DERIVATIONS OF TENSOR PRODUCTS OF NONASSOCIATIVE ALGEBRAS

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Abstract. Let R and S be nonassociative unital algebras. Assuming that either one of them is finite dimensional or both are finitely generated, we show that every derivation of $R \otimes S$ is the sum of derivations of the following three types: (a) $\operatorname{ad} u$ where u belongs to the nucleus of $R \otimes S$, (b) $L_z \otimes f$ where f is a derivation of S and z lies in the center of R, and (c) $g \otimes L_w$ where g is a derivation of R and w lies in the center of S.

1. Introduction

Let R and S be nonassociative algebras. What are natural examples of derivations of the tensor product algebra $R \otimes S$? First of all, just as in any algebra, every element u from the nucleus gives rise to the derivation $x \mapsto ux - xu$. Next, given a derivation f of S and an element z from the center of R, the map given by $x \otimes y \mapsto zx \otimes f(y)$ is a derivation of $R \otimes S$. Similarly, $x \otimes y \mapsto g(x) \otimes wy$ defines a derivation of $R \otimes S$ for every derivation g of R and every central element $w \in S$. The goal of this short paper is to prove that under rather mild assumptions – namely, both R and S are unital and either one of them is finite dimensional or both are finitely generated – every derivation of $R \otimes S$ is the sum of derivations of the three types just described. From the nature of this result, and the relative simplicity of its proof, one would expect that it is known; however, we have not been able to find it in the literature. Among related results, we first mention the one by Block [3, Theorem 7.1] which considers a similar situation, just that the assumption that R is unital is weakened and, on the other hand, S is assumed to be associative and commutative. See also [1] for some extensions of Block's theorem. Benkart and Osborn dealt with the special case where R is the (associative) matrix algebra $M_n(F)$ [2, Corollary 4.9]. Finally, in the case where both R and S are associative, the description of derivations of $R \otimes S$ can be (under some finiteness assumptions) obtained as a byproduct of results on Hochschild cohomology; see, for example, [7, Corollary 3.4].

In the next section we provide all definitions and prove a basic lemma. The third section is devoted to the main result, and in the last, fourth, section we record some corollaries.

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2. Preliminaries

Let A be a nonassociative (i.e., not necessarily associative) algebra over a field F. For $x, y, z \in A$ we write

$$[x, y, z] = (xy)z - x(yz).$$

The set

$$N(A) = \{n \in A \,|\, [n, A, A] = [A, n, A] = [A, A, n] = 0\}$$

is called the *nucleus* of A, and the set

$$Z(A) = \{z \in N(A) \,|\, zx = xz \text{ for all } x \in A\}$$

is called the *center* of A. Of course, A is associative if and only if $N(A) = A$, and in this case the center is simply the set of elements that commute with all elements in A. We will consider the case where $A = R \otimes S$, the tensor product of unital algebras R and S. It is therefore important to note that

$$N(R \otimes S) = N(R) \otimes N(S),$$

as one can readily check.

Recall that a linear map $d : A \to A$ is called a *derivation* if it satisfies

$$d(xy) = d(x)y + xd(y) \text{ for all } x, y \in A.$$

By $\mathrm{Der}(A)$ we denote the set of all derivations of A. Further, for every $u \in A$ we define $L_u, R_u, \mathrm{ad}\, u : A \to A$ by

$$L_u(x) = ux, \;\; R_u(x) = xu, \;\; \mathrm{ad}\, u = L_u - R_u.$$

Note that $\mathrm{ad}\, u \in \mathrm{Der}(A)$ if $u \in N(A)$. (If A is associative, such a derivation is said to be *inner*; in nonassociative algebras one defines inner derivations somewhat differently, cf. [8, p. 21]).

The following simple lemma will be needed in the proof of the main result.

Lemma 2.1. *Let R and S be nonassociative algebras, let d be a derivation of $R \otimes S$, and let $\{s_i \,|\, i \in I\}$ be a basis of S. Suppose that S is unital. Then for each $i \in I$ there exists a derivation d_i of R such that for every $x \in R$ we have*

$$d(x \otimes 1) = \sum_{i \in I} d_i(x) \otimes s_i \tag{2.1}$$

and $d_i(x) = 0$ for all but finitely many $i \in I$. Furthermore, if R is finitely generated, then $d_i = 0$ for all but finitely many i.

Proof. Of course, for every $x \in R$ there exist uniquely determined elements $d_i(x) \in R$ such that (2.1) holds and $d_i(x) = 0$ for all but finitely many $i \in I$. The linearity of d clearly implies the linearity of $d_i : R \to R$. Further,

$$d(xu \otimes 1) = d\big((x \otimes 1) \cdot (u \otimes 1)\big) = d(x \otimes 1)(u \otimes 1) + (x \otimes 1)d(u \otimes 1)$$

yields

$$\sum_{i \in I} \big(d_i(xu) - d_i(x)u - xd_i(u)\big) \otimes s_i = 0,$$

which implies that $d_i \in \operatorname{Der}(R)$. Finally, assume that R is generated by the set $\{r_1, \dots, r_m\}$. The set I_0 of all $i \in I$ such that $d_i(r_j) \neq 0$ for some $j \in \{1, \dots, m\}$ is finite. Clearly, $d_i = 0$ for every $i \in I \setminus I_0$. □

In general, there may be infinitely many nonzero derivations d_i of R such that for each $x \in R$ we have $d_i(x) = 0$ for all but finitely many i. For example, this holds for the partial derivatives $\frac{\partial}{\partial X_i}$ on $F[X_1, X_2, \dots]$. In such a case, given any elements $w_i \in Z(S)$ we have that

$$d = \sum_{i \in I} d_i \otimes L_{w_i}$$

is a derivation of $R \otimes S$.

3. Main result

We are now in a position to prove our main theorem.

Theorem 3.1. *Let R and S be nonassociative unital algebras. Suppose that either at least one of R and S is finite dimensional or they both are finitely generated. Then every derivation d of $R \otimes S$ can be written as*

$$d = \operatorname{ad} u + \sum_{j=1}^{p} L_{z_j} \otimes f_j + \sum_{i=1}^{q} g_i \otimes L_{w_i},$$

where $u \in N(R) \otimes N(S)$, $z_j \in Z(R)$, $w_i \in Z(S)$, $f_j \in \operatorname{Der}(S)$, and $g_i \in \operatorname{Der}(R)$.

Proof. Pick a basis $\{w_i \,|\, i \in I\}$ of $Z(S)$ and extend it to a basis of $\{w_i \,|\, i \in I\} \cup \{s_{i'} \,|\, i' \in I'\}$ of S. According to our assumption, either R is finitely generated or S is finite dimensional. Using Lemma 2.1 we see that in each of the two cases we may conclude that there exist finitely many $g_i, h_i \in \operatorname{Der}(R)$ such that, by a slight abuse of notation,

$$d(x \otimes 1) = \sum_{i=1}^{q} g_i(x) \otimes w_i + \sum_{i=1}^{l} h_i(x) \otimes s_i \tag{3.1}$$

for every $x \in R$. Analogously, we have

$$d(1 \otimes y) = \sum_{j=1}^{p} z_j \otimes f_j(y) + \sum_{j=1}^{m} r_j \otimes k_j(y) \tag{3.2}$$

for every $y \in S$ and some $f_j, k_j \in \operatorname{Der}(S)$ and some $z_j \in Z(R)$ and $r_j \in R$ which play a similar role as $w_i \in Z(S)$ and $s_i \in S$. Combining (3.1) and

(3.2) we obtain

$$\begin{aligned}
d(x\otimes y) =& d\big((x\otimes 1)\cdot(1\otimes y)\big)\\
=& d(x\otimes 1)(1\otimes y)+(x\otimes 1)d(1\otimes y)\\
=& \sum_{i=1}^{q} g_i(x)\otimes w_i y+\sum_{i=1}^{l} h_i(x)\otimes s_i y\\
&+\sum_{j=1}^{p} xz_j\otimes f_j(y)+\sum_{j=1}^{m} xr_j\otimes k_j(y).
\end{aligned}$$

Thus,

$$d=\sum_{i=1}^{q} g_i\otimes L_{w_i}+\sum_{i=1}^{l} h_i\otimes L_{s_i}+\sum_{j=1}^{p} L_{z_j}\otimes f_j+\sum_{j=1}^{m} R_{r_j}\otimes k_j \tag{3.3}$$

(here we have used that $L_{z_j}=R_{z_j}$ for $z_j\in Z(R)$). Since $g_i\otimes L_{w_i}$ and $L_{z_j}\otimes f_j$ are derivations of $R\otimes S$, so is

$$\delta := d-\sum_{j=1}^{p} L_{z_j}\otimes f_j-\sum_{i=1}^{q} g_i\otimes L_{w_i}.$$

By (3.3), we can write δ as

$$\delta=\sum_{i=1}^{l} h_i\otimes L_{s_i}+\sum_{j=1}^{m} R_{r_j}\otimes k_j.$$

The theorem will be proved by showing that $\delta=\operatorname{ad} u$ for some $u\in N(R)\otimes N(S)$.

Suppose that at least one h_i is nonzero. Without loss of generality we may assume that $\{h_1,\dots,h_s\}$ is a maximal linearly independent subset of $\{h_1,\dots,h_l\}$. Writing each h_i with $i>s$ as a linear combination of $h_1,\dots,h_s$ we see that $\sum_{i=1}^{l} h_i\otimes L_{s_i}$ can be rewritten as $\sum_{i=1}^{s} h_i\otimes L_{n_i}$ where n_i are linearly independent elements in $\operatorname{span}\{s_1,\dots,s_l\}$. Similarly, by assuming that $\{k_1,\dots,k_t\}$ is a maximal linearly independent subset of $\{k_1,\dots,k_m\}$ we can rewrite $\sum_{j=1}^{m} R_{r_j}\otimes k_j$ as $\sum_{j=1}^{t} R_{m_j}\otimes k_j$ where m_j are linearly independent elements in $\operatorname{span}\{r_1,\dots,r_m\}$. To summarize, we have

$$\delta=\sum_{i=1}^{s} h_i\otimes L_{n_i}+\sum_{j=1}^{t} R_{m_j}\otimes k_j, \tag{3.4}$$

where $h_1,\dots,h_s\in \operatorname{Der}(R)$ are linearly independent (or all zero), $k_1,\dots,k_t\in \operatorname{Der}(S)$ are linearly independent (or all zero), the elements $n_1,\dots,n_s\in S$ are linearly independent and such that

$$\operatorname{span}\{n_1,\dots,n_s\}\cap Z(S)=0, \tag{3.5}$$

and $m_1,\dots,m_t\in R$ are linearly independent and such that

$$\operatorname{span}\{m_1,\dots,m_t\}\cap Z(R)=0. \tag{3.6}$$

Let us express δ in a different way. Since h_i and k_j, as derivations, vanish on unity, we have

$$\begin{aligned}\delta(x\otimes y) &= \delta\big((1\otimes y)\cdot(x\otimes 1)\big)\\ &= (1\otimes y)\delta(x\otimes 1)+\delta(1\otimes y)(x\otimes 1)\\ &= \sum_{i=1}^{s} h_i(x)\otimes yn_i+\sum_{j=1}^{t} m_j x\otimes k_j(y).\end{aligned}$$

Thus,

$$\delta=\sum_{i=1}^{s} h_i\otimes R_{n_i}+\sum_{j=1}^{t} L_{m_j}\otimes k_j. \tag{3.7}$$

Combining both expressions of δ, (3.4) and (3.7), we will now show that $n_i\in N(S)$ for every i. This will be achieved by computing $\delta(x\otimes yv)$, where $x\in R$ and $y,v\in S$, in several ways. First, using (3.4) we obtain

$$\begin{aligned}\delta(x\otimes yv) &= \delta\big((1\otimes y)\cdot(x\otimes v)\big)\\ &= (1\otimes y)\delta(x\otimes v)+\delta(1\otimes y)(x\otimes v)\\ &= \sum_{i=1}^{s} h_i(x)\otimes y(n_i v)+\sum_{j=1}^{t} xm_j\otimes yk_j(v)+\sum_{j=1}^{t} m_j x\otimes k_j(y)v.\end{aligned}$$

On the other hand, using (3.7) we obtain

$$\begin{aligned}\delta(x\otimes yv) &= \delta\big((x\otimes y)\cdot(1\otimes v)\big)\\ &= \delta(x\otimes y)(1\otimes v)+(x\otimes y)\delta(1\otimes v)\\ &= \sum_{i=1}^{s} h_i(x)\otimes (yn_i)v+\sum_{j=1}^{t} m_j x\otimes k_j(y)v+\sum_{j=1}^{t} xm_j\otimes yk_j(v).\end{aligned}$$

Comparing these two expressions we get

$$\sum_{i=1}^{s} h_i(x)\otimes\big(y(n_i v)-(yn_i)v\big)=0$$

for all $x\in R$, $y,v\in S$. This can be written as

$$\sum_{i=1}^{s} h_i\otimes(L_y L_{n_i}-L_{yn_i})=0$$

for every $y\in S$. Since $h_1,\dots,h_m$ are linearly independent it follows, by a basic property of tensor products, that $L_y L_{n_i}-L_{yn_i}=0$ for every $y\in S$ and every i. That is,

$$[S,n_i,S]=0$$

for every i.

In the second step we use only (3.4). On the one hand, we have

$$\begin{aligned}\delta(x\otimes yv) &= \delta\big((x\otimes y)\cdot(1\otimes v)\big)\\ &= \delta(x\otimes y)(1\otimes v)+(x\otimes y)\delta(1\otimes v)\\ &= \sum_{i=1}^{s} h_i(x)\otimes(n_iy)v+\sum_{j=1}^{t} xm_j\otimes k_j(y)v+\sum_{j=1}^{t} xm_j\otimes yk_j(v).\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta(x\otimes yv) &= \sum_{i=1}^{s} h_i(x)\otimes n_i(yv)+\sum_{j=1}^{t} xm_j\otimes k_j(yv)\\ &= \sum_{i=1}^{s} h_i(x)\otimes n_i(yv)+\sum_{j=1}^{t} xm_j\otimes k_j(y)v+\sum_{j=1}^{t} xm_j\otimes yk_j(v).\end{aligned}$$

Comparing we obtain

$$\sum_{i=1}^{s} h_i(x)\otimes\big((n_iy)v-n_i(yv)\big)=0.$$

Similarly as above we see that this implies

$$[n_i,S,S]=0$$

for every i. Analogously we derive from (3.7) that

$$[S,S,n_i]=0.$$

Thus, $n_i\in N(S)$, $i=1,\dots,s$.

In a similar fashion one proves that $m_j\in N(R)$, $j=1,\dots,t$.

From (3.4) and (3.7) it follows that

$$\sum_{i=1}^{s} h_i\otimes \operatorname{ad} n_i=\sum_{j=1}^{t}\operatorname{ad} m_j\otimes k_j.$$

In view of (3.6), $\{\operatorname{ad} m_1,\dots,\operatorname{ad} m_t\}$ is a linearly independent set. Therefore each k_j is a linear combination of the $\operatorname{ad} n_i$'s (see, e.g., [4, Lemma 4.9]). Thus, there exist $\lambda_{ij}\in F$ such that

$$k_j=\sum_{i=1}^{s}\lambda_{ij}\operatorname{ad} n_i.$$

Consequently,

$$\sum_{i=1}^{s}\Big(h_i-\sum_{j=1}^{t}\lambda_{ij}\operatorname{ad} m_j\Big)\otimes\operatorname{ad} n_i=0.$$

Since, by (3.5), the set $\{\operatorname{ad} n_1,\dots,\operatorname{ad} n_m\}$ is linearly independent, it follows that

$$h_i=\sum_{j=1}^{t}\lambda_{ij}\operatorname{ad} m_j.$$

Accordingly, using (3.7) we have

$$\begin{aligned}\delta =&\sum_{i=1}^{s}\sum_{j=1}^{t}\lambda_{ij}\Big(\mathrm{ad}\, m_j\otimes R_{n_i}+L_{m_j}\otimes \mathrm{ad}\, n_i\Big)\\ =&\sum_{i=1}^{s}\sum_{j=1}^{t}\lambda_{ij}\Big(L_{m_j}\otimes L_{n_i}-R_{m_j}\otimes R_{n_i}\Big)\\ =&\sum_{i=1}^{s}\sum_{j=1}^{t}\lambda_{ij}\mathrm{ad}\,(m_j\otimes n_i),\\ =&\mathrm{ad}\,\Big(\sum_{i=1}^{s}\sum_{j=1}^{t}\lambda_{ij}m_j\otimes n_i\Big),\end{aligned}$$

which is the desired conclusion. □

4. Corollaries

We close this paper by three rather straightforward corollaries to Theorem 3.1. The first one considers the situation where there are no other derivations than those of the form $\mathrm{ad}\, u$ with u from the nucleus.

Corollary 4.1. *Let R and S be as in Theorem 3.1. If every derivation of R is of the form $\mathrm{ad}\, m$ for some $m\in N(R)$ and every derivation of S is of the form $\mathrm{ad}\, n$ for some $n\in N(S)$, then every derivation of $R\otimes S$ is of the form $\mathrm{ad}\, u$ for some $u\in N(R)\otimes N(S)$.*

Proof. If $g=\mathrm{ad}\, m$, $m\in N(R)$, and $w\in Z(S)$, then $g\otimes L_w=\mathrm{ad}\,(m\otimes w)$. Similarly, if $z\in Z(R)$ and $f=\mathrm{ad}\, n$, $n\in N(S)$, then $L_z\otimes f=\mathrm{ad}\,(z\otimes n)$. □

If R and S are associative, this corollary gets a simpler form: if both R and S have the property that all their derivations are inner, then so does $R\otimes S$. It would be interesting to find out whether or not this also holds without the finiteness assumptions.

Since the center of the matrix algebra $M_n(F)$ consists of scalar multiples of the identity matrix, and since every derivation of $M_n(F)$ is, as is well-known, inner, the following result by Benkart and Osborn follows immediately.

Corollary 4.2. [2, Corollary 4.9] *Let S be an arbitrary nonassociative unital algebra. Then every derivation d of $M_n(S)$ can be written as $d=\mathrm{ad}\, u+f^\sharp$ where $u\in M_n(N(S))$ and $f^\sharp$ is a derivation obtained by applying a derivation f of S to each matrix entry.*

The upper triangular matrix algebra $T_n(F)$ has the same properties, i.e., its center is trivial and all of its derivations are inner. Hence we have the following corollary.

Corollary 4.3. *Let S be an arbitrary nonassociative unital algebra. Then every derivation d of $T_n(S)$ can be written as $d=\mathrm{ad}\, u+f^\sharp$ where $u\in$*

$T_n(N(S))$ and $f^\sharp$ is a derivation obtained by applying a derivation f of S to each matrix entry.

Apparently, this corollary is known only in the case where S is associative [5, 6].

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