QUASI-IDENTITIES ON MATRICES AND THE CAYLEY-HAMILTON POLYNOMIAL

MATEJ BREŠAR, CLAUDIO PROCESI, AND ŠPELA ŠPENKO

Abstract. We consider certain functional identities on the matrix algebra M_n that are defined similarly as the trace identities, except that the "coefficients" are arbitrary polynomials, not necessarily those expressible by the traces. The main issue is the question of whether such an identity is a consequence of the Cayley-Hamilton identity. We show that the answer is affirmative in several special cases, and, moreover, for every such an identity P and every central polynomial c with zero constant term there exists $m \in \mathbb{N}$ such that the affirmative answer holds for $c^m P$. In general, however, the answer is negative. We prove that there exist antisymmetric identities that do not follow from the Cayley-Hamilton identity, and give a complete description of a certain family of such identities.

1. Introduction

Given a finite dimensional algebra A and an integer m (or ∞), let $\mathcal{C}$ be the commutative ring of polynomial functions on m copies of A and $\mathcal{C}\langle X\rangle$ the free algebra in m variables $X = \{x_1, \dots, x_m\}$. We call this algebra the algebra of *quasi-polynomials* of A. Elements of $\mathcal{C}\langle X\rangle$ can clearly be *evaluated* in A and then the *quasi-identities of* A are those quasi-polynomials that vanish at all evaluations (see Section 2).

In this paper we consider the fundamental case where $A = M_n(F)$, the algebra of matrices. We will see that quasi-identities appear in a natural way as linear relations among the noncommutative polynomial functions on A. In this sense the theory of quasi-identities is a worthwhile generalization to the theory of polynomial identities of A.

Quasi-identities appear as a class of *functional identities*. Let us give a brief background on this more general notion. A functional identity is an identical relation in a ring that, besides arbitrary elements that appear in a similar fashion as in a polynomial identity, also involves arbitrary functions which are considered as unknowns. The goal of the general functional identity theory is to describe these functions. It has been developed with applications in mind. Starting with the solution of a long-standing Herstein's problem on Lie isomorphisms in 1993 [10], functional identities have since turned out to be applicable to various problems in noncommutative algebra, nonassociative algebra, operator theory, functional analysis, and mathematical physics. We refer the reader to the book [11] for an account of functional identities and their applications.

Given a functional identity, one usually first finds its "obvious" solutions, i.e., those functions that satisfy this identity for formal reasons, independent of the structure of the ring in question.

Key words and phrases. Functional identity, quasi-polynomial, quasi-identity, Cayley-Hamilton identity, T-ideal, trace identity, polynomial identity, matrix algebra, algebra with trace, Azumaya algebra.

2010 *Math. Subj. Class.* Primary 16R60. Secondary 16R10, 16R30.

The first author was supported by ARRS Grant P1-0288.

These are called the *standard solutions*. A typical result states that either the standard solutions are in fact the only possible solutions or the ring has some special properties, like satisfying a polynomial identity of a certain degree related to the number of variables. The existing theory of functional identities, as surveyed in [11], thus gives definitive results for a large class of noncommutative rings, but, paradoxically, tells us nothing about the basic example of a noncommutative ring, i.e., the matrix algebra $M_n = M_n(F)$ (unless n is big enough). This is reflected in applications – one usually has to exclude M_n (for "small" n) in a variety of results whose proofs depend on the general theory of functional identities, although by the nature of these results one can conjecture that this exclusion is unnecessary (see [4, 5, 6, 8, 9] for typical examples). The problem with M_n is that it allows *nonstandard solutions*. Their description seems to be a much harder problem than the description of standard solutions. Moreover, it is not clear what methods could be of use.

To the best of our knowledge, the recent paper [13] is the first work giving complete results on functional identities on M_n. However, it treats only functional identities in one variable.

In this paper we restrict ourselves to the study of quasi-identities, which are important examples of functional identities in several variables. Quasi-polynomials, also called Beidar polynomials in some papers, were introduced in 2000 by Beidar and Chebotar [7], and have since played a fundamental role in the theory of functional identities and its applications. Standard solutions of quasi-identities can be very easily described: all coefficient functions must be 0 (cf. [11, Lemma 4.4]). The Cayley-Hamilton theorem gives rise to a basic example of a quasi-identity on the matrix algebra M_n with nonstandard solutions. We call it the Cayley-Hamilton identity. The main theme of this paper is the following question to which we have addressed ourselves:

Question. *Is every quasi-identity of M_n a consequence of the Cayley-Hamilton identity?* (A more accurate formulation will be given in the next section.)

An important motivation for this question is the well-known theorem, proved independently by Procesi [25] and Razmyslov [30], saying that the answer to such a question is positive for the related *trace identities*. Further, from the main result of [13] it is evident that nonstandard solutions of functional identities in one variable follow from the Cayley-Hamilton identity.

The main goal of this paper is to show that the answer to the above question is **negative** in general. The space $\mathfrak{I}_n/(Q_n)$ of quasi-identities modulo the subspace of those quasi-identities which follow from the Cayley-Hamilton identity is determined in two steps by Proposition 3.3 and by Corollary 3.6 through the exact sequence (4). This already points out the geometric nature of the question, one sees that $\mathfrak{I}_n/(Q_n)$ is an interesting invariant of the quotient map of the action of the projective linear group acting, by simultaneous conjugation, on the space of m-tuples of matrices. The space $\mathfrak{I}_n/(Q_n)$ appears as a module on the quotient variety, supported on the singular set, cf. Theorem 3.7. Still the complexity of this quotient map makes it difficult to describe this module and even to decide in a simple way if it is nonzero. Thus we restrict ourselves to describing a particular subspace of this module. We show that there exist antisymmetric quasi-identities of M_n of degree n^2 that are not a consequence of the Cayley-Hamilton identity. In fact in Theorem 4.7, which is the main result of the paper, we give a precise description of those quasi-identities which transform under the linear group as the adjoint representation. In order to achieve this result, we first have Theorem 4.5 of independent interest, which describes the way in which the adjoint representation of the simple

Lie algebra $\mathfrak{g}$ of traceless matrices sits in the exterior algebra $\bigwedge \mathfrak{g}$ of the same Lie algebra $\mathfrak{g}$. The discovery of this remarkable phenomenon has been the starting point for a general theorem for all simple Lie algebras, as shown in [16], and it also gives an insightful explanation of the basic theorem on identities of matrices, namely the Amitsur-Levitzki identity [28].

Outline. The paper is organised as follows. In Section 2 we collect notation, recall some basic facts of the theory of polynomial identities of matrices, and develop some basic formalism on functional identities of matrices.

In Section 3 we give the structure of the main object of study, the space $\mathfrak{I}_n/(Q_n)$ of quasi-identities modulo the subspace of those quasi-identities which follow from the Cayley-Hamilton identity, in terms of the basic exact sequence (4). From this we prove that for every quasi-identity P and every central polynomial c with zero constant term there exists $m \in \mathbb{N}$ such that $c^m P$ is a consequence of the Cayley-Hamilton identity. This implies that the module $\mathfrak{I}_n/(Q_n)$ is supported on the singular set.

Section 4 is the most technical part in which a detailed study of the antisymmetric quasi-identities in degree n^2 is performed and the main Theorem 4.7 is proved.

Finally, Section 5 is devoted to positive results to the above question in various special cases. Thereby we indicate that finding a quasi-identity that is not a consequence of the Cayley-Hamilton identity can not be achieved in a simple minded way.

We finish the paper by giving a positive solution to Specht problem for quasi-identities, i.e., we show that the T-ideal of quasi-identities is finitely generated.

2. Preliminaries

2.1. **Theory of identities.** Before we enter in the main theme of this paper let us quickly review some basic facts of the classical theory of identities. For more details we refer the reader to [17, 20, 27, 33].

2.1.1. *Polynomial identities.* Polynomial identities appear in the formalism of *universal algebra.* Whenever we have some category of algebras which admits *free algebras* one has the concept of identities in m variables (where m can also be ∞), of an algebra A. That is the ideal of the free algebra $\mathcal{F}_m$ in m variables x_k formed by those element which vanish for all evaluations of the variables x_k into elements $a_k \in A$. An ideal of identities is a T-ideal, i.e., an ideal of the free algebra closed under substitution of the variables x_k with elements H_i of the same free algebra. It is easily seen that a T-ideal $\mathcal{I} \subset \mathcal{F}_m$ is automatically the ideal of identities of an algebra, namely $\mathcal{F}_m/\mathcal{I}$. Recall that we say that a T-ideal $\mathcal{I}$ is generated as T-ideal by a subset I if it is the minimal T-ideal containing I. That is, it is generated as an ideal by all subsets obtained from I applying substitution of variables with elements of the free algebra.

Of particular interest is the case of noncommutative associative algebras over a field F, for which we assume, for simplicity, that

$$\operatorname{char}(F) = 0$$

(this assumption will be used throughout the paper without further mention). In this case the free algebra in m variables x_k is the usual algebra of noncommutative polynomials with basis the *words* in the variables x_k. For $m = \infty$ we set

$$X := \{x_k \mid k = 1, 2, \dots\}, \quad F\langle X\rangle \quad \text{the free algebra.} \tag{1}$$

In this case a particularly interesting example is the theory of polynomial identities of the algebra $M_n = M_n(F)$ of all $n \times n$ matrices over the field F. An implicit description of these identities is given through the *algebra of generic matrices.*

We fix an integer $n \geq 1$, and set

$$\mathcal{C} := F\big[x_{ij}^{(k)} \,|\, 1 \leq i, j \leq n, k = 1, 2, \ldots\big]. \tag{2}$$

This commutative polynomial ring is the algebra of polynomial functions on sequences of matrices. Inside the matrix algebra $M_n(\mathcal{C})$ we can define the *generic matrices* ξ_k where ξ_k is the matrix with entries the variables $x_{ij}^{(k)}$. It is then easily seen, since F is assumed to be infinite, that the ideal of polynomial identities of $M_n(F)$ is the kernel of the evaluation map $x_k \mapsto \xi_k$.

The F-subalgebra of $M_n(\mathcal{C})$ generated by the ξ_k, i.e., the image of the free algebra under this evaluation, is the free algebra in the ξ_k in the category of noncommutative algebras satisfying the identities of $M_n(F)$. This algebra has been extensively studied although a very precise description is available only for $n = 2$. We shall denote it by $F\langle \xi \,|\, n\rangle$ or just $F\langle \xi_k\rangle$ if the integer n is fixed, and call it the *algebra of generic matrices.*

2.1.2. *Trace identities.* When dealing with matrices in characteristic 0, it is useful to think that they form an algebra with a further unary operation the *trace*, $x \mapsto \mathrm{tr}(x)$. One can formalize this as follows.

An *algebra with trace* is an algebra $\mathfrak{R}$ equipped with an additional structure, that is a linear map $\mathrm{tr} : \mathfrak{R} \to \mathfrak{R}$ satisfying the following properties

$$\mathrm{tr}(ab) = \mathrm{tr}(ba), \quad a\,\mathrm{tr}(b) = \mathrm{tr}(b)\,a, \quad \mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$$

for all $a, b \in \mathfrak{R}$. The notion of a morphism between algebras with trace is then obvious and such algebras form a category which contains free algebras.

In this case the free algebra is the algebra of noncommutative polynomials with basis the *words* in the variables x_k but over the polynomial ring $\mathfrak{T}$ in the infinitely many variables $\mathrm{tr}(M)$, where M runs over all possible words considered equivalent under cyclic moves (i.e., $ab \sim ba$).

In this setting again the *trace identities* of matrices are the kernel of the evaluation of the free algebra into the generic matrices, but now the image is the subalgebra $\mathcal{T}_n\langle\xi_k\rangle$ of $M_n(\mathcal{C})$ generated by the generic matrices and the algebra $\mathcal{T}_n$, the image of $\mathfrak{T}$, generated by all traces of the monomials in the ξ_k.

It is a remarkable fact that in this setting both the trace identities and the *free* algebra $\mathcal{T}_n\langle\xi_k\rangle$ can be interpreted in the language of the first and second fundamental theorem for matrices (FFT and SFT).

We have the projective linear group $G := PGL(n, F)$ acting by conjugation on matrices and hence also on sequences of matrices, and we have (see [27, Chapter 11]):

Theorem 2.1. FFT: *The algebra $\mathcal{T}_n$ is the algebra of G-invariant polynomial functions on the space of sequences of matrices. The algebra $\mathcal{T}_n\langle\xi_k\rangle$ is the algebra of G-equivariant polynomial maps from sequences of matrices to matrices.*

SFT: *The ideal of trace identities on $n \times n$ matrices is generated, as a T-ideal, by the Cayley-Hamilton polynomial.*

Another way of stating the FFT is by noticing that G acts on $\mathcal{C}$ by $gf(x) := f(g^{-1}x)$ and on $M_n(F)$ by conjugation, hence it acts on $M_n(\mathcal{C}) = \mathcal{C} \otimes_F M_n(F)$ and we have

$$\mathcal{T}_n\langle \xi_k \rangle = M_n(\mathcal{C})^G, \quad \mathcal{T}_n = \mathcal{C}^G.$$

Notice that as soon as $n \geq 2$ the algebra $\mathcal{T}_n$ is the center of $\mathcal{R}_n$.

The FFT for matrices is essentially classical, as for the SFT, Procesi [25] and Razmyslov [30] proved that the T-ideal of trace identities of M_n is generated by $Q_n(x_1, \dots, x_n)$ and $\mathrm{tr}(Q_n(x_1, \dots, x_n)x_{n+1})$, where Q_n is the multilinear *Cayley-Hamilton polynomial*. Let us recall that the Cayley-Hamilton polynomial is

$$q_n = q_n(x_1) = x_1^n + \tau_1(x_1)x_1^{n-1} + \cdots + \tau(x_1).$$

As it is well-known, each $\tau_i(x_1)$ can be expressed (in characteristic 0) as a $\mathbb{Q}$-linear combination of the products of $\mathrm{tr}(x_1^j)$. Evaluating in $M_n(C)$ we have $\tau_1(\xi_1) = -\mathrm{tr}(\xi_1) = -(x_{11}^{(1)} + \cdots + x_{nn}^{(1)}), \dots, \tau_n(\xi_1) = (-1)^n \det(\xi_1)$. Now, $Q_n(x_1, \dots, x_n)$ denotes the multilinear version of $q_n(x_1)$ obtained by full polarization. Recall that it can be written as

$$Q_n := \sum_{\sigma \in S_{n+1}} \epsilon_\sigma \phi_\sigma(x_1, \dots, x_n) \tag{3}$$

where $\epsilon_\sigma = \pm 1$ denotes the sign of the permutation σ, while ϕ_σ is defined using the cycle decomposition of

$$\sigma = (i_1, \dots, i_{k_1})(j_1, \dots, j_{k_2}) \dots (u_1, \dots, u_h)(s_1, \dots s_k, n+1)$$

as

$$\phi_\sigma(x_1, \dots, x_n) = \mathrm{tr}(x_{i_1} \cdots x_{i_{k_1}})\mathrm{tr}(x_{j_1} \cdots x_{j_{k_2}}) \cdots \mathrm{tr}(x_{u_1} \cdots x_{u_h}) x_{s_1} \cdots x_{s_k}.$$

Thus, for example,

$$Q_2(x_1, x_2) = x_1x_2 + x_2x_1 - \mathrm{tr}(x_1)x_2 - \mathrm{tr}(x_2)x_1 + \mathrm{tr}(x_1)\mathrm{tr}(x_2) - \mathrm{tr}(x_1x_2).$$

Note that $Q_n(x_1, \dots, x_n)$ is symmetric, i.e., $Q_n(x_1, \dots, x_n) = Q_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for every permutation σ, and that $q_n(x_1) = \frac{1}{n!}Q_n(x_1, \dots, x_1)$. By a slight abuse of terminology, we will call both Q_n and q_n the Cayley-Hamilton polynomial, or, when associated with M_n, the *Cayley-Hamilton identity*. In view of the terminology introduced below, more accurate names in the setting of this paper may be the Cayley-Hamilton quasi-polynomial (resp. quasi-identity), but we omit "quasi" for simplicity.

2.1.3. *Central polynomials.* Recall that an element of the free algebra $F\langle X \rangle$ is a *central polynomial* for $n \times n$ matrices if it takes scalar values under any evaluation into matrices. It is then clear that the center, denoted $\mathcal{Z}_n$, of the algebra $F\langle \xi_k \rangle$ of generic matrices, is the image of the set of central polynomials. A basic discovery based on the existence of central polynomials found independently by Formanek [19] and Razmyslov [29] is that the center $\mathcal{Z}_n$ is rather large. Then fundamental theorems of PI theory tell us that the central quotient of $F\langle \xi_k \rangle$ and of $\mathcal{T}_n\langle \xi_k \rangle$ coincide and as soon as $n \geq 2$ give rise to a division algebra of rank n^2 over its center which is the field of quotients of both $\mathcal{Z}_n$ and $\mathcal{T}_n$.

The following theorem gathers together some known facts, but we recall them for completeness.

Theorem 2.2. *If $c \in \mathcal{Z}_n$ has zero constant term, then $F\langle \xi_k \rangle[c^{-1}] = \mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$ is a rank n^2 Azumaya algebra over its center $\mathcal{Z}_n[c^{-1}] = \mathcal{T}_n[c^{-1}]$. Moreover,*

$$M_n(\mathcal{C}[c^{-1}]) \cong \mathcal{C}[c^{-1}] \otimes_{\mathcal{Z}_n[c^{-1}]} \mathcal{T}_n\langle \xi_k \rangle[c^{-1}].$$

Proof. As it is well known and easy to see, c is an identity of M_{n-1}. Since c is invertible in $F\langle \xi_k \rangle[c^{-1}]$ and $\mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$, these two algebras cannot have nonzero quotients satisfying the identities of M_{n-1}. It follows from the Artin-Procesi theorem that $F\langle \xi_k \rangle[c^{-1}]$ and $\mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$ are Azumaya algebras over their centers of rank n^2. These centers are clearly $\mathcal{Z}_n[c^{-1}]$ and $\mathcal{T}_n[c^{-1}]$. By general properties, the reduced trace of $x \in F\langle \xi_k \rangle[c^{-1}]$ is just the trace of x considered as a matrix in $F\langle \xi_k \rangle[c^{-1}]$, and $F\langle \xi_k \rangle[c^{-1}]$ is closed under the reduced trace. Hence every element in $\mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$ is contained in $F\langle \xi_k \rangle[c^{-1}]$. Accordingly, $F\langle \xi_k \rangle[c^{-1}] = \mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$ and $\mathcal{Z}_n[c^{-1}] = \mathcal{T}_n[c^{-1}]$.

Recall a standard fact (see [2] and [3] or [34, Theorem 2.8]) that if $R \subseteq S$, R is an Azumaya algebra and the center $Z(R)$ of R is contained in the center $Z(S)$ of S, then $S \cong R \otimes_{Z(R)} R'$ where R' is the centralizer of R in S. Taking $\mathcal{T}_n\langle \xi_k \rangle[c^{-1}]$ for R and $M_n(\mathcal{C}[c^{-1}])$ for S we obtain the last assertion of the theorem. □

We should remark that this theorem has a geometric content. Let F be algebraically closed. If we fix the number of generic matrices to a finite number m, we have the action by simultaneous conjugation of $G := PGL_n(F)$ on the affine space $M_n(F)^m$. By geometric invariant theory the algebra $\mathcal{T}_n$ is the coordinate ring of the categorical quotient $M_n(F)^m//PGL_n(F)$, a variety parameterizing the closed orbits, which correspond to isomorphism classes of semisimple representations of dimension n of the free algebra in m generators (cf. [1]).

The action of the projective group G is free on the open set of irreducible representations and the complement of this open set is exactly the subvariety of m-tuples of matrices where all central polynomials with no constant term vanish. The Azumaya algebra property reflects this geometry. Except for the special case $m = n = 2$ the variety $M_n(F)^m//PGL_n(F)$ is smooth exactly on this open set and the quotient map $M_n(F)^m \to M_n(F)^m//PGL_n(F)$ is not flat over the singular set (cf. [24]). As we shall see these singularities are in some sense measured by the quasi-identities of matrices modulo those which are a consequence of the Cayley Hamilton identity, see the exact sequence (4). This will be described as a module $\mathfrak{I}_n/(Q_n)$ supported in the singular part of the quotient variety. On the other hand to prove that this module is indeed nontrivial is quite difficult and although we will show this, we only have a partial description of this phenomenon, the description of the *antisymmetric part of the module*.

2.2. **Quasi-identities.** The purpose of this section is to introduce the setting and record some easy results on the main theme of this paper, *quasi-identities*. Let us point out, first of all, that we will consider our problems exclusively on the algebra $M_n = M_n(F)$. Various problems on functional identities studied in [11] can be solved for quite general classes of rings, but the study of nonstandard solutions is of a different nature and confining to matrices seems natural in this context.

We will define a quasi-polynomial in a slightly different way than in [7] and [11]. Our definition is not restricted to the multilinear situation, and, on the other hand, is adjusted for applications to the matrix algebra M_n.

We use the notations of (1) and (2). A *quasi-polynomial* is an element of the algebra $\mathcal{C}\big\langle X\big\rangle := \mathcal{C}\big\langle X\big\rangle$, the free algebra in the variables X with coefficients in the polynomial algebra of functions on $M_n(F)^{|X|}$.

Thus, a quasi-polynomial is a polynomial in the noncommuting indeterminates x_k whose coefficients are ordinary polynomials in the commuting indeterminates $x_{ij}^{(k)}$, coordinates of the space $M_n(F)^{|X|}$. A quasi-polynomial P can be therefore uniquely written as

$$P = \sum \lambda_M M,$$

where M is a noncommutative monomial in the x_k's and λ_M is a commutative polynomial in the $x_{ij}^{(k)}$'s, that is a polynomial function on sequences of matrices. Of course, P depends on finitely many x_k's and finitely many $x_{ij}^{(k)}$'s. We can therefore write

$$P = P(x_{11}^{(1)}, \dots, x_{nn}^{(1)}, \dots, x_{11}^{(m)}, \dots, x_{nn}^{(m)}, x_1, \dots, x_m)$$

for some m. It is possible to put this setting in the framework of universal algebra, but we shall limit to the following easy facts.

2.2.1. *Substitutional rules.* Commutative indeterminates $x_{ij}^{(k)}$ have a *substitutional rule*, that is given as follows. We have a map $\Phi : x_k \mapsto \xi_k$ of $\mathcal{C}\big\langle X\big\rangle$ to $M_n(\mathcal{C})$ which maps x_k to the corresponding generic matrix and is the identity on $\mathcal{C}$, so for each choice of $H \in \mathcal{C}\big\langle X\big\rangle$ it makes sense to speak of $\Phi(H)_{ij}$, the (i, j) entry of $\Phi(H)$. The substitution in $\mathcal{C}\big\langle X\big\rangle$ should be understood as that one substitutes $x_k \mapsto H_k \in \mathcal{C}\big\langle X\big\rangle$ and simultaneously $x_{ij}^{(k)} \mapsto \Phi(H_k)_{ij}$. We define

Definition 2.3. A *T-ideal* of $\mathcal{C}\big\langle X\big\rangle$ as an ideal that is closed under all such substitutions.

Also, it is convenient to use a more suggestive notation and write $\lambda_M(x_1, \dots, x_m)$ for $\lambda_M(x_{11}^{(1)}, \dots, x_{nn}^{(1)}, \dots, x_{11}^{(m)}, \dots, x_{nn}^{(m)})$, and hence $P(x_1, \dots, x_m)$ for P.

We now define the *evaluation* of P at an m-tuple $A_1, \dots, A_m \in M_n$, $P(A_1, \dots, A_m)$, by substituting A_k for x_k and $a_{ij}^{(k)}$ for $x_{ij}^{(k)}$, where $A_k = (a_{ij}^{(k)})$.

Definition 2.4. If $P(A_1, \dots, A_m) = 0$ for all $A_1, \dots, A_m \in M_n$, then we say that P is a *quasi-identity* of M_n. We denote by $\mathfrak{I}_n$ the set of all quasi-identities of M_n.

The set $\mathfrak{I}_n$ of all quasi-identities of M_n clearly forms a T-ideal of $\mathcal{C}\big\langle X\big\rangle$. As for polynomial or trace identities, $\mathfrak{I}_n$ is the kernel of the $\mathcal{C}-$linear evaluation map from $\mathcal{C}\big\langle X\big\rangle$ to $M_n(\mathcal{C})$ mapping x_k to the generic matrix ξ_k.

Let I denote the identity of $M_n(\mathcal{C})$. For convenience we repeat the proof in

Lemma 2.5. *The algebra $\mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$ is isomorphic to the subalgebra $\mathcal{C}\langle\xi_k\rangle$ of $M_n(\mathcal{C})$ generated by all generic matrices $\xi_k = (x_{ij}^{(k)})$, $k = 1, 2, \dots$, and all λI, $\lambda \in \mathcal{C}$.*

Proof. Let $\Phi : \mathcal{C}\big\langle X\big\rangle \to M_n(\mathcal{C})$ be the homomorphism determined by $\Phi(x_k) = (x_{ij}^{(k)})$ and $\Phi(\lambda) = \lambda I$ for $\lambda \in \mathcal{C}$. It is immediate that $\ker \Phi \subseteq \mathfrak{I}_n$. Given $P = P(x_1, \dots, x_m) \in \mathfrak{I}_n$ we have $P(A_1, \dots, A_m) = 0$ for all $A_i \in M_n$. Since $\mathrm{char}(F) = 0$, and hence F is infinite, a standard argument shows that $\Phi(P) = 0$. Thus, $\ker \Phi = \mathfrak{I}_n$, and the result follows. $\square$

In fact the evaluation ρ of the free algebra with trace to generic matrices with traces factors through $\mathcal{C}\big\langle X\big\rangle$

$$\rho : \mathfrak{T}\langle X\rangle \xrightarrow{\pi} \mathcal{C}\big\langle X\big\rangle \to M_n(\mathcal{C})$$

by evaluating the trace monomials $\mathrm{tr}(x_{i_1}\cdots x_{i_m})$ into $M_n(\mathcal{C})$ but keeping fixed the free variables. The image of π is the algebra $\mathcal{T}_n\big\langle X\big\rangle$ of invariants of the algebra $\mathcal{C}\big\langle X\big\rangle$ with respect to the action of the projective group on the coefficients $\mathcal{C}$ and fixing the variables X.

Thus the image through π of a trace polynomial can also be viewed as a quasi-polynomial $\sum \lambda_M M$, but such that every λ_M is an invariant and thus can be expressed as a linear combination of the products of $\mathrm{tr}(x_{i_1}\cdots x_{i_m})$.

Every trace identity gives rise to a quasi-identity of M_n, but a nontrivial element of $\mathfrak{T}\langle X\rangle$ may very well map to 0 under π, so a nontrivial trace identity may correspond to a trivial quasi-identity. In view of the SFT for matrices we may again consider the quasi-polynomial arising from the Cayley-Hamilton theorem, therefore it is natural to look in this context for a possible analogue of the SFT for quasi-identities (cf. Theorem 2.1).

Definition 2.6. We shall say that a quasi-identity P of M_n is a *consequence of the Cayley-Hamilton identity* if P lies in the T-ideal of $\mathcal{C}\big\langle X\big\rangle$ generated by Q_n.

The question pointed out in the introduction thus asks the following:

Main question. *Is the T-ideal $\mathfrak{I}_n$ generated by Q_n?*
(Here we may replace Q_n by q_n, as q_n and Q_n generate the same T-ideal.)

As we have already remarked, the ideal of quasi-identities $\mathfrak{I}_n$ is the kernel of the evaluation map Φ of $\mathcal{C}\big\langle X\big\rangle$ into $M_n(\mathcal{C})$ mapping the variables to the generic matrices. In view of Lemma 2.5 we have a sequence of inclusion maps

$$F\langle\xi_k\rangle \subset \mathcal{T}_n\langle\xi_k\rangle \subset \mathcal{C}\langle\xi_k\rangle.$$

Our first remark is that, unlike $F\langle\xi_k\rangle \cong F\langle X\rangle/\mathrm{id}(M_n)$ and $\mathcal{T}_n\langle\xi_k\rangle$, $\mathcal{C}\langle\xi_k\rangle \cong \mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$ is not a domain. This can be deduced from Lemma 2.8 below, but let us, nevertheless, give a simple concrete example.

Example 2.7. Note that none of

$$P_1 = x_{12}^{(2)}x_1 - x_{12}^{(1)}x_2 + x_{12}^{(1)}x_{22}^{(2)} - x_{22}^{(1)}x_{12}^{(2)}$$

and

$$P_2 = x_{12}^{(2)}x_1 - x_{12}^{(1)}x_2 + x_{12}^{(1)}x_{11}^{(2)} - x_{11}^{(1)}x_{12}^{(2)}$$

lies in $\mathfrak{I}_2$, but P_1P_2 does.

The center of $\mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$ is isomorphic to $\mathcal{C}$, which is a domain. We may therefore form the algebra of central quotients of $\mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$, which consists of elements of the form αR where $R \in \mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$ and α lies in

$$\mathcal{K} := F\big(x_{ij}^{(k)} \,|\, 1 \le i, j \le n, k = 1, 2, \ldots\big),$$

the field of rational functions in $x_{ij}^{(k)}$ (cf. [33, p. 54]). In order to describe this $\mathcal{K}$-algebra, we invoke the Capelli polynomials

$$C_{2k-1}(x_1, \ldots, x_k, y_1, \ldots, y_{k-1}) := \sum_{\sigma \in S_k} \epsilon_\sigma x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots x_{\sigma(k-1)}y_{k-1}x_{\sigma(k)},$$

where ϵ_σ is the sign of the permutation σ. As it is well-known, C_{2n^2-1} is a polynomial identity of every proper subalgebra of $M_n(E)$ but not of $M_n(E)$ itself, for every field E [33, Theorem 1.4.8].

Lemma 2.8. *The algebra of central quotients of $\mathcal{C}\langle X\rangle/\mathfrak{I}_n$ is isomorphic to $M_n(\mathcal{K})$.*

Proof. Since C_{2n^2-1} is not a polynomial identity of $M_n(F)$, it is also not a polynomial identity of the $\mathcal{K}$-subalgebra of $M_n(\mathcal{K})$ generated by all generic matrices $(x_{ij}^{(k)})$, $k = 1, 2, \ldots$. But then this subalgebra is the whole algebra $M_n(\mathcal{K})$. Now we can apply Lemma 2.5. $\square$

We conclude this section with a small application. Define the image of $P = P(x_1, \ldots, x_m) \in \mathcal{C}\langle X\rangle$ as

$$\mathrm{im}(P) = \{P(A_1, \ldots, A_m) \mid A_1, \ldots, A_m \in M_n\}.$$

It is an open question which subsets of M_n can be images of noncommutative polynomials; cf. [21, 35]. Since, on the other hand, such an image is closed under conjugation it follows that among linear subspaces of M_n there are only four possibilities: $\{0\}$, the space of all scalar matrices, the space of all trace zero matrices, and M_n. The situation with quasi-polynomials is strikingly different.

Proposition 2.9. *For every linear subspace V of M_n there exists $P \in \mathcal{C}\langle X\rangle$ such that $\mathrm{im}(P) = V$.*

Proof. By taking the sums of quasi-polynomials in distinct indeterminates we see that it is enough to prove the theorem for the case where V is one-dimensional, $V = FA$ for some $A \in M_n$. According to Lemma 2.8, we may identify $x_{11}^{(1)}A \in M_n(\mathcal{K})$ with $\lambda^{-1}P_0$ where $0 \neq \lambda \in \mathcal{C}$ and $P_0 \in \mathcal{C}\langle X\rangle$. Hence $\mathrm{im}(P_0) \subseteq FA$. Picking an indeterminate $x_{ij}^{(k)}$ of which P_0 is independent we thus see that $P = x_{ij}^{(k)}P_0$ satisfies $\mathrm{im}(P) = FA$. $\square$

3. Quasi-identities and the Cayley-Hamilton identity

3.1. **Trace algebras and the Cayley-Hamilton identity.** We begin by reformulating our problem in the commutative algebra framework by using the result from [26]. Let us, therefore, recall the content of that paper. We have already seen in §2.1.2 the notion of an algebra with trace, in particular for any commutative algebra A we consider $M_n(A)$ with the usual trace.

For an algebra with trace R and a number $n \in \mathbb{N}$, we define the universal map into $n \times n$ matrices as a pair of a commutative algebra $\mathcal{A}_R$ and a morphism (of algebras with trace) $j : R \to M_n(\mathcal{A}_R)$ with the following universal property: for any other map (of algebras with trace) $f : R \to M_n(\mathcal{B})$ with $\mathcal{B}$ commutative there is a unique map $\bar{f} : \mathcal{A}_R \to \mathcal{B}$ of commutative algebras making the diagram commutative

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[dr, "f"'] & M_n(\mathcal{A}_R) \arrow[d, "M_n(\bar{f})"] \\ & M_n(\mathcal{B}) \end{tikzcd}$$

The existence of such a universal map is easily established, although in general it may be 0.

The main idea comes from category theory, that is, from representable functors. We take the functor from commutative algebras to sets which associates to a commutative algebra $\mathcal{B}$ the set of (trace preserving) morphisms $\hom(R, M_n(\mathcal{B}))$ and want to prove that it is representable, i.e., that there is a commutative algebra $\mathcal{A}_R$ and a natural isomorphism $\hom(R, M_n(\mathcal{B})) \cong \hom(\mathcal{A}_R, \mathcal{B})$. Then the identity map $1_{\mathcal{A}_R} \in \hom(\mathcal{A}_R, \mathcal{A}_R)$ corresponds to the universal map $j \in \hom(R, M_n(\mathcal{A}_R))$.

We have seen the category of algebras with trace has free algebras which are the usual free algebras in indeterminates x_k to which we add a commutative algebra $\mathfrak{T}$ of *formal traces*. Then we see that the commutative algebra associated to a free algebra is the polynomial algebra in indeterminates $x_{ij}^{(k)}$. The universal map maps x_k to the generic matrix with entries $x_{ij}^{(k)}$ and the formal traces $\operatorname{tr}(x_{i_1} x_{i_2} \dots x_{i_k})$ map to the traces of the corresponding monomials in the generic matrices. From a presentation of R as a quotient of a free algebra one obtains a presentation of $\mathcal{A}_R$ as a quotient of the ring of polynomials in the $x_{ij}^{(k)}$.

If we consider now algebras over a field F (which is of characteristic 0 by the above convention) we have that the group $G = GL_n(F)$ of invertible $n \times n$ matrices (in fact the projective group $PGL_n(F) = GL_n(F)/F^*$) acts on the algebra $\mathcal{A}_R$ and it also acts by conjugation on $M_n(F)$, so it acts diagonally on $M_n(\mathcal{A}_R)$. The main theorem of [26] says that

Theorem 3.1. *The image of j is the invariant algebra $M_n(\mathcal{A}_R)^G$ and the kernel of j is the trace-ideal generated by the evaluations of the formal Cayley-Hamilton expression for the given n. In particular, if R satisfies the n-th Cayley-Hamilton identity, then j is injective.*

3.1.1. *The trace on $\mathcal{C}\langle X\rangle$.* Now we will apply this theory to $\mathcal{C}\langle X\rangle$. For this we need to make it into an algebra with trace. For reasons that will soon become clear, let us write $\mathcal{C}_x$ for $\mathcal{C}$ and hence $\mathcal{C}_x\langle X\rangle$ until the end of this section.

Definition 3.2. We define the trace $\operatorname{tr} : \mathcal{C}_x\langle X\rangle \to \mathcal{C}_x$ as the $\mathcal{C}_x$-linear map satisfying $\operatorname{tr}(1) = n$ and mapping a monomial in the indeterminates x_k into the trace of the corresponding monomial in generic matrices ξ_k in the indeterminates $x_{ij}^{(k)}$.

In order to understand what is the universal map of this algebra with trace into $n \times n$ matrices we introduce a second polynomial algebra $\mathcal{C}_y = F\big[y_{ij}^{(k)} \mid 1 \le i, j \le n, k = 1, 2, \dots\big]$.

The group G acts on $\mathcal{C}_x$ and $\mathcal{C}_y$, and, by the FFT, the invariants are in both cases the invariants of matrices, that is the algebra generated by the traces of monomials. We identify the two algebras of invariants and call this algebra $\mathcal{T}_n$.

Now we set

$$\mathcal{A}_n := \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{C}_y,$$

and let $\xi_k := (y_{ij}^{(k)})$ denote the generic matrix in $M_n(\mathcal{C}_y)$. Note that the algebra $\mathcal{T}_n\langle \xi_k, k = 1, 2, \dots\rangle$ may be identified to the algebra $\mathcal{T}_n\langle \xi_k\rangle$ of equivariant maps studied in Theorem 2.1 and which has $\mathcal{T}_n$ as the center.

From now on let $j : \mathcal{C}\langle X\rangle \to M_n(\mathcal{A}_n)$ denote the $\mathcal{C}_x$-linear map which maps x_k to the generic matrix $\xi_k = (y_{ij}^{(k)})$, and let (Q_n) denote the T-ideal of $\mathcal{C}\big\langle X\big\rangle$ generated by Q_n. Since we are thinking of $\mathcal{C}_x$ as a coefficient ring, in the next proposition the action of G on $\mathcal{A}_n = \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{C}_y$, is by acting on the second factor $\mathcal{C}_y$. The action on $M_n(\mathcal{A}_n) = M_n(F) \otimes_F \mathcal{A}_n$ is the tensor product action.

Proposition 3.3. (1) *The map $j : \mathcal{C}_x\langle X\rangle \to M_n(\mathcal{A}_n) = M_n(\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{C}_y)$ is the universal map into matrices.*
(2) *The algebra $\mathcal{C}\langle X\rangle/(Q_n)$ is isomorphic to the algebra $M_n(\mathcal{A}_n)^G = \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle \xi_k\rangle$.*

Proof. By Theorem 3.1, (2) follows from (1) so it is enough to prove that j is the universal map.

Take an algebra with trace $\mathcal{B}$. Let us compute the representable functor $\hom(\mathcal{C}\langle X\rangle, M_n(\mathcal{B}))$. In order to give a homomorphism $\phi : \mathcal{C}\langle X\rangle \to M_n(\mathcal{B})$ in the category of algebras with trace, we have to choose arbitrary elements $a_{ij}^{(k)} \in \mathcal{B}$ for the images of the elements $x_{ij}^{(k)}$, and matrices $B_k = (b_{ij}^{(k)})$ for the images of the elements x_k.

Moreover, if we consider the matrices $A_k := (a_{ij}^{(k)})$ we need to impose that the trace of each monomial formed by the A_k equals the trace of the corresponding monomial formed by the B_k.

Now to give the $a_{ij}^{(k)}$ is the same as to give a homomorphism of $\mathcal{C}_x$ to $\mathcal{B}$, and to give the $b_{ij}^{(k)}$ is the same as to give a homomorphism of $\mathcal{C}_y$ to $\mathcal{B}$. The compatibility means that the restrictions of these two homomorphisms to the algebra $\mathcal{T}_n$, which is contained naturally in both copies, coincide. This is exactly the description of a homomorphism of $\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{C}_y$ to $\mathcal{B}$. Thus, j is indeed the universal map.

Next observe that the action of G is only on the factor $\mathcal{C}_y$. By Theorem 3.1 it follows that the kernel of j is equal to (Q_n). Thus, it remains to find $M_n(\mathcal{A}_n)^G$, the image of j. Note that $M_n(\mathcal{A}_n) = \mathcal{C}_x \otimes_{\mathcal{T}_n} M_n(\mathcal{C}_y)$ and that G acts trivially on $\mathcal{C}_x$ while on $M_n(\mathcal{C}_y)$ it is the action used in the universal map of the free algebra with trace (see [26] for details). By a standard argument on reductive groups we have $M_n(\mathcal{A}_n)^G = \mathcal{C}_x \otimes_{\mathcal{T}_n} M_n(\mathcal{C}_y)^G$, which is by the FFT equal to $\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle$. $\square$

Remark 3.4. The algebra $\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{C}_y$, a fiber product, contains a lot of the hidden combinatorics needed to understand functional identities. It appears to be a rather complicated object as pointed out by some experimental computations carried out by H. Kraft (whom we thank), which show that even for $n = 2$ as soon as the number of variables is ≥ 3 it is not an integral domain nor is it Cohen-Macaulay. This of course is due to the fact that the categorical quotient described by the inclusion $\mathcal{T}_n = \mathcal{C}^G$ has a rather singular behavior outside the open set parameterizing irreducible representations.

We have to introduce some more notation. As in the proof of the preceding proposition, let ξ_k stand for the generic matrix $(y_{ij}^{(k)})$. Analogously, we write η_k for the generic matrix $(x_{ij}^{(k)})$. There is a canonical homomorphism

$$\pi : \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle \to \mathcal{C}_x\langle \eta_k, k = 1, 2, \ldots\rangle,$$

$$\pi : \lambda \otimes f(\xi_1, \ldots, \xi_d) \mapsto \lambda f(\eta_1, \ldots, \eta_d)$$

(note that by Lemma 2.5 the latter algebra is nothing but $\mathcal{C}\big\langle X\big\rangle/\mathfrak{I}_n$).

Lemma 3.5. *A quasi-identity P of M_n is not a consequence of the Cayley-Hamilton identity if and only if $j(P)$ is a nonzero element of the kernel of π.*

Proof. Let $\Phi : \mathcal{C}\big\langle X\big\rangle \to \mathcal{C}_x\langle\eta_k, k = 1, 2, \ldots\rangle$ be the homomorphism from Lemma 2.5, i.e., $\Phi(x_k) = \eta_k$ and $\Phi(\lambda) = \lambda I$ for $\lambda \in \mathcal{C}_x$, and let $j : \mathcal{C}\langle X\rangle \to \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle$ be the universal map.

Note that $\pi j = \Phi$. Since, by Proposition 3.3, $\ker j$ is the T-ideal of $\mathcal{C}\langle X\rangle$ generated by Q_n, and $\ker \Phi = \mathfrak{I}_n$, this implies the assertion of the lemma. $\square$

Corollary 3.6. *The space $\mathfrak{I}_n/(Q_n)$, measuring quasi-identities modulo the ones deduced from Q_n, is isomorphic under the map induced by $j : \mathcal{C}\langle X\rangle/(Q_n) \to C_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle$ to the kernel of the map π. That is, we have an exact sequence*

$$0 \longrightarrow \mathfrak{I}_n/(Q_n) \xrightarrow{\ j\ } \mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle \xrightarrow{\ \pi\ } M_n(\mathcal{C}_x). \tag{4}$$

As an application of Theorem 2.2 we have the following theorem on quasi-identities.

Theorem 3.7. *Let P be a quasi-identity of M_n. For every central polynomial c of M_n with zero constant term there exists $m \in \mathbb{N}$ such that $c^m P$ is a consequence of the Cayley-Hamilton identity.*

Proof. Note that

$$(\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k\rangle)[c^{-1}] \cong \mathcal{C}_x[c^{-1}] \otimes_{\mathcal{T}_n[c^{-1}]} \mathcal{T}_n\langle\xi_k\rangle[c^{-1}] \cong M_n(\mathcal{C}_x[c^{-1}])$$

by Theorem 2.2 (the change of variables does not make any difference since $\mathcal{C}_x$ is canonically isomorphic to $\mathcal{C}_y$). This isomorphism is induced by π introduced before Lemma 3.5. Therefore $(\ker \pi)[c^{-1}] = 0$. Since every quasi-identity P lies in $\ker(\pi j)$ by Lemma 3.5, there exists $m \in \mathbb{N}$ such that $c^m P = 0$ in $\mathcal{C}_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle\xi_k, k = 1, 2, \ldots\rangle$, i.e., $c^m P$ is a consequence of the Cayley-Hamilton identity by Proposition 3.3. $\square$

We have seen that $\ker \pi$ measures the space of quasi-identities modulo the ones deduced from Q_n. This is in fact a $\mathcal{T}_n-$module and, as we shall see, it is nonzero. What the previous theorem tells us is that this module is *supported* in the closed set of non-irreducible representations.

4. Antisymmetric quasi-identities

4.1. **Antisymmetric identities derived from the Cayley-Hamilton identity.** By the *antisymmetrizer* we mean the operator that sends a multilinear expression $f(x_1, \ldots, x_h)$ into the antisymmetric expression $\frac{1}{h!}\sum_{\sigma\in S_h} \epsilon_\sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(h)})$, where ϵ_σ is the sign of σ. For example, applying the antisymmetrizer to the noncommutative monomial $x_1 \cdots x_h$ we get the standard polynomial of degree h, $S_h(x_1, \ldots, x_h) = \sum_{\sigma\in S_h} \epsilon_\sigma x_{\sigma(1)} \ldots x_{\sigma(h)}$, and up to scalar this is the only multilinear antisymmetric noncommutative polynomial of degree h. Further, applying the antisymmetrizer to the quasi-monomial $x^{(1)}_{i_1,j_1} \cdots x^{(k)}_{i_k,j_k} x_{k+1} \cdots x_{n^2}$ we get an antisymmetric quasi-polynomial, which is nonzero as long as the pairs (i_l, j_l) are pairwise different, and is, because of the antisymmetry, an identity of every proper subspace of M_n, in particular of the space of trace zero $n \times n$ matrices. Replacing each variable x_k by $x_k - \frac{1}{n}\mathrm{tr}(x_k)$, we thus get a quasi-identity of M_n. Our ultimate goal is to show that *not every such quasi-identity is a consequence of the Cayley-Hamilton identity*. For this we need several auxiliary results. We start by introducing the appropriate setting.

Let A be a finite dimensional F-algebra with basis e_i, and let V be a vector space over F. The set of multilinear antisymmetric functions from V^k to A is given by functions $F(v_1, \ldots, v_k) = \sum_i F_i(v_1, \ldots, v_k)e_i$ with $F_i(v_1, \ldots, v_k)$ multilinear antisymmetric functions from V^k to F, in other words $F_i(v_1, \ldots, v_k) \in \bigwedge^k V^*$. This space can be therefore identified with $\bigwedge^k V^* \otimes A$. Using the algebra structure of A we have a wedge product of these functions: for $F \in \bigwedge^h V^* \otimes A,\ H \in \bigwedge^k V^* \otimes A$ we define

$$F \wedge H(v_1, \dots, v_{h+k}) := \frac{1}{h!k!} \sum_{\sigma \in S_{h+k}} \epsilon_\sigma F(v_{\sigma(1)}, \dots, v_{\sigma(h)}) H(v_{\sigma(h+1)}, \dots, v_{\sigma(h+k)})$$

It is easily verified that with this product the algebra of multilinear antisymmetric functions from V to A is isomorphic to the tensor product algebra $\bigwedge V^* \otimes A$. We will apply this to $V = A = M_n$. Again the group $G = PGL(n, F)$ acts on these functions and it will be of interest to study the *invariant algebra*

$$A_n := (\bigwedge M_n^* \otimes M_n)^G.$$

If N_n denotes the Lie algebra of trace zero $n \times n$ matrices, the multilinear and antisymmetric trace expressions for such matrices can be identified with the invariants $(\bigwedge N_n^*)^G$ of $\bigwedge N_n^*$ under the action of G. By a result of Chevalley transgression [15] and Dynkin [18] this is the exterior algebra in the elements

$$T_h := \operatorname{tr}\big(S_{2h+1}(x_1, \dots, x_{2h+1})\big), \quad 1 \le h \le n-1.$$

In this subsection we will deal with $AT_n := (\bigwedge M_n^*)^G$ rather than with $(\bigwedge N_n^*)^G$. From this result it easily follows that, with a slight abuse of notation, the former is the exterior algebra in the elements $T_0 := \operatorname{tr}(S_1(x_1)), T_1, \dots, T_{n-1}$. We remark that we use only traces of the standard polynomials of odd degree since, as it is well-known, $\operatorname{tr}(S_{2h}(x_1, \dots, x_{2h})) = 0$ for every h, see [32].

The group G obviously acts by automorphisms, thus $A_n = (\bigwedge M_n^* \otimes M_n)^G$ is indeed an associative algebra. The main known fact that we shall use is (see, e.g., [23] or [31, Corollary 4.2]):

Proposition 4.1. *The dimension of A_n over F is $n2^n$.*

Inside A_n we have the identity map X which in the natural coordinates is the *generic matrix* $\sum_{h,k} x_{hk} e_{hk}$. By iterating the definition of wedge product we have the important fact (see also [28]):

Proposition 4.2. *As a multilinear function, each power $X^a := X^{\wedge a}$ equals the standard polynomial S_a.*

As a consequence we have $S_a \wedge S_b = S_{a+b}$ and by the Amitsur-Levitzki Theorem $X^{2n} = 0$. We summarize the rules:

$$S_a = X^a,\ T_h \wedge X = -X \wedge T_h,\ X^{2n} = 0,$$

where the powers of X should be understood with respect to the wedge product.

Remark 4.3. Note that the elements

$$T_{h_1} \wedge T_{h_2} \dots \wedge T_{h_i} \wedge X^k,$$

where $h_1 < h_2 < \dots < h_i$ and k is arbitrary, form a linear basis of the algebra of multilinear and antisymmetric expressions in noncommutative variables and their traces.

We can consider this algebra as the exterior algebra in the variables T_h, and a variable X in degree 1 which anticommutes with the T_i. We now factor out the ideal of elements of degree $> n^2$ and T_h for $h \geq n$, and thus obtain a symbolic algebra which we call $\mathcal{TA}_n$. The algebra A_n of multilinear antisymmetric invariant functions on matrices to matrices is a quotient of this algebra. We have to discover the identities that generate the corresponding ideal, as for instance the Amitsur-Levitzki identity $X^{2n} = 0$, which is the basic *even* identity. The next lemma points out the basic *odd* identity.

Lemma 4.4. *The element $O_n := nX^{2n-1} - \sum_{i=0}^{n-1} X^{2i} \wedge T_{n-i-1} \in \mathcal{TA}_n$ is an identity of M_n. Moreover, O_n is an antisymmetric trace identity of minimal degree.*

Proof. We know by the SFT that every trace identity is obtained from Q_n by substitution of variables and multiplication, hence any antisymmetric identity is obtained by first applying such a procedure obtaining a multilinear identity and then antisymmetrizing. Since Q_n is symmetric this procedure gives zero if we substitute two variables by two monomials of the same odd degree. In particular this means that we can keep at most one variable unchanged and we have to substitute the others with monomials of degree ≥ 2, thus the minimal identity that we can develop in this way is by substituting $x_2, \dots, x_n$ with distinct monomials M_i, $2 \leq i \leq n$, of degree 2 and then antisymmetrizing.

We use the formula (3). If a permutation σ contains a cycle $(i_1, \dots, i_k)$ in which neither 1 nor $n+1$ appear, substituting and alternating into the corresponding element $\phi_\sigma(x_1, \dots, x_n)$, we get that the antisymmetrization of the factor $\operatorname{tr}(M_{i_1} \dots M_{i_k})$ is zero as $\operatorname{tr}(X^{2i}) = \operatorname{tr}(S_{2i}) = 0$. Thus the only terms of (3) which give a contribution are the ones where either σ is a unique cycle and they contribute to $(-1)^n n! X^{2n-1}$ or the ones with two cycles, one containing 1 and the other $n+1$; such permutations can be described in the form

$$\sigma = (1, i_1, \dots, i_h)(i_{h+1}, \dots, i_{n-1}, n+1).$$

For each h there are exactly $(n-1)!$ of these and they all have the sign $(-1)^{n-1}$. The antisymmetrization of ϕ_σ after substitution gives

$$\operatorname{tr}(X^{2h+1}) X^{2(n-h-1)} = X^{2(n-h-1)} \wedge T_h,$$

and the claim follows. □

By AT_{n-1} we denote the subalgebra of AT_n generated by the $n-1$ elements T_i, $0 \leq i \leq n-2$. This is an exterior algebra and has dimension 2^{n-1}.

Theorem 4.5. *$A_n = (\bigwedge M_n^* \otimes M_n)^G$ is a free left module over the algebra AT_{n-1} with basis $\{1, X, \dots, X^{2n-1}\}$. The kernel of the canonical homomorphism from $\mathcal{TA}_n$ to $(\bigwedge M_n^* \otimes M_n)^G$ is generated by X^{2n} and O_n.*

Proof. We have that $\dim(\bigwedge M_n^* \otimes M_n)^G = n2^n$ by Proposition 4.1. Moreover, by Remark 4.3 and the FFT Theorem of invariant theory of matrices 2.1, we know that A_n as module over AT_n is generated by the elements $1, X, \dots, X^{2n-1}$.

Now consider the left submodule N of $(\bigwedge M_n^* \otimes M_n)^G$ generated over the algebra AT_{n-1} by the elements $1, X, \dots, X^{2n-1}$. Clearly $\dim(N) \leq (2n)2^{n-1} = n2^n$ and the equality holds if and only if N is a free module. By the dimension formula this is also equivalent to say that N coincides with $(\bigwedge M_n^* \otimes M_n)^G$.

So it is enough to show that N coincides with $(\bigwedge M_n^* \otimes M_n)^G$. For this it suffices to show that N is stable under multiplication by the *missing generator* T_{n-1}. Due to the commutation relations it is enough to use the right multiplication, which is an $AT_{n-1}-$linear map.

From the identity O_n we have

$$1 \wedge T_{n-1} = T_{n-1} = -\sum_{i=1}^{n-1} X^{2i} \wedge T_{n-i-1} + nX^{2n-1},$$

hence for all $i \geq 1$ we have

$$X^j \wedge T_{n-1} = -\sum_{i=1}^{n-[\frac{j}{2}]} X^{2i+j} \wedge T_{n-i-1},$$

which gives the matrix of such multiplication in this basis as desired. $\square$

4.2. **Antisymmetric identities that are not a consequence of the Cayley-Hamilton identity.** We have denoted by N_n the subspace of trace zero matrices and $G = PGL(n, F)$ acts on M_n and N_n by conjugation.

We work with the associative algebra $(\bigwedge N_n^* \otimes M_n)^G$ of G-equivariant antisymmetric multilinear functions from N_n to M_n. We let Y be the element of $N_n^* \otimes M_n = \hom(N_n, M_n)$ corresponding to the inclusion. We note that $Y = X - \frac{\operatorname{tr}(X)}{n}$.

It easily follows from Theorem 4.5 and the previous formula that also $(\bigwedge N_n^* \otimes M_n)^G$ is a free module on the powers Y^i, $0 \leq i \leq 2n-1$, over the exterior algebra in the $n-2$ generators $\operatorname{tr}(Y^{2i+1})$, $1 \leq i \leq n-2$, (note that $\operatorname{tr}(Y) = 0$).

Finally, we know that the element $\operatorname{tr}(Y^{2n-1})$ acts on this basis by the basic formula:

$$Y^j \wedge \operatorname{tr}(Y^{2n-1}) = -\sum_{i=1}^{n-[\frac{j}{2}]} Y^{2i+j} \wedge \operatorname{tr}(Y^{2n-2i-1}).$$

We now construct the formal algebra of symbolic expressions by adding to $(\bigwedge N_n^* \otimes M_n)^G$ a variable X with the rules

$$XY = -YX, \quad X\operatorname{tr}(Y^{2i+1}) = -\operatorname{tr}(Y^{2i+1})X.$$

We place X in degree 1 and factor out all elements of degree $> n^2$. We call this formal algebra $\tilde{A}_n$. Its connection to quasi-identities will be revealed below.

Consider now the algebra $\mathbb{F}_n := \bigwedge N_n^*[X]$ with again X in degree 1, $X^{2n} = 0$ and X anticommutes with the elements of degree 1, that is with N_n^*. We also impose that the expressions of degrees $> n^2$ are zero in $\mathbb{F}_n$. Each element of this algebra induces an antisymmetric multilinear functions from N_n to M_n and the elements that give rise to the zero function are exactly the antisymmetric multilinear quasi-identities on N_n. As above, we set $T_i = \operatorname{tr}(Y^{2i+1}) \in \bigwedge^{2i+1} N_n^*$. Let us first identify the subspace of $\mathbb{F}_n$ of quasi-identities deduced from Q_n.

Proposition 4.6. *The space of quasi-identities deduced from Q_n in $\mathbb{F}_n$ is the ideal generated by the element*

$$O_n := nX^{2n-1} - \sum_{i=0}^{n-2} X^{2i} \wedge T_{n-i-1}.$$

Proof. By definition a quasi-identity is deduced from Q_n if it is obtained by first substituting the variables in Q_n with monomials, and then multiplying by monomials and polynomials in the coordinates. If it is multilinear this procedure passes only through steps in which all substitutions are multilinear, as for antisymmetrizing we can first make it multilinear then antisymmetrize. Thus we see that the quasi-identities in $\mathbb{F}_n$ deduced from Q_n equal the ideal generated by the invariant antisymmetric quasi-identities deduced from Q_n. By Theorem 4.5 these are multiples of O_n, proving the result. (Note that we have slightly abused the notation – since we are dealing with trace zero matrices N_n we have $T_0 = 0$, unlike in Theorem 4.5.) $\square$

We set $J := O_n\mathbb{F}_n$ to be the ideal generated by the element O_n. We will concentrate on degree n^2 where we know that all formal expressions are identically zero as functions on N_n. We want to describe in the space of the quasi-identities of degree n^2, $\mathbb{F}_n[n^2]$, the subspace $J \cap \mathbb{F}_n[n^2]$ of the elements which are a consequence of Q_n.

4.2.1. *Restricting to an isotypic component.* Let us notice that the group G acts on $\mathbb{F}_n$ through its action on $\bigwedge N_n^*$ and fixing X. Namely, we have a representation

$$\mathbb{F}_n = \oplus_{i=0}^{2n-1}(\oplus_{j=0}^{n^2-i} \bigwedge^{j} N_n^*)X^i, \quad \mathbb{F}_n[n^2] = \oplus_{i=1}^{2n-1} \bigwedge^{n^2-i} N_n^* X^i. \tag{5}$$

We now restrict to the subspace stable under G and corresponding to the isotypic component of type N_n. This is motivated by the fact that the component of $\mathbb{F}_n[n^2]$ relative to X^2 is $\bigwedge^{n^2-2} N_n^* X^2$, which is visibly isomorphic to N_n as a representation. It is explicitly described as follows: the space $\bigwedge^{n^2-2} N_n^*$ of multilinear antisymmetric functions of n^2-2 matrix variables can be thought of as the span of the determinants of the maximal minors (of size n^2-2) of the $(n^2-2)\times(n^2-1)$ matrix whose i^{th} row are the coordinates of the i^{th} matrix variable X_i which is assumed to be of trace 0.

Let us denote by $\mathbb{G}_n[n^2]$ the isotypic component of type N_n in $\mathbb{F}_n[n^2]$, and by $\mathbb{G}_n[n^2]_{CH}$ the part of this component deducible from Q_n. We are now in a position to state our main result.

Theorem 4.7. *We have a direct sum decomposition*

$$\mathbb{G}_n[n^2] = \mathbb{G}_n[n^2]_{CH} \oplus \bigwedge^{n^2-2} N_n^* X^2.$$

In particular we have the following corollary.

Corollary 4.8. *The space $\bigwedge^{n^2-2} N_n^* X^2$ consists of quasi-identities which are not a consequence of the Cayley-Hamilton identity Q_n.*

Remark 4.9. Corollary 4.8 shows that there exist quasi-identities on N_n which are not a consequence of the Cayley-Hamilton identity Q_n. However, by substituting the variable x_k with $x_k - \frac{1}{n}\mathrm{tr}(x_k)$ we readily obtain quasi-identities on M_n that do not follow from Q_n. This corollary therefore answers our basic question posed in the introduction.

The following question now presents itself.

Question 4.10. What is the minimal degree of a quasi-identity of M_n that is not a consequence of the Cayley-Hamilton identity, and how many variables it involves?

Before engaging in the proof of Theorem 4.7 we need to develop some formalism.

First of all recall that for a reductive group G, a representation U, and an irreducible representation N, we have a canonical isomorphism

$$\hom_G(N,U) = (N^*\otimes U)^G, \quad j:(N^*\otimes U)^G\otimes N \xrightarrow{\cong} U_N;\ j[(\phi\otimes u)\otimes n]\mapsto \langle \phi \,|\, n\rangle u,$$

where U_N denotes the isotypic component of type N.

We want to apply this isomorphism to $U = \mathbb{F}_n$, or $\mathbb{F}_n[n^2]$ and $N = N_n \cong N_n^*$. In particular we have to start describing $(N_n\otimes\mathbb{F}_n)^G$. In fact it is necessary to work with

$$\tilde{A}_n = (M_n\otimes\mathbb{F}_n)^G = ((F\oplus N_n)\otimes\mathbb{F}_n)^G = \mathbb{F}_n^G\oplus(N_n\otimes\mathbb{F}_n)^G. \tag{6}$$

We have

$$\tilde{A}_n = \mathbb{F}_n^G\oplus(N_n\otimes\mathbb{F}_n)^G = \oplus_{i=0}^{2n-1}\oplus_{j=0}^{n^2-i}(\bigwedge^j N_n^*)^G X^i\oplus_{i=0}^{2n-1}\oplus_{j=0}^{n^2-i}(N_n\otimes\bigwedge^j N_n^*)^G X^i, \tag{7}$$

$$(M_n\otimes\mathbb{F}_n[n^2])^G = \mathbb{F}_n[n^2]^G\oplus(N_n\otimes\mathbb{F}_n[n^2])^G = \oplus_{i=1}^{2n-1}(\bigwedge^{n^2-i}N_n^*)^G X^i\oplus_{i=1}^{2n-1}(N_n\otimes\bigwedge^{n^2-i}N_n^*)^G X^i.$$

Now $\tilde{A}_n$ is still an algebra containing $\mathbb{F}_n^G$ as a subalgebra. This is the algebra described at the beginning of this subsection, where Y denoted the generic trace zero matrix.

Lemma 4.11. *We have*

$$(\mathbb{F}_n[n^2]\cap J)^G\oplus(N_n\otimes[\mathbb{F}_n[n^2]\cap J])^G = (M_n\otimes[\mathbb{F}_n[n^2]\cap J])^G = \tilde{A}_n[n^2]\cap(1\otimes O_n)\tilde{A}_n \tag{8}$$

and under the isomorphism $j:(N_n\otimes\mathbb{F}_n[n^2])^G\otimes N_n\to\mathbb{G}_n[n^2]$ the space $\mathbb{G}_n[n^2]_{CH}$ corresponds to $(N_n\otimes[\mathbb{F}_n[n^2]\cap J])^G\otimes N_n$.

Proof. We have $(M_n\otimes[\mathbb{F}_n[n^2]\cap J])^G = \tilde{A}_n[n^2]\cap(M_n\otimes J)^G$. Since O_n is G- invariant, O_n acts (by multiplication with $1\otimes O_n$) on the space of invariants $\tilde{A}_n = (M_n\otimes\mathbb{F}_n)^G$, that is

$$(M_n\otimes J)^G = (M_n\otimes O_n\mathbb{F}_n)^G = (1\otimes O_n)(M_n\otimes\mathbb{F}_n)^G = (1\otimes O_n)\tilde{A}_n,$$

proving (8).

By definition $\mathbb{G}_n[n^2]_{CH} = \mathbb{G}_n[n^2]\cap J = \mathbb{G}_n[n^2]\cap O_n\mathbb{F}_n$. Since by definition $\mathbb{G}_n[n^2]$ is the isotypic component of type $N_n\cong N_n^*$ in $\mathbb{F}_n[n^2]$, we have $(N_n\otimes\mathbb{F}_n[n^2])^G = (N_n\otimes\mathbb{G}_n[n^2])^G$. Thus clearly

$$(N_n\otimes\mathbb{G}_n[n^2]_{CH})^G = (N_n\otimes[\mathbb{G}_n[n^2]\cap J])^G = (N_n\otimes[\mathbb{F}_n[n^2]\cap J])^G.$$

□

On the other hand, the elements $T_i\in(\bigwedge^{2i+1}N_n^*)^G$ equal $\operatorname{tr}(Y^{2i+1})$, so in particular $T_{n-1}\in\mathbb{F}_n^G$ acts on $\tilde{A}_n$ as

$$T_{n-1} = nY^{2n-1} - \sum_{i=1}^{n-2} Y^{2i}\wedge T_{n-i-1}.$$

Thus, we have

Lemma 4.12. *On $\tilde{A}_n$ the element $1\otimes O_n$ acts by multiplying by*

$$\bar{O}_n := n(X^{2n-1}-Y^{2n-1}) - \sum_{i=1}^{n-2}(X^{2i}-Y^{2i})\wedge T_{n-i-1}.$$

Our goal is to understand $\mathbb{G}_n[n^2]_{CH}$. On the other hand, $\mathbb{F}_n[n^2]$ consists of all quasi-identities, hence $\mathbb{F}_n[n^2]^G$ is formed of trace identities and so it is contained in J. Thus, $(\mathbb{F}_n[n^2] \cap J)^G = \mathbb{F}_n[n^2]^G$ and from (8) we have

$$\tilde{A}_n[n^2] \cap \bar{O}_n \tilde{A}_n = \mathbb{F}_n[n^2]^G \oplus (N_n \otimes \mathbb{G}_n[n^2]_{CH})^G. \tag{9}$$

In order to study the isotypic component of type N_n in $\mathbb{F}_n[n^2] \cap J$ we therefore need to analyze

$$(M_n \otimes [\mathbb{F}_n[n^2] \cap J])^G = \tilde{A}_n[n^2] \cap \bar{O}_n \tilde{A}_n = \tilde{A}_n[n^2 - 2n + 1]\bar{O}_n. \tag{10}$$

Lemma 4.13. *We have a* monomial basis *in $\tilde{A}_n$ made of elements of the form $\mathcal{T} X^i Y^j$ where $\mathcal{T}$ is a product of some of the elements T_k, $1 \le k \le n-2$, written in the increasing order. Its degree is $i + j$ plus the sum of the $2k + 1$ for the T_k appearing in $\mathcal{T}$.*

Proof. This follows from (7) and Theorem 4.5. □

From (10) we need to understand the monomials in degree n^2 and $n^2 - 2n + 1$ which are bases of $\tilde{A}_n$, $\tilde{A}_n[n^2 - 2n + 1]$, respectively, and consider the matrix in these bases of multiplication by $\bar{O}_n$ as a map

$$\pi_n : \tilde{A}_n[n^2 - 2n + 1] \to \tilde{A}_n[n^2].$$

In order to understand the image of π_n we construct a linear function ρ on $\tilde{A}_n[n^2]$ defined on the monomial $M := \mathcal{T} X^i Y^j$ of degree n^2 as follows.

(1) If $\mathcal{T}$ does not contain at least two of the factors T_h, T_k, we set $\rho(M) = 0$.
(2) If $\mathcal{T}$ does not contain only one factor T_h, we set $\rho(M) = (-1)^{h+n}$.
(3) If $\mathcal{T}$ contains all the factors T_k and $2 \le i, j \le 2n - 2$ are even, we set $\rho(M) = n$; otherwise we set $\rho(M) = 0$.

We denote by $\mathcal{S}$ the ordered product of all T_k, $1 \le k \le n - 2$, an element of degree $n^2 - 2n$.

Proposition 4.14. *The image of π_n equals the kernel of ρ. Moreover,*

$$\tilde{A}_n[n^2] = \operatorname{im} \pi_n \oplus F\mathcal{S}X^2Y^{2n-2}.$$

Proof. First we prove that the image of π_n is contained in the kernel of ρ.

For this take any monomial $A = \mathcal{T} X^i Y^j \in \tilde{A}_n[n^2 - 2n + 1]$ and consider $A\bar{O}_n$.

i) Firstly, if $\mathcal{T}$ misses at least 3 of the elements T_i then all the terms in $A\bar{O}_n$ miss at least 2 of the elements T_i, thus ρ is 0 on all terms.

ii) Assume $\mathcal{T}$ misses two elements T_h, T_k. The terms $A\, n(X^{2n-1} - Y^{2n-1})$ in $A\bar{O}_n$ then miss at least 2 of the elements T_i, so ρ is 0 on these terms. The remaining nonzero terms come from $B := -[A(X^{2(n-1-h)} - Y^{2(n-1-h)}) \wedge T_h + A(X^{2(n-1-k)} - Y^{2(n-1-k)}) \wedge T_k]$. Observe first that $\sum_{i=1}^{n-2} 2i + 1 = n^2 - 2n$ and so the degree of $\mathcal{T}$ is $n^2 - 2n - 2(h + k) - 2$. The degree of A is $n^2 - 2n + 1$, so that $i + j = 2h + 2k + 3$. We may assume $h > k$, $i \ge j$.

If $i+2(n-1-h) < 2n$ (and hence $j+2(n-1-h) < 2n$), then $-A(X^{2(n-1-h)} - Y^{2(n-1-h)}) \wedge T_h$ is the difference of two monomials on which ρ attains the same value, so on this expression ρ vanishes, same for k.

If $i + 2(n - 1 - h) \ge 2n$, i.e., $i \ge 2h + 2$, and hence $j \le 2k + 1 < 2h + 1$, we have

$$\begin{aligned} B &= AY^{2(n-1-h)} \wedge T_h + AY^{2(n-1-k)} \wedge T_k = A \wedge T_h Y^{2(n-1-h)} + A \wedge T_k Y^{2(n-1-k)} \\ &= (-1)^{i+j}(\mathcal{T} \wedge T_h X^i Y^{2(n-1-h)+j} + \mathcal{T} \wedge T_k X^i Y^{2(n-1-k)+j}). \end{aligned}$$

When we place $\mathcal{T} \wedge T_h$ in the increasing order we multiply by $(-1)^u$ where u is the number of factors of index $> h$. Since we have $n-2$ factors, $(-1)^u = (-1)^{n-2-h}$ and the value of ρ on the first term is $(-1)^{i+j}(-1)^u(-1)^{k+n} = (-1)^{i+j+h+k}$. For the second term the number of terms we have to exchange is the number of terms of index bigger than k minus 1 so we get the sign $-(-1)^{i+j+h+k}$ and the two terms cancel.

iii) Assume $\mathcal{T}$ misses only one element T_h. In this case the degree of $\mathcal{T}$ is $n^2-2n-2h-1$, thus $i+j = 2h+2$. The two terms $A\,n(X^{2n-1} - Y^{2n-1})$ are 0 unless either $i = 0$ or $j = 0$, since we are assuming $i \geq j$ this implies $j = 0, i = 2h+2$ and

$$A\,n(X^{2n-1} - Y^{2n-1}) = -n\mathcal{T}X^{2h+2}Y^{2n-1}. \tag{11}$$

The other contribution to the product is $-A(X^{2(n-1-h)} - Y^{2(n-1-h)}) \wedge T_h$.

If we have $2(n-1-h)+i < 2n$, then on this contribution ρ vanishes. In this case $i \neq 2h+2$, so the contribution (11) does not appear.

If $2(n-1-h)+i \geq 2n$, i.e. $i \geq 2h+2$, we have $i = 2h+2, j = 0$. The product $A\bar{O}_n$ is 0 unless $2h+2 < 2n$, in this case it equals

$$-n\mathcal{T}X^{2h+2}Y^{2n-1} + (-1)^{n-h}\mathcal{S}X^{2h+2}Y^{2(n-1-h)}, \tag{12}$$

where as before we have used $\mathcal{T} \wedge T_h = (-1)^{n-h}\mathcal{S}$. By definition the value of ρ on $\mathcal{T}X^{2h+2}Y^{2n-1}$ is $(-1)^{n+h}$, as for $\mathcal{S}X^{2h+2}Y^{2(n-1-h)}$ we have $2n+2 \geq 2$ and also $2(n-1-h) \geq 2$, thus the value of ρ on it equals n. The value on the sum is therefore $-n(-1)^{n+h} + (-1)^{n-h}n = 0$.

iv) Finally we consider the case in which $\mathcal{T}$ is the product of all the T_i's. In this case $i+j = 1$, and since we assume $i \geq j$ we have $i = 1, j = 0$. The only possible terms in the product are in $A\,n(X^{2n-1} - Y^{2n-1}) = -n\mathcal{T}XY^{2n-1}$. By definition ρ is 0 on this term.

We now want to prove that the image of π_n coincides with the kernel of ρ. For this we have to show that the image of π_n has codimension 1. It is enough to show that adding a single vector to the image of π_n we obtain the entire space. We define V to be the space spanned by $\mathcal{S}X^2Y^{2n-2}$ and $\mathrm{im}(\pi_n)$. We want to show that $V = \tilde{A}_n[n^2]$. In the case iv) we have already seen that $\mathcal{S}XY^{2n-1}, \mathcal{S}X^{2n-1}Y$ belong to the image of π_n.

Claim 1. *For every h we have $\mathcal{S}X^{2h+1}Y^{2(n-h)-1} \in \mathrm{im}(\pi_n)$, $\mathcal{S}X^{2h}Y^{2(n-h)} \in V$.*

To prove this claim, consider T_h of degree $2h+1$. We may remove T_h from $\mathcal{S}$ obtaining a product $\mathcal{S}^{(h)}$ and take the element

$$A := \mathcal{S}^{(h)}X^{2h+1}Y \in \tilde{A}_n[n^2-2n+1].$$

We have

$$A\bar{O}_n = \pm\mathcal{S}X^{2h+1}Y(X^{2(n-h-1)} - Y^{2(n-h-1)}) = \pm(\mathcal{S}X^{2n-1}Y - \mathcal{S}X^{2h+1}Y^{2n-2h-1}).$$

Since $\mathcal{S}X^{2n-1}Y \in \mathrm{im}(\pi_n)$ we deduce $\mathcal{S}X^{2h+1}Y^{2n-2h-1} \in \mathrm{im}(\pi_n)$.

For the other case consider $A := \mathcal{S}^{(n-h)}X^2Y^{2(n-h)} \in \tilde{A}_n[n^2-2n+1]$. Then

$$A\bar{O}_n = \pm\mathcal{S}X^2Y^{2(n-h)}(X^{2(h-1)} - Y^{2(h-1)}) = \pm(\mathcal{S}X^{2h}Y^{2(n-h)} - \mathcal{S}X^2Y^{2n-2}).$$

Since $\mathcal{S}X^2Y^{2n-2} \in V$ we have $\mathcal{S}X^{2h}Y^{2(n-h)} \in V$.

Claim 2. *If $\mathcal{T}X^iY^j \in V$, also $\mathcal{T}X^jY^i \in V$.*

By definition $\mathrm{im}(\pi_n)$ is invariant under the exchange of X, Y, while V is obtained from $\mathrm{im}(\pi_n)$ by adding $\mathcal{S}X^2Y^{2n-2}$, but by Claim 1 we also have $\mathcal{S}X^{2n-2}Y^2 \in V$. This proves Claim 2.

Claim 3. *All monomials $\mathcal{T}X^iY^j \in \tilde{A}_n[n^2]$, where $\mathcal{T}$ misses one element T_h, are in V.*

We must have $i+j = 2(n+h)+1$. Apply (12) and Claim 1 to deduce that $\mathcal{T}X^{2h+2}Y^{2n-1} \in V$. By Claim 2 also $\mathcal{T}X^{2n-1}Y^{2h+2}$ belongs to V. We may assume $i \geq j$ by Claim 2. It thus suffices to consider only the case $i > 2h+2$. Note that in the case $h = n-2$, we have $i+j = 4n-3$, thus $i = 2n-1, j = 2n-2$, so in this case the previous argument establishes the claim.

Consider now the case $h < n-2$. We first consider the case $i = 2n-2$. (Note that the case $i = 2n-1$ has been considered above.) Take $A := \mathcal{T}^{(n-2)}X^{2n-2}Y^{2h+1}$, where $\mathcal{T}^{(n-2)}$ denotes the element obtained from $\mathcal{T}$ by removing T_{n-2}. Then

$$A\bar{O}_n = \pm\mathcal{T}X^{2n-2}Y^{2h+3} \pm S^{(n-2)}X^{2n-2}Y^{2n-1}.$$

As $S^{(n-2)}X^{2n-2}Y^{2n-1}$ has already been proven to belong to V, $TX^{2n-2}Y^{2h+3} \in V$. We now prove by the decreasing induction that $\mathcal{T}X^iY^{2(n+h)+1-i}$ lies in V for $i > 2h+2$. Take $A := \mathcal{T}^{(n-2)}X^iY^{2h+2n-i-1} \in \tilde{A}_n[n^2-2n+1]$. We have

$$\begin{aligned} A\bar{O}_n =& \mathcal{T}^{(n-2)}X^iY^{2h+2n-i-1}\bar{O}_n \\ =& \mathcal{T}^{(n-2)}X^iY^{2h+2n-i-1}[-(X^2-Y^2)\wedge T_{n-2} - (X^{2(n-1-h)} - Y^{2(n-1-h)})\wedge T_h] \\ =& \pm\,\mathcal{T}(X^{i+2}Y^{2h+2n-i-1} - X^iY^{2h+2n-i+1}) \in V \end{aligned}$$

in case $i < 2n-2$. Since by the induction hypothesis $\mathcal{T}X^{i+2}Y^{2h+2n-i-1} \in V$, it follows that $\mathcal{T}X^iY^{2h+2n-i+1} \in V$, and we have the desired result.

Claim 4. *All monomials $\mathcal{T}X^iY^j \in \tilde{A}_n[n^2]$, where $\mathcal{T}$ miss $m \geq 2$ elements T_h, are in V.*

Assume that $\mathcal{T}$ misses elements $T_{h_1}, \dots, T_{h_m}$. Let us denote $s = \sum_{i=1}^m (2h_i+1)$. We first show that $\mathcal{T}X^{2n-1}Y^{s+1} \in \mathrm{im}(\pi_n)$,$\mathcal{T}X^{2n-2}Y^{s+2} \in \mathrm{im}(\pi_n)$. Since $m \geq 2$ and $s \leq 2(2n-1)-2n = 2n-2$, $\mathcal{T}$ cannot miss $\mathcal{T}_{n-2}$. Denote by $\mathcal{T}^{(n-2)}$ the element obtained from $\mathcal{T}$ by removing T_{n-2}. As all monomials in $\mathcal{T}^{(n-2)}\bar{O}_n$ miss at least two elements T_{k_1}, T_{k_2}, they cannot miss T_{n-2} by the previous argument, thus we have $\mathcal{T}^{(n-2)}X^{2n-1}Y^{s-1}\bar{O}_n = \pm\mathcal{T}X^{2n-1}Y^{s+1}$. The same argument shows that $\mathcal{T}X^{2n-2}Y^{s+2} \in \mathrm{im}(\pi_n)$. Arguing by the decreasing induction we may assume that $\mathcal{T}X^{2n-k+2}Y^{s+k-2} \in \mathrm{im}(\pi_n)$. We have

$$\mathcal{T}^{(n-2)}X^{2n-k}Y^{s+k-2}\bar{O}_n = \pm(\mathcal{T}X^{2n-k+2}Y^{s+k-2} - \mathcal{T}X^{2n-k}Y^{s+k}) \in \mathrm{im}(\pi_n).$$

By the induction hypothesis $\mathcal{T}X^{2n-k+2}Y^{s+k-2} \in \mathrm{im}(\pi_n)$ and thus $\mathcal{T}X^{2n-k}Y^{s+k} \in \mathrm{im}(\pi_n)$. □

Proof of Theorem 4.7. Note that $\mathcal{S}X^2Y^{2n-2}$, which is not in the image of π_n by Proposition 4.14, is a generator of the 1-dimensional space $(N_n \otimes \bigwedge^{n^2-2} N_n^* X^2)^G \subset (N_n \otimes \mathbb{F}_n[n^2])^G$. The decomposition of $A_n[n^2]$ from Proposition 4.14 thus induces the decomposition $\mathbb{G}_n[n^2] = \mathbb{G}_n[n^2]_{CH} \oplus \bigwedge^{n^2-2} N_n^* X^2$. □

5. Quasi-identities that follow from the Cayley-Hamilton identity

In this final section we collect several results on quasi-identities of matrices and the Cayley-Hamilton identity, and give a positive solution for the Specht problem for quasi-identities of matrices.

5.1. **Quasi-identities and local linear dependence.** Let $\mathfrak{R}$ be an F-algebra. Noncommutative polynomials $f_1, \ldots, f_t \in F\langle x_1, \ldots, x_m\rangle$ are said to be $\mathfrak{R}$-*locally linearly dependent* if the elements $f_1(r_1, \ldots, r_m), \ldots, f_t(r_1, \ldots, r_m)$ are linearly dependent in $\mathfrak{R}$ for all $r_1, \ldots, r_m \in \mathfrak{R}$. This concept has actually appeared in Operator Theory [14], and was recently studied from the algebraic point of view in [12]. We will see that it can be used in the study of quasi-identities.

Recall that C_m stands for the Capelli polynomial. The following well-known result (see, e.g., [33, Theorem 7.6.16]) was used in [12] as a basic tool.

Theorem 5.1. *Let $\mathfrak{R}$ be a prime algebra. Then $a_1, \ldots, a_t \in \mathfrak{R}$ are linearly dependent over the extended centroid of $\mathfrak{R}$ if and only if $C_{2t-1}(a_1, ..., a_t, r_1, ..., r_{t-1}) = 0$ for all $r_1, \ldots, r_{t-1} \in \mathfrak{R}$.*

By using a similar approach as in the proof of [12, Theorem 3.1], just by applying Theorem 5.1 to the algebra of generic matrices instead of to the free algebra $F\langle X\rangle$, we get the following characterization of M_n-local linear dependence through the central polynomials (cf. §2.1.3).

Theorem 5.2. *Noncommutative polynomials $f_1, \ldots, f_t$ are M_n-locally linearly dependent if and only if there exist central polynomials $c_1, \ldots, c_t$, not all polynomial identities, such that $\sum_{i=1}^t c_i f_i$ is a polynomial identity of M_n.*

Proof. By Theorem 5.1, the condition that $f_1, \ldots, f_t$ are M_n-locally linearly dependent is equivalent to the condition that

$$H := C_{2t-1}(f_1, \ldots, f_t, y_1, \ldots, y_{t-1})$$

is a polynomial identity of M_n. Since M_n and the algebra $F\langle \xi_k\rangle$ of $n \times n$ generic matrices satisfy the same polynomial identities, this is the same as saying that H is a polynomial identity of $F\langle \xi_k\rangle$. Using Theorem 5.1 once again we see that this is further equivalent to the condition that $f_1, \ldots, f_t$, viewed as elements of $F\langle \xi_k\rangle$, are linearly dependent over the extended centroid of $F\langle \xi_k\rangle$. Since $F\langle \xi_k\rangle$ is a prime PI-algebra, its extended centroid is the field of fractions of the center of $F\langle \xi_k\rangle$; the latter can be identified with central polynomials, and hence the desired conclusion follows. □

Corollary 5.3. *If noncommutative polynomials $f_0, f_1, \ldots, f_t$ are M_n-locally linearly dependent, while $f_1, \ldots, f_t$ are M_n-locally linearly independent, then there exist central polynomials $c_0, c_1, \ldots, c_t$, such that c_0 is nontrivial and $\sum_{i=0}^t c_i f_i$ is a polynomial identity of M_n.*

Later, in Remark 5.6, we will show that this result can be used to obtain an alternative proof of a somewhat weaker version of Theorem 3.7.

Lemma 5.4. *If a quasi-polynomial $\sum_{i=1}^t \lambda_i M_i$ is a quasi-identity of M_n, then either each $\lambda_i = 0$ or $M_1, \ldots, M_t$ are M_n-locally linearly dependent (and hence satisfy the conclusion of Theorem 5.2).*

Proof. We may assume that $\lambda_i = \lambda_i(x_1, \ldots, x_m)$ and $M_i = M_i(x_1, \ldots, x_m)$. The set W of all m-tuples $(A_1, \ldots, A_m) \in M_n^m$ such that $\lambda_i(A_1, \ldots, A_m) = 0$ for all $i = 1, \ldots, t$ is closed in the Zariski topology of $F^{n^2 m}$. Similarly, the set Z of all m-tuples $(A_1, \ldots, A_m) \in M_n^m$ such that $M_1(A_1, \ldots, A_m), \ldots, M_t(A_1, \ldots, A_m)$ are linearly dependent is also closed - namely, the linear dependence can be expressed through zeros of a polynomial by Theorem 5.1. If $Z = M_n^m$, then $M_1, \ldots, M_t$ are M_n-locally linearly dependent. Assume therefore that $Z \neq M_n^m$. Suppose that $W \neq M_n^m$. Then, since $F^{n^2 m}$ is irreducible (as $\operatorname{char}(F) = 0$), the complements of W

and Z in M_n^m have a nonempty intersection. This means that there exist $A_1, \dots, A_m \in M_n^m$ such that $\lambda_i(A_1, \dots, A_m) \neq 0$ for some i and $M_1(A_1, \dots, A_m), \dots, M_t(A_1, \dots, A_m)$ are linearly independent. However, this is impossible since $\sum_{i=1}^t \lambda_i M_i$ is a quasi-identity. Thus, $W = M_n^m$, i.e., each $\lambda_i = 0$. $\square$

We conclude this subsection with a theorem giving a condition under which a quasi-identity is a consequence of the Cayley-Hamilton identity.

Theorem 5.5. *Let $P = \lambda_0 M_0 + \sum_{i=1}^t \lambda_i M_i \in \mathfrak{I}_n$. If $M_1, \dots, M_t$ are M_n-locally linearly independent, then P is a consequence of the Cayley-Hamilton identity.*

Proof. We may assume that some $\lambda_i \neq 0$, and so $M_0, M_1, \dots, M_t$ are M_n-locally linearly dependent by Lemma 5.4. Theorem 5.2 tells us that there exist central polynomials c_0, c_i, not all trivial, such that $c_0 M_0 + \sum_{i=1}^t c_i M_i$ is a polynomial identity. Multiplying this identity with λ_0 and then comparing it with the quasi-identity $c_0 P$ it follows that $\sum_{i=1}^t (c_0\lambda_i - c_i\lambda_0) M_i \in \mathfrak{I}_n$. Lemma 5.4 implies that $c_0\lambda_i = c_i\lambda_0$ for every i. Let us write $\lambda_i = \lambda_i^G \lambda_i'$ where λ_i^G is the product of all irreducible factors of λ_i that are invariant under the conjugation by $G = GL_n(F)$, and λ_i' is the product of the remaining irreducible factors of λ_i. Hence $c_i \lambda_0^G \lambda_0' = c_0 \lambda_i^G \lambda_i'$ and therefore $\lambda_0' = \lambda_i'$ for every i. We thus have

$$P = \lambda_0'\big(\lambda_0^G M + \sum_{i=1}^t \lambda_i^G M_i\big).$$

Since λ_i^G are invariant under G, they are trace polynomials. Therefore the desired conclusion follows from the SFT. $\square$

Remark 5.6. Theorem 3.7 in particular tells us that for every quasi-identity P of M_n there exists a nontrivial central polynomial c with zero constant term such that cP is a consequence of the Cayley-Hamilton identity. Let us give an alternative proof of that, based on local linear dependence and the SFT.

We first remark that the condition that $c \in F\langle X\rangle$ is a central polynomial can be expressed as that there exists $\alpha_c \in \mathcal{C}$ such that $c - \alpha_c \in \mathfrak{I}_n$. Actually, $c - \alpha_c$ is a trace identity since $\alpha_c = \frac{1}{n}\mathrm{tr}(c)$. Therefore the SFT implies that for every central polynomial c of M_n there exists $\alpha_c \in \mathcal{C}$ such that $c - \alpha_c$ is a quasi-identity of M_n contained in the T-ideal of $\mathcal{C}\big\langle X\big\rangle$ generated by Q_n.

Now take an arbitary $P \in \mathfrak{I}_n$, and let us prove that c with the aforementioned property exists. Write $P = \sum_{i=1}^{n^2} \lambda_i x_i + \sum \lambda_M M$ where each M in the second summation is different from $x_1, \dots, x_{n^2}$. We proceed by induction on the number of summands d in the second summation. If $d = 0$, then $P = 0$ by Lemma 5.4 (since $x_1, \dots, x_{n^2}$ are M_n-locally linearly independent). Let $d > 0$. Pick M_0 such that $M_0 \notin \{x_1, \dots, x_{n^2}\}$ and $\lambda_{M_0} \neq 0$. Note that $M_0, x_1, \dots, x_{n^2}$ are M_n-locally linearly dependent, while $x_1, \dots, x_{n^2}$ are M_n-locally linearly independent. Thus, by Corollary 5.3 there exist central polynomials $c_0, c_1, \dots, c_{n^2}$ such that c_0 is nontrivial and $f := c_0 M_0 + \sum_{i=1}^{n^2} c_i x_i$ is an identity of M_n. Let $\alpha_i \in \mathcal{C}$, $i = 0, 1, \dots, n^2$, be such that $c_i - \alpha_i$ is a quasi-identity lying in the T-ideal generated by Q_n. Let us define $P' := \alpha_0 P - \lambda_{M_0}\alpha_0 M_0 - \lambda_{M_0}\sum_{i=1}^{n^2} \alpha_i x_i$. Writing each α_i as $c_i - (c_i - \alpha_i)$ we see that P' is a quasi-identity. Note that P' involves $d-1$ summands not lying in $\mathcal{C}x_i$, $i = 1, \dots, n^2$. Therefore the induction assumption yields the existence of a nonzero central polynomial c' such that $c'P'$ lies in the T-ideal generated by Q_n. Setting $c = c_0c'$ we thus have $c \neq 0$ and

$$\begin{aligned} cP =&(c_0-\alpha_0)c'P + \alpha_0 c'P = (c_0-\alpha_0)c'P + c'P' + \lambda_{M_0}c'\big(\alpha_0 M_0 + \sum_{i=1}^{n^2}\alpha_i x_i\big) \\ =&(c_0-\alpha_0)c'P + c'P' - \lambda_{M_0}c'\big((c_0-\alpha_0)M_0 + \sum_{i=1}^{n^2}(c_i-\alpha_i)x_i\big) + \lambda_{M_0}c'f. \end{aligned}$$

The T-ideal generated by Q_n contains $c_i - \alpha_i$, $0 \le i \le n^2$, $c'P'$, as well as f according to the SFT. Hence it also contains cP.

5.2. **Some special cases.** The purpose of this subsection is to examine several simple situations, which should in particular serve as an evidence of the delicacy of the problem of finding quasi-identities that are not a consequence of the Cayley-Hamilton identity.

We begin with quasi-polynomials $p = p(x)$ of one variable, i.e., $p(x) = \sum_{i=0}^m \lambda_i(x)x^i$. Here we could refer to results on more general functional identities of one variable in [13], but an independent treatment is very simple.

Proposition 5.7. *If a quasi-polynomial of one variable $p(x)$ is a quasi-identity of M_n, then there exists a quasi-polynomial $r(x)$ such that $p(x) = r(x)q_n(x)$.*

Proof. Let $p(x) = \sum_{i=0}^m \lambda_i(x)x^i$. The proof is by induction on m. Since $1, x, \dots, x^{n-1}$ are M_n-locally linearly independent, we may assume that $m \ge n$ by Lemma 5.4. Note that $p(x) - \lambda_m(x)x^{m-n}q_n(x)$ is a quasi-identity for which the induction assumption is applicable. Therefore $p(x) - \lambda_m(x)x^{m-n}q_n(x) = r_1(x)q_n(x)$ for some $r_1(x)$, and hence $p(x) = \big(\lambda_m(x)x^{m-n} + r_1(x)\big)q_n(x)$. □

What about quasi-polynomials of two variables? At least for 2×2 matrices, the answer comes easily.

Proposition 5.8. *If a quasi-polynomial of two variables $P = P(x, y)$ is a quasi-identity of M_2, then P is a consequence of the Cayley-Hamilton identity.*

Proof. It is an easy exercise to show that any quasi-polynomial of two variables $P = P(x, y)$ can be written as $P = \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy + R$ where R lies in the T-ideal generated by Q_2. Thus, if P is a quasi-identity, then each $\lambda_i = 0$ by Lemma 5.4, so that $P = R$ is a consequence of the Cayley-Hamilton identity. □

Now one may wonder what should be the degree of a quasi-identity that may not follow from the Cayley-Hamilton identity. We first record a simple result which is a byproduct of the general theory of functional identities.

Proposition 5.9. *If $\sum \lambda_M M$ is a quasi-identity of M_n and $\deg(\lambda_M) + \deg(M) < n$ for every M, then each $\lambda_M = 0$.*

Proof. Apply, for example, [11, Corollary 2.23, Lemma 4.4]. □

The question arises what can be said about quasi-identities of degree n. The multilinearization process works for the quasi-polynomials just as it works for the ordinary noncommutative polynomials. The multilinear quasi-polynomials therefore deserve a special attention. By saying that $P = P(x_1, \dots, x_n)$ is *multilinear* of degree n we mean, of course, that P consists of summands of the form $\lambda(x_{i_1}, \dots, x_{i_k})x_{i_{k+1}} \cdots x_{i_n}$ where $\{1, \dots, n\}$ is the disjoint union of $\{i_1, \dots, i_k\}$ and $\{i_{k+1}, \dots, i_m\}$, and $\lambda(x_{i_1}, \dots, x_{i_k}) \in \mathcal{C}$ is multilinear, i.e., it is a linear combination of monomials of the form $x^{(i_1)}_{s_1t_1} x^{(i_2)}_{s_2t_2} \cdots x^{(i_k)}_{s_kt_k}$. A basic example is the Cayley-Hamilton polynomial Q_n.

Theorem 5.10. *Every multilinear quasi-identity of M_n of degree n is a scalar multiple of Q_n.*

Proof. Let $S_{n,k}$, $1 \le k \le n$, denote the set of all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$. For convenience we also set $S_{n,0} = S_n$. Note that a multilinear quasi-polynomial P of degree n can be written as

$$P(x_1, \dots, x_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_{n,k}} \lambda_{k\sigma}(x_{\sigma(1)}, \dots, x_{\sigma(k)}) x_{\sigma(k+1)} \cdots x_{\sigma(n)}$$

(here, $\lambda_{0\sigma}$ are scalars). By e_{ij} we denote the matrix units in M_n.

We assume that P is a quasi-identity, and proceed by a series of claims.

Claim 1. *For all $\sigma \in S_{n,k}$, $1 \le k \le n$, and all distinct $1 \le i_1, \dots, i_k \le n$, we have*

$$\lambda_{k\sigma}(e_{11}, \dots, e_{kk}) = \lambda_{k\sigma}(e_{i_1i_1}, \dots, e_{i_ki_k}) = -\lambda_{k-1,\sigma}(e_{11}, \dots, e_{k-1,k-1}).$$

The proof is by induction on k. First, take $0 \le j \le n-1$ and substitute

$$e_{1+j,1+j}, e_{1+j,2+j}, e_{2+j,3+j}, \dots, e_{n-1+j,n+j}$$

(with addition modulo n) for $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ in P. Considering the coefficient at $e_{1+j,n+j}$ we get $\lambda_{0\sigma} + \lambda_{1\sigma}(e_{1+j,1+j}) = 0$. This implies the truth of Claim 1 for $k = 1$. Let $k > 1$ and take $\sigma \in S_{n,k}$. Choose a subset of $\{1, \dots, n\}$ with $k-1$ elements, $\{i_{n-k+2}, \dots, i_n\}$, and let $\{j_1, \dots, j_{n-k+1}\}$ be its complement. Let us substitute

$$e_{i_{n-k+2},i_{n-k+2}}, \dots, e_{i_n,i_n},\ e_{j_1,j_1}, e_{j_1,j_2}, e_{j_2,j_3}, \dots, e_{j_{n-k},j_{n-k+1}}$$

for $x_{\sigma(1)}, \dots, x_{\sigma(n)}$, respectively, in P. Similarly as above, this time by considering the coefficient at $e_{j_1,j_{n-k+1}}$, we obtain

$$\lambda_{k-1,\sigma}(e_{i_{n-k+2},i_{n-k+2}}, \dots, e_{i_n,i_n}) + \lambda_{k\sigma}(e_{i_{n-k+2},i_{n-k+2}}, \dots, e_{i_n,i_n}, e_{j_1,j_1}) = 0.$$

The desired conclusion follows from the induction hypothesis.

Claim 2. *For all $\sigma, \tau \in S_{n,k}$, $0 \le k \le n-1$, and all distinct $1 \le i_1, \dots, i_k \le n$, we have*

$$\lambda_{k\sigma}(e_{11}, \dots, e_{kk}) = \lambda_{k\tau}(e_{i_1i_1}, \dots, e_{i_ki_k}).$$

Evaluating P at $e_{11}, \dots, e_{nn}$ results in

$$\lambda_{n-1,\sigma_i}(e_{11}, \dots, e_{i-1,i-1}, e_{i+1,i+1}, \dots, e_{nn}) = \lambda_{n-1,\mathrm{id}}(e_{11}, \dots, e_{n-1,n-1})$$

for all $1 \le i \le n-1$, where σ_i stands for the cycle $(i\ i+1\ \dots\ n)$. Accordingly, since $S_{n,n-1}$ consists of id and all σ_i, $1 \le i \le n-1$, the case $k = n-1$ follows from Claim 1. We may now assume that $k < n-1$ and that Claim 2 holds for $k+1$. Take $\sigma \in S_{n,k}$. If $\sigma \in S_{n,k+1}$ then

$$\lambda_{k\sigma}(e_{11}, \dots, e_{kk}) = -\lambda_{k+1,\sigma}(e_{11}, \dots, e_{k+1,k+1})$$

by Claim 1. If $\sigma \notin S_{n,k+1}$ there exists $1 \le i \le k$ such that $\sigma(k+1) < \sigma(i)$. Substituting

$$e_{11}, \ldots, e_{kk}, e_{k+1,k+1}, e_{k+1,k+2}, e_{k+2,k+3}, \ldots, e_{n-1,n}$$

for $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ in P we infer that for a certain permutation τ (specifically, $\tau = \sigma \circ (k+1\ k\ \ldots\ i+1\ i)$) we have

$$\lambda_{k\sigma}(e_{11}, \ldots, e_{kk}) = -\lambda_{k+1,\tau}(e_{11}, \ldots, e_{i-1,i-1}, e_{k+1,k+1}, e_{i+1,i+1}, \ldots, e_{k-1,k-1}).$$

Since every $\lambda_{k\sigma}(e_{11}, \ldots, e_{kk})$ is associated to an evaluation of $\lambda_{k+1,\tau}$, Claim 2 follows by the induction hypothesis and Claim 1.

Claim 3. $P = \lambda_{0,id} Q_n$.

By Claim 2 we have $\lambda_{0\sigma} = \lambda_{0\tau}$ for all $\sigma, \tau \in S_n$. Accordingly, $R := P - \lambda_{0,id} Q_n$ does not involve summands of the form $\mu x_{\sigma(1)} \cdots x_{\sigma(n)}$, $\mu \in F$, and can be therefore written as

$$R(x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{\sigma \in S_{n,k}} \mu_{k\sigma}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) x_{\sigma(k+1)} \cdots x_{\sigma(n)}.$$

We must prove that $R = 0$, i.e., each $\mu_{k\sigma} = 0$. We proceed by induction on k. For $k = 0$ this holds by the hypothesis, so let $k > 0$. It suffices to show that $\mu_{k\sigma}(e_{i_1 j_1}, \ldots, e_{i_k j_k}) = 0$ for arbitrary matrix units $e_{i_1 j_1}, \ldots, e_{i_k j_k}$. Choose distinct $l_1, \ldots, l_{n-k}$ such that $l_s \ne i_t$ for all s, t. Substitute

$$e_{i_1 j_1}, \ldots, e_{i_k j_k}, e_{l_1 l_2}, e_{l_2 l_3}, \ldots, e_{l_{n-k} i_1}$$

for $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ in P. There is only one way to factorize $e_{l_1 i_1}$ as a product of at most $n-k$ chosen matrix units, i.e., $e_{l_1 i_1} = e_{l_1 l_2} e_{l_2 l_3} \cdots e_{l_{n-k} i_1}$. By induction hypothesis it thus follows that $\mu_{k\sigma}(e_{i_1 j_1}, \ldots, e_{i_k j_k}) = 0$. □

5.3. **Specht problem for quasi-identities.** In this paragraph we finally prove a version of Specht problem for quasi-identities, i.e., we show that $\mathfrak{I}_n$ is finitely generated as a T-ideal. As it is well-known, Kemer [22] has shown that for polynomial identities such a question has a positive answer (in characteristic 0). In our case the answer is also positive since we can apply the classical method of *primary covariants* of Capelli and Deruyts (dressed up as Cauchy formula and highest weights), cf. [27, Chapter 3] to which we also refer for the statements used in the proof.

Theorem 5.11. *The ideal $\mathfrak{I}_n$ of all quasi-identities of M_n is finitely generated, as a T-ideal, by elements which depend on at most $2n^2$ variables.*

Proof. First of all, if we impose the Cayley-Hamilton identity, we are reduced to study the problem for the space $\mathcal{C}\langle X\rangle/(Q_n)$ isomorphic to $\ker \pi \subset C_x \otimes_{\mathcal{T}_n} \mathcal{T}_n\langle \xi_k\rangle \subset M_n(C_x \otimes_{\mathcal{T}_n} C_y)$. Instead of considering all possible we consider only linear substitutions of variables, that is, we consider all these spaces as representations of the infinite linear group.

Now we use the language of symmetric algebras; the space C_x equals $S[M_n^* \otimes V]$ where $V = \oplus_{i=1}^{\infty} F e_i$ is an infinite dimensional vector space over which the infinite linear group G_∞ acts.

By Cauchy's formula we have

$$S[M_n^* \otimes V] = \oplus_\lambda S_\lambda(M_n^*) \otimes S_\lambda(V),$$

where λ runs over all partitions with at most n^2 columns and $S_\lambda(V)$ is the corresponding Schur functor. By representation theory the tensor product $S_\lambda(V)\otimes S_\mu(V)$ of two such representations is a sum of representations $S_\gamma(V)$ where γ runs over partitions with at most $2n^2$ columns.

Hence we have that $C_x \otimes_{\mathcal{T}_n} C_y$, which is a quotient of $C_x \otimes_F C_y$, is a sum of representations $S_\lambda(M_n^*)\otimes S_\mu(V)$ where μ has at most $2n^2$ columns. Now any representation $S_\gamma(V)$ is irreducible under G_∞ and generated by a highest weight vector. If γ has k columns such a highest weight vector on the other hand lies in $S_\gamma(\oplus_{i=1}^k Fe_i)$.

This means that under linear substitution of variables any element in $M_n(C_x \otimes_{\mathcal{T}_n} C_y)$, and hence also in $\ker \pi$, is obtained from elements depending on at most $2n^2$ variables.

Finally, if we restrict the number of variables to a finite number m, the corresponding space $\ker \pi_m$ is a finitely generated module over a finitely generated algebra, and the claim follows. □

Acknowledgement. The authors would like to thank Hanspeter Kraft for an enlightening conversation and for providing some computer assisted computations.

References

[1] M. Artin, On Azumaya algebras and finite-dimensional representations of rings, *J. Algebra* **11** (1969), 532–563.
[2] M. Auslander, O. Goldman, Maximal orders, *Trans. Amer. Math. Soc.* **97** (1960), 1–24.
[3] M. Auslander, O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.
[4] Yu. Bahturin, M. Brešar, Lie gradings on associative algebras, *J. Algebra* **321** (2009), 264–283.
[5] Yu. Bahturin, M. Brešar, I. Shestakov, Jordan gradings on associative algebras, *Algebras Repr. Th.* **14** (2011), 113–129.
[6] K. I. Beidar, M. Brešar, M. A. Chebotar, W. S. Martindale, On Herstein's Lie map conjectures, I, *Trans. Amer. Math. Soc.* **353** (2001), 4235–4260.
[7] K. I. Beidar, M. A. Chebotar, On functional identities and d-free subsets of rings II, *Comm. Algebra* **28** (2000), 3953–3972.
[8] K. I. Beidar, M. A. Chebotar, On Lie-admissible algebras whose commutator Lie algebras are Lie subalgebras of prime associative algebras, *J. Algebra* **233** (2000), 675-703.
[9] K. I. Beidar, Y. Fong, On additive isomorphisms of prime rings preserving polynomials, *J. Algebra* **217** (1999), 650-667.
[10] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525–546.
[11] M. Brešar, M. A. Chebotar, W. S. Martindale 3rd, *Functional identities*, Birkhäuser Verlag, 2007.
[12] M. Brešar, I. Klep, A local-global principle for linear dependence of noncommutative polynomials, *Israel J. Math.* **193** (2013), 71–82.
[13] M. Brešar, Š. Špenko, Functional identities in one variable, *J. Algebra* **401** (2014), 234–244.
[14] J. F. Camino, J. W. Helton, R. E. Skelton, J. Ye, Matrix inequalities: a symbolic procedure to determine convexity automatically, *Int. Eq. Oper. Th.* **46** (2003), 399–454.
[15] C. Chevalley, The Betti numbers of the exceptional Lie groups, in "Proc. International Congress of Mathematicians 1950", Vol. II, 21–24.
[16] C. De Concini, P. Papi, C. Procesi, The adjoint representation inside the exterior algebra of a simple Lie algebra, arXiv:1311.4338.
[17] V. Drensky, E. Formanek, *Polynomial identity rings*, Advanced Course in Methematics, CRM Barcelona, Birkhäuser Verlag, Basel (2004).
[18] E. B. Dynkin, Homologies of compact Lie groups, *Amer. Math. Soc. Transl.* **12** (1959), 251–300.
[19] E. Formanek, Central polynomials for matrix rings, *J. Algebra* **32** (1972), 129–132.

[20] E. Formanek, *The polynomial identities and invariants of $n \times n$ matrices*, Amer. Math. Soc., 1991.
[21] A. Kanel-Belov, S. Malev, L. H. Rowen, The images of non-commutative polynomials evaluated on 2×2 matrices, *Proc. Amer. Math. Soc.* **140** (2012), 465–478.
[22] A. R. Kemer, Finite basability of identities of associative algebras, *Algebra and Logic* **26** (1987), 362–397.
[23] B. Kostant, Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = End\, V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$, *Adv. Math.* **125** (1997), 275–350.
[24] L. Le Bruyn, C. Procesi, Etale local structure of matrix invariants and concomitants, *Algebraic groups Utrecht 1986*, Springer Lecture Notes 1271, 143–175 (1987).
[25] C. Procesi, The invariant theory of $n \times n$ matrices, *Adv. Math.* **19** (1976), 306–381.
[26] C. Procesi, A formal inverse to the Cayley-Hamilton theorem, *J. Algebra* **107** (1987), 63–74.
[27] C. Procesi, *Lie groups: An approach through invariants and representations*, Springer Universitext, 2007.
[28] C. Procesi, On the theorem of Amitsur–Levitzki, arXiv:1308.2421 (to appear in *Israel Journal of Mathematics*).
[29] Yu. P. Razmyslov, A certain problem of Kaplansky, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 483–501.
[30] Yu. P. Razmyslov, Identities with trace in full matrix algebras over a field of characteristic zero, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 723-756.
[31] M. Reeder, Exterior powers of the adjoint representation, *Canad. J. Math.* **49** (1997), 133–159.
[32] S. Rosset, A new proof of the Amitsur-Levitski identity, *Israel J. Math.* **23** (1976), 187–188.
[33] L. H. Rowen, *Polynomial identities in ring theory*, Academic Press, 1980.
[34] D. J. Saltman, *Lectures on division algebras*, Amer. Math. Soc., 1999.
[35] Š. Špenko, On the image of a noncommutative polynomial, *J. Algebra* **377** (2013), 298–311.

M. Brešar, Faculty of Mathematics and Physics, University of Ljubljana, and Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
E-mail address: matej.bresar@fmf.uni-lj.si

C. Procesi, Dipartimento di Matematica, Sapienza Università di Roma, Italy
E-mail address: procesi@mat.uniroma1.it

Š. Špenko, Institute of Mathematics, Physics, and Mechanics, Ljubljana, Slovenia
E-mail address: spela.spenko@imfm.si