CHARACTERIZING HOMOMORPHISMS, DERIVATIONS, AND MULTIPLIERS IN RINGS WITH IDEMPOTENTS

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Abstract. In certain rings containing noncentral idempotents we characterize homomorphisms, derivations, and multipliers by their actions on elements satisfying some special conditions. For example, we consider the condition that an additive map h between rings $\mathcal{A}$ and $\mathcal{B}$ satisfies $h(x)h(y)h(z) = 0$ whenever $x, y, z \in \mathcal{A}$ are such that $xy = yz = 0$. As an application we obtain some new results on local derivations and local multipliers. In particular, we prove that if $\mathcal{A}$ is a prime ring containing a nontrivial idempotent, then every local derivation from $\mathcal{A}$ into itself is a derivation.

1. Introduction

Let $\mathcal{A}$ be a ring and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that a *local derivation* is an additive map $d : \mathcal{A} \to \mathcal{M}$ such that for every $x \in \mathcal{A}$ there exists a derivation $d_x : \mathcal{A} \to \mathcal{M}$ such that $d(x) = d_x(x)$. The standard problem, initiated by Kadison [14] and Larson and Sourour [15], is to find conditions implying that a local derivation is actually a derivation. A number of papers dealing with this problem have been published [1, 2, 6, 8, 10, 13, 18, 19, 20, 21, 23, 24, 25, 26]. There is also a related, yet somewhat simpler problem on *local multipliers* (they are defined below), see for example [7, 9, 12, 13, 22].

It is quite common to study local derivations and multipliers in algebras that contain many idempotents, in a sense that the linear span of all idempotents is "large". The main novelty of this paper is that we shall deal with the subring generated by all idempotents instead of the span. This will make it possible for us to obtain a new type of results, giving definitive conclusions under very mild assumptions concerning the existence of idempotents. For example, we will show that every local derivation from a prime ring $\mathcal{A}$ into itself is a derivation provided that $\mathcal{A}$ contains only one nontrivial idempotent (Corollary 3.9); moreover, if $\mathcal{A}$ is either a simple ring with a nontrivial idempotent or a ring of $n \times n$ matrices, $n \geq 2$, over any unital ring, then every local derivation from $\mathcal{A}$ into an arbitrary $\mathcal{A}$-bimodule is a derivation (Corollary 3.8). The existence of nontrivial idempotents is

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certainly necessary in these results since there exist local derivations that are not derivations even on some fields [14].

The results on local derivations and multipliers will be actually obtained as corollaries to our main results in which we shall study maps satisfying certain more general conditions. These conditions also cover some other conditions that have been studied in the literature, notably those from the recent paper [3] by Chebotar, Ke and Lee. In fact, generalizing the results from [3] is another purpose of this paper.

In order to present these conditions we introduce and fix some notation. Throughout the paper, $\mathcal{A}$ and $\mathcal{B}$ will be rings, $\mathcal{M}$ will be an arbitrary $\mathcal{A}$-bimodule, and $\mathcal{L}$ will be an arbitary left $\mathcal{A}$-module. These modules will play an entirely formal role in this paper. The only assumption that we shall occasionaly need is that they are unital. In general we do not even require that $\mathcal{A}$ and $\mathcal{B}$ have unities, but in some of our results this will be necessary.

Let $h : \mathcal{A} \to \mathcal{B}$, $d : \mathcal{A} \to \mathcal{M}$ and $f : \mathcal{A} \to \mathcal{L}$ be additive maps. We shall consider the following conditions:

$$(h1)\quad xy = yz = 0 \Longrightarrow h(x)h(y)h(z) = 0 \quad \text{for all } x, y, z \in \mathcal{A};$$
$$(d1)\quad xy = yz = 0 \Longrightarrow xd(y)z = 0 \quad \text{for all } x, y, z \in \mathcal{A};$$
$$(h2)\quad xy = 0 \Longrightarrow h(x)h(y) = 0 \quad \text{for all } x, y \in \mathcal{A};$$
$$(d2)\quad xy = 0 \Longrightarrow d(x)y + xd(y) = 0 \quad \text{for all } x, y \in \mathcal{A};$$
$$(f2)\quad xy = 0 \Longrightarrow xf(y) = 0 \quad \text{for all } x, y \in \mathcal{A}.$$

Note that homomorphisms satisfy the conditions $(h1)$ and $(h2)$, derivations satisfy the conditions $(d1)$ and $(d2)$, and multipliers satisfy the condition $(f2)$. The obvious problem, which is the main issue of this paper, is to show that in appropriate settings the converses are true.

The condition $(h1)$ turns out to be the most general one, i. e. the other ones can be, as we shall see, reduced to this one. Maps satisfying the conditions $(h2)$ and $(d2)$, which are clearly special cases of maps satisfying $(h1)$ and $(d1)$, respectively, were studied in [3] as well as in many other papers. We refer to [3] and [4] for history and references.

There is a simple link between "local maps" and these conditions. If $d : \mathcal{A} \to \mathcal{M}$ is a local derivation, then for all $x, y, z \in \mathcal{A}$ we have

$$xd(y)z = xd_y(y)z = d_y(xy)z - d_y(x)yz,$$

and hence $xd(y)z = 0$ in case when $xy = yz = 0$. Thus, local derivations satisfy $(d1)$, and similarly we see that local multipliers satisfy $(f2)$. Unfortunately, local automorphisms, which have also been thoroughly studied in the literature, do not always satisfy either $(h1)$ or $(h2)$, so for them our results are not directly applicable.

The results that we shall state as theorems are of formal nature, they hold in arbitrary rings and their (bi)modules. In corollaries we will show what do these theorems yield for two special classes of rings: rings that are generated by idempotents, and prime rings with nontrivial idempotents. We shall conclude the paper with an example justifying the necessity of certain assumptions in our results on prime rings.

2. Remarks on rings with idempotents

We continue by introducing some notation which will be fixed throughout the paper. By $\mathcal{E}$ we denote the set of all idempotents in $\mathcal{A}$, by $\mathcal{R}$ we denote the subring of $\mathcal{A}$ generated by $\mathcal{E}$, and by $\mathcal{I}$ we denote the ideal of $\mathcal{A}$ generated by $[\mathcal{E},\mathcal{A}]$; here, $[\,.\,,\,.\,]$ denotes the commutator.

The main goal of this preliminary section is to state a simple lemma which is of some independent interest. In particular it shows that the existence of merely one noncentral idempotent in $\mathcal{A}$ implies that $\mathcal{R}$ contains a nonzero ideal. The lemma could be derived by following the arguments in Herstein's theory of Lie ideals [11], but we shall give a short direct proof instead. Nevertheless it should be mentioned that the idea of dealing with $[\mathcal{E},\mathcal{A}]$ is well-known, cf. [11, p. 19].

Lemma 2.1. *$\mathcal{I}\subseteq\mathcal{R}$.*

Proof. Let $e\in\mathcal{E}$ and $x\in\mathcal{A}$. Then the elements $e+ex-exe$ and $e+xe-exe$ are also idempotents, and so their difference, $r=[e,x]$, lies in $\mathcal{R}$. By a direct computation one can check that

$$zr=[e,z[e,r]]-[e,z][e,r],\ \ rw=[e,[e,r]w]-[e,r][e,w],$$
$$zrw=[e,z[e,r]w]-[e,z][e,rw]-[e,zr][e,w]+2[e,z]r[e,w]$$

for all $z,w\in\mathcal{A}$. Since $[e,\mathcal{A}]\subseteq\mathcal{R}$, this proves that $\mathcal{I}\subseteq\mathcal{R}$. □

We shall obtain particularly nice and definitive results for those rings $\mathcal{A}$ in which $\mathcal{R}=\mathcal{A}$. Lemma 2.1 makes it possible for us to list a few examples of such rings:

(A) $\mathcal{A}$ is a simple ring containing a nontrivial idempotent;
(B) $\mathcal{A}$ is a unital ring containing an idempotent e_0 such that the ideals generated by e_0 and $1-e_0$, respectively, are both equal to $\mathcal{A}$;
(C) $\mathcal{A}=M_n(\mathcal{B})$, the ring of all $n\times n$ matrices over any unital ring $\mathcal{B}$, where $n\geq 2$.

By a nontrivial idempotent we mean an idempotent different from 0 and 1. Since simple rings (as well as more general prime rings) cannot contain nontrivial central idempotents, the fact that rings of the type (A) satisfy $\mathcal{R}=\mathcal{A}$ follows directly from Lemma 2.1.

Now suppose that $\mathcal{A}$ satisfies the condition (B). Then in particular we have $\sum_i x_i(1-e_0)y_i=e_0$ for some $x_i,y_i\in\mathcal{A}$, and so $e_0=\sum_i[e_0,x_i](1-e_0)y_i\in\mathcal{I}$. Similarly we see that $1-e_0\in\mathcal{I}$. Consequently, $1\in\mathcal{I}$ and so $\mathcal{I}=\mathcal{R}=\mathcal{A}$.

Finally, if $\mathcal{A}$ is of the type (C), then it is easy to see that $\mathcal{A}$ satisfies (B). Indeed, one can take, for example, the matrix unit E_{11} for e_0.

3. Conditions $(h1)$ and $(d1)$

We shall first consider the condition $(h1)$. As we shall see, the results on the condition $(d1)$ will follow from those on the condition $(h1)$.

3.1. Condition $(h1)$.

The following theorem is of crucial importance for this paper; all other results will be, sometimes directly and sometimes by following the method, derived from this one.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings and let $h : \mathcal{A} \to \mathcal{B}$ be an additive map satisfying $(h1)$ and $h(1) = 1$. Then the restriction of h to $\mathcal{R}$ is a homomorphism. Moreover,*

$$h(rxs) + h(r)h(x)h(s) = h(rx)h(s) + h(r)h(xs) \quad \text{for all } r, s \in \mathcal{R},\ x \in \mathcal{A}. \tag{1}$$

Proof. Let $e, f \in \mathcal{E}$ and $x \in \mathcal{A}$. Using

$$\begin{aligned}
&(1-e) \cdot exf = exf \cdot (1-f) = 0,\\
&e \cdot (1-e)xf = (1-e)xf \cdot (1-f) = 0,\\
&(1-e) \cdot ex(1-f) = ex(1-f) \cdot f = 0,\\
&e \cdot (1-e)x(1-f) = (1-e)x(1-f) \cdot f = 0,
\end{aligned}$$

we infer from $(h1)$ and $h(1) = 1$ that

$$\begin{aligned}
&\big(1-h(e)\big) \cdot h(exf) \cdot \big(1-h(f)\big) = 0,\\
&h(e) \cdot h((1-e)xf) \cdot (1-h(f)) = 0,\\
&(1-h(e)) \cdot h(ex(1-f)) \cdot h(f) = 0,\\
&h(e) \cdot h((1-e)x(1-f)) \cdot h(f) = 0.
\end{aligned}$$

For convenience we rewrite these identities as follows:

$$\begin{aligned}
h(exf) &= h(e)h(exf) + h(exf)h(f) - h(e)h(exf)h(f),\\
-h(e)h(xf) &= -h(e)h(exf) - h(e)h(xf)h(f) + h(e)h(exf)h(f),\\
-h(ex)h(f) &= -h(e)h(ex)h(f) - h(exf)h(f) + h(e)h(exf)h(f),\\
h(e)h(x)h(f) &= h(e)h(ex)h(f) + h(e)h(xf)h(f) - h(e)h(exf)h(f).
\end{aligned}$$

Note that the sum of the right-hand sides of these four identities is 0. Therefore, the sum of the left-hand sides must be 0 too. This proves the following special case of (1):

$$h(exf) + h(e)h(x)h(f) = h(ex)h(f) + h(e)h(xf) \quad \text{for all } e, f \in \mathcal{E},\ x \in \mathcal{A}. \tag{2}$$

In particular, by setting $x = 1$ we see that $h(ef) = h(e)h(f)$ for all $e, f \in \mathcal{E}$. Moreover, since by (2) we have

$$\begin{aligned}
&h(e_1e_2\dots e_n)\\
=\ &h(e_1e_2\dots e_{n-1})h(e_n) + h(e_1)h(e_2\dots e_n) - h(e_1)h(e_2\dots e_{n-1})h(e_n)
\end{aligned}$$

it follows by induction on n that $h(e_1e_2\dots e_n) = h(e_1)h(e_2)\dots h(e_n)$ for all $e_1, e_2\dots, e_n \in \mathcal{E}$. This shows that h is a homomorphism on $\mathcal{R}$.

In order to prove (1) we introduce, for any $a \in \mathcal{A}$, the sets

$$\mathcal{S}_a = \{r \in \mathcal{R} \mid h(rxa) + h(r)h(x)h(a) = h(rx)h(a) + h(r)h(xa) \text{ for all } x \in \mathcal{A}\},$$
$$\mathcal{T}_a = \{s \in \mathcal{R} \mid h(axs) + h(a)h(x)h(s) = h(ax)h(s) + h(a)h(xs) \text{ for all } x \in \mathcal{A}\}.$$

Clearly, both $\mathcal{S}_a$ and $\mathcal{T}_a$ are additive subgroups of $\mathcal{R}$. Moreover, given $r, r' \in \mathcal{S}_a$ we have

$$h(rr'xa) = h(r(r'x)a) = h(rr'x)h(a) + h(r)h(r'xa) - h(r)h(r'x)h(a),$$

and hence, since

$$h(r'xa) = h(r'x)h(a) + h(r')h(xa) - h(r')h(x)h(a)$$

and $h(r)h(r') = h(rr')$,

$$h(rr'xa) = h(rr'x)h(a) + h(rr')h(xa) - h(rr')h(x)h(a).$$

This shows that $rr' \in \mathcal{S}_a$. That is, $\mathcal{S}_a$ is a subring of $\mathcal{R}$. Similarly, $\mathcal{T}_a$ is a subring. By (2) we have $\mathcal{E} \subseteq \mathcal{S}_f$ for every $f \in \mathcal{E}$. Since $\mathcal{S}_f$ is a subring it follows that $\mathcal{R} \subseteq \mathcal{S}_f$. That is to say, $f \in \mathcal{T}_r$ for every $f \in \mathcal{E}$ and every $r \in \mathcal{R}$, i.e. $\mathcal{E} \subseteq \mathcal{T}_r$, and hence $\mathcal{R} \subseteq \mathcal{T}_r$ since $\mathcal{T}_r$ is a subring. But this means that (1) holds true. $\square$

We remark that the first part of the proof leading to (2) is similar to Shulman's argument in [23, pp. 68-69], which, however, was used in a different context.

Corollary 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings and let $h : \mathcal{A} \to \mathcal{B}$ be an additive map satisfying $(h1)$ and $h(1) = 1$. If $\mathcal{R} = \mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)), then h is a homomorphism.*

Corollary 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings and let $h : \mathcal{A} \to \mathcal{B}$ be a bijective additive map satisfying $(h1)$ and $h(1) = 1$. If $\mathcal{A}$ is a prime ring containing a nontrivial idempotent, then h is an isomorphism.*

Proof. Since $\mathcal{I} \subseteq \mathcal{R}$ by Lemma 2.1, h is a homomorphism on $\mathcal{I}$ by Theorem 3.1. In particular, for all $u, v \in \mathcal{I}$ and $x \in \mathcal{A}$ we have $h(uxv) = h(ux)h(v)$ and $h(uxv) = h(u)h(xv)$, and so (1) implies that

$$h(uxv) = h(ux)h(v) = h(u)h(xv) = h(u)h(x)h(v) \quad \text{for all } u, v \in \mathcal{I},\ x \in \mathcal{A}. \tag{3}$$

We claim that $h(\mathcal{I})b_0 = 0$, where $b_0 \in \mathcal{B}$, implies $b_0 = 0$. Indeed, since h is surjective we have $b_0 = h(a_0)$ for some $a_0 \in \mathcal{A}$, and so (3) yields $h(\mathcal{I}a_0\mathcal{I}) = 0$. Accordingly, $\mathcal{I}a_0\mathcal{I} = 0$ and hence, since $\mathcal{A}$ is prime and $\mathcal{I} \neq 0$ in view of our assumption, we have $a_0 = 0$, and so $b_0 = 0$. Similarly we see that $b_0h(\mathcal{I}) = 0$ implies $b_0 = 0$.

Now, since by (3) we have $h(\mathcal{I})\big(h(xv) - h(x)h(v)\big) = 0$, $x \in \mathcal{A}$, $v \in \mathcal{I}$, it follows that $h(xv) = h(x)h(v)$. Therefore, for $x, y \in \mathcal{A}$ and $v \in \mathcal{I}$ we have $h(xyv) = h(xy)h(v)$, and on the other hand, $h(xyv) = h(x)h(yv) =$

$h(x)h(y)h(v)$. Thus, $\big(h(xy)-h(x)h(y)\big)h(\mathcal{I})=0$, and so $h(xy)=h(x)h(y)$. □

3.2. **Condition** $(d1)$**.** Problems on derivations can be often reduced to similar ones on homomorphisms through the trick that we are now going to apply. Given an $\mathcal{A}$-bimodule $\mathcal{M}$, the set of all matrices of the form

$$\begin{pmatrix} x & m \\ 0 & x \end{pmatrix}, \qquad x\in\mathcal{A},\, m\in\mathcal{M},$$

forms a ring under the usual matrix operations. We denote this ring by $\mathcal{B}$. Given a map $d:\mathcal{A}\to\mathcal{M}$ we define $h:\mathcal{A}\to\mathcal{B}$ by

$$h(x)=\begin{pmatrix} x & d(x) \\ 0 & x \end{pmatrix}.$$

Note that d is a derivation if and only if h is a homomorphism; moreover, d satisfies $(d1)$ if and only h satisfies $(h1)$, and d satisfies the condition (4) below if and only if h satisfies (1). Of course, $d(1)=0$ if and only if $h(1)=1$. Therefore, the first part of the next theorem follows directly from Theorem 3.1.

Theorem 3.4. *Let $\mathcal{A}$ be a unital ring, let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and let $d:\mathcal{A}\to\mathcal{M}$ be an additive map satisfying $(d1)$ and $d(1)=0$. Then the restriction of d to $\mathcal{R}$ is a derivation. Moreover,*

$$d(rxs)+rd(x)s=d(rx)s+rd(xs) \quad \text{for all } r,s\in\mathcal{R},\ x\in\mathcal{A}, \tag{4}$$

and

$$\mathcal{I}\big(d(xy)-d(x)y-xd(y)\big)\mathcal{I}=0 \quad \text{for all } x,y\in\mathcal{A}. \tag{5}$$

Proof. We only have to prove (5). Pick $u\in\mathcal{I}$, $s\in\mathcal{R}$, and $x\in\mathcal{A}$. Then $ux\in\mathcal{I}\subseteq\mathcal{R}$ and so, since d is a derivation on $\mathcal{R}$, we have $d(uxs)=d(ux)s+uxd(s)$. On the other hand, since $u,s\in\mathcal{R}$ it follows from (4) that $d(uxs)=d(ux)s+ud(xs)-ud(x)s$. Comparing the last two identities we get

$$ud(xs)=ud(x)s+uxd(s) \quad \text{for all } u\in\mathcal{I},\ x\in\mathcal{A},\ s\in\mathcal{R}. \tag{6}$$

Now pick $y\in\mathcal{A}$ and $v\in\mathcal{I}$. Since $v\in\mathcal{R}$ it follows from (6) that

$$ud(xyv)=ud((xy)v)=ud(xy)v+uxyd(v).$$

On the other hand, since $yv,v\in\mathcal{I}\subseteq\mathcal{R}$ and $ux\in\mathcal{I}$, (6) yields

$$\begin{aligned} ud(xyv) &= ud(x(yv))=ud(x)yv+uxd(yv)\\ &= ud(x)yv+uxd(y)v+uxyd(v). \end{aligned}$$

Comparing these two identities we obtain (5). □

Corollary 3.5. *Let $\mathcal{A}$ be a unital ring, let $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule, and let $d:\mathcal{A}\to\mathcal{M}$ be an additive map satisfying $(d1)$ and $d(1)=0$. If $\mathcal{R}=\mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)), then d is a derivation.*

Corollary 3.6. *Let $\mathcal{A}$ be a unital ring and let $d : \mathcal{A} \to \mathcal{A}$ be an additive map satisfying* $(d1)$ *and* $d(1) = 0$. *If $\mathcal{A}$ is a prime ring containing a nontrivial idempotent, then d is a derivation.*

Now let $d : \mathcal{A} \to \mathcal{M}$ be a local derivation. Then d satisfies $(d1)$. If $\mathcal{A}$ and $\mathcal{M}$ are unital, then it is easy to see that we also have $d(1) = 0$ and so Theorem 3.4 can be used. Otherwise, we consider the ring $\mathcal{A}_1$ obtained by adjoining a unity to $\mathcal{A}$. Setting $1m = m = m1$ for every $m \in \mathcal{M}$, $\mathcal{M}$ then becomes a unital $\mathcal{A}_1$-bimodule. Extend d to $\mathcal{A}_1$ by defining $d(1) = 0$ and note that d is a local derivation on $\mathcal{A}_1$. Therefore, the following is true:

Remark 3.7. *The conclusion of Theorem 3.4 holds for local derivations $d : \mathcal{A} \to \mathcal{M}$ even when $\mathcal{A}$ and $\mathcal{M}$ are not unital.*

In particular we thus see that every ring $\mathcal{A}$ with a noncentral idempotent contains a nonzero ideal $\mathcal{I}$ such that every local derivation $d : \mathcal{A} \to \mathcal{M}$ is a derivation on $\mathcal{I}$.

A special case of (4) where r and s are idempotents was known for local derivations before; it appears in the proof of [23, Theorem 1], and it has been extensively used also in [8] and [21], for example. The work on this paper actually begun by observing that (4) can be extended from the case where r and s are idempotents to the case where they are products of idempotents. While the proof of this is fairly easy, this more general identity seems to be much stronger and more useful. Let us record two of its (indirect) consequences, analogues of Corollaries 3.5 and 3.6.

Corollary 3.8. *Suppose that $\mathcal{R} = \mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)). Then every local derivation $d : \mathcal{A} \to \mathcal{M}$ is a derivation.*

Corollary 3.9. *Let $\mathcal{A}$ be a prime ring containing a nontrivial idempotent. Then every local derivation $d : \mathcal{A} \to \mathcal{A}$ is a derivation.*

4. Conditions $(h2)$, $(d2)$ and $(f2)$

It is obvious that the condition $(h2)$ implies the condition $(h1)$, and that $(d2)$ implies $(d1)$. Therefore, the results of the previous section already give some conclusions for maps satisfying $(h2)$ and $(d2)$. However, by a direct approach we will be able to obtain somewhat stronger results. In particular, we will avoid the assumption that our rings are unital, and we will obtain short proofs and slight improvements of the main results from the recent paper [3].

4.1. **Condition** $(h2)$.

It was observed in [3] that $(h2)$ implies that h satisfies

$$h(xe)h(z) = h(x)h(ez) \quad \text{for all } e \in \mathcal{E},\ x, z \in \mathcal{A}. \tag{7}$$

Indeed, just note that $(x - xe) \cdot ez = 0$ and $xe \cdot (z - ez) = 0$, and so $h(x - xe) \cdot h(ez) = 0$ and $h(xe) \cdot h(z - ez) = 0$, from which (7) follows. This identity played an important role in [3], and its special cases appear also in [4]. However, we shall use it in a somewhat different way than the authors did in those two papers.

Theorem 4.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be an additive map satisfying* $(h2)$*. Then*

$$h(xr)h(z) = h(x)h(rz) \quad \text{for all } r \in \mathcal{R},\ x, z \in \mathcal{A}, \tag{8}$$

and

$$h(wt)h(z)h(x)h(y) = h(w)h(t)h(z)h(xy) \\ \text{for all } t \in \mathcal{A}^2\mathcal{I},\ x, y, z, w \in \mathcal{A}. \tag{9}$$

Proof. Let

$$\mathcal{T} = \{r \in \mathcal{A} \,|\, h(xr)h(z) = h(x)h(rz) \text{ for all } x, z \in \mathcal{A}\}.$$

Given $r, r' \in \mathcal{T}$ and $x, z \in \mathcal{A}$ we have

$$h(xrr')h(z) = h(xr)h(r'z) = h(x)h(rr'z),$$

showing that $\mathcal{T}$ is a subring of $\mathcal{A}$. Since $\mathcal{E} \subseteq \mathcal{T}$ by (7), it follows that $\mathcal{R} \subseteq \mathcal{T}$. This proves (8).

Now let $u \in \mathcal{I}$ and $x, y, z, w, w', w'' \in \mathcal{A}$. Then $uzx, w''uz, w'w''u \in \mathcal{I} \subseteq \mathcal{R}$ and so using (8) we get

$$\begin{aligned} & h(w)h(w')h(w'')h(uzxy) = h(w)h(w')h(w''uzx)h(y) \\ = \quad & h(w)h(w'w''uz)h(x)h(y) = h(ww'w''u)h(z)h(x)h(y). \end{aligned}$$

On the other hand, since $uz, w''u \in \mathcal{R}$, we have

$$\begin{aligned} & h(w)h(w')h(w'')h(uzxy) = h(w)h(w')h(w''uz)h(xy) \\ = \quad & h(w)h(w'w''u)h(z)h(xy). \end{aligned}$$

Comparing we get

$$h(ww'w''u)h(z)h(x)h(y) = h(w)h(w'w''u)h(z)h(xy),$$

which proves (9). □

Corollary 4.2. *Suppose that $\mathcal{R} = \mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)). If an additive map $h : \mathcal{A} \to \mathcal{B}$ satisfies* $(h2)$*, then $h(xy)h(z) = h(x)h(yz)$ for all $x, y, z \in \mathcal{A}$.*

Corollary 4.3. *Let $\mathcal{A}$ be such that $\mathcal{A}^2\mathcal{I} \neq 0$ and let $\mathcal{B}$ be prime. If a bijective additive map $h : \mathcal{A} \to \mathcal{B}$ satisfies* $(h2)$*, then there is λ in the extended centroid of $\mathcal{B}$ such that $h(xy) = \lambda h(x)h(y)$ for all $x, y \in \mathcal{A}$.*

Proof. Since h is surjective and $\mathcal{B}$ is prime, we have $h(x)h(y) \neq 0$ for some $x, y \in \mathcal{A}$. Therefore, by a well-known result of Martindale [17, Theorem 2 (a)], it follows from (9) that for each pair of elements $w \in \mathcal{A}$ and $t \in \mathcal{A}^2\mathcal{I}$, $h(w)h(t)$ and $h(wt)$ are linearly dependent over the extended centroid $\mathcal{C}$ of $\mathcal{B}$. Fixing w and t such that $h(w)h(t) \neq 0$ (such exist since $\mathcal{A}^2\mathcal{I} \neq 0$ and $\mathcal{B}$ is prime) we thus have $h(wt) = \lambda h(w)h(t)$ for some $\lambda \in \mathcal{C}$. But then (9) implies that $h(w)h(t)h(z)\big(h(xy) - \lambda h(x)h(y)\big) = 0$ for all $x, y, z \in \mathcal{A}$, and hence the desired conclusion follows from the primeness of $\mathcal{B}$. □

The technical assumption that $\mathcal{A}^2\mathcal{I} \neq 0$ is just slightly more restrictive than the one that $\mathcal{I} \neq 0$, i.e. that $\mathcal{A}$ contains a noncentral idempotent. For example, in unital rings or in (semi)prime rings, these two assumptions are equivalent. Moreover, a careful inspection of the proof shows that in the prime ring case assuming even less is sufficient: it is enough to require the assumption from [3] that the maximal right ring of quotients $\mathcal{Q}$ of $\mathcal{A}$ contains a nontrivial idempotent e such that $e\mathcal{A} \cup \mathcal{A}e \subseteq \mathcal{A}$. Therefore, Corollary 4.3 generalizes [3, Theorem 1]. In particular it shows that the assumption that $\deg(\mathcal{B}) \geq 3$ can be removed in this theorem. The proof of Corollary 4.3 is also simpler and in particular it avoids the use of the theory of functional identities.

If $\mathcal{A}$ is a unital ring, then some of our formulae can be simplified. Indeed, set $\alpha = h(1) \in \mathcal{B}$. Letting $x = z = 1$ in (8) we see that $h(r)\alpha = \alpha h(r)$ for every $r \in \mathcal{R}$. Accordingly, for every $u \in \mathcal{I}$ and $z \in \mathcal{A}$ we have

$$\alpha h(u)h(z) = \alpha h(1u)h(z) = \alpha^2 h(uz) = \alpha h(uz)\alpha = h(1)h(uz)\alpha = h(u)h(z)\alpha.$$

Therefore, setting $w = 1$ in (9) it follows that

$$h(t)h(z)\big(\alpha h(xy) - h(x)h(y)\big) = 0 \quad \text{for all } t \in \mathcal{I},\ x, y, z \in \mathcal{A}.$$

If $\mathcal{B}$ is prime, h is bijective and $\mathcal{I} \neq 0$, then it follows that $\alpha h(xy) = h(x)h(y)$ for all $x, y \in \mathcal{A}$, from which we infer that $\lambda = \alpha^{-1}$.

4.2. **Condition** $(d2)$**.** Obvious examples of maps satisfying $(d2)$ are derivations and multiplications by central elements, and of course their sums.

Given $d : \mathcal{A} \to \mathcal{M}$, we define $\mathcal{B}$ and h in the same way as in Subsection 3.2. Note that d satisfies $(d2)$ if and only if h satisfies $(h2)$. Consequently, by a straightforward computation one can check that Theorem 4.1 implies

Theorem 4.4. *Let $d : \mathcal{A} \to \mathcal{M}$ be an additive map satisfying $(d2)$. Then*

$$d(xr)z + xrd(z) = d(x)rz + xd(rz) \quad \textit{for all } r \in \mathcal{R},\ x, z \in \mathcal{A}, \tag{10}$$

and

$$wtz\big(d(xy) - d(x)y - xd(y)\big) = \big(d(wt) - d(w)t - wd(t)\big)zxy \quad \textit{for all } t \in \mathcal{A}^2\mathcal{I},\ x, y, z, w \in \mathcal{A}. \tag{11}$$

If $\mathcal{R} = \mathcal{A}$ then (10) can be read as

$$d(xy)z + xyd(z) = d(x)yz + xd(yz) \quad \text{for all } x, y, z \in \mathcal{A}. \tag{12}$$

Suppose that $\mathcal{A}$ is a unital ring and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule. Then setting $x = z = 1$ in (12) it follows that $\lambda = d(1)$ lies in $\mathcal{Z}(\mathcal{M})$, the center of $\mathcal{M}$, and hence, by setting $z = 1$, it follows that $\delta : x \mapsto d(x) - \lambda x$ is a derivation. Thus we have

Corollary 4.5. *Suppose that $\mathcal{R} = \mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)). If an additive map $d : \mathcal{A} \to \mathcal{M}$ satisfies $(d2)$, then d satisfies (12). Moreover, if both $\mathcal{A}$ and $\mathcal{M}$ are unital, then $\lambda = d(1) \in \mathcal{Z}(\mathcal{M})$ and there is a derivation $\delta : \mathcal{A} \to \mathcal{M}$ such that $d(x) = \lambda x + \delta(x)$ for all $x \in \mathcal{A}$.*

A simple modification of the proof of Corollary 4.3 gives

Corollary 4.6. *Let $\mathcal{A}$ be a prime ring containing a nontrivial idempotent. If an additive map $d : \mathcal{A} \to \mathcal{A}$ satisfies* ($d2$)*, then there is λ in the extended centroid of $\mathcal{A}$ and a derivation δ from $\mathcal{A}$ into the central closure of $\mathcal{A}$ such that $d(x) = \lambda x + \delta(x)$ for all $x \in \mathcal{A}$.*

In case when $\mathcal{A}$ is unital, this conclusion can be simplified: note that $\lambda = d(1) \in \mathcal{Z}(\mathcal{A})$ and δ maps $\mathcal{A}$ into itself.

Corollary 4.6 improves [3, Theorem 2]; here as well we could replace the assumption that $\mathcal{A}$ contains a nontrivial idempotent by a somewhat milder one that $\mathcal{Q}$ contains a nontrivial idempotent e such that $e\mathcal{A} \cup \mathcal{A}e \subseteq \mathcal{A}$ (see the comments following Corollary 4.3).

4.3. **Condition ($f2$).** Recall that an additive map $f : \mathcal{A} \to \mathcal{L}$ is called a (right) *multiplier* if $f(xy) = xf(y)$ for all $x, y \in \mathcal{A}$. For example, the (right) multiplications $x \mapsto xm$, where m is a fixed element in $\mathcal{L}$, are multipliers, and in fact these are obviously the only multipliers in the case where $\mathcal{A}$ is unital. By a *local multiplier* we of course mean an additive map $f : \mathcal{A} \to \mathcal{L}$ such that for every $x \in \mathcal{A}$ there is a multiplier $f_x : \mathcal{A} \to \mathcal{L}$ such that $f(x) = f_x(x)$. Clearly, every local multiplier satisfies ($f2$).

A natural question is: are additive maps satisyfing ($f2$) necessarily multipliers? If we wish to deal with arbitrary modules, then we have to take into account that in the case when the module multiplication is trivial, i.e. $\mathcal{A}\mathcal{L} = 0$, every additive map $f : \mathcal{A} \to \mathcal{L}$ satisfies ($f2$). Therefore, our goal will be to prove that ($f2$) implies that f satisfies a weaker condition than being a multiplier, namely,

$$xf(yz) = xyf(z) \quad \text{for all } x, y, z \in \mathcal{A}. \tag{13}$$

Of course, under a very mild assumption that for every $m \in \mathcal{L}$, $\mathcal{A}m = 0$ implies $m = 0$, (13) is equivalent to the condition that f is a multiplier. In particular this assumption is fulfilled in the case when $\mathcal{A}$ and $\mathcal{L}$ are unital.

We make a left $\mathcal{A}$-module $\mathcal{L}$ an $\mathcal{A}$-bimodule by defining $\mathcal{L}\mathcal{A} = 0$ (cf. [13]). In this setting, the conditions ($d2$) and ($f2$) are equivalent. Therefore, the results from the previous subsection are applicable.

Theorem 4.7. *Let $f : \mathcal{A} \to \mathcal{L}$ be an additive map satisfying* ($f2$)*. Then*

$$xf(rz) = xrf(z) \quad \textit{for all } r \in \mathcal{R},\ x, z \in \mathcal{A}, \tag{14}$$

and

$$\mathcal{A}\mathcal{I}\big(f(xy) - xf(y)\big) = 0 \quad \textit{for all } x, y \in \mathcal{A}. \tag{15}$$

Proof. Note that (14) follows from (10). Given $x, y, z \in \mathcal{A}$ and $u \in \mathcal{I}$, we see from (14) that $zf(uxy) = zuf(xy)$, and on the other hand, $zf(uxy) = zuxf(y)$. Comparing we obtain (15). □

Corollary 4.8. *Suppose that $\mathcal{R} = \mathcal{A}$ (e.g., if $\mathcal{A}$ is as in (A), (B), or (C)). If an additive map $f : \mathcal{A} \to \mathcal{L}$ satisfies* ($f2$) *(in particular, if f is a local multiplier), then f satisfies (13).*

This corollary considerably extends [7, Theorem 3].

The notion of a local multiplier from $\mathcal{A}$ into itself is closely related to the notion of a *right ideal preserving map* (cf. [7]). By this we mean a map $f : \mathcal{A} \to \mathcal{A}$ such that $f(\mathcal{K}) \subseteq \mathcal{K}$ for every right ideal $\mathcal{K}$ of $\mathcal{A}$. If f is a right ideal preserving map, then for every $x \in \mathcal{A}$ the element $f(x)$ must lie in the right ideal generated by x; that is, there exist $a_x \in \mathcal{A}$ and an integer n_x such that $f(x) = xa_x + n_x x$. But then f is a local multiplier. Conversely, a local multiplier from $\mathcal{A}$ into itself is a right ideal preserving map provided that $\mathcal{K}\mathcal{A} = \mathcal{K}$ for every right ideal $\mathcal{K}$ of $\mathcal{A}$ (this condition is trivially satisfied if $\mathcal{A}$ is unital).

Corollary 4.9. *Let $\mathcal{A}$ be a prime ring containing a nontrivial idempotent. If an additive map $f : \mathcal{A} \to \mathcal{A}$ satisfies* $(f2)$ *(in particular, if f is a local multiplier or f is a right ideal preserving map), then f is a multiplier.*

5. An example

If $\mathcal{A}$ is a division ring, then every additive map from $\mathcal{A}$ into $\mathcal{A}$ is trivially right ideal preserving (and hence a local multiplier). This simple fact clearly explains why the last results do not hold in general prime (or simple) rings, i.e. it justifies the assumption that our rings must contain nontrivial idempotents. Since nontrivial examples of local multipliers generate nontrivial examples of local derivations (by introducing the trivial multiplication $\mathcal{L}\mathcal{A} = 0$), it formally also justifies this assumption in some of our other results. Admittedly, this argument is a bit artificial since the bimodule so constructed in particular cannot be unital. However, as already mentioned in the introduction, there are other examples. In his pioneering paper [14] Kadison gave an example, which he attributed to Jensen, of a local derivation that is not a derivation on the field of rational functions. Further, it is a fact that there exist division rings in which every nonzero inner derivation is surjective [5, 16]. It is clear that every additive map from such a ring into itself that maps the center into 0 is a local derivation, but of course it is only seldom a derivation.

The goal of this last section is to present an example of a map which is neither a derivation nor a multiplier but it satisfies several conditions that have been treated. It is defined on a prime (even primitive) ring having plenty of idempotents, and maps into a suitably chosen bimodule. In particular this example justifies our confinement to maps from rings into themselves in corollaries concerning prime rings.

In a note added in proof Kadison [14] mentioned that Kaplansky has found local derivations on the algebra $\mathbb{C}[x]/[x^3]$ that are not derivations. We did not see Kaplansky's example, but just the fact that it exists has encouraged us to search for examples in the algebra that we shall introduce in the next paragraph. We remark that the algebra $\mathbb{C}[x]/[x^3]$ itself is clearly not sufficient for our purposes since it is not even semiprime.

Let V be an infinite dimensional vector space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $\mathcal{F}$ be the algebra of all finite rank linear operators on V. Further, let A be a linear operator on V such that $A^3 \in \mathcal{F}$ and I, A, A^2 are linearly independent modulo $\mathcal{F}$ (one can easily find such an operator via the matrix representation). Let $\mathcal{A}$ be the algebra generated by I, A and $\mathcal{F}$. Clearly $\mathcal{F}$ is an ideal of $\mathcal{A}$. Considering $\mathcal{A}$ and $\mathcal{F}$ as $\mathcal{A}$-bimodules, we can construct the quotient bimodule $\mathcal{M} = \mathcal{A}/\mathcal{F}$. Define $d : \mathcal{A} \to \mathcal{M}$ by

$$d(F + \lambda_0 I + \lambda_1 A + \lambda_2 A^2) = \lambda_1 A + \mathcal{F}$$

for all $F \in \mathcal{F}$ and $\lambda_i \in \mathbb{F}$. We claim that d has the following properties:

- d is a local derivation but not a derivation,
- d is a local multiplier but not a multiplier,
- d satisfies $(d2)$ and $d(1) = 0$ but is not a derivation,
- d satisfies $(f2)$ but is not a multiplier.

Since $d(A^2) = 0$ while $d(A)A = Ad(A) = A^2 + \mathcal{F}$, d is certainly neither a derivation nor a multiplier.

Pick $x = F_0 + \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \in \mathcal{A}$ and define $d_x : \mathcal{A} \to \mathcal{M}$ by $d_x = 0$ if $\alpha_1 = 0$, and

$$d_x(F + \lambda_0 I + \lambda_1 A + \lambda_2 A^2) = \lambda_1 A + 2(\lambda_2 - \alpha_1^{-1}\alpha_2\lambda_1)A^2 + \mathcal{F}$$

if $\alpha_1 \neq 0$. Note that d_x is a derivation and $d(x) = d_x(x)$. Thus, d is a local derivation. Further, define $f_x : \mathcal{A} \to \mathcal{M}$ by $f_x = 0$ if $\alpha_1 = 0$,

$$f_x(F + \lambda_0 I + \lambda_1 A + \lambda_2 A^2) = (F + \lambda_0 I + \lambda_1 A + \lambda_2 A^2)(1 - \alpha_1^{-1}\alpha_2 A + \mathcal{F})$$

if $\alpha_1 \neq 0$ and $\alpha_0 = 0$, and

$$f_x(F+\lambda_0 I+\lambda_1 A+\lambda_2 A^2) = (F+\lambda_0 I+\lambda_1 A+\lambda_2 A^2)(\alpha_0^{-1}\alpha_1 A-\alpha_0^{-2}\alpha_1^2 A^2+\mathcal{F})$$

if $\alpha_0 \neq 0$. Note that in any case f_x is a multiplier and $d(x) = f_x(x)$. So, d is a local multiplier.

Let $x = F_0 + \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \in \mathcal{A}$ and $y = F_0 + \beta_0 I + \beta_1 A + \beta_2 A^2 \in \mathcal{A}$ be such that $xy = 0$. Note that this is possible only in the case when one of the following conditions is fulfilled: $\alpha_0 = \alpha_1 = \alpha_2 = 0$, $\alpha_0 = \alpha_1 = \beta_0 = 0$, $\alpha_0 = \beta_0 = \beta_1 = 0$, or $\beta_0 = \beta_1 = \beta_2 = 0$. In any case we have $d(x)y = xd(y) = 0$. In particular, d satisfies $(d2)$ and $(f2)$.

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