ZERO PRODUCT PRESERVING MAPS ON $C^1[0,1]$

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Abstract. The main result of the paper characterizes continuous bilinear maps ϕ from $C^1[0,1] \times C^1[0,1]$ into a Banach space X with the property that $\phi(f,g) = 0$ whenever $fg = 0$. This is applied to the study of zero product preserving operators on $C^1[0,1]$, and operators on $C^1[0,1]$ satisfying a version of the condition of the locality of an operator.

1. Introduction

The recent paper [2] introduces the class of Banach algebras A with the following property: Every continuous bilinear map ϕ from $A \times A$ into an arbitrary Banach space X such that $\phi(a,b) = 0$ whenever $ab = 0$, satisfies the condition $\phi(ab,c) = \phi(a,bc)$ for all $a,b,c \in A$. If A is unital, then this condition is equivalent to the one that $\phi(a,b) = P(ab)$ for all a,b and some continuous linear operator $P : A \to X$; indeed, one defines P by $P(a) = \phi(a,1)$. It turns out that this class of Banach algebras is quite large. In particular it includes C^*-algebras, group algebras, and Banach algebras generated by idempotents. Further, it is shown in [2] that a variety of problems, which were previously considered only in some special algebras, can be handled in this class of algebras. Most of these problems concern linear operators, but they can be reduced to bilinear ones having the aforementioned property. Let us just mention one typical example in order to illustrate this idea. A linear operator T between Banach algebras A and B is said to be *zero product preserving* if for all $a,b \in A$, $ab = 0$ implies $Ta \cdot Tb = 0$. The problem of describing such operators has been thoroughly studied in the literature. Now, if A belongs to our class, then by considering $\phi(a,b) = Ta \cdot Tb$ one immediately realizes that T satisfies $T(ab) \cdot Tc = Ta \cdot T(bc)$ for all $a,b,c \in A$. Under some additional assumptions one can then derive that T is a weighted homomorphism.

One of the most notable examples of Banach algebras that *does not* belong to this class is $C^1[0,1]$, the algebra of continuously differentiable functions

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from $[0,1]$ to $\mathbb{C}$. Indeed, the bilinear map $\phi : C^1[0,1] \times C^1[0,1] \to C[0,1]$ defined by $\phi(f,g) = fg'$ satisfies the property that $\phi(f,g) = 0$ whenever $fg = 0$ (namely, $fg = 0$ yields $f'g + fg' = 0$, hence $f^2g' = 0$ and so $fg' = 0$), but clearly does not satisfy $\phi(fg,h) = \phi(f,gh)$ for all $f,g,h \in C^1[0,1]$. Other typical examples are maps $(f,g) \mapsto f'g$ and $(f,g) \mapsto f'g'$. Therefore, it seems to be a challenging problem to describe the general form of a continuous bilinear map ϕ from $C^1[0,1] \times C^1[0,1]$ into a Banach space X such that $fg = 0$ implies $\phi(f,g) = 0$. We solve this problem in Section 2. The result is that ϕ can be expressed as $\phi(f,g) = P(fg) + Q(fg') + R(f'g')$ where P, Q, R are continuous linear operators (the term involving $f'g$ is missing simply because $f'g = (fg)' - fg'$). The rest of the paper is devoted to applications of this result. In Section 3 we consider operators from $C^1[0,1]$ into a left Banach A-module X with the property that $fg = 0$ implies $f \cdot Tg = 0$; this condition can be viewed as an algebraic version of the condition that an operator is local. Section 4 is devoted to zero product preserving operators on $C^1[0,1]$.

The methods used in this paper are applicable to the algebra $C^n[0,1]$ with $n > 1$. However, considering these algebras would make the paper rather lengthy, without bringing really new ideas. Therefore we shall restrict ourselves to the $C^1[0,1]$ case.

2. Zero product preserving bilinear maps on $C^1[0,1]$

As usual, we endow $C^1[0,1]$ with the norm $\|\cdot\|_1$ given by $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ for each $f \in C^1[0,1]$. Then $C^1[0,1]$ becomes a Banach algebra which is generated by polynomials (see e.g. [3, Theorem 4.4.1]). By $\mathbf{1}$ and $\mathbf{x}$ we denote the functions on $[0,1]$ given by

$$\mathbf{1}(t) = 1, \quad \mathbf{x}(t) = t, \quad (t \in [0,1]).$$

Further, we will make use of the continuous linear operator

$$V \colon C[0,1] \to C^1[0,1], \ (Vf)(t) = \int_0^t f(s)ds, \quad (f \in C[0,1], \ t \in [0,1]).$$

By X we denote an arbitrary Banach space.

Theorem 2.1. *Let $\phi \colon C^1[0,1] \times C^1[0,1] \to X$ be a continuous bilinear map satisfying*

$$f,g \in C^1[0,1], \ fg = 0 \quad \Rightarrow \quad \phi(f,g) = 0.$$

Then there exist continuous linear operators $P \colon C^1[0,1] \to X$ and $Q, R \colon C[0,1] \to X$ such that

$$\phi(f,g) = P(fg) + Q(fg') + R(f'g') \tag{1}$$

for all $f,g \in C^1[0,1]$.

Proof. The proof consists in proving that the continuous linear operators R, S, T defined by

$$P \colon C^1[0,1] \to X, \ Pf = \phi(f, \mathbf{1})$$

and

$$Q, R\colon C[0,1] \to X,\ Qf = \phi(\mathbf{1}, Vf) - \phi(Vf, \mathbf{1}),$$
$$Rf = \phi(Vf, \mathbf{x}) - \phi(\mathbf{x}\cdot Vf, \mathbf{1}) + \phi(V^2 f, \mathbf{1}) - \phi(\mathbf{1}, V^2 f)$$

satisfy (1). By using the continuity and the density of the polynomials on $C^1[0,1]$, it suffices to check that (1) holds for all monomials $f = \mathbf{x}^m$ and $g = \mathbf{x}^n$ with $m, n \in \mathbb{N} \cup \{0\}$. In the case where either $m = 0$ or $n = 0$ this is easily checked. Therefore, in the sequel we restrict our attention to the case $m, n \geq 1$ and thus we shall prove the truthfulness of the identity

$$\begin{aligned} \phi\left(\mathbf{x}^m, \mathbf{x}^n\right) = {} & \phi\left(\mathbf{x}^{m+n}, \mathbf{1}\right) \\ & + \phi\left(\mathbf{1}, \tfrac{n}{m+n}\mathbf{x}^{m+n}\right) - \phi\left(\tfrac{n}{m+n}\mathbf{x}^{m+n}, \mathbf{1}\right) \\ & + \phi\left(\tfrac{mn}{m+n-1}\mathbf{x}^{m+n-1}, \mathbf{x}\right) - \phi\left(\tfrac{mn}{m+n-1}\mathbf{x}^{m+n}, \mathbf{1}\right) \\ & + \phi\left(\tfrac{mn}{(m+n)(m+n-1)}\mathbf{x}^{m+n}, \mathbf{1}\right) - \phi\left(\mathbf{1}, \tfrac{mn}{(m+n)(m+n-1)}\mathbf{x}^{m+n}\right). \end{aligned} \tag{2}$$

To this end we first observe that the map ϕ gives rise to a continuous linear operator Φ on the projective tensor product $A = C^1[0,1] \otimes_\pi C^1[0,1]$ defined through

$$\Phi(f \otimes g) = \phi(f, g), \quad (f, g \in C^1[0,1]).$$

Every element in A can be thought of as a function in $C([0,1] \times [0,1])$ by defining

$$(f \otimes g)(s,t) = f(s)g(t), \quad (f, g \in C^1[0,1],\ s, t \in [0,1]). \tag{3}$$

We claim that $\Phi(H) = 0$ whenever $H \in A$ is such that

$$\operatorname{supp}(H) \cap \{(s,s)\colon\ s \in [0,1]\} = \emptyset. \tag{4}$$

Let $H \in A$ be satisfying (4), write $H = \sum_{n=1}^\infty f_n \otimes g_n$ with $f_n, g_n \in C^1[0,1]$, let $\delta > 0$ be such that

$$\delta \leq |t - r| + |s - r|, \quad ((s,t) \in \operatorname{supp}(H),\ r \in [0,1]),$$

and let $p \in \mathbb{N}$ be such that $4/p < \delta$. For $k = 0, 1, \ldots, p$, consider the open subset U_k of $[0,1]$ defined by

$$U_k = \left\{t \in [0,1]\colon\ \left|t - \frac{k}{p}\right| < \frac{1}{p}\right\} \quad (k = 0, \ldots, p).$$

Since $\cup_{k=o}^p U_k = [0,1]$, it follows that there exist smooth functions $\omega_0, \ldots, \omega_p$ on $[0,1]$ with

$$\omega_0 + \cdots + \omega_p = \mathbf{1} \tag{5}$$

and $\operatorname{supp}(\omega_k) \subset U_k$ for $k = 0, \ldots, p$. It is easily seen that

$$\operatorname{supp}(H) \cap (U_j \times U_k) = \emptyset$$

whenever U_j and U_k are such that $U_j \cap U_k \neq \emptyset$. This later property, together with (5) implies that

$$H = H\sum_{j=0}^{p}\sum_{k=0}^{p}\omega_j\otimes\omega_k =$$

$$\sum_{U_j\cap U_k=\emptyset} H\big(\omega_j\otimes\omega_k\big) = \sum_{U_j\cap U_k=\emptyset}\sum_{n=1}^{\infty}\big(f_n\omega_j\big)\otimes\big(g_n\omega_k\big).$$

Finally, we observe that

$$\Phi(H) = \sum_{U_j\cap U_k=\emptyset}\sum_{n=1}^{\infty}\phi\big(f_n\omega_j, g_n\omega_k\big) = 0,$$

because

$$\operatorname{supp}(f_n\omega_j)\cap\operatorname{supp}(g_n\omega_k)\subset\operatorname{supp}(\omega_j)\cap\operatorname{supp}(\omega_k)\subset U_j\cap U_k.$$

Let (σ_ϵ) be the family of 2π-periodic functions introduced in [1, Lemma 2.1] as follows. Let $0<\epsilon<\pi/2$ and define $\sigma_\epsilon\colon\mathbb{R}\to\mathbb{R}$ by

$$\sigma_\epsilon(t) = \begin{cases} 0, & -\pi < t \le -2\epsilon; \\ -2\epsilon - t, & -2\epsilon < t \le -\epsilon; \\ t, & -\epsilon < t \le \epsilon; \\ 2\epsilon - t, & \epsilon < t \le 2\epsilon; \\ 0, & 2\epsilon < t \le \pi; \end{cases} \qquad \sigma_\epsilon(t+2\pi) = \sigma_\epsilon(t)\ (t\in\mathbb{R}).$$

Then

$$\widehat{\sigma_\epsilon}(0) = 0,\quad \widehat{\sigma_\epsilon}(k) = \frac{i}{\pi k^2}[\sin(2k\epsilon) - 2\sin(k\epsilon)]\quad (k\in\mathbb{Z}\setminus\{0\}),$$

where, as usual $\widehat{\sigma_\epsilon}$ stands for the Fourier transform of σ_ϵ.

We recall a standard fact of the classical Fourier analysis that the function $\int_0^t \sigma_\epsilon(s)ds$ satisfies

$$\int_0^t \sigma_\epsilon(s)ds = \sum_{k\neq 0}\frac{\widehat{\sigma_\epsilon}(k)}{ik}e^{ikt} - \sum_{k\neq 0}\frac{\widehat{\sigma_\epsilon}(k)}{ik},\quad (t\in\mathbb{R}), \tag{6}$$

where the first series in (6) converges uniformly on $\mathbb{R}$, and the right side of (6) is the Fourier series of the function on the left side. Then the function $\int_0^t\int_0^s \sigma_\epsilon(r)drds$ satisfies

$$\int_0^t\int_0^s \sigma_\epsilon(r)drds + \left(\sum_{k\neq 0}\frac{\widehat{\sigma_\epsilon}(k)}{ik}\right)t = \sum_{k\neq 0}\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2}e^{ikt} - \sum_{k\neq 0}\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2},\quad (t\in\mathbb{R}), \tag{7}$$

where the first series in (7) converges uniformly on $\mathbb{R}$, and the right side of (7) is the Fourier series of the function on the left side. We define

$$\tau_\epsilon\colon\mathbb{R}\to\mathbb{R},\ \tau_\epsilon(t) = 6\int_0^t\int_0^s \sigma_\epsilon(r)drds,\quad (t\in\mathbb{R}).$$

We also define

$$\rho_\epsilon\colon [0,1]\times[0,1]\to\mathbb{C},\ \rho_\epsilon(s,t)=\tau_\epsilon(s-t),\quad (s,t\in[0,1]).$$

It is immediate to see that $\rho_\epsilon(s,t)=(s-t)^3$ whenever $s,t\in[0,1]$ are such that $|s-t|\le\epsilon$. On the other hand, we have

$$\rho_\epsilon(s,t)=\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2}e^{iks}e^{-ikt}-\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2}-\left(\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{ik}\right)(s-t),\ \ (s,t\in[0,1]),$$

so that

$$\begin{aligned}\rho_\epsilon =&\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2}\exp\big(ik(\cdot)\big)\otimes\exp\big(-ik(\cdot)\big)\\ &-\left(\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{(ik)^2}\right)\mathbf{1}\otimes\mathbf{1}-\left(\sum_{k\neq 0}6\frac{\widehat{\sigma_\epsilon}(k)}{ik}\right)(\mathbf{x}\otimes\mathbf{1}-\mathbf{1}\otimes\mathbf{x})\in A\end{aligned}$$

and

$$\begin{aligned}\|\rho_\epsilon\|_A\le&\sum_{k\neq 0}6\frac{|\widehat{\sigma_\epsilon}(k)|}{k^2}\left\|\exp\big(ik(\cdot)\big)\right\|_1\left\|\exp\big(-ik(\cdot)\big)\right\|_1\\ &+\left(\sum_{k\neq 0}6\frac{|\widehat{\sigma_\epsilon}(k)|}{k^2}\right)\|\mathbf{1}\|_1\|\mathbf{1}\|_1\\ &+\left(\sum_{k\neq 0}6\frac{|\widehat{\sigma_\epsilon}(k)|}{|k|}\right)(\|\mathbf{x}\|_1\|\mathbf{1}\|_1+\|\mathbf{1}\|_1\|\mathbf{x}\|_1)\\ =&\sum_{k\neq 0}6\frac{|\widehat{\sigma_\epsilon}(k)|}{k^2}(1+|k|)^2+\sum_{k\neq 0}6\frac{|\widehat{\sigma_\epsilon}(k)|}{k^2}+\sum_{k\neq 0}24\frac{|\widehat{\sigma_\epsilon}(k)|}{|k|}\\ \le&54\sum_{k\neq 0}|\widehat{\sigma_\epsilon}(k)|=\mu(\epsilon),\end{aligned}\tag{8}$$

where $\mu\colon\mathbb{R}\to\mathbb{R}$ is the function defined by

$$\mu(t)=54\sum_{k\neq 0}\frac{|\sin(2kt)-2\sin(kt)|}{\pi k^2}.\tag{9}$$

Since the series in the right side of (9) converges uniformly on $\mathbb{R}$, it follows that μ is continuous.

Of course, proving (2) is equivalent to proving that

$$\Phi(F)=0,\tag{10}$$

where $F\in A$ is given by

$$\begin{aligned}F=&\mathbf{x}^m\otimes\mathbf{x}^n-\mathbf{x}^{m+n}\otimes\mathbf{1}-\tfrac{n}{m+n}\mathbf{1}\otimes\mathbf{x}^{m+n}\\ &+\tfrac{n}{m+n}\mathbf{x}^{m+n}\otimes\mathbf{1}-\tfrac{mn}{m+n-1}\mathbf{x}^{m+n-1}\otimes\mathbf{x}\end{aligned}$$

$$+\tfrac{mn}{m+n-1}\mathbf{x}^{m+n}\otimes\mathbf{1}-\tfrac{mn}{(m+n)(m+n-1)}\mathbf{x}^{m+n}\otimes\mathbf{1}+\tfrac{mn}{(m+n)(m+n-1)}\mathbf{1}\otimes\mathbf{x}^{m+n}.$$

Of course, on account of the identification given in (3), F can be thought of as the polynomial in two variables given by

$$\begin{aligned}F(s,t)=&s^mt^n-s^{m+n}-\tfrac{n}{m+n}t^{m+n}+\tfrac{n}{m+n}s^{m+n}-\tfrac{mn}{m+n-1}s^{m+n-1}t\\&+\tfrac{mn}{m+n-1}s^{m+n}-\tfrac{mn}{(m+n)(m+n-1)}s^{m+n}+\tfrac{mn}{(m+n)(m+n-1)}t^{m+n}.\end{aligned}$$

It is straightforward to check that

$$F(s,s)=0,\ \frac{\partial F(s,s)}{\partial t}=0,\ \frac{\partial^2 F(s,s)}{\partial t^2}=0,\quad (s\in\mathbb{R})$$

and so

$$F(s,t)=(s-t)^3G(s,t) \tag{11}$$

for some polynomial G. We now define $F_\epsilon\in A$ by $F_\epsilon=\rho_\epsilon G$. On account of (11), we have

$$\operatorname{supp}(F-F_\epsilon)\cap\{(s,s):\ s\in[0,1]\}=\emptyset.$$

Consequently, $\Phi(F)=\Phi(F_\epsilon)$. On the other hand, according to (8), we have

$$\|\Phi(F_\epsilon)\|\le\|\Phi\|\|F_\epsilon\|_A\le\|\Phi\|\|G\|_A\|\rho_\epsilon\|_A\le\|\Phi\|\|G\|_A\mu(\epsilon).$$

Since μ is continuous and $\mu(0)=0$, it follows that

$$\|\Phi(F)\|=\lim_{\epsilon\to 0}\|\Phi(F_\epsilon)\|=0$$

and therefore that $\Phi(F)=0$, which completes the proof. □

3. Local-like operators on $C^1[0,1]$

An operator T from a subalgebra A of a function algebra $C(\Omega)$ into $C(\Omega)$ is said to be *local* if the support of f contains the support of Tf. Under mild restrictions this is equivalent to the condition that for all $f,g\in A$,

$$fg=0\quad\Rightarrow\quad f\cdot Tg=0; \tag{12}$$

see [5, Lemma 1]. The purpose of this section is to consider the condition (12) for an operator T from $C^1[0,1]$ into a left Banach $C^1[0,1]$-module X. Let us mention in this context also a result by Johnson who defined the notion of a local operator from a function algebra into its left Banach module in a somewhat different way, and showed that every such operator T from $C_0(\mathbb{R})$ into an essential left Banach $C_0(\mathbb{R})$-module is a multiplier [4, Proposition 3.1], i.e. it satisfies $T(fg)=f\cdot Tg$ for all $f,g\in C_0(\mathbb{R})$.

Of course, multipliers certainly satisfy (12). Since $C^1[0,1]$ contains $\mathbf{1}$, every multiplier $T:C^1[0,1]\to X$ is of the form $Tf=f\cdot\xi$ for some $\xi\in X$. But there can be other operators from $C^1[0,1]$ into X satisfying (12). For instance, a classical result by Peetre [6] implies that $T:C^1[0,1]\to C[0,1]$ is a local operator if and only if there exist $h,k\in C[0,1]$ such that $T(f)=fh+f'k$ for all $f\in C^1[0,1]$; in the abstract situation where the role of $C[0,1]$ is replaced by X, we can thus consider the map $T(f)=f\cdot\xi+f'\cdot\eta$ which also satisfies (12), provided of course that $f'\cdot\eta$ makes sense (say, if

X is also a $C[0,1]$-module). However, more complicated examples can be constructed:

Example 3.1. Let $X = C^1[0,1]^*$ be the dual of $C^1[0,1]$. Then X is a left Banach $C^1[0,1]$-module via $(f\cdot\eta)(h) = \eta(fh)$, $\eta \in X$, $f,h \in C^1[0,1]$. Define $T : C^1[0,1] \to X$ by

$$(Tf)(h) = \int_0^1 f'h'$$

Since $fg = 0$ implies $fg' = f'g' = 0$, one can easily check that T is a continuous operator satisfying (12). If $f \in C^2[0,1]$, then we clearly have

$$(Tf)(h) = f'(1)h(1) - f'(0)h(0) - \int_0^1 f''h$$

In particular, for all $f \in C^3[0,1]$ we thus have

$$Tf = f'\cdot\eta + f''\cdot\zeta,$$

where $\eta, \zeta \in X$ are defined by

$$\eta(h) = h(1) - h(0), \quad \zeta(h) = -\int_0^1 h$$

This example nicely illustrates the next theorem.

Theorem 3.2. *Let X be a unital left Banach $C^1[0,1]$-module and let $T : C^1[0,1] \to X$ be a continuous linear operator. Then the following two conditions are equivalent:*

(i) for all $f,g \in C^1[0,1]$, $fg = 0$ implies $f\cdot Tg = 0$;

(ii) there exist $\xi,\eta,\zeta \in X$ such that

$$Tf = f\cdot\xi + f'\cdot\eta + f''\cdot\zeta \quad (f \in C^3[0,1]).$$

Proof. Let us first show that (i) implies (ii). So assume that T satisfies (12). Set $\xi = T\mathbf{1}$. Replacing T by the map $f \mapsto Tf - f\cdot\xi$ we see that there is no loss of generality in assuming that $T\mathbf{1} = 0$, i.e. $\xi = 0$. Now we set

$$\eta = T\mathbf{x}, \quad \zeta = \frac{1}{2}T\mathbf{x}^2 - \mathbf{x}\cdot\eta.$$

Our goal is to show that $Tf = f'\cdot\eta + f''\cdot\zeta$ for all $f \in C^3[0,1]$. Since the polynomials are dense in the Banach algebra $C^3[0,1]$ (see e.g. [3, Theorem 4.4.1]), it suffices to show that this identity holds only for $f = \mathbf{x}^n$, $n \geq 0$. Thus, we have to show that

$$T\mathbf{x}^n = n\mathbf{x}^{n-1}\cdot\eta + n(n-1)\mathbf{x}^{n-2}\cdot\zeta \quad (n \geq 3). \tag{13}$$

By Theorem 2.1 there exist continuous linear operators $P\colon C^1[0,1] \to X$ and $Q,R\colon C[0,1] \to X$ such that

$$f\cdot Tg = P(fg) + Q(fg') + R(f'g') \quad (f,g \in C^1[0,1]). \tag{14}$$

Setting $g = \mathbf{1}$ in (14) it follows immediately that $P = 0$. Next, setting $f = \mathbf{1}$ we see that $Tg = Qg'$ for all $g \in C^1[0,1]$. Accordingly, $TVh = Qh$ for all $h \in C[0,1]$, and so (14) becomes

$$f \cdot Tg = TV(fg') + R(f'g') \quad (f, g \in C^1[0,1]). \tag{15}$$

Setting $f = \mathbf{x}$ and $g = \mathbf{x}^n$ in (15) we get

$$\mathbf{x} \cdot T\mathbf{x}^n - \frac{n}{n+1} T\mathbf{x}^{n+1} = nR\mathbf{x}^{n-1} \tag{16}$$

On the other hand, setting $f = \mathbf{x}^n$ and $g = \mathbf{x}$ in (15) we get

$$\mathbf{x}^n \cdot \eta - \frac{1}{n+1} T\mathbf{x}^{n+1} = nR\mathbf{x}^{n-1} \tag{17}$$

Comparing (16) and (17) we arrive at

$$\mathbf{x} \cdot T\mathbf{x}^n - \frac{n}{n+1} T\mathbf{x}^{n+1} = \mathbf{x}^n \cdot \eta - \frac{1}{n+1} T\mathbf{x}^{n+1},$$

that is

$$\frac{n-1}{n+1} T\mathbf{x}^{n+1} = \mathbf{x} \cdot T\mathbf{x}^n - \mathbf{x}^n \cdot \eta.$$

From this identity (13) follows immediately by induction on n.

Next we prove that (ii) implies (i). Observe that it is enough to prove the following claim: for every $f \in C^1[0,1]$ and every $\varepsilon > 0$ there exists a smooth function F on $[0,1]$ such that the zero set of F contains the zero set of f and

$$\|F - f\|_{C^1[0,1]} < \varepsilon.$$

Then we can find a sequence $\{f_n\}_{n=1}^{\infty}$ of smooth functions on $[0,1]$ which in the C^1 sense converges to f and such that the zero set of every function f_n contains the zero set of f.

From the nature of this claim one might suspect that it is already known. However, we were unable to find any reference for it and therefore we include its proof.

Let $f \in C^1[0,1]$. If f is identically equal to 0, the claim is trivial. Also, if f has no zeros, the claim follows from the fact that polynomials are dense in $C^1[0,1]$. So henceforth we will assume that $f \not\equiv 0$ and that it has zeros on $[0,1]$.

Let Z_f be the intersection of the zero sets of f and f'. It is a closed subset of $[0,1]$ and every zero of f outside of Z_f is isolated. Also, given a neighbourhood U of Z_f there are only finitely many zeros of f outside U.

Let $\varepsilon > 0$ and let

$$U_\varepsilon = \{|f| < \varepsilon, |f'| < \varepsilon\}.$$

We choose ε so small that U_ε is a proper subset of $[0,1]$. The set U_ε is an open subset of $[0,1]$ and hence it is at most countable union of pairwise disjoint intervals $\{I_j\}_{j=1}^{\infty}$, where each I_j is either an open interval or a semiclosed interval of the form $[0,b)$ or $(a,1]$. Let

$$d = \operatorname{dist}(Z_f, [0,1] \setminus U_\varepsilon) > 0.$$

Without changing the notation we replace U_ε with an open subset which we get as the union of all those intervals I_j for which $Z_f \cap I_j \neq \emptyset$. There are only finitely many such intervals and their union contains Z_f.

Let $I_j \subseteq U_\varepsilon$ be an open interval. The length of I_j is greater or equal to $2d$ and we define $\widetilde{I}_j = (a_j, b_j) \subseteq I_j$ as the largest open subinterval such that the distance of each end point of $\widetilde{I}_j$ from Z_f is exactly $\frac{d}{2}$. Finally, let $J_j = [a_j + \frac{d}{4}, b_j - \frac{d}{4}]$. If $I_j \subseteq U_\varepsilon$ is a semiclosed interval these definitions are appropriately modified. We denote the union of $\widetilde{I}_j$-s by $\widetilde{U}_\varepsilon$ and the union of J_j-s by V_ε. Then $Z_f \subset V_\varepsilon$.

Let φ be a nonnegative smooth function on $\mathbb{R}$ such that $\varphi = 0$ outside $[-1, 1]$, $\varphi > 0$ on $(-1, 1)$ and

$$\int_{-\infty}^{\infty} \varphi(y)\, dy = 1.$$

We define

$$\chi(x) = \frac{4}{d} \int_{[0,1]\setminus \widetilde{U}_\varepsilon} \varphi(4\frac{x-y}{d})\, dy$$

Then χ is a smooth function on $\mathbb{R}$ with values between 0 and 1.

Since φ is a nonnegative smooth function on $\mathbb{R}$ and since the set $[0, 1] \setminus \widetilde{U}_\varepsilon$ has no isolated points, we have that $\chi(x) = 0$ if and only if

$$\varphi(4\frac{x-y}{d}) = 0 \qquad \text{for all } y \in [0, 1] \setminus \widetilde{U}_\varepsilon.$$

That is, $\chi(x) = 0$ if and only if $|x - y| \geq \frac{d}{4}$ for all $y \in [0, 1] \setminus \widetilde{U}_\varepsilon$, which means that the distance $d(x, [0, 1] \setminus \widetilde{U}_\varepsilon) \geq \frac{d}{4}$. Therefore $\chi = 0$ exactly on V_ε and so also on Z_f. On the other hand we have that $\chi(x) = 1$ if and only if the whole support of $\varphi(4\frac{x-y}{d})$ lies in $[0, 1] \setminus \widetilde{U}_\varepsilon$, that is, the distance $d(x, \widetilde{U}_\varepsilon) \geq \frac{d}{4}$. Hence $\chi = 1$ outside U_ε.

Let us define $F = \chi f$. Then

$$|F - f| = |\chi - 1||f| < \varepsilon$$

because $\chi = 1$ outside U_ε and $|f| < \varepsilon$ on U_ε. We also have

$$|F' - f'| \leq |\chi - 1||f'| + |\chi'||f|.$$

The first term is estimated as above. For the second term we should observe the following. From the definition of function χ we have

$$|\chi'| \leq \frac{M}{d}.$$

Here M is a constant which does not depend on d. Also, χ' can be different from 0 only on the set of those points x from $U_\varepsilon \setminus V_\varepsilon$ such that $d(x, V_\varepsilon) \leq \frac{d}{2}$.

Hence for every such point x there exists a point x_0 from Z_f so that $|x - x_0| \leq d$. Since $|f'| < \varepsilon$ on U_ε we get

$$|f(x)| = |f(x) - f(x_0)| = |\int_{x_0}^{x} f'(y)\, dy| \leq \varepsilon\, d.$$

So

$$|\chi' f| \leq \frac{M}{d}\varepsilon\, d = M\,\varepsilon.$$

Hence F is a C^1 function on $[0,1]$ such that

$$\|F - f\|_{C^1[0,1]} \leq (2 + M)\varepsilon$$

and the zero set of F equals the union of V_ε and the isolated zeros of f outside V_ε.

From here on we will assume that the zero set of f consists of finitely many isolated zeros and finitely many closed intervals J_j. Let V be the union of J_j-s. Let $K = (a, b)$ be an open interval from $[0,1] \setminus V$ such that its end points a and b belong to V. We know that there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which in the C^1 sense on $[0,1]$ converges to f. Let us observe their restrictions to $[a, b]$. Adding appropriate linear polynomials, that is,

$$P_n(x) - P_n(a) - \frac{x - a}{b - a}(P_n(b) - P_n(a)),$$

we may assume that $P_n(a) = P_n(b) = 0$. Further, adding appropriate cubic polynomials we may also assume that $P_n'(a) = P_n'(b) = 0$. Finally, we may add appropriate polynomials so that every zero of f on (a, b) is also a zero of P_n and still the sequence $\{P_n\}_{n=1}^{\infty}$ in the C^1 sense converges to f on $[a, b]$. If K is a semiclosed interval the argument is similar.

Gluing P_n-s with the zero function on V we get C^1 functions f_n on $[0,1]$ which have zeros at all zeros of f, which in the C^1 sense converge to f and which are smooth at all points of $[0,1]$ except at the boundary points of J_j-s.

Let $\varepsilon > 0$. With the same procedure as in the beginning of the proof of the claim we first find a smooth function $\widetilde{\chi}$ which equals 0 in a neighbourhood of V and such that

$$\|\widetilde{\chi} f - f\|_{C^1[0,1]} < \frac{\varepsilon}{2}.$$

Then for n large enough we have

$$\|\widetilde{\chi} f_n - \widetilde{\chi} f\|_{C^1[0,1]} \leq \|\widetilde{\chi}\|_{C^1[0,1]} \|f_n - f\|_{C^1[0,1]} < \frac{\varepsilon}{2}$$

and so the smooth function $F = \widetilde{\chi} f_n$ does the required approximation.

□

4. Zero product preserving operators on $C^1[0,1]$

This section is devoted to zero product preserving operators T from $C^1[0,1]$ into another Banach algebra A. It should be mentioned here that such operators on function algebras are more commonly known as *disjointness preserving operators*, *separating operators* or *Lamperti operators*. Anyway, we shall keep the terminology from [2] which is more standard in algebraic and noncommutative setting. For historic comments and references about these operators we refer the reader to [2] and [5].

In our basic result we consider the most general situation where A is an arbitrary (commutative) Banach algebra and there are no other restrictions on T besides that it preserves zero products.

Theorem 4.1. *Let A be a commutative Banach algebra and let $T : C^1[0,1] \to A$ be a continuous linear zero product preserving operator. Then there exist $a, b \in A$ such that*

$$c^2 \cdot Tf = a \cdot \Psi f + b \cdot \Psi f' \quad (f \in C^1[0,1]) \tag{18}$$

where $c = (T\mathbf{x})^2 - T\mathbf{1} \cdot T\mathbf{x}^2$ and $\Psi : C[0,1] \to A$ is a continuous linear operator satisfying $c \cdot \Psi(fg) = \Psi f \cdot \Psi g$ for all $f, g \in C[0,1]$.

Proof. Theorem 2.1 tells us that

$$Tf \cdot Tg = P(fg) + Q(fg') + R(f'g') \quad (f, g \in C^1[0,1]), \tag{19}$$

where $P \colon C^1[0,1] \to A$ and $Q, R \colon C[0,1] \to A$ are continuous linear operators. Let

$$u = T\mathbf{1}, \quad w = T\mathbf{x}.$$

First set $f = \mathbf{1}$ in (19), and then $g = \mathbf{1}$; since u commutes with Tf it follows easily that $Pf = u \cdot Tf$ and $Q = 0$. Thus, we have

$$Tf \cdot Tg - u \cdot T(fg) = R(f'g') \quad (f, g \in C^1[0,1]). \tag{20}$$

Writing $f = \mathbf{x}$ and $g = \mathbf{x}^{r+1}$ in (20) we obtain

$$R(\mathbf{x}^r) = \frac{1}{r+1}\Big(w \cdot T\mathbf{x}^{r+1} - u \cdot T\mathbf{x}^{r+2}\Big). \tag{21}$$

Therefore, setting $f = \mathbf{x}^k$ and $g = \mathbf{x}^l$ in (20) it follows from (21) that

$$\begin{aligned} T\mathbf{x}^k \cdot T\mathbf{x}^l &= u \cdot T\mathbf{x}^{k+l} + klR\mathbf{x}^{k+l-2} \\ &= u \cdot T\mathbf{x}^{k+l} + \frac{kl}{k+l-1}\Big(w \cdot T\mathbf{x}^{k+l-1} - u \cdot T\mathbf{x}^{k+l}\Big). \end{aligned}$$

Thus,

$$T\mathbf{x}^k \cdot T\mathbf{x}^l = \frac{1}{k+l-1}\Big(klw \cdot T\mathbf{x}^{k+l-1} - (k-1)(l-1)u \cdot T\mathbf{x}^{k+l}\Big). \tag{22}$$

Now define $\Psi : C[0,1] \to A$ by

$$\Psi f = w \cdot TVf - u \cdot T(\mathbf{x} \cdot Vf).$$

In particular,

$$\Psi\mathbf{x}^n = \frac{1}{n+1}\Big(w \cdot T\mathbf{x}^{n+1} - u \cdot T\mathbf{x}^{n+2}\Big) \quad (n \geq 0). \tag{23}$$

Our goal is to show that $c \cdot \Psi(fg) = \Psi f \cdot \Psi g$ for all $f, g \in C[0,1]$. Since Ψ is continuous, it suffices to show that

$$c \cdot \Psi(\mathbf{x}^{n+m}) = \Psi\mathbf{x}^n \cdot \Psi\mathbf{x}^m \quad (n, m \geq 0). \tag{24}$$

We have

$$\begin{aligned}
& \Psi\mathbf{x}^n \cdot \Psi\mathbf{x}^m \\
= \; & \frac{1}{(n+1)(m+1)}\Big(w \cdot T\mathbf{x}^{n+1} - u \cdot T\mathbf{x}^{n+2}\Big)\Big(w \cdot T\mathbf{x}^{m+1} - u \cdot T\mathbf{x}^{m+2}\Big) \\
= \; & \frac{1}{(n+1)(m+1)}\Big(w^2 \cdot T\mathbf{x}^{n+1} \cdot T\mathbf{x}^{m+1} - uw \cdot T\mathbf{x}^{n+2} \cdot T\mathbf{x}^{m+1} \\
& -uw \cdot T\mathbf{x}^{n+1} \cdot T\mathbf{x}^{m+2} + u^2 \cdot T\mathbf{x}^{n+2} \cdot T\mathbf{x}^{m+2}\Big).
\end{aligned}$$

Using (22) it follows that

$$\begin{aligned}
& (n+1)(m+1)\Psi\mathbf{x}^n \cdot \Psi\mathbf{x}^m \\
= \; & \frac{(n+1)(m+1)}{n+m+1} w^3 \cdot T\mathbf{x}^{n+m+1} - \frac{nm}{n+m+1} uw^2 \cdot T\mathbf{x}^{n+m+2} \\
& - \frac{(n+2)(m+1)}{n+m+2} uw^2 \cdot T\mathbf{x}^{n+m+2} + \frac{(n+1)m}{n+m+2} u^2 w \cdot T\mathbf{x}^{n+m+3} \\
& - \frac{(n+1)(m+2)}{n+m+2} uw^2 \cdot T\mathbf{x}^{n+m+2} + \frac{n(m+1)}{n+m+2} u^2 w \cdot T\mathbf{x}^{n+m+3} \\
& + \frac{(n+2)(m+2)}{n+m+3} u^2 w \cdot T\mathbf{x}^{n+m+3} - \frac{(n+1)(m+1)}{n+m+3} u^3 \cdot T\mathbf{x}^{n+m+4}.
\end{aligned}$$

By a straightforward computation one can check that this yields

$$\begin{aligned}
\Psi\mathbf{x}^n \cdot \Psi\mathbf{x}^m = \; & \frac{1}{n+m+1} w^3 \cdot T\mathbf{x}^{n+m+1} \\
& - \frac{3n+3m+4}{(n+m+1)(n+m+2)} uw^2 \cdot T\mathbf{x}^{n+m+2} \\
& + \frac{3n+3m+8}{(n+m+2)(n+m+3)} u^2 w \cdot T\mathbf{x}^{n+m+3} \\
& - \frac{1}{n+m+3} u^3 \cdot T\mathbf{x}^{n+m+4}.
\end{aligned} \tag{25}$$

Let us now consider the left-hand side of (24). We have

$$\begin{aligned}
& (n+m+1)c \cdot \Psi(\mathbf{x}^{n+m}) \\
= \; & (n+m+1)w^2 \cdot \Psi\mathbf{x}^{n+m} - (n+m+1)u \cdot T\mathbf{x}^2 \cdot \Psi\mathbf{x}^{n+m} \\
= \; & w^3 \cdot T\mathbf{x}^{n+m+1} - uw^2 \cdot T\mathbf{x}^{n+m+2} \\
& -uw \cdot T\mathbf{x}^2 \cdot T\mathbf{x}^{n+m+1} + u^2 \cdot T\mathbf{x}^2 \cdot T\mathbf{x}^{n+m+2}.
\end{aligned}$$

Using (22) we now get

$$\begin{aligned} & c\cdot\Psi(\mathbf{x}^{n+m}) \\ = \; & \frac{1}{n+m+1}w^3\cdot T\mathbf{x}^{n+m+1} - \frac{1}{n+m+1}uw^2\cdot T\mathbf{x}^{n+m+2} \\ & -\frac{2(n+m+1)}{(n+m+1)(n+m+2)}uw^2\cdot T\mathbf{x}^{n+m+2} \\ & +\frac{n+m}{(n+m+1)(n+m+2)}u^2w\cdot T\mathbf{x}^{n+m+3} \\ & +\frac{2(n+m+2)}{(n+m+1)(n+m+3)}u^2w\cdot T\mathbf{x}^{n+m+3} \\ & -\frac{1}{n+m+3}u^3\cdot T\mathbf{x}^{n+m+4}. \end{aligned}$$

Comparing this result with (25) one easily checks that (24) holds.

Now set $a = u\cdot c$ and $b = w\cdot c - u\cdot\Psi\mathbf{x}$, and let us show that (18) holds. It suffices to consider the case where $f = \mathbf{x}^n$, $n \geq 0$. We have

$$\begin{aligned} & a\cdot\Psi\mathbf{x}^n + b\cdot\Psi(\mathbf{x}^n)' \\ = \; & u\cdot c\cdot\Psi\mathbf{x}^n + nc\cdot w\cdot\Psi\mathbf{x}^{n-1} - nu\cdot\Psi\mathbf{x}\cdot\Psi\mathbf{x}^{n-1}. \end{aligned}$$

Since $\Psi\mathbf{x}\cdot\Psi\mathbf{x}^{n-1} = c\cdot\Psi\mathbf{x}^n$ by (24), it follows that

$$\begin{aligned} & a\cdot\Psi\mathbf{x}^n + b\cdot\Psi(\mathbf{x}^n)' \\ = \; & c\Big((1-n)u\cdot\Psi\mathbf{x}^n + nw\cdot\Psi\mathbf{x}^{n-1}\Big) \\ = \; & c\Big(\frac{1-n}{n+1}u\cdot\big(w\cdot T\mathbf{x}^{n+1} - u\cdot T\mathbf{x}^{n+2}\big) + w\cdot\big(w\cdot T\mathbf{x}^n - u\cdot T\mathbf{x}^{n+1}\big)\Big) \\ = \; & c\Big(\frac{-2n}{n+1}uw\cdot T\mathbf{x}^{n+1} - \frac{1-n}{n+1}u^2\cdot T\mathbf{x}^{n+2} + w^2\cdot T\mathbf{x}^n\Big). \end{aligned}$$

Since $w^2 = c + u\cdot T\mathbf{x}^2$, we see by using (22) that

$$\begin{aligned} w^2\cdot T\mathbf{x}^n &= c\cdot T\mathbf{x}^n + u\cdot T\mathbf{x}^2\cdot T\mathbf{x}^n \\ &= c\cdot T\mathbf{x}^n + \frac{1}{n+1}u\cdot\big(2nw\cdot T\mathbf{x}^{n+1} - (n-1)u\cdot T\mathbf{x}^{n+2}\big). \end{aligned}$$

Returning to the previous identity, it now clearly follows that

$$a\cdot\Psi\mathbf{x}^n + b\cdot\Psi(\mathbf{x}^n)' = c^2\cdot T\mathbf{x}^n,$$

as desired. □

Remark 4.2. The assumption in Theorem 4.1 that A is commutative can be replaced by a milder assumption that $T\mathbf{1}$ commutes with Tf for every $f \in C^1[0,1]$. Namely, inspecting the beginning of the proof we see that under this assumption one can derive (20), from which it clearly follows that the range of T is commutative. But then one can assume with no loss of generality that A is commutative.

How useful is the information given in Theorem 4.1? It depends on c. The most favorable situation is when c is invertible in A. Then the conclusion of the theorem can be stated as

$$Tf = g \cdot \Phi f + h \cdot \Phi f' \tag{26}$$

for some $g, h \in A$ and an algebra homomorphism $\Phi : C[0,1] \to A$; namely, we define $\Phi f = c^{-1} \cdot \Psi f$, $g = c^{-1}a$ and $h = c^{-1}b$. This of course characterizes zero product preserving operators. Let us point out a special case when this occurs (also taking into account Remark 4.2).

Corollary 4.3. *Let A be an arbitrary unital Banach algebra and let $T : C^1[0,1] \to A$ be a continuous linear zero product preserving operator such that $T\mathbf{1} = 0$ and $T\mathbf{x} = 1$. Then there exist a continuous algebra homomorphism $\Psi : C[0,1] \to A$ such that $Tf = \Psi f'$ for all $f \in C^1[0,1]$.*

The other extreme case is when $c^2 = 0$ (in particular, if $c = 0$). Then Theorem 4.1 is meaningless. Unfortunately, this can occur. Just consider any operator T such that $Tf \cdot Tg = 0$ for all $f, g \in C^1[0,1]$. Not only that $c = 0$ in this case, but (26) does not necessarily hold. By taking direct sums of algebras one can find more subtle examples. All these suggest that in this generality, i.e. when A is an arbitrary algebra, the characterization of arbitrary zero product preserving operators seems to be beyond reach. Thus one might consider some special algebras A. In the next corollary we will handle the case where $A = \ell^\infty(S)$, the Banach algebra of all bounded functions on a non-empty set S. We will basically show that (26) holds in this case.

Corollary 4.4. *Let $T : C^1[0,1] \to \ell^\infty(S)$ be a continuous linear zero product preserving operator. Then there exist functions $g, h \in \ell^\infty(S)$ and a function $\mu : S \to [0,1]$ such that*

$$(Tf)(s) = g(s)f(\mu(s)) + h(s)f'(\mu(s)) \quad (f \in C^1[0,1], s \in S).$$

Proof. Let us first consider a continuous linear zero product preserving operator T from $C^1[0,1]$ into $\mathbb{C}$ (thus, T satisfies the condition that $Tf = 0$ or $Tg = 0$ whenever $fg = 0$). As we shall see, the general case can be easily reduced to this one. First we note that if c from Theorem 4.1 is not 0, then T takes the form (26) for some homomorphism $\Phi : C^1[0,1] \to \mathbb{C}$. Since every nonzero homomorphism from $C^1[0,1]$ into $\mathbb{C}$ is an evaluation functional, the desired conclusion that $Tf = gf(\mu) + hf'(\mu)$ for some $g, h \in \mathbb{C}$ and $\mu \in [0,1]$ follows immediately in this case. So we may assume that $c = 0$, that is, $uT\mathbf{x}^2 = (T\mathbf{x})^2$, where $u = T\mathbf{1}$. Suppose that $u = 0$. Then also $w = T\mathbf{x} = 0$. Returning back to the beginning of the proof of Theorem 4.1, we see from (22) that $(T\mathbf{x}^k)^2 = 0$. Hence $T\mathbf{x}^k = 0$ and so $T = 0$. Therefore we may assume that $u \neq 0$. From (24) it clearly follows that $\Psi = 0$, and so, by (23), $uT\mathbf{x}^{n+2} = T\mathbf{x}T\mathbf{x}^{n+1}$. A simple induction argument shows that this implies $u^{k-1}T\mathbf{x}^k = (T\mathbf{x})^k$ for all $k \geq 1$. Consequently, the functional $\Theta : C^1[0,1] \to \mathbb{C}$ given by $\Theta f = u^{-1}Tf$ satisfies $\Theta\mathbf{x}^{n+m} = \Theta\mathbf{x}^n\Theta\mathbf{x}^m$ and so it is a nonzero homomorphism. But then there exists $\mu \in [0,1]$ such that $\Theta f = f(\mu)$, and hence $Tf = uf(\mu)$.

Now consider the general case with $T : C^1[0,1] \to \ell^\infty(S)$. For every $s \in S$, the operator $f \mapsto (Tf)(s)$ is a zero product preserving operator from $C^1[0,1]$ into $\mathbb{C}$. Therefore, by what we have just proved, there exist $g(s), h(s) \in \mathbb{C}$ and $\mu(s) \in \mathbb{C}$ such that $(Tf)(s) = g(s)f(\mu(s)) + h(s)f'(\mu(s))$. It only remains to show that the functions $s \mapsto g(s)$ and $s \mapsto h(s)$ are bounded. But this follows immediately by first setting $f = \mathbf{1}$ and then $f = \mathbf{x}$ in the above formula. □

Corollary 4.4 is of course also applicable to zero product preserving operators from $C^1[0,1]$ into algebras of continuous bounded functions. One might then wonder whether (or better, where) the functions g, h and μ are continuous in this case. But we shall not consider this question here. Let us refer the reader to the paper by Kantrowitz and Neumann [5] which considers a zero product preserving continuous linear operator T which is defined on $C^m(\Omega)$, where $m \geq 0$ and Ω is an open subset of $\mathbb{R}^n$, and maps into $C(\Gamma)$ where Γ is a locally compact Hausdorff space. Although there are some differences in the general setting, Kantrowitz and Neumann have treated in [5] a more general problem and Corollary 4.4 is not really surprising in view of their achievements. Yet this corollary is of some interest because of its proof which does not use standard tools of the theory of zero product preserving operators between function algebras such as support points.

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