CONTINUOUS COMMUTING FUNCTIONS ON MATRIX ALGEBRAS

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Abstract. If a continuous function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ satisfies $f(x)x = xf(x)$ for all $x \in M_n(\mathbb{C})$, then there exist functions $a_0, a_1, \ldots, a_{n-1} : M_n(\mathbb{C}) \to \mathbb{C}$ such that $f(x) = \sum_{j=0}^{n-1} a_j(x)x^j$ for all $x \in M_n(\mathbb{C})$. Moreover, a_j are continuous on the set of all non-derogatory matrices.

Dedicated to Vladimir Sergeichuk on the occasion of his 70th birthday

1. Introduction

A function f from an algebra A to itself is said to be *commuting* if

$$f(x)x = xf(x)$$

for all $x \in A$. The problem of describing such functions has been studied by many authors over the last six decades, and has in particular played a key role in the development of the theory of functional identities and, especially, of its applications. We refer the reader to the survey paper [1] and Chapters 5-8 of the book [2] for history and motivation.

In the framework of functional identities, f is usually assumed to be the trace of an m-linear function $F : A^m \to A$ (meaning that $f(x) = F(x, \ldots, x)$ for all $x \in A$). The desired conclusion, then, is that f is of the form $f(x) = \sum_{j=0}^{m} a_j(x)x^j$, $x \in A$, where a_j is the trace of an $(m-j)$-linear function and maps A into its center $Z(A)$. In a series of papers by different authors, started in the early 1990's, it was shown that this holds for quite general algebras A (see [1, 2] for details). However, the techniques of functional identities do not work well in low dimensional algebras, and so, paradoxically, the case where A is the matrix algebra $M_n(F)$ (with F a field of characteristic 0) was covered only rather recently [3]. The proof was based on the methods of commutative algebra.

In this short paper, we address the question of describing a commuting function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ under the sole assumption that f is continuous. This is, of course, much weaker than requiring that f is the trace of an m-linear function. However, assuming continuity makes it possible to approach the problem from a fresh perspective, using methods that are essentially different from those employed in [1, 2] as well as in [3]. What to expect under this assumption? It is tempting to conjecture that f has to be of the form

$$f(x) = a_0(x)1 + a_1(x)x + \cdots + a_{n-1}(x)x^{n-1} \tag{1.1}$$

for all $x \in M_n(\mathbb{C})$, where $a_0, a_1, \ldots, a_{n-1} : M_n(\mathbb{C}) \to \mathbb{C}$ are continuous functions. Unfortunately, this is not exactly true. Actually, it does turn out that f takes

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the form (1.1), but we cannot claim that the a_j's are continuous on the whole set $M_n(\mathbb{C})$. We will prove, however, that they are continuous on the subset of all non-derogatory matrices, and, moreover, provide an example showing that for *every* derogatory matrix $d \in M_n(\mathbb{C})$ there exists a continuous commuting function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that if $a_j : M_n(\mathbb{C}) \to \mathbb{C}$ are *any* functions that satisfy (1.1) for all $x \in M_n(\mathbb{C})$, then at least one of them is discontinuous at d.

2. Example

A matrix $x \in M_n(\mathbb{C})$ is called *non-derogatory* [4] if and only if one (and hence all) of the following equivalent conditions holds:

- the characteristic and minimal polynomial of x coincide (i.e., $1, x, \ldots, x^{n-1}$ are linearly independent),
- for each eigenvalue λ of x, there is exactly one Jordan block with the eigenvalue λ in the Jordan normal form of x,
- each matrix $y \in M_n(\mathbb{C})$ that commutes with x is of the form $y = p(x)$ for some polynomial $p \in \mathbb{C}[X]$.

We denote by $N_n(\mathbb{C}) \subset M_n(\mathbb{C})$ the set of all non-derogatory matrices. The set $N_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$ since $N_n(\mathbb{C})$ contains all $n \times n$ matrices having n distinct eigenvalues, and the set of all such matrices is dense in $M_n(\mathbb{C})$. The set $N_n(\mathbb{C})$ is also open in $M_n(\mathbb{C})$. Indeed, $x_0 \in N_n(\mathbb{C})$ if and only if the matrices $1, x_0, \ldots, x_0^{n-1}$ are linearly independent. Since a small enough perturbation of a linearly independent n-tuple of vectors is again linearly independent, there is an open neighbourhood of x_0 such that every member of this neighbourhood belongs to $N_n(\mathbb{C})$.

Let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a commuting map. Then, for every $x \in N_n(\mathbb{C})$ there exists a polynomial $p_x \in \mathbb{C}[X]$ (depending on x) such that $f(x) = p_x(x)$. Of course, we may assume that $\deg p_x \leq n-1$, and since $1, x, \ldots, x^{n-1}$ are linearly independent, there exist uniquely determined functions $a_0, a_1, \ldots, a_{n-1} : N_n(\mathbb{C}) \to \mathbb{C}$ such that (1.1) holds for all $x \in N_n(\mathbb{C})$. Assume additionally that f is continuous. Since $N_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, it seems plausible at first glance that the functions a_j are continuous and can be continuously extended to $M_n(\mathbb{C})$ so that (1.1) holds for all $x \in M_n(\mathbb{C})$. However, it turns out that this is not entirely true. To show this, the natural idea is to find functions $a_0, a_1, \ldots, a_{n-1} : M_n(\mathbb{C}) \to \mathbb{C}$ that are continuous on $N_n(\mathbb{C})$ and behave badly when x tends to some derogatory matrix d, while the function $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by (1.1) is continuous at d.

Example 2.1. Let $n \geq 2$ and let $d \in M_n(\mathbb{C})$ be a derogatory matrix with minimal polynomial $m(X) = \sum_{j=0}^{r} \lambda_j X^j \in \mathbb{C}[X]$, where $1 \leq r < n$ and $\lambda_r = 1$. Define $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{\|x-d\|}} m(x) & \text{if} \quad x \neq d \\ 0 & \text{if} \quad x = d \end{cases},$$

where $\| \cdot \|$ is a submultiplicative norm on $M_n(\mathbb{C})$. Then f is commuting and continuous on $M_n(\mathbb{C}) \setminus \{d\}$. Let us show that it is also continuous at d. To this end, note that for every $h \in M_n(\mathbb{C})$,

$$m(d+h) = \sum_{j=0}^{r} \lambda_j (d+h)^j$$

can be written as a sum of matrices of the form

$$\lambda_j d^{k_1} h^{\ell_1} d^{k_2} h^{\ell_2} \cdots d^{k_s} h^{\ell_s},$$

where $k_i, \ell_i \geq 0$, $\sum_{i=1}^{s} k_i + \ell_i = j \leq r$, and, since $m(d) = 0$, at least one ℓ_i is positive. This implies that there exist nonnegative constants $c_1, c_2, \ldots, c_r = 1$ such that

$$\|m(d+h)\| \leq c_1\|h\| + c_2\|h\|^2 + \cdots + c_r\|h\|^r.$$

Consequently,

$$\lim_{h\to 0} \|f(d+h)\| = \lim_{h\to 0} \left\| \frac{1}{\sqrt{\|h\|}} m(d+h) \right\|$$

$$\leq \lim_{h\to 0} \frac{1}{\sqrt{\|h\|}} \left(c_1\|h\| + c_2\|h\|^2 + \cdots + c_r\|h\|^r \right) = 0,$$

proving that f is continuous on $M_n(\mathbb{C})$.

Now let $a_j : M_n(\mathbb{C}) \to \mathbb{C}$, $j = 0, 1, \ldots, n-1$, be any functions satisfying (1.1) for every $x \in M_n(\mathbb{C})$. Since $1, x, \ldots, x^{n-1}$ are linearly independent if $x \in N_n(\mathbb{C})$, we have

$$a_r(x) = \frac{1}{\sqrt{\|x - d\|}}$$

for every $x \in N_n(\mathbb{C})$. As $N_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, it follows that a_r is discontinuous at d.

3. Main Theorem

We start with a technical lemma.

Lemma 3.1. *Let $\mathcal{U}$ be a unitary space (a finite-dimensional complex inner product space), $\mathcal{W} \subset \mathcal{U}$ an open subset, and $x_0 \in \mathcal{W}$. Let m be a positive integer, $u_0, u_1, \ldots, u_m : \mathcal{W} \to \mathcal{U}$ continuous functions, and $b_0, b_1, \ldots, b_m : \mathcal{W} \to \mathbb{C}$ scalar-valued functions. Assume that*

$$h(x) = b_0(x)u_0(x) + b_1(x)u_1(x) + \ldots + b_m(x)u_m(x), \quad x \in \mathcal{W},$$

is a continuous function from $\mathcal{W}$ to $\mathcal{U}$ such that $h(x_0) = 0$. Finally suppose that the vectors $u_0(x_0), u_1(x_1), \ldots, u_m(x_0)$ are linearly independent. Then we have

$$\lim_{x\to x_0} b_j(x) = 0$$

for every $j = 0, 1, \ldots, m$.

Proof. Since $u_0(x_0), u_1(x_1), \ldots, u_m(x_0)$ are linearly independent, we can find an orthonormal system of vectors $e_0, e_1, \ldots, e_m$ and a bijective linear map $T : \mathcal{U} \to \mathcal{U}$ such that

$$Tu_j(x_0) = e_j$$

for all $j = 0, 1, \ldots, m$. Replacing the functions $h, u_0, \ldots, u_m : \mathcal{W} \to \mathcal{U}$ by the functions $Th, Tu_0, \ldots, Tu_m$, respectively, we see that there is no loss of generality in assuming that $u_j(x_0) = e_j$ for all $j = 0, 1, \ldots, m$. It follows that

$$\lim_{x\to x_0} u_j(x) = e_j, \quad j = 0, 1, \ldots, m.$$

For each $x \in \mathcal{W}$ and each $j \in \{0, 1, \ldots, m\}$ there exists a unique scalar $\lambda_j(x) \in \mathbb{C}$ and a unique vector $z_j(x)$ orthogonal to e_j such that

$$u_j(x) = \lambda_j(x)e_j + z_j(x).$$

Then

$$\lambda_j(x) = \langle u_j(x), e_j \rangle,$$

and therefore

$$\lim_{x\to x_0} \lambda_j(x) = 1 \text{ and } \lim_{x\to x_0} z_j(x) = 0, \quad j = 0,1,\ldots,m.$$

Since all norms on a finite-dimensional vector space are equivalent, there exists a positive real number c such that for every $(m+1)$-tuple of complex numbers $\gamma_0, \gamma_1, \ldots, \gamma_m$ we have

$$\sum_{j=0}^{m} |\gamma_j| \le c \sqrt{\sum_{j=0}^{m} |\gamma_j|^2}.$$

Let ε be any positive real number less than $\frac{1}{1+c}$. By the above we can find a neighbourhood $\mathcal{V}$ of x_0 in $\mathcal{W}$ such that for every $x \in \mathcal{V}$ and every $j \in \{0,1,\ldots,m\}$ we have

- $\|h(x)\| < \varepsilon$, and
- $1 - \varepsilon < \lambda_j(x)$, and
- $\|z_j(x)\| < \varepsilon$.

Hence, for every $x \in \mathcal{V}$ we have

$$\begin{aligned}
\varepsilon > \|h(x)\| = \left\| \sum_{j=0}^{m} b_j(x) u_j(x) \right\| &= \left\| \sum_{j=0}^{m} b_j(x)\lambda_j(x) e_j + \sum_{j=0}^{m} b_j(x) z_j(x) \right\| \\
&\ge \left\| \sum_{j=0}^{m} b_j(x)\lambda_j(x) e_j \right\| - \left\| \sum_{j=0}^{m} b_j(x) z_j(x) \right\| \\
&\ge \sqrt{\sum_{j=0}^{m} |b_j(x)|^2 |\lambda_j(x)|^2} - \sum_{j=0}^{m} |b_j(x)| \, \|z_j(x)\| \\
&\ge (1-\varepsilon) \sqrt{\sum_{j=0}^{m} |b_j(x)|^2} - \varepsilon \sum_{j=0}^{m} |b_j(x)| \\
&\ge (1-\varepsilon) \sqrt{\sum_{j=0}^{m} |b_j(x)|^2} - c\varepsilon \sqrt{\sum_{j=0}^{m} |b_j(x)|^2}.
\end{aligned}$$

It follows that for every $x \in \mathcal{V}$ we have

$$\sqrt{\sum_{j=0}^{m} |b_j(x)|^2} < \frac{\varepsilon}{1 - \varepsilon(1+c)},$$

and consequently,

$$\lim_{x\to x_0} b_j(x) = 0, \quad j = 0,\ldots,m,$$

as desired. $\square$

We are now in a position to prove our main result.

Theorem 3.2. *Let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a continuous commuting function. Then there exist functions $a_0, a_1, \dots, a_{n-1} : M_n(\mathbb{C}) \to \mathbb{C}$ that are continuous on the set $N_n(\mathbb{C})$ of all non-derogatory matrices and satisfy*

$$f(x) = a_0(x)1 + a_1(x)x + \dots + a_{n-1}(x)x^{n-1}$$

for all $x \in M_n(\mathbb{C})$.

Proof. The bulk of the proof is showing that

$$f(x) \in \operatorname{span}\{1, x, x^2, \dots, x^{n-1}\}, \quad x \in M_n(\mathbb{C}). \tag{3.1}$$

From this the desired conclusion follows. Indeed, with the help of the axiom of choice we see that (3.1) implies that there exist functions $a_j : M_n(\mathbb{C}) \to \mathbb{C}$ satisfying $f(x) = \sum_{j=0}^{n-1} a_j(x)x^j$ for all $x \in M_n(\mathbb{C})$. All we need to prove is that they are continuous on $N_n(\mathbb{C})$. Since all norms on $M_n(\mathbb{C})$ are equivalent, it is enough to show this for the case where $M_n(\mathbb{C})$ is a unitary space with the inner product given by

$$\langle x, y\rangle = \operatorname{tr}(xy^*), \quad x, y \in M_n(\mathbb{C}).$$

Choose $x_0 \in N_n(\mathbb{C})$. Define a continuous function $h : N_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$h(x) = b_0(x)u_0(x) + b_1(x)u_1(x) + \dots + b_{n-1}(x)u_{n-1}(x), \quad x \in N_n(\mathbb{C}),$$

where

$$b_j(x) = a_j(x) - a_j(x_0) \text{ and } u_j(x) = x^j, \quad j = 0, 1, \dots, n-1.$$

Since $N_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$, an application of Lemma 3.1 gives $\lim_{x\to x_0} b_j(x) = 0$, so a_j is continuous at x_0.

Thus, from now on we focus on proving (3.1).

We will need the following observation. Let k be a positive integer, $z \in M_k(\mathbb{C})$, and $\Lambda \subset \mathbb{C}$ a finite subset. Then for every positive real number ε we can find a matrix $w \in M_k(\mathbb{C})$ such that w has k distinct eigenvalues, each eigenvalue of w belongs to $\mathbb{C} \setminus \Lambda$, and $\|z - w\| < \varepsilon$. To verify this we only need to apply Schur's theorem stating that every complex square matrix is unitarily similar to an upper triangular matrix. As unitary similarity does not affect the norm we may assume with no loss of generality that z is already upper triangular. But then the eigenvalues of z are exactly its diagonal entries and by a sufficiently small perturbation of the diagonal entries of z we can get w with the desired properties.

In the proof we will identify $n \times n$ matrices with linear operators acting on $\mathbb{C}^n$. Our first claim is that for every $x \in M_n(\mathbb{C})$ and every subspace $U \in \mathbb{C}^n$ that is invariant under x, the subspace U is invariant under the operator $f(x)$ as well.

Indeed, with respect to the direct sum decomposition $\mathbb{C}^n = U \oplus U^\perp$, the operator x has the matrix representation

$$x = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$$

and using the above observation it is possible to find a sequence (x_m) of matrices

$$x_m = \begin{bmatrix} x_{m,1} & x_2 \\ 0 & x_{m,3} \end{bmatrix}$$

converging to x such that both $x_{m,1}$ and $x_{m,3}$ are matrices with all eigenvalues of algebraic multiplicity one, and moreover, the intersection of the spectra of $x_{m,1}$

and $x_{m,3}$ is empty for every positive integer m. It follows that each x_m is non-derogatory and consequently, each $f(x_m)$ is a polynomial in x_m. In particular, for every positive integer m the matrix $f(x_m)$ is of the form

$$f(x_m) = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

and by continuity of f, the same must be true for $f(x)$. Equivalently, U is invariant under $f(x)$, as desired.

It follows that if $x \in M_n(\mathbb{C})$ is similar to a block diagonal matrix, that is, if there exists an invertible $s \in M_n(\mathbb{C})$ such that

$$x = s \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_p \end{bmatrix} s^{-1},$$

where x_1 is a $k_1 \times k_1$ matrix, x_2 is a $k_2 \times k_2$ matrix, ... , and x_p is a $k_p \times k_p$ matrix, $k_1 + \cdots + k_p = n$, then

$$f(x) = s \begin{bmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & * \end{bmatrix} s^{-1},$$

where the $*$'s stand for some $k_1 \times k_1$ matrix, some $k_2 \times k_2$ matrix, ... , and some $k_p \times k_p$ matrix.

For a positive integer k and a complex number λ we denote by $j(\lambda, k)$ the $k \times k$ Jordan block with eigenvalue λ, that is,

$$j(\lambda, k) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}.$$

A $k \times k$ matrix w commutes with $j(0, k)$ if and only if it commutes with $j(\lambda, k)$ for every complex number λ, and this is equivalent to the condition that w is an upper triangular Toeplitz matrix

$$w = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & \dots & \mu_k \\ 0 & \mu_1 & \mu_2 & \dots & \mu_{k-1} \\ 0 & 0 & \mu_1 & \dots & \mu_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_1 \end{bmatrix}$$

for some $\mu_1, \dots, \mu_k \in \mathbb{C}$.

Thus, if $x \in M_n(\mathbb{C})$ has the Jordan canonical form

$$x = s \begin{bmatrix} j(\lambda_1, k_1) & 0 & \dots & 0 \\ 0 & j(\lambda_2, k_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & j(\lambda_p, k_p) \end{bmatrix} s^{-1},$$

$k_1 + \cdots + k_p = n$, then

$$f(x) = s \begin{bmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & * \end{bmatrix} s^{-1},$$

where the $*$'s stand for some $k_1 \times k_1$ upper triangular Toeplitz matrix, some $k_2 \times k_2$ upper triangular Toeplitz matrix,..., and some $k_p \times k_p$ upper triangular Toeplitz matrix.

Note that in general $\lambda_1, \dots, \lambda_p$ are not distinct.

Our next claim is that if $a, b \in M_{p+q}(\mathbb{C})$ are operators

$$a = \begin{bmatrix} j(\lambda, p) & 0 \\ 0 & j(\lambda, q) \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix},$$

where b_1 and b_2 are upper triangular Toeplitz matrices, such that every subspace $U \subset \mathbb{C}^n$ that is invariant under a is also invariant under b, then there exists a polynomial $p \in \mathbb{C}[X]$ such that $b = p(a)$. Indeed, we may assume with no loss of generality that $p \geq q$ and then we have

$$b = \begin{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_p \\ 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{p-1} \\ 0 & 0 & \alpha_1 & \dots & \alpha_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \dots & \beta_q \\ 0 & \beta_1 & \beta_2 & \dots & \beta_{q-1} \\ 0 & 0 & \beta_1 & \dots & \beta_{q-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta_1 \end{bmatrix} \end{bmatrix}$$

and we need to prove that $\alpha_j = \beta_j$, $j = 1, \dots, q$. Let $\xi_1, \dots, \xi_p, \xi_{p+1}, \dots, \xi_{p+q}$ denote the standard basis of $\mathbb{C}^{p+q}$. Since span $\{\xi_1 + \xi_{p+1}\}$ is an invariant subspace under a, it has to be invariant also under b yielding that $\alpha_1 = \beta_1$. We observe next that span $\{\xi_1 + \xi_{p+1}, \xi_2 + \xi_{p+2}\}$ is also invariant under a, and hence under b. It follows that $\alpha_2 = \beta_2$. After q steps we get all the desired equalities.

A straightforward argument extends the above statement from the direct sum of two Jordan blocks with the same eigenvalue to the direct sum of any number of Jordan blocks with the same eigenvalue. Hence, if $x \in M_n(\mathbb{C})$ whose eigenvalues are $\{\lambda_1, \dots, \lambda_k\}$ can be written as

$$x = s \begin{bmatrix} x_{\lambda_1} & 0 & \dots & 0 \\ 0 & x_{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{\lambda_k} \end{bmatrix} s^{-1},$$

where x_{λ_j}, $j = 1, \dots, k$, is the direct sum of all Jordan blocks belonging to the eigenvalue λ_j, then there exist polynomials $p_1, \dots, p_k \in \mathbb{C}[X]$ such that

$$f(x) = s \begin{bmatrix} p_1(x_{\lambda_1}) & 0 & \dots & 0 \\ 0 & p_2(x_{\lambda_2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_k(x_{\lambda_k}) \end{bmatrix} s^{-1}.$$

We need to show that we actually have $f(x) = p(x)$ for some polynomial p. The problem that the polynomials $p_1, \dots, p_k$ are not necessarily equal can be resolved by a dimension argument. To see this we observe that the linear space $\mathcal{M}$ of all matrices of the form $q(x)$, where q is any polynomial, is a subspace of the linear space $\mathcal{N}$ of all matrices of the form

$$s \begin{bmatrix} q_1(x_{\lambda_1}) & 0 & \dots & 0 \\ 0 & q_2(x_{\lambda_2}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_k(x_{\lambda_k}) \end{bmatrix} s^{-1},$$

where $q_1, \dots, q_k \in \mathbb{C}[X]$ are any polynomials. To conclude our proof we need to show that $\mathcal{M} = \mathcal{N}$ and to verify this it is enough to check that

$$\dim \mathcal{M} = \dim \mathcal{N}. \tag{3.2}$$

The dimension of $\mathcal{M}$ equals the degree of the minimal polynomial of x, which is equal to $r_1 + \dots + r_k$, where r_j, $j = 1, \dots, k$, is the size of the largest Jordan block corresponding to the eigenvalue λ_j.

On the other hand, we have

$$r_j = \dim \mathcal{N}_j = \dim\{q(x_{\lambda_j}) : q \in \mathbb{C}[X]\}$$

and since $\mathcal{N}$ is isomorphic to the direct sum of linear spaces $\mathcal{N}_j$, $j = 1, \dots, k$, we conclude that (3.2) is true. With this, (3.1) is proved. $\square$

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