COMPUTING THE MAXIMAL ALGEBRA OF QUOTIENTS OF A LIE ALGEBRA

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Abstract. The maximal algebra of quotients of a semiprime Lie algebra was introduced recently by M. Siles Molina. In the present paper we answer some natural questions concerning this concept, and describe maximal algebras of quotients of certain Lie algebras that arise from associative algebras.

1. Introduction

The theory of algebras (or rings) of quotients of associative algebras (rings) has a rich history and is still an active research area. In the recent paper [22] the fourth author initiated the study of algebras of quotients of Lie algebras. Adapting some ideas from the associative (and also Jordan [18]) context, she introduced the notion of a general (abstract) algebra of quotients of a Lie algebra, and also, as a special concrete example, the notion of the maximal algebra of quotients $Q_m(L)$ of a semiprime Lie algebra L. The reason for this name is that every algebra of quotients of L can be embedded into $Q_m(L)$.

The introductory paper [22] was followed by [6, 20], and the present paper is the fourth one in the series. While the preceding papers mostly considered abstract properties of algebras of quotients, our main objective is to compute $Q_m(L)$ for some Lie algebras L. Specifically, we are interested in Lie algebras of the form $L = A^-/Z$ where A^- is the Lie algebra associated to a prime associative algebra A and Z is the center of A, and in Lie algebras of the form $L = K/Z_K$ where K is the Lie algebra of skew elements of a prime associative algebra with involution and Z_K is its center.

In section 2 we gather together basic definitions and elementary properties needed throughout the paper. Section 3 begins with some observations concerning the question of whether $Q_m(I)$, where I is an essential ideal of a Lie algebra L, is equal to $Q_m(L)$. These are applied to the question of when is $Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A))$, where A is a prime algebra and $\mathrm{Der}(A)$ is the Lie algebra of all derivations of A. The answer that we obtain is used in subsequent sections. In section 4 we compute $Q_m(A^-/Z)$ - it turns out that (under a very mild technical assumption) it is equal to a certain Lie algebra that arises from derivations from nonzero ideals of A into A. Its definition is a bit too technical to be stated here; let us just mention that this Lie algebra lies between $\mathrm{Der}(A)$ and $\mathrm{Der}(Q_s(A))$, where $Q_s(\,.\,)$ denotes the symmetric Martindale algebra of quotients. So if A is such that $A = Q_s(A)$ (for example, if A is simple), then we have $Q_m(A^-/Z) = \mathrm{Der}(A)$. Section 5 yields similar results for K/Z_K - the analogy with the A^-/Z case is perfect, the only difference is that we have to deal only with derivations δ that preserve $*$ (in the sense that $\delta(x^*) = \delta(x)^*$). The main tool in both sections 4 and 5

2000 *Mathematics Subject Classification.* Primary 17B60. Secondary 16W25, 16W10.

Key words and phrases. Algebra of quotients, Lie algebra, derivation, prime algebra, semiprime algebra.

is the description of Lie derivations on various Lie subalgebras of (associative) prime algebras [1, 2] (see also the recent book [5]). Finally, in section 6 we consider the question of whether $Q_m((Q_m(L))$ is equal to $Q_m(L)$. We show that in certain special situations this is true, namely, if L is a simple algebra or if $L = A^-/Z$ where A is either a simple algebra (satisfying a minor technical assumption) or an affine PI prime algebra (i.e. a finitely generated prime algebra which satisfies a polynomial identity). In general, however, it is not true that $Q_m((Q_m(L))$ agrees with $Q_m(L)$. More concretely, we show that $Q_m(Q_m(A^-/Z)) \supsetneq Q_m(A^-/Z)$, where $A = K[\,t\,][x, y \mid xy = tyx]$. This algebra was already used by Passman in [19] to settle a similar question for $Q_s(\,.\,)$. That is, he proved that this algebra is such that $Q_s(Q_s(A)) \supsetneq Q_s(A)$. Our approach requires an analysis of derivations on $Q_s(A)$ in order to show that A is also suitable for our purposes.

2. Preliminaries

Throughout the paper we consider Lie and associative algebras, and we tacitly assume that all of them are algebras over a fixed commutative unital ring of scalars Φ. Lie algebras will be usually denoted by L, and associative ones by A. For convenience we assume that all our algebras are *2-torsion-free* (i. e. $2x \neq 0$ for every nonzero x in an algebra), although this assumption is not always necessary. We will use it without further mention. For associative algebras we do not assume that they must be unital. Let us also mention that for commutative algebras our results are either trivially true or trivially false, so we are only interested in the noncommutative ones.

Now we recall some definitions and introduce the basic notation.

A Φ-module L together with a bilinear map $[\,,\,]\colon L \times L \to L$, denoted by $(x, y) \mapsto [x, y]$ (called the *bracket of* x and y), is said to be a *Lie algebra over* Φ if the following axioms are satisfied: (i) $[x, x] = 0$, and (ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (the so-called *Jacobi identity*). Let X be a subset of L. The set

$$\mathrm{Ann}(X) = \mathrm{Ann}_L(X) = \{a \in L \mid [a, x] = 0 \text{ for every } x \in X\}$$

is called the *annihilator of* X *in* L. It is easy to check (by using the Jacobi identity) that $\mathrm{Ann}(X)$ is an ideal of L when X is also an ideal of L. In the special situation that $X = L$, $\mathrm{Ann}(L)$ is called the *center* of L and will be denoted by Z_L.

In case A is an associative algebra and X is a subset of A, the *annihilator of* X *in* A is defined as

$$\mathrm{Ann}(X) = \mathrm{Ann}_A(X) = \{a \in A \mid ax = 0 = xa \text{ for every } x \in X\}.$$

It will be clear from the context whether $\mathrm{Ann}(X)$ denotes the annihilator in the associative or in the Lie algebra setting. Note that $\mathrm{Ann}(X)$ is an ideal of A whenever X is an ideal of A.

We say that a Lie algebra L is *semiprime* if for every nonzero ideal I of L, $[I, I] \neq 0$. In the sequel we shall usually denote $[I, I]$ by I^2. It is easy to see that L is semiprime if and only if $I \cap \mathrm{Ann}(I) = 0$ for all ideals I of L [22, Lemma 1.2 (i)]. Next, L is said to be *prime* if for every nonzero ideals I, J of L, $[I, J] \neq 0$. An ideal I of L is said to be *essential* if its intersection with any nonzero ideal is again a nonzero ideal. If L is semiprime, then an ideal I of L is essential if and only if $\mathrm{Ann}(I) = 0$ [22, Lemma 1.2 (ii)]. It is easy to see that in this case I^2 is also an essential ideal. Further, the intersection of essential ideals is clearly again an essential ideal. Note also that a nonzero ideal of a prime algebra is automatically essential.

One defines the notions of semiprimeness, primeness, and essentiality of an ideal for associative algebras in exactly the same way as for Lie algebras, just that of course the bracket must be replaced by the associative product.

Every associative algebra A gives rise to a Lie algebra A^- by considering the same module structure and the bracket given by $[x, y] = xy - yx$. Ideals of A^- are called *Lie ideals* of A; so, a Φ-submodule U of A is a Lie ideal if $[U, A] \subseteq U$. Clearly, if I is an ideal of A, then it is also a Lie ideal of A. A more important example is $[I, A]$, the linear span of all $[y, x]$ with $y \in I$ and $x \in A$, which is a Lie ideal (but not necessarily an ideal) of A. Note that Z_{A^-} agrees with the associative center Z of A, and clearly it is a Lie ideal of A. So we can form the Lie algebra A^-/Z. We will be primarily interested in this type of Lie algebras, and in Lie algebras that arise from algebras with involution: If A has an involution $*$, then the set of its *skew elements*

$$K = K_A = \{x \in A \mid x^* = -x\}$$

is a subalgebra of A^-. We will consider the Lie algebra K/Z_K.

Let B be a subalgebra of an algebra A. A linear map $\delta\colon B \to A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in B$. By a derivation of A we simply mean a derivation from A into A. Let $\mathrm{Der}(A)$ denote the set of all derivations of A. Clearly $\mathrm{Der}(A)$ becomes a Φ-module by defining operations in the natural way, and moreover, it becomes a Lie algebra if we define the bracket by $[\delta, \mu] = \delta\mu - \mu\delta$, $\delta, \mu \in \mathrm{Der}(A)$. Any element x of A determines a map $\mathrm{ad}\, x\colon A \to A$ defined by $\mathrm{ad}\, x(y) = [x, y]$ which is a derivation of A. For every Lie ideal U of A, the restriction of the map $\mathrm{ad} : A \to \mathrm{Der}(A)$ to U,

$$\begin{array}{ccc} U & \to & \mathrm{Der}(A) \\ y & \mapsto & \mathrm{ad}\, y \end{array}$$

defines a Lie algebra homomorphism with kernel $\mathrm{Ann}_U(A)$, which allows us to identify the algebra $U/\mathrm{Ann}_U(A)$ with the subalgebra $\mathrm{ad}\,(U)$ of $\mathrm{Der}(A)$. For any $y \in U$ and $\delta \in \mathrm{Der}(A)$, $[\delta, \mathrm{ad}\, y] = \mathrm{ad}\, \delta(y)$, hence $\mathrm{ad}\,(U)$ is an ideal of $\mathrm{Der}(A)$ whenever $\delta(U) \subseteq U$ for every $\delta \in \mathrm{Der}(A)$. The ideal $\mathrm{ad}\,(A)$ of $\mathrm{Der}(A)$ is usually denoted by $\mathrm{Inn}(A)$; the elements of $\mathrm{Inn}(A)$ are called *inner derivations* of A. Note that $A^-/Z \cong \mathrm{Inn}(A)$.

Derivations are defined analogously in the Lie algebra context. So, if M is a subalgebra of a Lie algebra L, then a linear map $\delta\colon M \to L$ is called a derivation if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in M$. By $\mathrm{Der}(L)$ we will denote the Lie algebra of all derivations from L into L. Incidentally, if δ is a derivation of an associative algebra A, then it is also a derivation of the Lie algebra A^-. The converse is not true in general. Derivations of A^- are called *Lie derivations* of A. For example, every linear map from A into the center of A that vanishes on $[A, A]$ is a Lie derivation.

Various constructions of algebras of quotients of associative algebras are known. In the present paper we shall come across to one of them. A *symmetric Martindale algebra of quotients* of a prime algebra A is an algebra $Q_s(A)$ satisfying the following properties:

(i) A is a subalgebra of $Q_s(A)$;

(ii) for every $q \in Q_s(A)$ there exists a nonzero ideal I of A such that $qI \cup Iq \subseteq A$;

(iii) for evey $q \in Q_s(A)$ and every nonzero ideal I of A, $qI = 0$ or $Iq = 0$ implies $q = 0$;

(iv) if I is a nonzero ideal of A and $f\colon I_A \to A_A$ and $g\colon {}_AI \to {}_AA$ are such that $xf(y) = g(x)y$ for all $x, y \in I$, then there exists $q \in Q$ such that $f(x) = qx$ and $g(x) = xq$ for all $x \in I$.

It is a fact that $Q_s(A)$ exists and is characterized up to isomorphism through these four properties. Its center C is a field called the *extended centroid* of A. We refer the reader to [3] and to [13] for an account on these concepts.

We also have to define the concept of the *degree* of a prime algebra A. The reason for this is that algebras of certain low degrees must be excluded in the results on Lie derivations [1, 2] that we are going to apply. On the other hand, we shall need to use results that appear in [4, 7, 14, 15], which also require degree restrictions. For every $x \in A$ we define $\deg(x)$ as the degree of algebraicity of x over the extended centroid $\mathcal{C}$, provided that x is algebraic. If x is not algebraic, then we define $\deg(x) = \infty$. Further we define $\deg(A) = \sup\{\deg(x) \mid x \in A\}$. It is well-known that $\deg(A) < \infty$ if and only if A is a PI algebra. Furthermore, it is known that $\deg(A) = n < \infty$ if and only if A satisfies the standard polynomial identity of degree $2n$, but does not satisfy any polynomial identity of degree $< 2n$, and this is further equivalent to the condition that A can be embedded into the matrix algebra $M_n(F)$ for some field F (say, one can take the algebraic closure of $\mathcal{C}$ for F), but cannot be embedded into $M_{n-1}(K)$ for any commutative algebra K.

We now recall the definition of what is the main object of this paper. Let L be a subalgebra of a Lie algebra Q. We say that Q is an *algebra of quotients* of L if for every nonzero $q \in Q$ there exists an ideal I of L such that $\mathrm{Ann}_L(I) = 0$ and $0 \neq [I, q] \subseteq L$ (see [22, Proposition 2.15]). We are interested in a particular algebra of quotients, the so-called maximal one. We now have to confine ourselves to the case where L is a semiprime algebra. The definition is based on derivations from essential ideals of L into L. We first define that two pairs (δ, I), (μ, J), where I, J are essential ideals of L and $\delta\colon I \to L$, $\mu\colon J \to L$ are derivations, are equivalent if δ and μ agree on some essential ideal contained in $I \cap J$. This is an equivalence relation. Denote by δ_I the equivalence class determined by (δ, I). The set of all such classes becomes a Lie algebra if we define addition, scalar multiplication, and bracket as follows:

$$\delta_I + \mu_J = (\delta + \mu)_{I\cap J}, \quad \alpha(\delta_I) = (\alpha\delta)_I, \quad [\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I\cap J)^2}.$$

(see [22, Theorem 3.4]). This Lie algebra is called the *maximal algebra of quotients* of L, and will be denoted by $Q_m(L)$. One may identify L with a subalgebra of $Q_m(L)$ via the embedding $x \mapsto \mathrm{ad}\, x_L$. It turns out that the following three properties characterize $Q_m(L)$ up to an isomorphism: (i) for every $q \in Q$ there exists an essential ideal I of L such that $[I, q] \subseteq L$, (ii) $[q, I] \neq 0$ for every nonzero $q \in Q$ and every essential ideal I of L, and (iii) for every essential ideal I of L and any derivation $\delta\colon I \to L$ there exists $q \in Q$ such that $\delta(x) = [q, x]$ for all $x \in I$ (see [22, Theorem 3.8]). As already mentioned, every algebra of quotients of L can be embedded into $Q_m(L)$ (see [22, Proposition 3.6]). We remark that one can easily show that $Q_m(L) = L$ if L is a finite dimensional semisimple Lie algebra [22, Lemma 3.9].

3. The maximal Lie algebra of quotients of an essential ideal

The purpose of this section is to consider the problem of whether $Q_m(I)$ is isomorphic to $Q_m(L)$, for an ideal I of a semiprime Lie algebra L. Of course, this question only makes sense if we assume that I itself is a semiprime algebra, so that $Q_m(I)$ exists at all. Under this assumption we will give a positive answer provided that L satisfies a certain additional condition.

We remark that similar questions have been studied also in the associative context, but apparently they can be solved more easily (see e. g. [3, Proposition 2.1.10]).

Following [17], we say that a Lie algebra L is *strongly semiprime* (resp. *strongly prime*) if:

(i) L is semiprime (prime).

(ii) For each n, given $0 \neq U_n \lhd \dots \lhd U_2 \lhd U_1 \lhd L$ there exists $0 \neq W \lhd L$ such that $W \subseteq U_n$.

We shall use SSP (or SP) as shorthand for strong semiprimeness (respectively, strong primeness). We will also say that U_n as in the definition above is an *n-subideal*. Of course, 1-subideals are just ideals.

In [17] this concept was introduced for any nonassociative algebra, but here we are interested only in Lie algebras. The proof of the following lemma is included in the proof of [17, Theorem 6.2].

Lemma 3.1. *A Lie algebra L is SSP (SP) if and only if*

(i) *L is semiprime (prime), and*

(ii) *given $0 \neq U_2 \lhd U_1 \lhd L$, there exists $0 \neq W \lhd L$ such that $W \subseteq U_2$.*

Lemma 3.2. *Let L be an SSP Lie algebra. Then, for any n-subideal U_n of L there exists an ideal $\widetilde{U}_n$ of L, which is the largest ideal $\widetilde{U}_n$ of L contained in U_n. If U_i is essential in U_{i-1}, $i = 2, \dots n$, and U_1 is essential in L, then $\widetilde{U}_n$ is an essential ideal of L.*

Proof. The first assertion is obvious: one just defines $\widetilde{U}_n$ as the sum of all ideals of L contained in U_n. Assume now that U_i is essential in U_{i-1} and U_1 is essential in L. This implies that $I \cap U_n \neq 0$ for every nonzero ideal I of L. Suppose that $I \cap \widetilde{U}_n = 0$. Since L is an SSP Lie algebra, $I \cap U_n$ contains a nonzero ideal J of L. By hypothesis, $J \cap \widetilde{U}_n = 0$, and $\widetilde{U}_n + J$ is an ideal of L bigger than $\widetilde{U}_n$ and contained in U_n, which contradicts the maximality of $\widetilde{U}_n$. □

A different proof of the previous lemma can be obtained from [17, Remark 1.2] and [22, Lemma 2.11].

Theorem 3.3. *Let I be an essential ideal of an SSP Lie algebra L. Then $Q_m(I)$ is the maximal algebra of quotients of L, i. e. $Q_m(I) \cong Q_m(L)$.*

Proof. Notice that I viewed as an algebra is SSP (see [17, Remark 2.11]), so we can consider $Q_m(I)$. Define

$$\begin{array}{cccc} \varphi: & Q_m(L) & \to & Q_m(I) \\ & \delta_J & \mapsto & \delta_{(J\cap I)^2} \end{array}$$

The map φ is well-defined: Since $\mathrm{Ann}_I(J \cap I) \subseteq \mathrm{Ann}_L(J \cap I) = 0$, this means that $J \cap I$ is an essential ideal of I. Hence $(J \cap I)^2$ is also an essential ideal of I. Finally, note that δ maps $(J \cap I)^2$ into I.

It is straightforward to verify that φ is a Lie algebra monomorphism. To see the surjectivity take $\gamma_{I'} \in Q_m(I)$ with I' an essential ideal of I. By Lemma 3.2 there exists an essential ideal J of L contained in I'. Then, for $\gamma_J \in Q_m(L)$ we have $\varphi(\gamma_J) = \gamma_{(I\cap J)^2} = \gamma_{I'}$ and the proof is complete. □

Remark 3.4. Let A be a semiprime algebra. For every Lie ideal I of A, $Z_I = I \cap Z$ since $[y, I] = 0$, with $y \in I$, implies $y \in Z$; indeed, this follows from [8, Sublemma, p. 5] which states an element y in a semiprime algebra A must lie in the center of A if $[y,[y,x]] = 0$ for all $x \in A$. Moreover, we have the following isomorphism

$$I/Z_I = I/(I \cap Z) \cong (I + Z)/Z.$$

Let A be an algebra. Then the Lie algebra A^- is isomorphic to $K_{A\oplus A^0}$ and hence A^-/Z is isomorphic to $K_{A\oplus A^0}/Z_{K_{A\oplus A^0}}$, where A^0 denotes the opposite algebra of A, and $A \oplus A^0$ is endowed with the exchange involution. This fact allows us to use the results from [17], as follows.

It is proved in [12, Lemma 6 and Theorem 4] ([12, Lemma 4 and Theorem 2]) that $\mathrm{Inn}(A)$ and $\mathrm{Der}(A)$ are semiprime (resp. prime) Lie algebras. Taking into account the considerations above, apply [17, Theorem 6.2] to obtain that $A^-/Z \cong \mathrm{Inn}(A)$ is an SSP (resp. SP) Lie algebra. Let us record this observation.

Proposition 3.5. *Let A be a semiprime (resp. prime) algebra. Then $A^-/Z \cong \mathrm{Inn}(A)$ is an SSP (resp. SP) Lie algebra.*

Corollary 3.6. *Let A be a semiprime algebra. Then:*

$$Q_m([A,\, A]/Z_{[A,\, A]}) \cong Q_m(A^-/Z).$$

Proof. Applying Remark 3.4 we have $Z_{[A,\, A]} = [A,\, A] \cap Z$ from which it immediately follows that the map determined by $[x,\, y]+Z_{[A,\, A]} \mapsto [x,\, y]+Z$ is a well-defined Lie algebra monomorphism from $[A,\, A]/Z_{[A,\, A]}$ into A^-/Z. Identifying $[A,\, A]/Z_{[A,\, A]}$ with its image, we can regard it as an ideal of A^-/Z. We will prove now that $[A,\, A]/Z_{[A,\, A]}$ is essential in A^-/Z. To this end, given $a \in A \setminus Z$ it is enough to show that $[a,\, A] \nsubseteq Z_{[A,\, A]}$ (see [22, Lemma 1.2 (ii)]). Since $a \notin Z$ by [8, Sublemma, p. 5] it follows that $[a,\, [a,\, A]] \neq 0$; this means that $[a,\, A] \nsubseteq Z$ which implies that $[a, A] \nsubseteq Z_{[A,\, A]}$, as desired. In view of Proposition 3.5, the conclusion follows directly from Theorem 3.3. □

Corollary 3.7. *Let A be a prime algebra. If $\mathrm{Der}(A)$ is SP then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

Proof. Apply Theorem 3.3. □

The last result needs the assumption that $\mathrm{Der}(A)$ is strongly prime. It does not seem clear how to verify whether this condition is fulfilled. Below we give a criterion based only on the ideal lattice of A. First we gather together several lemmas.

The proof of the following lemma is included in the proof of [20, Theorem 2.12].

Lemma 3.8. *Let A be a semiprime algebra. Then A^-/Z is semiprime and for every essential ideal I of A, $(I + Z)/Z$ is an essential ideal of A^-/Z.*

Remark 3.9. By the previous lemma, an essential ideal I of a noncommutative semiprime algebra A cannot be central.

Lemma 3.10. *Let A be a semiprime algebra and let $\widetilde{I}$ be a nonzero ideal of $\mathrm{Inn}(A)$. Then there exists an ideal U of A such that $0 \neq \mathrm{ad}\,([U, A]) \subseteq \widetilde{I}$.*

Proof. It is easy to see that $I = \{x \in A \mid \operatorname{ad} x \in \widetilde{I}\}$ is a noncentral Lie ideal of A (use [8, Sublemma, p. 5]). Apply [10, Theorem 5] to find a nonzero ideal U of A satisfying $0 \neq [U, A] \subseteq I$, that is, U is an ideal of A such that $0 \neq \operatorname{ad}([U, A]) \subseteq \widetilde{I}$ and the lemma is proved. □

Lemma 3.11. *Let A be a prime algebra. Assume that for every nonzero ideal U of A there exists a nonzero ideal $\widetilde{U}$ of* $\operatorname{Der}(A)$ *such that $\widetilde{U} \subseteq \operatorname{ad}([U, A])$. Then* $\operatorname{Der}(A)$ *is an SP Lie algebra.*

Proof. Let $0 \neq \widetilde{I} \lhd \widetilde{J} \lhd \operatorname{Der}(A)$. Apply [22, Lemma 2.13] to obtain that $\operatorname{Der}(A)$ is an algebra of quotients of $\widetilde{J} \cap \operatorname{Inn}(A)$. Hence, given $0 \neq \delta \in \widetilde{I} \subseteq \operatorname{Der}(A)$ there exists $x \in A$ satisfying $0 \neq \operatorname{ad} x \in \widetilde{J} \cap \operatorname{Inn}(A)$ and $[\delta, \operatorname{ad} x] \neq 0$. Since $\widetilde{I}$ is an ideal of $\widetilde{J}$, $[\delta, \operatorname{ad} x] \in \widetilde{I}$ and $[\delta, \operatorname{ad} x] = \operatorname{ad} \delta(x) \in \operatorname{Inn}(A)$, therefore $\widetilde{I} \cap \operatorname{Inn}(A) \neq 0$. Consider $0 \neq \widetilde{I} \cap \operatorname{Inn}(A) \lhd \widetilde{J} \cap \operatorname{Inn}(A) \lhd \operatorname{Inn}(A)$. Since $\operatorname{Inn}(A)$ is an SP Lie algebra, there exists a nonzero ideal $\widetilde{K}$ of $\operatorname{Inn}(A)$ contained in $\widetilde{I} \cap \operatorname{Inn}(A)$. Apply Lemma 3.10 to find a nonzero ideal U of A such that $0 \neq \operatorname{ad}([U, A]) \subseteq \widetilde{K}$. Now, by the hypothesis there exists a nonzero ideal $\widetilde{U}$ of $\operatorname{Der}(A)$ satisfying $\widetilde{U} \subseteq \operatorname{ad}([U, A]) \subseteq \widetilde{I}$, as desired. □

Theorem 3.12. *Let A be a prime algebra. Then the following conditions are equivalent:*

(i) $\operatorname{Der}(A)$ *is SP.*
(ii) *Every nonzero ideal of A contains a nonzero ideal of A invariant under every element of* $\operatorname{Der}(A)$.

Moreover, if these conditions hold, then

$$Q_m(A^-/Z) \cong Q_m(\operatorname{Der}(A)).$$

Proof. Identify A^-/Z with $\operatorname{Inn}(A)$.

(i) ⇒ (ii). Let I be a nonzero ideal of A. By Remark 3.9, $\operatorname{ad}(I)$ is a nonzero ideal of $\operatorname{Inn}(A)$. Consider $0 \neq \operatorname{ad}(I) \lhd \operatorname{Inn}(A) \lhd \operatorname{Der}(A)$; by the hypothesis there exists $0 \neq \widetilde{I} \lhd \operatorname{Der}(A)$ contained in $\operatorname{ad}(I)$. It is clear that $J := \sum_{\delta \in \widetilde{I}} A\delta(A)A$ is a nonzero ideal of A. Moreover, J is indeed invariant under every element of $\operatorname{Der}(A)$. In fact, for $x, y, z \in A$, $\delta \in \widetilde{I}$, $\mu \in \operatorname{Der}(A)$ we have $\mu(x\delta(y)z) = \mu(x)\delta(y)z + x\mu\delta(y)z + x\delta(y)\mu(z) = \mu(x)\delta(y)z + x[\mu, \delta](y)z + x\delta\mu(y)z + x\delta(y)\mu(z) \in J$ since $\delta, [\mu, \delta] \in \widetilde{I}$. This shows that $\mu(J) \subseteq J$ for every $\mu \in \operatorname{Der}(A)$. Finally, taking into account that $\widetilde{I} \subseteq \operatorname{ad}(I)$ we have $J \subseteq A\widetilde{I}(A)A \subseteq A[I, A]A \subseteq I$.

(ii) ⇒ (i). To prove the strong primeness of $\operatorname{Der}(A)$ we will use Lemma 3.11. Let us therefore consider $0 \neq \operatorname{ad}([U, A]) \lhd \operatorname{Inn}(A) \lhd \operatorname{Der}(A)$, for U an ideal of A. By the hypothesis, there exists a nonzero ideal J of A, which is contained in U and is invariant under every element of $\operatorname{Der}(A)$. Since $\operatorname{ad}([J, A])$ is contained in $\operatorname{ad}([U, A])$, the proof will be complete by showing that $\operatorname{ad}([J, A])$ is a nonzero ideal of $\operatorname{Der}(A)$. It is straightforward to verify that $\operatorname{ad}([J, A])$ is an ideal of $\operatorname{Der}(A)$. The containment $\operatorname{ad}([J, A]) \subseteq \operatorname{ad}([U, A])$ is obvious. The ideal $[J, A]$ is noncentral; otherwise, apply the fact that Z is a prime ideal of A^- (see [12, Lemma 4]) to obtain $J \subseteq Z$, which is impossible by Remark 3.9. Thus, $[J, A] \nsubseteq Z$ and therefore $\operatorname{ad}([J, A]) \neq 0$.

The last assertion follows directly from Corollary 3.7. □

Example 3.13. If A is a prime algebra such that every nonzero ideal I of A contains a nonzero idempotent ideal J, then $\operatorname{Der}(A)$ is SP. This follows from Theorem 3.12 together with the fact that $J = J^2$ implies $\delta(J) = \delta(J^2) \subseteq J$ for every $\delta \in \mathrm{Der}(A)$. In particular, this holds if A is a prime von Neumann regular algebra or, more generally, if A is an exchange algebra with zero Jacobson radical.

We conclude this section by a straightforward corollary to Theorem 3.12.

Corollary 3.14. *Let A be a simple algebra. Then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

4. The maximal Lie algebra of quotients of A^-/Z

Our aim in this section is to give a description of $Q_m(A^-/Z)$, where A is a (semi)prime algebra. Recall that the elements of the maximal algebra of quotients of a Lie algebra arise from partial derivations defined on essential Lie ideals. Since in this particular case A^-/Z comes from an associative algebra A, it seems natural to consider instead associative derivations that are defined on essential associative ideals. With this idea in mind we proceed to introduce a new Lie algebra.

Given essential ideals I, J of A and associative derivations $\delta\colon I \to A$, $\mu\colon J \to A$, we say that the pairs (δ, I), (μ, J) are equivalent if δ and μ agree on some essential ideal contained in $I \cap J$. One can easily show that this is an equivalence relation. Denote by δ_I the class determined by (δ, I) and by $\mathrm{Der_m}(A)$ the set of all equivalence classes.

Lemma 4.1. *Let A be a semiprime algebra. Then $\mathrm{Der_m}(A)$ is a Lie algebra under the following operations:*

$$\delta_I + \mu_J = (\delta+\mu)_{I\cap J}, \quad \alpha(\delta_I) = (\alpha\delta)_I, \quad [\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I\cap J)^2}.$$

Proof. The only not entirely obvious part in proving that $\mathrm{Der_m}(A)$ is a Lie algebra is to show that the Lie bracket is well defined on $\mathrm{Der_m}(A)$. Let δ_I, $\mu_J \in \mathrm{Der_m}(A)$; for every $u, v \in I \cap J$ we have $[\delta,\mu](uv) = \delta\mu(uv) - \mu\delta(uv) = \delta((\mu u)v + u(\mu v)) - \mu((\delta u)v + u(\delta v))$, which makes sense because $(\mu u)v, u(\mu v), (\delta u)v, u(\delta v) \in I \cap J$. Since δ and μ are derivations, $[\delta, \mu]\colon (I \cap J)^2 \to A$ is a derivation too. □

Lemma 4.2. *Let A be a semiprime algebra and let Q be a subalgebra of $Q_s(A)$ that contains A. If $\delta\colon Q \to Q_s(A)$ is a derivation such that $\delta|_A = 0$, then $\delta = 0$.*

Proof. Suppose on the contrary that $\delta(q) \neq 0$ for some $q \in Q$. Since $Q_s(A)$ is a left quotient algebra of A, there exists $a \in A$ satisfying $aq \in A$ and $a\delta(q) \neq 0$. By the hypothesis, $0 = \delta(aq) = \delta(a)q + a\delta(q) = a\delta(q)$, which is a contradiction. □

Lemma 4.3. *If A is a prime algebra, then*

$$\mathrm{Der}(A) \subseteq \mathrm{Der_m}(A) \subseteq \mathrm{Der}(Q_s(A)).$$

Proof. Define

$$\begin{array}{rll}\phi\colon & \mathrm{Der}(A) & \to \mathrm{Der_m}(A)\\ & \delta & \mapsto \delta_A\end{array}$$

It is straightforward to verify that ϕ is a well-defined Lie algebra homomorphism. To prove the injectivity, take $\delta \in \mathrm{Der}(A)$ such that $\delta_A = 0$; this means that $\delta(I) = 0$ for some nonzero ideal I of A. Since $Q_s(I) = Q_s(A)$ (see [3, Proposition 2.1.10]) applying Lemma 4.2 to $I \subseteq A \subseteq Q_s(I)$ we obtain that $\delta = 0$, as desired.

Let δ_I be in $\mathrm{Der}_{\mathrm{m}}(A)$, with I a nonzero ideal of A and $\delta\colon I \to A$ a derivation. Apply [3, Proposition 2.5.1] to extend δ uniquely to a derivation δ' of $Q_s(A)$. Consider

$$\varphi\colon \begin{array}{ccc} \mathrm{Der}_{\mathrm{m}}(A) & \to & \mathrm{Der}(Q_s(A)) \\ \delta_I & \mapsto & \delta' \end{array} \tag{4.1}$$

If $\delta_I = \mu_J$, then there exists a nonzero ideal U of A contained in $I \cap J$ such that $\delta|_U = \mu|_U$. Extend δ and μ to derivations δ' and μ', respectively, of $Q_s(A)$. Since $\delta'|_U = \delta|_U = \mu|_U = \mu'|_U$ and $Q_s(A) = Q_s(U)$ (see [3, Proposition 2.1.10]), by Lemma 4.2 applied to $U \subseteq A \subseteq Q_s(U)$ we obtain that $\delta'|_A = \mu'|_A$, and again by Lemma 4.2 it follows that $\delta' = \mu'$, which proves that φ is well-defined. Finally, note that φ is a Lie algebra monomorphism. □

If I is a Lie ideal of A, we denote by $\overline{I}$ the ideal $(I + Z)/Z$ of A^-/Z.

Lemma 4.4. *Let U be a Lie ideal of a semiprime algebra A such that $\overline{U}$ is an essential ideal of A^-/Z. Then the associative subalgebra $\langle U\rangle$ of A generated by U contains an essential ideal of A.*

Proof. First we show that $\langle U\rangle$ contains a nonzero ideal of A. It is clear that $[\langle U\rangle, U] \subseteq \langle U\rangle$. Moreover, $[\langle U\rangle, U] \neq 0$; otherwise $[x, U] = 0$ for every $x \in U$. This would imply, by [8, Sublemma, p. 5], $x \in Z$ and, consequently, $U \subseteq Z$, a contradiction. Therefore [10, Theorem 3] yields our claim.

Hence, let I be a nonzero ideal contained in $\langle U\rangle$. Since the sum of all ideals contained in $\langle U\rangle$ is again an ideal contained in U, there is no loss of generality in assuming that I is the largest ideal contained in $\langle U\rangle$.

We will show that I is an essential ideal of A. First, we see that $\mathrm{Ann}(I)\cap U \subseteq Z$. Otherwise, by [10, Theorem 3], there exists a nonzero ideal J of A contained in $\langle \mathrm{Ann}(I)\cap U\rangle \subseteq \mathrm{Ann}(I)\cap \langle U\rangle$. Since $I \cap \mathrm{Ann}(I) = 0$ because A is semiprime, $I \subsetneq I \oplus J \subseteq \langle U\rangle$, which contradicts the maximality of I. Since $\overline{U}$ is an essential ideal of A^-/Z, $\mathrm{Ann}_{A^-/Z}(\overline{U}) = 0$. Note that $[\mathrm{Ann}(I), U] \subseteq \mathrm{Ann}(I)\cap U \subseteq Z$ implies $(\mathrm{Ann}(I) + Z)/Z = 0$, that is, $\mathrm{Ann}(I) \subseteq Z \subseteq U$. Now, $I \subseteq I \oplus \mathrm{Ann}(I) \subseteq \langle U\rangle$ and the maximality of I imply $\mathrm{Ann}(I) = 0$, hence I is an essential ideal of A. □

Proposition 4.5. *Let A be a semiprime algebra. Define*

$$\varphi\colon \begin{array}{ccc} \mathrm{Der}_{\mathrm{m}}(A) & \to & Q_m(A^-/Z) \\ \delta_I & \mapsto & \bar{\delta}_{\overline{I}} \end{array}$$

where

$$\bar{\delta}\colon \begin{array}{ccc} \overline{I} & \to & A^-/Z \\ \bar{y} & \mapsto & \overline{\delta(y)} \end{array}$$

Then φ is a Lie algebra homomorphism with kernel $\{\delta_I \in \mathrm{Der}_{\mathrm{m}}(A) \mid \delta(I) \subseteq Z\}$.

Proof. The map $\bar{\delta}$ is well-defined. Indeed, taking into account Lemma 3.8 we see that it is enough to show that for I an essential ideal of A, and $\delta \in \mathrm{Der}(I, A)$, $y \in I \cap Z$ implies $\delta(y) \in Z$. Note that for every $x \in I$ we have $[\delta(y), x] = \delta([y, x]) - [y, \delta(x)] = 0$. But this yields $\delta(y) \in Z$. Namely, only central elements can commute with every element from an essential ideal. Indeed, $[a, u] = 0$ for every $u \in I$ yields $[a, x]u = [a, xu] = 0$ for all $x \in A$ and $u \in I$, and hence $[a, x] = 0$ since I is essential.

It is easy to see that φ is a well-defined Lie algebra homomorphism. Let us now compute its kernel. First we show that if $\delta_I \in \mathrm{Der}_m(A)$ is such that $\delta(J) \subseteq Z$ for some essential ideal J of A contained in I, then $\delta(I) \subseteq Z$. For $x \in I$ and $u \in J$ we have $xu \in J$, and so $\delta(u), \delta(xu) \in Z$. Accordingly, $\delta(x)u = \delta(xu) - x\delta(u)$ commutes with x, that is, $\delta(x)ux = x\delta(x)u$. Replacing u by uy, where $u \in J$ and $y \in A$, it follows that $\delta(x)uyx = x\delta(x)uy = \delta(x)uxy$. Thus, $\delta(x)u[x,y] = 0$ for all $x \in I$, $y \in A$ and $u \in J$. Linearizing this identity we get $\delta(x)u[z,y] + \delta(z)u[x,y] = 0$ for all $x, z \in I$, $y \in A$, $u \in J$. Consequently, for $u, v \in J$, $x, z \in I$ and $y \in A$ we have $\delta(x)u[z,y]v\delta(x)u[z,y] = -\delta(x)u[z,y]v\delta(z)u[x,y] \in \delta(x)J[x,y] = 0$. Since J is essential, $aJa = 0$ with $a \in A$ implies $a = 0$. Therefore, $\delta(x)u[z,y] = 0$ for all $u \in J$, $x, z \in I$, $y \in A$. In particular, $[\delta(x), z]J[\delta(x), z] = 0$ for all $x, z \in I$, which yields $[\delta(x), z] = 0$. Since elements commuting with all elements from an essential ideal of A must lie in the center of A, it follows that $\delta(x) \in Z$, as desired.

Denote by T the set $\{\delta_I \in \mathrm{Der_m}(A) \mid \delta(I) \subseteq Z\}$. Clearly, T is contained in the kernel of φ. For the converse containment, suppose $\bar{\delta}_{\overline{I}} = 0$ for δ_I an element in $\mathrm{Der_m}(A)$. Then there exists an essential ideal $\overline{U}$ of A^-/Z, contained in $\overline{I}$, such that $\bar{\delta}(\overline{U}) = 0$. Consider $V := \pi^{-1}(\overline{U}) \cap I$, for $\pi\colon A \to A^-/Z$ the canonical projection. The ideal $\overline{V}$ is essential because $\overline{U}$ and $\overline{I}$ are. By Lemma 4.4, there is an essential ideal J of A such that $J \subseteq \langle V + Z\rangle \subseteq I + \langle Z\rangle$. For an element x in the essential ideal $I \cap J$ of A, $\bar{\delta}(\bar{x}) \in \bar{\delta}(\overline{U}) = 0$, that is, $\delta(I \cap J) \subseteq Z$. By what was proved in the preceding paragraph it now follows that $\delta_I \in T$. □

Lemma 4.6. *Let A be a prime noncommutative algebra, I an ideal of A and $\delta\colon I \to A$ a derivation. If $\delta(I) \subseteq Z$ then $\delta = 0$.*

Proof. Suppose that $\delta(I) \subseteq Z$ and let $u \in I$. Then $u^2 \in I$, so $\delta(u^2) \in Z$, that is, $2u\delta(u) \in Z$. Given $x \in A$ we have that $0 = [2u\delta(u), x] = 2\delta(u)[u, x]$ and since A is prime, this implies that $u \in Z$ or $\delta(u) = 0$. Thus, for every $u \in I$ we have either $u \in Z$ or $\delta(u) = 0$. Taking into account Remark 3.9 there exists $y \in I \setminus Z$ and so $\delta(y) = 0$. Now take $v \in I$. If $v \notin Z$, then $\delta(v) = 0$, and if $v \in Z$, then $v + y \notin Z$ whence $\delta(v + y) = 0$. Therefore $\delta(v) = 0$ in any case. □

Theorem 4.7. *Let A be a prime algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then* $\mathrm{Der_m}(A) \cong Q_m(A^-/Z)$.

Proof. Consider the map φ in Proposition 4.5. Its injectivity is proved by Lemma 4.6. Let us prove the surjectivity. Let $\bar{\delta}_{\overline{J}}$ be in $Q_m(A^-/Z)$, with $\overline{J}$ a nonzero ideal of A^-/Z and $\bar{\delta}\colon \overline{J} \to A^-/Z$ a derivation. Let $\pi\colon A \to A^-/Z$ be the canonical projection. Note that $\overline{J}$ can be represented as J/Z where $J = \pi^{-1}(\overline{J})$ is a noncentral Lie ideal of A. Define $\delta\colon J \to A^-/Z$ by $\delta = \bar{\delta}\pi$. It is clear that δ is a derivation in the sense of [2]. In view of the assumptions on the degree and the characteristic, we are now in a position to apply [2, Theorem 1.3]. Picking any set-theoretic map $\gamma\colon J \to A$ such that $\overline{\gamma(x)} = \delta(x)$ for every $x \in J$, it follows that there exists a derivation $d\colon \langle J\rangle \to \langle J \cup \gamma(J)\rangle\mathcal{C} + \mathcal{C}$, where $\mathcal{C}$ is the extended centroid of A, and a map $\mu\colon J \to \mathcal{C}$ such that $d(x) = \gamma(x) + \mu(x)$ for all $x \in J$. As above, here $\langle S\rangle$ denotes the subalgebra generated by the set S.

For $x, y \in J$ we have $d([x,y]) = [d(x), y] + [x, d(y)] = [\gamma(x), y] + [x, \gamma(y)]$ since $\mu(J) \subseteq \mathcal{C}$. This shows that $d([J,J]) \subseteq J$, which in turn implies $d(\langle [J,J]\rangle) \subseteq \langle J\rangle \subseteq A$. As $[J,J]$ is a noncentral Lie ideal of A, there exists a nonzero ideal I of A contained in $\langle [J,J]\rangle$ (cf. the first step of the proof of Lemma 4.4). Note that d_I is an element of $\mathrm{Der_m}(A)$, and that $\varphi(d_I) = \overline{\delta}_{\overline{J}}$. This concludes the proof. □

Corollary 4.8. *Let A be a prime algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *If* $A = Q_s(A)$, *then*

$$Q_m(A^-/Z) \cong \mathrm{Der}(A).$$

Proof. By Lemma 4.3 we obtain that $\mathrm{Der}(A) \cong \mathrm{Der_m}(A)$ and applying Theorem 4.7 it follows that $\mathrm{Der_m}(A) \cong Q_m(A^-/Z)$, as desired. □

Corollary 4.9. *Let A be a simple algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)) \cong \mathrm{Der}(A).$$

Proof. Apply Corollary 3.14 to show that $Q_m(\mathrm{Der}(A)) \cong Q_m(A^-/Z)$ and Corollary 4.8 to have that $Q_m(A^-/Z) \cong \mathrm{Der}(A)$, which completes the proof. □

In our final corollary we will extend Corollary 4.8 by considering prime algebras A such that $Q_s(A) = AZ^{-1}$, i. e. every element in $Q_s(A)$ is of the form $\frac{a}{\lambda}$, where $a \in A$ and $\lambda \in Z$. However, we have to add the assumption that A is affine, i. e. generated by a finite number of elements.

Corollary 4.10. *Let A be an affine prime algebra such that $Q_s(A) = AZ^{-1}$ and either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then*

$$Q_m(A^-/Z) \cong \mathrm{Der}(Q_s(A)).$$

Proof. Consider the monomorphism $\varphi\colon \mathrm{Der_m}(A) \to \mathrm{Der}(Q_s(A))$ defined in (4.1). In order to check that φ is surjective it is enough to show that given δ in $\mathrm{Der}(Q_s(A))$ there exists a nonzero ideal I of A such that $\delta(I) \subseteq A$. Indeed, if this was true, then we could consider $\delta_I \in \mathrm{Der_m}(A)$ and then applying Lemma 4.2 for the case $I \subseteq A \subseteq Q_s(A) = Q_s(I)$ conclude that $\delta = \varphi(\delta_I)$.

So pick $\delta \in \mathrm{Der}(Q_s(A))$. Let $x_1, \ldots, x_n$ be generators of A. According to our assumption, for each $i = 1, \ldots, n$ we have $\delta(x_i) = \frac{y_i}{\lambda_i}$ for some $y_i \in A$, $\lambda_i \in Z$. Set $\lambda = \prod_{i=1}^n \lambda_i \in Z$. It is clear that $\delta(A) \subseteq \sum_{i=1}^n A\delta(x_i)A$, which in turn implies that $\lambda\delta(A) \subseteq A$. Accordingly, $\delta(\lambda^2 x) = 2\lambda\delta(\lambda)x + \lambda^2\delta(x) \in A$ for every $x \in A$. That is, δ maps the ideal $I = \lambda^2 A \neq 0$ of A into A. □

5. The maximal Lie algebra of quotients of K/Z_K

The purpose of the current section is to obtain results on the maximal algebra of quotients of the Lie algebra K/Z_K that arises from an associative algebra with involution. Our line of argument benefits from the approach developed in sections 3 and 4, although the proofs do not carry over verbatim.

In particular, we have to take into account whether the involution is of the first kind or of the second kind (see below). It is also natural to restrict our attention to the Lie algebra $\mathrm{SDer}(A)$ of those derivations that commute with the involution $*$ and to construct a Lie algebra $\mathrm{SDer}_m(A)$ similar to $\mathrm{Der}_m(A)$ as in Section 3, considering partial derivations defined on $*$-ideals (i.e. ideals invariant under $*$). The main results are then parallel to Theorem 3.12 and Theorem 4.7.

Let A be a semiprime algebra with involution $*$. Then $*$ induces an involution on $\mathcal{C}$, the extended centroid of A. It is said that the involution on A is *of the first kind* if $\mathcal{C} \cap K = 0$; otherwise it is said to be *of the second kind*, that is, $\mathcal{C} \cap K \neq 0$.

The set

$$\mathrm{SDer}(A) := \{\delta \in \mathrm{Der}(A) \mid \delta(x^*) = \delta(x)^* \text{ for all } x \in A\}$$

is a Lie subalgebra of $\mathrm{Der}(A)$. As usual, we will denote by $\mathrm{ad}\,(K)$ the Lie algebra of derivations $\mathrm{ad}\, x \colon A \to A$ with x in K.

Lemma 5.1. *Let A be a semiprime algebra with involution $*$. Then:*

(i) $\mathrm{ad}\,(K) \subseteq \mathrm{Inn}(A) \cap \mathrm{SDer}(A)$.
(ii) $\delta(K) \subseteq K$ *for every* $\delta \in \mathrm{SDer}(A)$.
(iii) $\mathrm{ad}\,(K)$ *is an ideal of* $\mathrm{SDer}(A)$.

Proof. (i). For every $a \in K$ and $x \in A$, $((\mathrm{ad}\, a)x)^* = [a, x]^* = [x^*, a^*] = [a, x^*] = (\mathrm{ad}\, a)(x^*)$. This implies $\mathrm{ad}\, a \in \mathrm{SDer}(A)$.

(ii). Let δ be in $\mathrm{SDer}(A)$. For every $x \in K$, $\delta(x)^* = \delta(x^*) = \delta(-x) = -\delta(x)$. This shows $\delta(K) \subseteq K$.

(iii). For $a \in K$ and $\delta \in \mathrm{SDer}(A)$ we have $[\delta, \mathrm{ad}\, a] = \mathrm{ad}\, \delta(a)$, which, together with condition (ii), implies (iii). □

The following result is a generalization of [4, Lemma 2.9].

Lemma 5.2. *Let A be a prime algebra with involution $*$ of the first kind such that* $\deg(A) > 2$. *If $t \in K$ and $[t, K] = 0$, then $t = 0$.*

Proof. By [15, Lemma 2], the subalgebra generated by $[K, K]$ contains a nonzero ideal I of A. For $t \in K$ satisfying $[t, K] = 0$, use induction and the identity $[a, bc] = [a, b]c + b[a, c]$ which holds for all $a, b, c \in A$ to show that $[t, I] = 0$. Now, apply [4, Lemma 2.5] to obtain $t \in K \cap Z = 0$, as desired. □

Lemma 5.3. *Let A be a prime algebra with involution $*$. Then:*

(i) *If $*$ is of the second kind, then $Z_K = Z \cap K$ and $\delta(Z_K) \subseteq Z_K$ for every* $\delta \in \mathrm{SDer}(A)$.
(ii) *If $*$ is of the first kind and* $\deg(A) > 2$, *then $Z_K = Z \cap K = 0$.*

Proof. (i) follows taking into account Lemma 5.1(ii) and applying [12, Lemma 2 (ii)] and [4, Theorem 2.13]. To prove (ii) it is enough to apply Lemma 5.2. □

Remark 5.4. Let A be a prime algebra with involution $*$. The map

$$\begin{array}{ccc} K & \to & \mathrm{ad}\,(K) \\ x & \mapsto & \mathrm{ad}\, x \end{array}$$

is a Lie algebra epimorphism with kernel Z_K; this allows to identify K/Z_K with the ideal $\mathrm{ad}\,(K)$ of $\mathrm{SDer}(A)$. If the involution is of the first kind and $\deg(A) > 2$, it is in fact an isomorphism, by Lemma 5.3 (ii).

On the other hand, for every ideal I of K, the restriction of the map above to I, that is,

$$\begin{array}{ccc} I & \to & \mathrm{ad}\,(K) \\ y & \mapsto & \mathrm{ad}\, y \end{array}$$

is a Lie algebra homomorphism with kernel $Z_I = I \cap Z_K$, if the involution is of the second kind, or zero, if it is of the first kind and $\deg(A) > 2$. Indeed, $[y, I] = 0$, with $y \in I$, implies

$[y,[y,K]] = 0$. Then apply [4, Theorem 2.13] or Lemma 5.2 to have $y \in Z_K$ or $y = 0$. Moreover,

$$I/Z_I = I/(I \cap Z_K) \cong (I + Z_K)/Z_K \lhd K/Z_K.$$

Lemma 5.5. *Let A be a prime algebra with involution $*$ such that* $\deg(A) > 2$. *Then* $[I \cap K, K] \neq 0$ *for every nonzero $*$-ideal I of A.*

Proof. Consider a nonzero $*$-ideal I of A and suppose $I \cap K \subseteq Z_K = Z \cap K$. By Remark 3.9 we have $I \nsubseteq Z$. Hence, there exists $x \in I$ such that $x \notin Z$. By the hypothesis, $[x, I \cap K] \subseteq Z$ and taking into account [14, Theorem 2] if $*$ is of the second kind or [14, Theorem 3] if it is of the first kind we obtain $I \subseteq Z$, which is a contradiction. □

Lemma 5.6. *Let A be a prime algebra with involution $*$ such that* $\deg(A) > 4$. *Then, for every nonzero ideal $\widetilde{I}$ of* $\mathrm{ad}\,(K)$ *there exists a $*$-ideal U of A such that* $0 \neq \mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{I}$.

Proof. The set $I := \{x \in K \mid \mathrm{ad}\, x \in \widetilde{I}\}$ is an ideal of K not contained in Z_K and, therefore, it is not contained in Z. Apply [7, Theorem 1] if $*$ is of the second kind or [7, Theorem 5 and Lemma 7] if it is of the first kind to find a nonzero $*$-ideal U of A satisfying $[U \cap K, K] \subseteq I$, that is, $\mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{I}$. Note that $[U \cap K, K] \nsubseteq Z_K$ as otherwise, $U \cap K \subseteq Z_K$, which contradicts Lemma 5.5. □

By [17, Theorem 6.2] we have that, if A is a prime algebra with involution, $K/Z_K \cong \mathrm{ad}\,(K)$ is an SP Lie algebra. Moreover, by [11, Theorem 2], the Lie algebra $\mathrm{SDer}(A)$ is prime. As in section 3, the question of whether $\mathrm{SDer}(A)$ is SP is more delicate and is related to the ideal structure of A.

The proof of the following result is similar to that of Lemma 3.11, applying in this case Lemma 5.6 instead of Lemma 3.10.

Lemma 5.7. *Let A be a prime algebra with involution $*$ and such that* $\deg(A) > 4$. *Assume that for every $*$-ideal U of A there exists a nonzero ideal $\widetilde{U}$ of* $\mathrm{SDer}(A)$ *such that* $\widetilde{U} \subseteq \mathrm{ad}\,[U \cap K, K]$. *Then* $\mathrm{SDer}(A)$ *is an SP Lie algebra.*

Let A be an algebra and $\widetilde{I}$ a nonzero ideal of $\mathrm{Der}(A)$. In the proof of Theorem 3.12 we have shown that $J := \sum_{\delta \in \widetilde{I}} A\delta(A)A$ is a nonzero ideal of A invariant under every element of $\mathrm{Der}(A)$. If A has an involution $*$ and $\widetilde{I}$ is a nonzero ideal of $\mathrm{SDer}(A)$ then J, defined as above, is a $*$-ideal of A invariant under every element of $\mathrm{SDer}(A)$. Combining this with Lemmas 5.5 and 5.7 the proof of the following theorem is similar to that of Theorem 3.12, and therefore we omit the details.

Theorem 5.8. *Let A be a prime algebra with involution $*$ and such that* $\deg(A) > 4$. *Then the following conditions are equivalent:*

(i) $\mathrm{SDer}(A)$ *is SP.*
(ii) *Every nonzero $*$-ideal of A contains a nonzero $*$-ideal of A invariant under every element of* $\mathrm{SDer}(A)$.

Moreover, if the previous conditions are satisfied we have

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)).$$

With Theorem 5.8 at hand, the proof of the following corollary is a repetition of the proof of Corollary 3.14.

Corollary 5.9. *Let A be a prime algebra with involution $*$ such that $\deg(A) > 4$. If A is a $*$-simple algebra, then:*

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)).$$

We now turn to the question of having a good description of the Lie algebra $Q_m(K/Z_K)$, in the case that A is prime and has an involution. As already mentioned, to this end we shall introduce a new Lie algebra, whose definition is based on partial $*$-preserving derivations.

If I, J are nonzero $*$-ideals of A and $\delta\colon I \to A$, $\mu\colon J \to A$ are derivations of A which preserve $*$, we say that the two pairs (δ, I), (μ, J), are equivalent if δ and μ agree on some nonzero $*$-ideal contained in $I \cap J$. It is easy to verify that this is an equivalence relation. Denote by δ_I the equivalence class determined by (δ, I) and by $\mathrm{SDer_m}(A)$ the set of these equivalence classes.

Lemma 5.10. *Let A be a prime algebra with involution. Then $\mathrm{SDer_m}(A)$ is a Lie algebra under the natural operations defined as in Lemma 4.1.*

It is known that every involution $*$ defined on a prime algebra A can be lifted uniquely to an involution, also denoted by $*$, on $Q_s(A)$ (see [3, Proposition 2.5.1]). Thus, in this case, it makes sense to consider $\mathrm{SDer}(Q_s(A))$. The following result is analogous to Lemma 4.3. In order to prove it, it is enough to show that every $\delta \in \mathrm{SDer}(A)$ can be uniquely extended to a derivation δ' in $\mathrm{SDer}(Q_s(A))$. This can be shown by standard methods. Basically, it follows from the fact that $Q_s(A)$ is an algebra of left quotients of A, coupled with the fact that every derivation on A can be extended uniquely to a derivation of $Q_s(A)$ ([3, Proposition 2.5.1]).

Lemma 5.11. *If A is a prime algebra with involution, then*

$$\mathrm{SDer}(A) \subseteq \mathrm{SDer_m}(A) \subseteq \mathrm{SDer}(Q_s(A)).$$

Our aim now is to construct an isomorphism between the Lie algebra $\mathrm{SDer_m}(A)$ defined in Lemma 5.10 and the maximal algebra of quotients of the Lie algebra K/Z_K. Retain the notation $\langle X\rangle$ for the subalgebra of an algebra A generated by a set X.

Lemma 5.12. *Let A be a prime algebra with involution $*$ such that $\deg(A) > 4$, and let U be an ideal of K such that $U \nsubseteq Z_K$. Then the algebra $\langle U\rangle$ contains a nonzero $*$-ideal of A.*

Proof. Clearly $\langle U\rangle^* = \langle U\rangle$, and $\langle U\rangle \nsubseteq Z$ since $U \nsubseteq Z_K$. On the other hand, note that $[\langle U\rangle, K] \subseteq \langle U\rangle$. This follows by an induction argument using the identity $[uv, x] = u[v, x] + [u, x]v$, for every $u, v, x \in A$. Next, apply [15, Theorem 2] to obtain the desired conclusion. □

Lemma 5.13. *Let A be a prime algebra with involution $*$ such that $\deg(\mathrm{A}) > 2$. If δ_I is an element of $\mathrm{SDer_m}(A)$ such that $\delta(I \cap K) \subseteq Z_K$ then $\delta_I = 0$.*

Proof. It is well known that $\deg(I) = \deg(A)$ (because A is prime, see, e.g. [3, Theorem 6.4.1]). Therefore [16, Theorem 3] applies to show that, since $[x, \delta(x)] = 0$ for any $y \in K_I$ by our assumption and $\deg(I) > 2$, necessarily $\delta_I = 0$. □

Theorem 5.14. *Let A be a prime algebra with involution $*$ such that $\deg(A) > 4$. Then $\mathrm{SDer}_m(A) \cong Q_m(K/Z_K)$.*

Proof. Consider

$$\begin{array}{rccc} \varphi\colon & \mathrm{SDer}_m(A) & \to & Q_m(K/Z_K) \\ & \delta_I & \mapsto & \bar{\delta}_{\overline{I}} \end{array}$$

where $\overline{I} = ((I \cap K) + Z_K)/Z_K$ and

$$\begin{array}{rccc} \bar{\delta}\colon & \overline{I} & \to & K/Z_K \\ & \bar{y} & \mapsto & \overline{\delta(y)} \end{array}$$

The map $\bar{\delta}$ is well-defined. To see this, it is enough to check, by Lemma 5.5, that $\delta((I \cap K) \cap Z_K) \subseteq Z_K$, whenever I is a nonzero $*$-ideal of A and $\delta \in \mathrm{SDer}(I, A)$. By Lemma 5.1, if $y \in I \cap Z_K$ we have $y \in Z$ and arguing as in the proof of Proposition 4.5 we obtain $\delta(y) \in Z$. Consequently $\delta(y) \in \delta(K) \cap Z \subseteq Z_K$.

It is easy to see that φ is a well-defined Lie algebra homomorphism. We first prove it is one-to-one. Let δ_I be an element in $\mathrm{SDer}_m(A)$ such that $\bar{\delta}_{\overline{I}} = 0$. Then there exists a nonzero ideal $\overline{J} := J/Z_K$ of K/Z_K contained in $\overline{I}$ such that $\bar{\delta}(\overline{J}) = 0$. Consider $J_1 := \pi^{-1}(\overline{J}) \cap I$, where $\pi\colon K \to K/Z_K$ is the canonical projection, and note that the ideal $(J_1 + Z_K)/Z_K$ is nonzero because $\overline{J}$ and $\overline{I}$ are nonzero. By Lemma 5.12, there is a nonzero $*$-ideal U of A such that $U \subseteq \langle J_1 + Z_K\rangle \subseteq (I \cap K) + \langle Z_K\rangle$. Since $\bar{\delta}(\bar{u}) \in \bar{\delta}(\overline{J}) = 0$ for any element u in $(U \cap I) \cap K$, we see that $\delta((U \cap I) \cap K) \subseteq Z_K$ and, by Lemma 5.13, we conclude $\delta_I = 0$.

Now we show that φ is surjective. Let $\bar{\delta}_{\overline{J}}$ be in $Q_m(K/Z_K)$, with $\overline{J}$ a nonzero ideal of K/Z_K and $\bar{\delta}\colon \overline{J} \to K/Z_K$ a derivation. Note that $\overline{J}$ can be represented as J/Z_K where $J = \pi^{-1}(\overline{J})$ is a noncentral ideal of K. Define $\delta\colon J \to A^-/Z$ by $\delta = i\bar{\delta}\pi$, where $i\colon K/Z_K \to A^-/Z$ is given by $i(\bar{x}) = \bar{x} \in A^-/Z$. Since $Z_K = Z \cap K$ (see Lemma 5.1) it is straightforward to verify that i is a Lie algebra monomorphism. On the other hand, it is clear that δ is a Lie derivation in the sense of [1] and that K satisfies the conditions in [1, Theorem 3.2]. Therefore, take any set-theoretic map $\gamma\colon J \to K$ such that $\overline{\gamma(x)} = \delta(x)$ for every $x \in J$ (note that we may actually choose γ with image contained in K because $\delta(J) \subseteq K/Z_K$), and then it follows that there exists a derivation $d\colon \langle J\rangle \to \langle J \cup \gamma(J)\rangle\mathcal{C} + \mathcal{C}$, where $\mathcal{C}$ is the extended centroid of A, and a map $\mu\colon J \to \mathcal{C}$ such that $d(x) = \gamma(x) + \mu(x)$ for all $x \in J$.

For $x, y \in J$ we have

$$d([x, y]) = [d(x), y] + [x, d(y)] = [\gamma(x), y] + [x, \gamma(y)]$$

since $\mu(J) \subseteq \mathcal{C}$. This shows that $d([J, J]) \subseteq [K, J] \subseteq J$, which in turn implies $d(\langle [J, J]\rangle) \subseteq \langle J\rangle \subseteq A$. Apply Lemma 5.12 to the ideal $[J, J]$ of K (which is not contained in Z_K) to find a nonzero $*$-ideal I of A contained in $\langle [J, J]\rangle$. Note that d_I is an element of $\mathrm{SDer_m}(A)$. Finally, since $\mu(I) \subseteq K \cap \mathcal{C} = K \cap A \cap \mathcal{C} = K \cap Z = Z_K$ (by using [22, Lemma 1.3 (i)] and Lemma 5.1) it follows that $\varphi(d_I) = \bar{\delta}_{\overline{J}}$. This concludes the proof. □

Corollary 5.15. *Let A be a prime algebra with involution $*$ such that* $\deg(A) > 4$. *If* $A = Q_s(A)$, *then*

$$Q_m(K/Z_K) \cong \mathrm{SDer}(A).$$

Proof. By Lemma 5.11 we obtain that $\mathrm{SDer}(A) \cong \mathrm{SDer_m}(A)$ and applying Theorem 5.14 it follows that $\mathrm{SDer_m}(A) \cong Q_m(K/Z_K)$, as desired. □

Corollary 5.16. *Let A be a simple algebra with involution such that* $\deg(A) > 4$. *Then:*

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)) \cong \mathrm{SDer}(A).$$

Proof. Apply Corollary 5.9 to obtain that $Q_m(\mathrm{SDer}(A)) \cong Q_m(K/Z_K)$. Corollary 5.15 implies $Q_m(K/Z_K) \cong \mathrm{SDer}(A)$. □

6. Max-closed algebras

This final section is devoted to the problem of whether taking the maximal algebra of quotients is a closure operation, that is, if $Q_m(Q_m(L)) = Q_m(L)$ holds for every semiprime Lie algebra L. Notice that this question makes sense since $Q_m(L)$ is also semiprime ([22, Proposition 2.7 (ii)]). Although in some interesting special cases the answer is positive, we will prove that in general the containment $Q_m(L) \subseteq Q_m(Q_m(L))$ is strict. This justifies the terminology in the definition below.

Definition 6.1. We say that a semiprime Lie algebra L is *max-closed* if $Q_m(Q_m(L)) = Q_m(L)$.

In the next three results we present various examples of max-closed Lie algebras. The first one follows immediately from Corollary 4.9.

Corollary 6.2. *Let A be a simple algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then A^-/Z is max-closed.*

Theorem 6.3. *If L is a simple Lie algebra, then* $Q_m(L) \cong \mathrm{Der}(L)$ *is an SP Lie algebra and L is max-closed.*

Proof. In view of the simplicity of L we clearly have $Q_m(L) \cong \mathrm{Der}(L)$. Moreover, these two Lie algebras are prime by [22, Proposition 2.7 (ii)].

We claim that L is isomorphic to the smallest nonzero ideal of $\mathrm{Der}(L)$. Indeed, since $Z_L = 0$ we have $L \cong \mathrm{ad}(L) \lhd \mathrm{Der}(L)$. Identify L with $\mathrm{ad}(L)$ and consider $0 \neq \widetilde{U} \lhd \mathrm{Der}(L)$. Taking into account the simplicity of L and that $0 \neq \widetilde{U} \cap L \lhd L$ we obtain $\widetilde{U} \cap L = L$, which implies $L \subseteq \widetilde{U}$.

For $0 \neq \widetilde{I} \lhd \widetilde{J} \lhd \mathrm{Der}(L)$ apply what we have proved to obtain $L \subseteq \widetilde{J}$. We claim that $U = \widetilde{I} \cap L$ is a nonzero ideal of L. In fact, $[U, L] \subseteq L$ and $[U, L] \subseteq [U, \widetilde{J}] \subseteq \widetilde{I}$, which implies $[U, L] \subseteq \widetilde{I} \cap L = U$. To show that $U \neq 0$, consider $0 \neq \delta \in \widetilde{I}$. Since $Z_L = 0$ there exists $x \in L$ such that $0 \neq \mathrm{ad}\,\delta(x) \in L$. Moreover, $\mathrm{ad}\,\delta(x) = [\delta, \mathrm{ad}\,x] \in \widetilde{I}$; hence, $0 \neq \mathrm{ad}\,\delta(x) \in U$. Thus, U is a nonzero ideal of a simple Lie algebra L, so that $L = U \subseteq \widetilde{I}$. From Lemma 3.1 we now see that $\mathrm{Der}(L)$ is an SP Lie algebra.

It remains to show that L is max-closed. We have $Q_m(Q_m(L)) \cong Q_m(\mathrm{Der}(L))$. Since L is a nonzero ideal of an SP Lie algebra $\mathrm{Der}(L)$, it follows from Theorem 3.3 that $Q_m(L) \cong Q_m(\mathrm{Der}(L))$. □

Theorem 6.4. *Let A be a prime affine PI algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$, *and let J be a noncentral Lie ideal of A. Then the Lie algebra $J/(J \cap Z)$ is max-closed.*

Proof. Recall that $A^-/Z \cong \mathrm{Inn}(A)$ is an SP Lie algebra (see the discussion before Corollary 3.6). Accordingly, applying Theorems 3.3 and 4.7 it follows that $Q_m(J/(J \cap Z)) \cong Q_m(A^-/Z) \cong \mathrm{Der_m}(A)$. It is well-known that A, as a prime PI algebra, satisfies $Q_s(A) = AZ^{-1}$, and moreover, that $Q_s(A)$ is a simple algebra (see e. g. [21, Theorem 1.7.9] or [9, Theorem 1.4.3] from which this can be easily derived). Therefore we infer from Corollary 4.10 that $\mathrm{Der_m}(A) \cong \mathrm{Der}(Q_s(A))$. On the other hand, Corollary 4.9 shows that $Q_m(\mathrm{Der}(Q_s(A))) \cong \mathrm{Der}(Q_s(A))$, and the proof is thereby complete. □

We will finish this paper by finding an example of a Lie algebra which is not max-closed. The algebra A we shall deal with is the one that Passman used in [19] to show that $Q_s(\,.\,)$ is not a closure operation.

Let K be a field and let $A = K[\,t\,][x, y \mid xy = tyx]$. In [19] the following properties of A were established:

(i) A is a domain with center $Z = K[\,t\,]$;
(ii) $Q_s(A) = K(t)[x, y \mid xy = tyx]$;
(iii) $Q_s(Q_s(A)) = K(t)[x^{-1}, x, y^{-1}, y \mid xy = tyx]$.

We shall make use of (iii) in the proof below, but not in an explicit way.

Theorem 6.5. *Let $A = K[\,t\,][x, y \mid xy = tyx]$. Then the Lie algebra A^-/Z is not max-closed.*

Proof. We shall write Q for $Q_s(A)$. Note that the conditions of Corollary 4.10 are again fulfilled. Therefore, this corollary together with Theorem 4.7 shows that

$$Q_m(A^-/Z) \cong \mathrm{Der_m}(A) \cong \mathrm{Der}(Q).$$

Therefore it is enough to prove that $Q_m(\mathrm{Der}(Q)) \supsetneq \mathrm{Der}(Q)$.

Note that $Qx = xQ = QxQ$; this will be frequently used in the sequel without mention. We also remark that Q is the vector space direct sum of Qx and $\sum_{i=0}^{\infty} K(t)y^i$.

Let δ be a derivation of Q. Since $xy = tyx$ it follows that $\delta(x)y + x\delta(y) = \delta(t)yx + t\delta(y)x + ty\delta(x)$, and hence $\delta(x)y - ty\delta(x) \in Qx$. Writing $\delta(x) = qx + \sum_{i=0}^{m} \lambda_i(t)y^i$, where $q \in Q$ and $\lambda_i(t) \in K(t)$, it follows that $\sum_{i=0}^{m} \lambda_i(t)y^{i+1} - \sum_{i=0}^{m} t\lambda_i(t)y^{i+1} \in Qx$. That is, $\sum_{i=0}^{m}(1-t)\lambda_i(t)y^{i+1} \in Qx$. But then $\sum_{i=0}^{m}(1-t)\lambda_i(t)y^{i+1} = 0$ and hence $\lambda_i(t) = 0$ for each i. This proves that $\delta(x) \in Qx$, which in turn implies $\delta(Qx) \subseteq Qx$. Thus, Qx is invariant under every derivation of Q.

Let I be the linear span of all inner derivations of the form $\mathrm{ad}\,(\delta_1 \dots \delta_n(x))$, where $n \in \mathbb{N}$ and $\delta_1, \dots, \delta_n \in \mathrm{Der}(Q)$. We claim that I is a nonzero Lie ideal of $\mathrm{Der}(Q)$. Indeed, for every $\delta \in \mathrm{Der}(Q)$ we have

$$[\delta, \mathrm{ad}\,(\delta_1 \dots \delta_n(x))] = \mathrm{ad}\,(\delta\delta_1 \dots \delta_n(x)) \in I,$$

showing that I is an ideal, and $\mathrm{ad}\,(\mathrm{ad}\,y(x)) = \mathrm{ad}\,[y, x]$, and so $I \neq 0$. Define $\Delta\colon I \to \mathrm{Der}(Q)$ by $\Delta(d) = [\mathrm{ad}\,x^{-1}, d]$ for every $d \in I$, so that

$$\Delta(\mathrm{ad}\,(\delta_1 \dots \delta_n(x))) = [\mathrm{ad}\,x^{-1}, \mathrm{ad}\,(\delta_1 \dots \delta_n(x))] = \mathrm{ad}\,[x^{-1}, \delta_1 \dots \delta_n(x)];$$

this makes sense since $\delta_1 \dots \delta_n(x) \in Qx$ by what was proved in the preceding paragraph. Clearly Δ is a derivation. This allows us to consider $\Delta_I \in Q_m(\mathrm{Der}(Q))$. We claim that Δ_I is not in $\mathrm{Der}(Q)$. Suppose this was not true. Then $\Delta_I = \mathrm{ad}\,\delta_{\mathrm{Der}(Q)}$ for some $\delta \in \mathrm{Der}(Q)$. This means that there exists a nonzero ideal J of $\mathrm{Der}(Q)$ contained in I and such that $\Delta|_J = (\mathrm{ad}\,\delta)|_J$. It is easy to see that derivations defined on I which agree on a nonzero ideal J contained in I, must agree on the entire I. Thus, $\Delta = (\mathrm{ad}\,\delta)|_I$. That is,

$$[\mathrm{ad}\,x^{-1}, \mathrm{ad}\,(\delta_1 \dots \delta_n(x))] = [\delta, \mathrm{ad}\,(\delta_1 \dots \delta_n(x))] = \mathrm{ad}\,(\delta\delta_1 \dots \delta_n)(x)$$

for all $\delta_1, \dots, \delta_n \in \mathrm{Der}(Q)$. In particular,

$$\mathrm{ad}\,[x^{-1}, [y, x]\,] = [\mathrm{ad}\,x^{-1}, \mathrm{ad}\,[y, x]\,] = \mathrm{ad}\,(\delta([y, x])),$$

which implies $[x^{-1},[y,x]\,]-\delta([y,x])\in Z_Q=K(t)$. Since δ, as a derivation of Q, leaves Qx invariant, it follows that $[x^{-1},[y,x]\,]\in Qx+K(t)$. However,

$$[x^{-1},[y,x]\,]=x^{-1}(yx-xy)-(yx-xy)x^{-1}=t^{-1}y-y-y+ty=(t^{-1}+t-2)y,$$

a contradiction. □

Acknowledgments

Part of this work was carried out during visits of the second, third and fourth authors to the University of Maribor and of the first author to the Universidad de Málaga and the Universitat Autònoma de Barcelona. They thank these host centers for their warm hospitality and support. The first author was partially supported by the grant P1-0288 from ARRS. The second author was partially supported by the DGI MEC-FEDER through Project MTM2005-00934 and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The third author was supported by grants by the Junta de Andalucía (BOJA n. 120 de 21 de junio de 2004 and "Ayuda para estancia en centros de excelencia extranjeros"); the third and fourth authors by the Spanish MEC and Fondos FEDER jointly, through projects MTM2004-06580-C02-02, and by the Junta de Andalucía, PAI project FQM-1215, and all four authors by the "Plan Andaluz de Investigación y Desarrollo Tecnológico", project FQM-336.

The authors would also like to thank the referee for several useful comments and suggestions.

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