L} = \mathcal{Z} + \mathcal{K}$, and $\mathcal{M}_0 = \mathcal{A}_0$ implies that $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$ or $\mathcal{L} = \mathcal{A}$. There are some differences compared to the case just treated, but the necessary modifications are quite obvious. Therefore we omit the details. ■

The above results make it possible for us to describe Lie skew-ideals in full matrix algebras with involution. The description depends on the kind of an involution.

Theorem 5.15. *Let $\mathcal{A}$ be a full matrix algebra with involution of the first kind, and let $\mathcal{L}$ be a Lie skew-ideal of $\mathcal{A}$. If $\dim_{\mathbb{F}} \mathcal{A} \neq 4, 16$, then $\mathcal{L}$ is either 0, $\mathcal{Z}$, $\mathcal{K}$, $[\mathcal{S}, \mathcal{K}]$, $\mathcal{S}$, $\mathcal{Z} + \mathcal{K}$, $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$.*

We omit details of the proof. Let us just mention that using the description of involutions on a full matrix algebra over an algebraically closed field, a scalar extension argument reduces the general situation to the two cases considered in Lemmas 5.12 and 5.14.

Theorem 5.16. *Let $\mathcal{A}$ be a full matrix algebra with involution of the second kind, and let $\mathcal{L}$ be a Lie skew-ideal of $\mathcal{A}$. Then $\mathcal{L}$ is either 0, $\mathcal{Z}$, $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$.*

Proof. Since $*$ is of the second kind, there exists $z \in \mathcal{Z} = \mathbb{F}$ such that $w = z - z^* \neq 0$. Thus w is nonzero skew-symmetric element in $\mathcal{Z}$. Pick $x \in \mathcal{L}$ and $a \in \mathcal{A}$. We can write $a = s + k$ where $s \in \mathcal{S}$ and $k \in \mathcal{K}$; indeed, we take $s = \frac{a+a^*}{2}$, $k = \frac{a-a^*}{2}$. Clearly, $ws \in \mathcal{K}$ and so $[x, ws] \in \mathcal{L}$, and of course also $[x, k] \in \mathcal{L}$. But then $[x, a] = w^{-1}[x, ws] + [x, k] \in \mathcal{L}$. This proves that $[\mathcal{L}, \mathcal{A}] \subseteq \mathcal{L}$; that is, $\mathcal{L}$ is a Lie ideal of $\mathcal{A}$. Now apply Lemma 5.3. ■

5.3.2. Classification of polynomials according to their $*$-images. We now turn to the classification problem for polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. Note that the notion of cyclic equivalence extends readily to the free $*$-algebra.

Lemma 5.17. *Let $f = f(X_1, \dots, X_n, X_1^*, \dots, X_n^*) \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. If f is linear in X_n, then there exist polynomials $g = g(X_1, \dots, X_{n-1}, X_1^*, \dots, X_{n-1}^*) \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ and $g' = g'(X_1, \dots, X_{n-1}, X_1^*, \dots, X_{n-1}^*) \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$ such that $f \overset{\text{cyc}}{\sim} gX_n + X_n^* g'$.*

Proof. The proof is basically the same as the proof of Lemma 5.6. It suffices to consider the case where f is a monomial. If $f = mX_n m'$ then use $mX_n m' - m'mX_n = [mX_n, m']$, and if $f = mX_n^* m'$ then use $mX_n^* m' - X_n^* m'm = [m, X_n^* m']$. ■

Our aim now is to obtain versions of Theorem 5.7 for polynomials in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. The situation is easier for involutions of the second kind, where Lie skew ideals coincide with Lie ideals.

Theorem 5.18. *Let $\mathcal{A}$ be a full matrix algebra with involution of the second kind, let $f \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$, and let us write $\mathcal{L} := \operatorname{span} f(\mathcal{A})$. Then exactly one of the following four possibilities holds:*

(i) *f is an identity of $\mathcal{A}$; in this case $\mathcal{L} = 0$;*
(ii) *f is a central polynomial of $\mathcal{A}$; in this case $\mathcal{L} = \mathcal{Z}$;*
(iii) *f is not an identity of $\mathcal{A}$, but is cyclically equivalent to an identity of $\mathcal{A}$; in this case $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$;*
(iv) *f is not a central polynomial of $\mathcal{A}$ and is not cyclically equivalent to an identity of $\mathcal{A}$; in this case $\mathcal{L} = \mathcal{A}$.*

For an involution of the first kind the situation is somewhat more complicated since Theorem 5.15 yields eight possible classes.

For the ease of exposition we introduce some notation to be used in the next theorem. Let $\mathcal{A}$ be an algebra endowed with a (fixed) involution $*$. By $\operatorname{Id}(\mathcal{A})$ we denote the set of all polynomial identities of $\mathcal{A}$ in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. At this point it seems appropriate to mention that if an algebra satisfies a nontrivial identity in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$, then it also satisfies a nontrivial identity in $\mathbb{F}\langle \underline{X} \rangle$ [Ami2]; this is why in the $*$-algebra context we confine ourselves to the (usual) PI algebras. Next, by $\operatorname{Cen}(\mathcal{A})$ we denote the set of all central polynomials of $\mathcal{A}$ in $\mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$. Note that $\operatorname{Id}(\mathcal{A})$ and $\operatorname{Cen}(\mathcal{A})$ depend on the involution chosen.

Theorem 5.19. *Let $\mathcal{A}$ be a full matrix algebra with involution of the first kind, let $f \in \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle$, and let us write $\mathcal{L} := \operatorname{span} f(\mathcal{A})$. If $\dim_{\mathbb{F}} \mathcal{A} \neq 1, 4, 16$, then exactly one of the following eight possibilities holds:*

(i) $f \in \operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = 0$;*
(ii) $f \in \operatorname{Cen}(\mathcal{A})$*; in this case $\mathcal{L} = \mathcal{Z}$;*
(iii) $f \in \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$ *and* $f \notin \operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = \mathcal{K}$;*
(iv) $f \in \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Cen}(\mathcal{A})$ *and* $f \notin \operatorname{Cen}(\mathcal{A})$*; in this case $\mathcal{L} = \mathcal{Z} + \mathcal{K}$;*
(v) $f \in \operatorname{Sym} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*,* $f \notin \operatorname{Id}(\mathcal{A})$ *and f is cyclically equivalent to an element of* $\operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = [\mathcal{S}, \mathcal{K}]$;*
(vi) $f \in \operatorname{Sym} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*,* $f \notin \operatorname{Cen}(\mathcal{A})$ *and f is not cyclically equivalent to an element of* $\operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = \mathcal{S}$;*
(vii) $f \notin \operatorname{Sym} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*,* $f \notin \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*, and $f + f^*$ is cyclically equivalent to an element of* $\operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$;*
(viii) $f \notin \operatorname{Sym} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*,* $f \notin \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$*,* $f \notin \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Cen}(\mathcal{A})$ *and $f + f^*$ is not cyclically equivalent to an element of* $\operatorname{Id}(\mathcal{A})$*; in this case $\mathcal{L} = \mathcal{A}$.*

Proof. We start by remarking that $\mathcal{L}$ is a Lie skew-ideal of $\mathcal{A}$ by Theorem 5.11. Therefore $\mathcal{L}$ is either 0, $\mathcal{Z}$, $\mathcal{K}$, $[\mathcal{S}, \mathcal{K}]$, $\mathcal{S}$, $\mathcal{Z} + \mathcal{K}$, $[\mathcal{A}, \mathcal{A}]$ or $\mathcal{A}$ by Theorem 5.15.

We divide the proof into two parts, (a) and (b), depending on whether or not $f + f^*$ is cyclically equivalent to an element of $\operatorname{Id}(\mathcal{A})$.

(a) Assume that $f + f^*$ is cyclically equivalent to an identity. Then $f = \frac{f+f^*}{2} + \frac{f-f^*}{2}$ is a sum of an identity, commutators, and a skew-symmetric polynomial, and hence $f(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}] + \mathcal{K} \subseteq [\mathcal{A}, \mathcal{A}] + \mathcal{K}$. The reader can easily verify $\mathcal{K} = [\mathcal{K}, \mathcal{K}]$. This forces $f(\mathcal{A}) \subseteq [\mathcal{A}, \mathcal{A}]$, and consequently $\mathcal{L} \subseteq [\mathcal{A}, \mathcal{A}]$.

Recall from the proof of Theorem 5.7 that $\mathcal{Z} \cap [\mathcal{A}, \mathcal{A}] = 0$. Therefore $\mathcal{L}$ is neither $\mathcal{Z}$, $\mathcal{Z} + \mathcal{K}$, $\mathcal{S}$ nor $\mathcal{A}$. Thus $\mathcal{L} \in \{0, \mathcal{K}, [\mathcal{S}, \mathcal{K}], [\mathcal{A}, \mathcal{A}]\}$. If f itself is an identity, then of course (i) holds. Now suppose f is not an identity. If $f \in \operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$, then (iii) holds. If $f \in \operatorname{Sym} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Id}(\mathcal{A})$, then (v) holds. Otherwise (vii) holds. Let us also point out that f cannot belong to $\operatorname{Skew} \mathbb{F}\langle \underline{X}, \underline{X}^* \rangle + \operatorname{Cen}(\mathcal{A})$ if (vii) occurs.

(b) Now assume that $f+f^*$ is not cyclically equivalent to an identity. Let us first show that $\mathcal{L}\not\subseteq[\mathcal{A},\mathcal{A}]$. Suppose this is not true, that is, suppose $f(\mathcal{A})\subseteq[\mathcal{A},\mathcal{A}]$. As a skew-symmetric polynomial, $f-f^*$ automatically satisfies $(f-f^*)(\mathcal{A})\subseteq\mathcal{K}\subseteq[\mathcal{A},\mathcal{A}]$. But then $s=f+f^*=2f-(f-f^*)$ has the same property, i.e., $s(\mathcal{A})\subseteq[\mathcal{A},\mathcal{A}]$. Suppose that s is linear in X_n. Then Lemma 5.17 tells us that there exist $g=g(X_1,\dots,X_{n-1},X_1^*,\dots,X_{n-1}^*)$ and $g'=g'(X_1,\dots,X_{n-1},X_1^*,\dots,X_{n-1}^*)$ in $\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$ such that $s\overset{\text{cyc}}{\sim}gX_n+X_n^*g'$. It is clear that then $(gX_n+X_n^*g')(\mathcal{A})\subseteq[\mathcal{A},\mathcal{A}]$. Pick $a_1,\dots,a_{n-1}\in\mathcal{A}$ and set

$$b=g(a_1,\dots,a_{n-1},a_1^*,\dots,a_{n-1}^*),$$
$$c=g'(a_1,\dots,a_{n-1},a_1^*,\dots,a_{n-1}^*).$$

Then $bx+x^*c\in[\mathcal{A},\mathcal{A}]$ for all $x\in\mathcal{A}$, and hence also for all $x\in\mathcal{A}$. Consequently,

$$(b+c^*)x=(bx+x^*c)+(c^*x-x^*c)\in[\mathcal{A},\mathcal{A}]+\mathcal{K}=[\mathcal{A},\mathcal{A}].$$

Thus $w\mathcal{A}\subseteq[\mathcal{A},\mathcal{A}]$ where $w=b+c^*$. As in the proof of Theorem 5.7 we see that this yields $w=0$, i.e.,

$$g(a_1,\dots,a_{n-1},a_1^*,\dots,a_{n-1}^*)+g'(a_1,\dots,a_{n-1},a_1^*,\dots,a_{n-1}^*)^*=0.$$

Since the a_i's are arbitrary elements in $\mathcal{A}$, this means that $g+g'^*\in\mathrm{Id}(\mathcal{A})$. Thus

$$s\overset{\text{cyc}}{\sim}gX_n+X_n^*g'=(-h^*X_n+X_n^*g')+(g+g'^*)X_n\in\operatorname{Skew}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle+\mathrm{Id}(\mathcal{A}).$$

Since $s=f+f^*\in\operatorname{Sym}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$ and since both $\operatorname{Skew}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$ and $\mathrm{Id}(\mathcal{A})$ are invariant under $*$, we now arrive at the contradiction that s is cyclically equivalent to an element in $\mathrm{Id}(\mathcal{A})$. Recall that this was derived under the assumption that s is linear in X_n. The general case can be reduced to this one in the same way as in the proof of Theorem 5.7. Therefore we have indeed $\mathcal{L}\not\subseteq[\mathcal{A},\mathcal{A}]$.

We now know that $\mathcal{L}\in\{\mathcal{Z},\mathcal{S},\mathcal{Z}+\mathcal{K},\mathcal{A}\}$. If $f\in\mathrm{Cen}(\mathcal{A})$, then (ii) holds. Suppose now that f is not a central polynomial. If $f\in\operatorname{Skew}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle+\mathrm{Cen}(\mathcal{A})$, then (iv) holds. If $f\in\operatorname{Sym}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle+\mathrm{Id}(\mathcal{A})$, then (vi) must hold. Otherwise we have (viii).

Due to the construction of the cases (i) - (viii) it is clear that they are exhaustive and mutually exclusive. ■

5.3.3. Tracial $*$-Nullstellensätze. We are now in a position to give the tracial Nullstellensätze for free $*$-algebras:

Corollary 5.20. *Let $d\neq 1,2,4$, and let $f\in\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$ be a polynomial in n variables. Fix an involution $*$ on $M_d(\mathbb{F})$. If it is of the first kind, assume that $f\in\operatorname{Sym}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$. Then $\operatorname{tr}(f(A_1,\dots,A_n,A_1^*,\dots,A_n^*))=0$ for all $A_i\in M_d(\mathbb{F})$ if and only if f is cyclically equivalent to an identity of $M_d(\mathbb{F})$.*

Corollary 5.21 (cf. Theorem 2.1 in [KS]). *Let $f\in\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$ be a polynomial in n variables. Fix an involution $*$ on $M_d(\mathbb{F})$. If it is of the first kind, assume that $f\in\operatorname{Sym}\mathbb{F}\langle\underline{X},\underline{X}^*\rangle$. Then $\operatorname{tr}(f(A_1,\dots,A_n,A_1^*,\dots,A_n^*))=0$ for all $A_i\in M_d(\mathbb{F})$ and all $d\geq 2$ if and only if $f\overset{\text{cyc}}{\sim}0$.*

5.4. Bounded operators on a Hilbert space

Using a similar line of ideas we were able to determine the span of values of polynomials in certain algebras appearing in operator theory in [BK2]. As a sample let us give the result for bounded operators $\mathcal{B}(\mathcal{H})$ and compact operators $\mathcal{K}(\mathcal{H})$ on an infinite dimensional Hilbert space.

Theorem 5.22. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then*

$$\operatorname{span} f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$$

for every nonconstant polynomial $f \in \mathbb{C}\langle \underline{X} \rangle$.

Theorem 5.23. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then*

$$\operatorname{span} f(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$$

for every nonzero polynomial $f \in \mathbb{C}\langle \underline{X} \rangle$ with zero constant term.

The main new ingredients needed for these proofs are:

- the algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ are isomorphic to the tensor product of themselves with an arbitrary full matrix algebra;
- in $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ every element is a sum of commutators;
- if $\mathcal{L}$ is both a noncentral Lie ideal and a subalgebra of a simple algebra $\mathcal{A}$, then $\mathcal{L} = \mathcal{A}$.

The first fact is folklore. The second one is not obvious; for $\mathcal{B}(\mathcal{H})$ this is a result of Halmos [Hal], and for $\mathcal{K}(\mathcal{H})$ this is due to Pearcy and Topping [PT]. The third one is a result by Herstein [Her, Theorem 1.2].

5.5. The multilinear tracial Nullstellensatz

This subsection contains our new result, a multilinear tracial Nullstellensatz with constraints. It is, in our view, a surprising result, in that it uses the theory of polynomial identities (inherently dimension-dependent) to prove a dimension-free statement.

Important elements in $\mathbb{F}\langle \underline{X} \rangle$ are *Capelli polynomials* C_{2n-1} defined by

$$C_{2n-1}(X_1, \ldots, X_{2n-1}) = \sum_{\sigma \in S_n} (-1)^\sigma X_{\sigma(1)} X_{n+1} X_{\sigma(2)} X_{n+2} \ldots X_{\sigma(n-1)} X_{2n-1} X_{\sigma(n)}.$$

They can be used to characterize linear dependence of elements in algebras. The following result, originally due to Razmyslov, is a special case of [BMM, Theorem 2.3.7].

Theorem 5.24. *Let $\mathcal{A}$ be a centrally closed prime algebra. Then $a_1, \ldots, a_n$ are linearly dependent if and only if $C_{2n-1}(a_1, \ldots, a_n, r_1, \ldots, r_{n-1}) = 0$ for all $r_i \in \mathcal{A}$.*

We refer the reader to [BMM] for the notion of a centrally closed prime algebra. The only important fact for us, however, is that $\mathbb{F}\langle \underline{X} \rangle$ is such an algebra. This follows, for example, from [BMM, Theorem 2.4.4]. Let us also remark that the "only if" part of Theorem 5.24 actually holds for every algebra. The "if" part is the nontrivial one.

Theorem 5.25. *Let $f, f_1, \ldots, f_m \in \mathbb{F}\langle \underline{X} \rangle$ be multilinear polynomials in the same variables $X_1, \ldots, X_n$. Suppose that for all $d \geq 1$ and all $\underline{A} \in M_d(\mathbb{F})^n$ the following holds:*

$$\operatorname{tr}(f_1(\underline{A})) = \ldots = \operatorname{tr}(f_m(\underline{A})) = 0 \quad \Rightarrow \quad \operatorname{tr}(f(\underline{A})) = 0. \tag{5.7}$$

Then f is cyclically equivalent to a linear combination of $f_1, \ldots, f_m$.

Proof. By Lemma 5.6, every multilinear polynomial in variables $X_1, \ldots, X_n$ is cyclically equivalent to a polynomial of the form gX_n where g is a multilinear polynomial in $X_1, \ldots, X_{n-1}$. Therefore there is no loss of generality in assuming that f is actually equal to gX_n, and f_i is equal to $g_i X_n$ for every i, where $g, g_1, \ldots, g_m$ are multilinear polynomials in $X_1, \ldots, X_{n-1}$. Under this assumption we will actually show that f is a linear combination of $f_1, \ldots, f_m$. Without loss of generality we may also assume that $f_1, \ldots, f_m$ are linearly independent.

We temporarily fix $d \geq 1$ and $A_1, \ldots, A_{n-1} \in M_d(\mathbb{F})$. Let us set

$$B = g(A_1, \ldots, A_{n-1}), \; B_i = g_i(A_1, \ldots, A_{n-1}).$$

According to our assumption we see that for every $T \in M_d(\mathbb{F})$,

$$\operatorname{tr}(B_1 T) = \ldots = \operatorname{tr}(B_m T) = 0 \quad \Rightarrow \quad \operatorname{tr}(BT) = 0.$$

This shows that if T is orthogonal to $B_1, \ldots, B_m$ with respect to the inner product $\langle S, T \rangle = \operatorname{tr}(ST^*)$, then T is orthogonal to B. Hence it follows that B lies in the linear span of $B_1, \ldots, B_m$. Applying the "only if" part of Theorem 5.24 for $\mathcal{A} = M_d(\mathbb{F})$ it follows that

$$C_{2m+1}(B, B_1, \ldots, B_m, R_1, \ldots, R_m) = 0$$

for all $R_1, \ldots, R_m \in M_d(\mathbb{F})$. Recalling the definition of B, B_i we see that this actually means that

$$C_{2m+1}(f, f_1, \ldots, f_m, X_{n+1}, \ldots, X_{n+m})$$

is an identity of $M_d(\mathbb{F})$ for any $d \geq 1$. It is well known that a nonzero polynomial cannot be an identity of $M_d(\mathbb{F})$ for every $d \geq 1$. Therefore

$$C_{2m+1}(f, f_1, \ldots, f_m, X_{n+1}, \ldots, X_{n+m}) = 0.$$

As this is an identity in the free algebra, we may replace X_i by any other member in $\mathbb{F}\langle \underline{X} \rangle$. Accordingly,

$$C_{2m+1}(f, f_1, \ldots, f_m, h_1, \ldots, h_m) = 0$$

for all $h_1, \ldots, h_m \in \mathbb{F}\langle \underline{X} \rangle$. We may now use the "if" part of Theorem 5.24 for $A = \mathbb{F}\langle \underline{X} \rangle$, and conclude that $f, f_1, \ldots, f_m$ are linearly dependent. As $f_1, \ldots, f_m$ are linearly independent by assumption, this yields the desired result. ■

Example 5.26. An obvious attempt at a strengthening of Theorem 5.25 fails. Let

$$f_1 = X_1 X_2 - X_2 X_1 - 1.$$

For every $d \in \mathbb{N}$ and $A_1, A_2 \in M_d(\mathbb{F})$, $\operatorname{tr}(f_1(A_1, B_1)) = d \neq 0$. Thus (5.7) holds for all $f \in \mathbb{F}\langle \underline{X} \rangle$. However not every f is cyclically equivalent to a multiple of f_1. For instance, consider $f = X_1$. If

$$f = \sum_i [p_i, q_i] + \lambda f_1 \tag{5.8}$$

for some $p_1, q_i \in \mathbb{F}\langle \underline{X} \rangle$ and $\lambda \in \mathbb{F}$, then setting all variables but X_1 to 0, (5.8) yields $X_1 = f = -\lambda \in \mathbb{F}$, a contradiction.

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Matej Brešar[1] and Igor Klep[2]
Faculty of Mathematics and Physics, University of Ljubljana, and
Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
e-mail: `matej.bresar@fmf.uni-lj.si`
e-mail: `igor.klep@fmf.uni-lj.si`