DERIVATIONS PRESERVING QUASINILPOTENT ELEMENTS

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Abstract. We consider a Banach algebra A with the property that, roughly speaking, sufficiently many irreducible representations of A on nontrivial Banach spaces do not vanish on all square zero elements. The class of Banach algebras with this property turns out to be quite large – it includes C^*-algebras, group algebras on arbitrary locally compact groups, commutative algebras, $L(X)$ for any Banach space X, and various other examples. Our main result states that every derivation of A that preserves the set of quasinilpotent elements has its range in the radical of A.

1. Introduction

Let A be a Banach algebra. The spectrum of an element a in A will be denoted by $\sigma(a)$. By $Q = Q_A$ we denote the set of all quasinilpotent elements in A, i.e., $Q = \{q \in A \,|\, \sigma(q) = \{0\}\}$, and by $\mathrm{rad}(A)$ we denote the (Jacobson) radical of A. Recall that $\mathrm{rad}(A) = \{q \in A \,|\, qA \subseteq Q\}$.

Let d be a derivation of A. If is well-known that $d(A) \subseteq \mathrm{rad}(A)$ if A is commutative; under the assumption that d is continuous this was proved by Singer and Wermer [11], and without this assumption considerably later by Thomas [12]. This result has been extended to noncommutative algebras in various directions. For instance, Le Page [8] proved that $d(A) \subseteq Q$ implies $d(A) \subseteq \mathrm{rad}(A)$ in case d is an inner derivation. For a general derivation d this was established somewhat later by Turovskii and Shulman [13] (and independently in [10]). In [4] it was proved that $d(A) \subseteq \mathrm{rad}(A)$ in case there exists $M > 0$ such that $r(d(x)) \leq M r(x)$ for all $x \in A$, where $r(\,.\,)$ stands for the spectral radius. Katavolos and Stamatopoulos [9] showed that if d is an inner derivation implemented by a quasinilpotent element, then $d(Q) \subseteq Q$ implies $d(A) \subseteq \mathrm{rad}(A)$.

Does $d(Q) \subseteq Q$ implies $d(A) \subseteq \mathrm{rad}(A)$ for an arbitrary derivation d of A? This question seems natural since the condition $d(Q) \subseteq Q$ with d arbitrary covers all conditions from the preceding paragraph. However, in general the answer is negative since Q can be $\{0\}$ even when A is noncommutative [7], and in such a case every nonzero inner derivation of A gives rise to a counterexample. One is therefore forced to confine to special classes of Banach algebras. Our main result, Theorem 4.1, states that the answer to the above question is positive in case A has the property β from Definition 2.1 below. There are some obvious examples of algebras with this property, say commutative algebras and $L(X)$ with X a Banach space. Our main point, however, is that the so-called algebras with the property $\mathbb{B}$, introduced in the recent paper [1], also have the property β (Example 2.4). The class of algebras for which the above question

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has a positive answer is therefore rather large, in particular it contains C^*-algebras, group algebras on arbitrary locally compact groups, and Banach algebras generated by idempotents.

2. The property β

We will deal with the class of Banach algebras having the following property.

Definition 2.1. A Banach algebra A is said to have the *property* β if there exists a family of continuous irreducible representations $(\pi_i)_{i\in I}$ of A on Banach spaces X_i such that

(a) $\bigcap_i \ker \pi_i = \mathrm{rad}(A)$.
(b) If $\dim X_i \geq 2$, then there exists $q \in A$ such that $q^2 = 0$ and $\pi(q) \neq 0$.

Example 2.2. Every commutative Banach algebra obviously has the property β.

Example 2.3. For every Banach space X, the algebra of all bounded linear operators on X, $L(X)$, has the property β. Indeed, just take $\pi = 1$ and a nonzero finite rank nilpotent for q. More generally, a primitive Banach algebra with nonzero socle has the property β.

Example 2.4. A Banach algebra A is said to have the *property* $\mathbb{B}$ if every continuous bilinear map $\varphi\colon A \times A \to X$, where X is an arbitrary Banach space, with the property that for all $a, b \in A$,

$$ab = 0 \implies \varphi(a, b) = 0,$$

necessarily satisfies

$$\varphi(ab, c) = \varphi(a, bc) \quad (a, b, c \in A).$$

The class of Banach algebras with the property $\mathbb{B}$ is quite large. It includes C^*-algebras, group algebras on arbitrary locally compact groups, Banach algebras generated by idempotents, and topologically simple Banach algebras containing a nontrivial idempotent. Furthermore, this class is stable under the usual methods of constructing Banach algebras. For details we refer the reader to [1].

We claim that

$$A \text{ has the property } \mathbb{B} \implies A \text{ has the property } \beta.$$

Indeed, take a continuous irreducible representation π of a Banach algebra A with the property $\mathbb{B}$ on a Banach space X with $\dim(X) \geq 2$. It is enough to show that there exist $a, b \in A$ such that

$$ab = 0,\ \pi(a) \neq 0,\ \pi(b) \neq 0.$$

Namely, since $\pi(A)$ is a prime algebra, we can then find $c \in A$ such that $\pi(b)\pi(c)\pi(a) \neq 0$. Hence $q = bca$ satisfies $q^2 = 0$ and $\pi(q) \neq 0$, as required in Definition 2.1. Assume, therefore, that such a and b do not exist. That is, for all $a, b \in A$, $ab = 0$ implies $\pi(a) = 0$ or $\pi(b) = 0$. Then the continuous bilinear mapping

$$\varphi\colon A \times A \to L(X)\widehat{\otimes}L(X), \quad \varphi(a, b) = \pi(a) \otimes \pi(b) \quad (a, b \in A)$$

satisfies the condition $ab = 0 \implies \varphi(a, b) = 0$. Consequently, we have

$$\pi(a)\pi(b) \otimes \pi(c) = \pi(a) \otimes \pi(b)\pi(c) \quad (a, b, c \in A).$$

Let $\xi, \zeta \in X \setminus \{0\}$. There exist $a, b \in A$ such that $\pi(a)\xi = \zeta$ and $\pi(b)\zeta = \xi$. Then $\pi(a)\pi(b) \otimes \pi(a) = \pi(a) \otimes \pi(b)\pi(a)$ and both $\pi(a)$ and $\pi(b)\pi(a)$ are different from zero. This implies that there exists $\lambda \in \mathbb{C}$ such that $\pi(a) = \lambda\pi(b)\pi(a)$. Hence

$$\zeta = \pi(a)\xi = \lambda\pi(b)\pi(a)\xi = \lambda\pi(b)\zeta = \lambda\xi.$$

From this we conclude that $\dim(X) = 1$, a contradiction.

Example 2.5. Let A have the property β and let $(\pi_i)_{i\in I}$ be the corresponding representations. The following constructions will be used later.

(1) The quotient Banach algebra $A/\mathrm{rad}(A)$ also has the property β. Indeed, for every $i \in I$ the representation π_i drops to an irreducible representation ϖ_i of the quotient Banach algebra $A/\mathrm{rad}(A)$ on X_i by defining

$$\varpi_i(a + \mathrm{rad}(A)) = \pi_i(a) \quad (a \in A).$$

It is clear that $(\varpi_i)_{i\in I}$ satisfies the required properties.

(2) Assume that A does not have an identity element. Let $A_{\mathbf{1}}$ be the Banach algebra formed by adjoining an identity to A, so that $A_{\mathbf{1}} = \mathbb{C}\mathbf{1} \oplus A$. For every $i \in I$, the representation π_i lifts to an irreducible representation ϖ_i of $A_{\mathbf{1}}$ on X_i by defining

$$\varpi_i(\alpha\mathbf{1} + a)\xi \;=\; \alpha\xi + \pi_i(a)\xi \quad (\alpha \in \mathbb{C},\, a \in A,\, \xi \in X_i).$$

Further, we adjoin the 1-dimensional representation $\varpi(\alpha\mathbf{1} + a) = \alpha$ $(\alpha \in \mathbb{C}, a \in A)$ to the family $(\varpi_i)_{i\in I}$. Then the resulting family satisfies the requirements of Definition 2.1. That is, $A_{\mathbf{1}}$ has the property β.

3. Tools

The purpose of this section is to gather together the results needed for the proof of Theorem 4.1 below. We start with a simple lemma which indicates that it is enough to consider the condition $d(Q) \subseteq Q$ on semisimple Banach algebras.

Lemma 3.1. *Let A be a Banach algebra and let d be a derivation of A such that $d(Q) \subseteq Q$. Then $d(\mathrm{rad}(A)) \subseteq \mathrm{rad}(A)$ and the derivation D of the semisimple Banach algebra $A/\mathrm{rad}(A)$, defined by $D(x + \mathrm{rad}(A)) = d(x) + \mathrm{rad}(A)$, satisfies $D(Q_{A/\mathrm{rad}(A)}) \subseteq Q_{A/\mathrm{rad}(A)}$.*

Proof. Write $\mathcal{R}$ for $\mathrm{rad}(A)$. Then $\big(d(\mathcal{R})+\mathcal{R}\big)/\mathcal{R}$ is a two-sided ideal of the semisimple Banach algebra $A/\mathcal{R}$. Since $d(Q) \subseteq Q$, it follows that $d(\mathcal{R}) \subseteq Q$ and so $\big(d(\mathcal{R}) + \mathcal{R}\big)/\mathcal{R}$ consists of quasinilpotent elements of $A/\mathcal{R}$. Therefore $\big(d(\mathcal{R}) + \mathcal{R}\big)/\mathcal{R} = \{0\}$, that is, $d(\mathcal{R}) \subseteq \mathcal{R}$.

On account of [6, Proposition 1.5.29(i)], we have $Q_{A/\mathcal{R}} = Q_A/\mathcal{R}$ and this clearly implies that $D(Q_{A/\mathcal{R}}) \subseteq Q_{A/\mathcal{R}}$. □

We need two standard results on Banach algebra derivations (see, e.g., [6, Proposition 2.7.22(ii) and Theorem 5.2.28(iii)]).

Theorem 3.2. *Let d be a derivation on a Banach algebra A.*

(1) *(Sinclair) If d is continuous, then $d(P) \subseteq P$ for each primitive ideal P of A.*
(2) *(Johnson and Sinclair) If A is semisimple, then d is automatically continuous.*

Our main tool is the Jacobson density theorem together with its extensions. First we state a version of this theorem which includes Sinclair's generalization involving invertible elements (see, e.g., [2, Theorem 4.2.5, Corollary 4.2.6]).

Theorem 3.3. *Let π be a continuous irreducible representation of a unital Banach algebra A on a Banach space X. If $\xi_1, \dots, \xi_n$ are linearly independent elements in X, and $\eta_1, \dots, \eta_n$ are arbitrary elements in X, then there exists $a \in A$ such that $\pi(a)\xi_i = \eta_i$, $i = 1, \dots, n$. Moreover, if $\eta_1, \dots, \eta_n$ are linearly independent, then a can be chosen to be invertible.*

The next theorem is basically [3, Theorem 4.6], but stated in the analytic setting (alternatively, one can use [5, Theorem 3.6] together with Theorem 3.2).

Theorem 3.4. *Let d be a continuous derivation on a Banach algebra A, and let π be a continuous irreducible representation of A on a Banach space X. The following statements are equivalent:*

(i) *There does not exist a continuous linear operator $T : X \to X$ such that $\pi(d(x)) = T\pi(x) - \pi(x)T$ for all $x \in A$.*
(ii) *If $\xi_1, \dots, \xi_n$ are linearly independent elements in X, and $\eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_n$, are arbitrary elements in X, then there exists $a \in A$ such that*

$$\pi(a)\xi_i = \eta_i \quad \textit{and} \quad \pi(d(a))\xi_i = \zeta_i, \;\; i = 1, \dots, n.$$

4. Main theorem

We now have enough information to prove the main result of the paper.

Theorem 4.1. *Let A be a Banach algebra with the property β, and let Q be the set of its quasinilpotent elements. If a derivation d of A satisfies $d(Q) \subseteq Q$, then $d(A) \subseteq \operatorname{rad}(A)$.*

Proof. We first assume that A is semisimple and has an identity element. Obviously $d(\mathbf{1}) = 0$. On account of Theorem 3.2, d is continuous and leaves the primitive ideals of A invariant.

Take an irreducible representation π of A on a Banach space X such as in Definition 2.1. We have to show that $\pi(d(A)) = \{0\}$.

Suppose first that $\dim X = 1$. Then $P = \ker \pi$ has codimension 1 in A, so that $A = \mathbb{C}\mathbf{1} \oplus P$. Hence $d(A) \subseteq P$, which gives $\pi(d(A)) = \{0\}$.

We now assume that $\dim X \geq 2$. According to Definition 2.1, there exists $q \in A$ such that $q^2 = 0$ and $\pi(q) \neq 0$. Let $\rho \in X$ be such that

$$\omega := \pi(q)\rho \neq 0.$$

Note that ω and ρ are linearly independent for $\pi(q)^2 = 0$. Also,

$$\pi(q)\omega = 0.$$

We now consider two cases.

Case 1. Let us first consider the possibility where conditions of Theorem 3.4 are fulfilled. Then there exists $a \in A$ such that

$$\pi(a)\rho = 0, \; \pi(a)\omega = 0, \; \pi(d(a))\rho = \omega, \; \pi(d(a))\omega = -\rho + \pi(d(q))\rho,$$

and

$$\pi(a)\pi(d(q))\rho = 0$$

(if $\pi(d(q))\rho$ lies in the linear span of ρ and ω, then this follows from the first two identities). Note that for any $n \geq 2$,

$$\pi(d(a^n))\rho = \pi(d(a))\pi(a)^{n-1}\rho + \cdots + \pi(a)^{n-1}\pi(d(a))\rho = 0,$$

and, similarly,

$$\pi(d(a^n))\omega = 0.$$

Both formulas trivially also hold for $n = 0$. Consequently,

$$\pi(d(e^a))\rho = \pi\Big(d\Big(\sum_{n=0}^{\infty}\frac{1}{n!}a^n\Big)\Big)\rho = \sum_{n=0}^{\infty}\frac{1}{n!}\pi(d(a^n))\rho = \pi(d(a))\rho = \omega.$$

Similarly,

$$\pi(d(e^a))\omega = \pi(d(a))\omega = -\rho + \pi(d(q))\rho.$$

By assumption, $d(e^{-a}qe^a) \in Q$, and hence also $e^a d(e^{-a}qe^a)e^{-a} \in Q$. Expanding $d(e^{-a}qe^a)$ according to the derivation law, and also using $e^a d(e^{-a}) + d(e^a)e^{-a} = d(\mathbf{1}) = 0$, it follows that

$$b := -d(e^a)e^{-a}q + d(q) + qd(e^a)e^{-a} \in Q.$$

However,

$$\begin{aligned}\pi(b)\rho =& -\pi(d(e^a))\pi(e^{-a})\pi(q)\rho + \pi(d(q))\rho + \pi(q)\pi(d(e^a))\pi(e^{-a})\rho\\ =& -\pi(d(e^a))\pi(e^{-a})\omega + \pi(d(q))\rho + \pi(q)\pi(d(e^a))\rho\\ =& -\pi(d(e^a))\omega + \pi(d(q))\rho + \pi(q)\omega\\ =&\rho,\end{aligned}$$

implying that $1 \in \sigma(\pi(b)) \subseteq \sigma(b)$ – a contradiction. This first possibility therefore cannot occur.

Case 2. We may now assume that there exists a continuous linear operator $T : X \to X$ such that

$$\pi(d(x)) = T\pi(x) - \pi(x)T$$

for each $x \in A$. Suppose there exists $\xi \in X$ such that ξ and $\eta := T\xi$ are linearly independent. By Theorem 3.3 then there is an invertible $a \in A$ such that $\pi(a)\rho = -\eta$ and $\pi(a)\omega = \xi$. Put $c := d(aqa^{-1})$. Note that $c \in Q$ since $aqa^{-1} \in Q$. However,

$$\begin{aligned}\pi(c)\xi =&\big(T\pi(a)\pi(q)\pi(a)^{-1} - \pi(a)\pi(q)\pi(a)^{-1}T\big)\xi\\ =&T\pi(a)\pi(q)\omega - \pi(a)\pi(q)\pi(a)^{-1}\eta\\ =&\pi(a)\pi(q)\rho = \pi(a)\omega = \xi,\end{aligned}$$

and hence $1 \in \sigma(\pi(c)) \subseteq \sigma(c)$. This is a contradiction, so $T\xi$ and ξ are linearly dependent for every $\xi \in X$. It is easy to see that this implies that T is a scalar multiple of the identity, whence $\pi(d(A)) = 0$.

Finally, we consider the case when A is an arbitrary Banach algebra. On account of Lemma 3.1, $d(\mathrm{rad}(A)) \subseteq \mathrm{rad}(A)$ and therefore d drops to a derivation D on the semisimple Banach algebra $A/\mathrm{rad}(A)$ with the property that $D(Q_{A/\mathrm{rad}(A)}) \subseteq Q_{A/\mathrm{rad}(A)}$. According to Example 2.5, $A/\mathrm{rad}(A)$ has the property β. If this Banach algebra already has an identity element, then we apply what has previously been proved to show that $D(A/\mathrm{rad}(A)) = \{0\}$ and hence that $d(A) \subseteq \mathrm{rad}(A)$. If $A/\mathrm{rad}(A)$ does not have an identity element, then we consider its unitization B (considered in Example 2.5) and we extend D to a derivation Δ of B by defining $\Delta(\mathbf{1}) = 0$. It is clear that $Q_B = Q_{A/\mathrm{rad}(A)}$. Therefore $\Delta(Q_B) \subseteq Q_B$. We thus get $\Delta(B) = \{0\}$, which implies that $D(A/\mathrm{rad}(A)) = \{0\}$ and therefore that $d(A) \subseteq \mathrm{rad}(A)$. □

Remark 4.2. From the proof of Theorem 4.1 it is evident that in the case where A is semisimple, the assumption that $d(Q) \subseteq Q$ can be replaced by a milder assumption that $d(q) \in Q$ for every square zero element $q \in A$.

Corollary 4.3. *Let A be a C^*-algebra and let Q be the set of its quasinilpotent elements. If a derivation d of A satisfies $d(Q) \subseteq Q$, then $d = 0$.*

Corollary 4.4. *Let G be a locally compact group and let Q be the set of the quasinilpotent elements of $L^1(G)$. If a derivation d of $L^1(G)$ satisfies $d(Q) \subseteq Q$, then $d = 0$.*

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