FUNCTIONAL IDENTITIES ON MATRIX ALGEBRAS

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ABSTRACT. Complete solutions of functional identities $\sum_{k \in K} F_k(\overline{x}_m^k)x_k = \sum_{l \in L} x_l G_l(\overline{x}_m^l)$ on the matrix algebra $M_n(\mathbb{F})$ are given. The nonstandard parts of these solutions turn out to follow from the Cayley-Hamilton identity.

1. INTRODUCTION

A functional identity on a ring R is an identical relation that involves arbitrary elements from R along with functions which are considered as unknowns. The fundamental example of such an identity is

$$(1.1) \qquad \sum_{k \in K} F_k(\overline{x}_m^k)x_k = \sum_{l \in L} x_l G_l(\overline{x}_m^l) \quad \text{for all } x_1 \ldots, x_m \in R,$$

where $\overline{x}_m^k = (x_1, \ldots, \hat{x}_k, \ldots, x_m)$, K and L are subsets of $\{1, \ldots, m\}$, and F_k, G_l are arbitrary functions from R^{m-1} to R. The description of these functions, which is the usual goal when facing (1.1), has turned to be applicable to various mathematical problems. We refer the reader to [BCM07] for an account of the theory of functional identities and its applications.

Up until very recently functional identities have been studied only in rings in which (1.1) has only so-called *standard solutions* (see Subsection 4.1 for the definition). This excludes the basic case where $R = M_n(\mathbb{F})$, the algebra of $n \times n$ matrices over a field $\mathbb{F}$, unless $|K| \leq n$ and $|L| \leq n$. Various applications of the general theory of functional identities therefore do not cover $M_n(\mathbb{F})$ with small n, although they yield definitive results for large classes of algebras whose dimension is either infinite or finite but big enough.

The recent papers [BS14] and [BPS14] give, to the best of our knowledge, the first complete results on functional identities of $M_n(\mathbb{F})$. The first one considers functional identities in one variable, and the second one considers quasi-identities; i.e., identities of the type (1.1) with $L = \varnothing$ and each F_k being a sum of scalar-valued functions multiplied by noncommutative monomials. In the present paper we take a step further and consider the general functional identity (1.1). We will give a full description of the functions F_k and G_l under the natural assumption that they are multilinear. Thus, in some sense we can say now that functional identities are finally understood in both finite and infinite dimensional context.

The Cayley-Hamilton theorem yields the fundamental example of a functional identity of $M_n(\mathbb{F})$ which does not have only standard solutions. We call it the *Cayley-Hamilton identity*.

From the results in [BS14] it is evident that all nonstandard solutions of functional identities in one variable of $M_n(\mathbb{F})$ follow from the Cayley-Hamilton identity, while the main result of [BPS14] shows that there exist quasi-identities of $M_n(\mathbb{F})$ which are not a consequence of the Cayley-Hamilton identity; in more technical terms, the T-ideal of all quasi-identities is not generated by the Cayley-Hamilton identity. The results in this paper will show that the nonstandard parts of the solutions of (1.1) follow from the Cayley-Hamilton identity. (This does not contradict the result of [BPS14] since the T-ideal of quasi-identities is a proper subset of the set of identities treated here.) Our main idea of handling (1.1) is to interpret this functional identity coordinate-wise and then apply the theory of syzygies (in fact, functional identities may be viewed as a kind of "noncommutative syzygies").

The paper is organized as follows. In Section 2 we give some remarks on generalized polynomial identities, which are, in the context of matrix algebras, more general than the functional identities (1.1). Our main results are obtained in Sections 3 and 4. In Section 3 we describe all solutions of the one-sided functional identities, i.e., those with $K = \varnothing$ or $L = \varnothing$, by applying the result on generators of syzygies on generic matrices from [Onn94]. In Section 4, which is basically independent of Section 3, we show that every solution of the two-sided functional identity (1.1) is *standard modulo one-sided identities*. Roughly speaking, this means that it can be expressed by standard solutions and solutions of one-sided identities. A precise definition is given in Subsection 4.1. We end the paper with an application to the theory of commuting functions.

2. Generalized polynomial identities

It is well-known that the T-ideal of trace identities of $M_n(\mathbb{F})$ is generated, as a T-ideal, by the Cayley-Hamilton identity [Pro07, p. 444]. In this sense every polynomial identity is a consequence of the Cayley-Hamilton identity. We have addressed ourselves the question whether a similar statement holds for the generalized polynomial identities. As we shall see, the answer is positive in the char$(\mathbb{F}) = 0$ case, and the proof is not difficult. From the nature of this result one would expect that it should be known, but we were unable to find it in the literature.

Let us recall the necessary definitions on generalized polynomial identities (for more details see [BMM96]). Let $x = \{x_1, x_2, \dots\}$ be a set of noncommuting indeterminates, and let $\mathbb{F}\langle x \rangle$ be the free algebra on X. Consider the free product $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ over $\mathbb{F}$. Its elements are sometimes called *generalized polynomials*. Informally they can be viewed as sums of expressions of the form $a_{i_0} x_{j_1} a_{i_1} \dots a_{i_{k-1}} x_{j_k} a_{i_k}$ where $a_{i_\ell} \in M_n(\mathbb{F})$. Given $f = f(x_1, \dots, x_k) \in M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ and $b_1, \dots, b_k \in M_n(\mathbb{F})$, we define the *evaluation* $f(b_1, \dots, b_k)$ in the obvious way. We say that f is a *generalized polynomial identity* (GPI) of $M_n(\mathbb{F})$ if $f(b_1, \dots, b_k) = 0$ for all $b_i \in M_n(\mathbb{F})$. For example, if e is a rank one idempotent in $M_n(\mathbb{F})$, then $[ex_1e, ex_2e]$ is readily a GPI. We remark that every identity of the form (1.1) can be interpreted as a GPI. This is because every multilinear function $F : M_n(\mathbb{F})^{m-1} \to M_n(\mathbb{F})$ is equal to a sum of functions of the form $(b_1, \dots, b_{m-1}) \mapsto a_{i_0} b_{j_1} a_{i_1} \dots a_{i_{m-2}} b_{j_{m-1}} a_{i_{m-1}}$ where $\{j_1, \dots, j_{m-1}\} = \{1, \dots, m-1\}$.

An ideal I of $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ is said to be a *T-ideal* if $f(x_1, \dots, x_k) \in I$ and $g_1, \dots, g_k \in M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ implies $f(g_1, \dots, g_k) \in I$. The set of all GPI's of $M_n(\mathbb{F})$ is clearly a T-ideal. The next proposition was obtained, in some form, already in [Lit31], and later extended to

considerably more general rings by Beidar (see e.g. [BMM96]). We will give a short alternative proof. Let us first recall the following elementary fact: If an algebra B contains a set of $n \times n$ matrix units; i.e., a set of elements e_{ij}, $1 \leq i, j \leq n$, satisfying

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \sum_{i=1}^{n} e_{ii} = 1,$$

then B is isomorphic to the matrix algebra $M_n(A)$ where $A = e_{11}Be_{11}$.

Proposition 2.1 ([BMM96, Proposition 6.5.5]). *Let e be a rank one idempotent in $M_n(\mathbb{F})$.*

(1) *If $|\mathbb{F}| = \infty$, then the T-ideal of all GPI's of $M_n(\mathbb{F})$ is generated by $[ex_1e, ex_2e]$.*
(2) *If $|\mathbb{F}| = q$, then the T-ideal of all GPI's of $M_n(\mathbb{F})$ is generated by $[ex_1e, ex_2e]$ and $(ex_1e)^q - ex_1e$.*

Proof. Pick a set of matrix units e_{ij}, $1 \leq i, j \leq n$, of $M_n(\mathbb{F})$ so that $e_{11} = e$. As $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ contains $M_n(\mathbb{F})$ as a subalgebra, it also contains all e_{ij}. Accordingly, $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle \cong M_n(A)$ where $A = e_{11}\big(M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle\big)e_{11}$. Since $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ is generated by the elements

$$e_{si}x_ke_{jt} = e_{s1}(e_{1i}x_ke_{j1})e_{1t},$$

it follows that A is generated by the elements

$$x_{ij}^{(k)} := e_{1i}x_ke_{j1}, \quad 1 \leq i, j \leq n, \ k = 1, 2, \dots$$

Note that A is actually the free algebra on the set $\overline{x} := \{x_{ij}^{(k)} \,|\, 1 \leq i, j \leq n, \ k = 1, 2, \dots \}$.

Let $\mathrm{Hom}_{M_n}(M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle, M_n(\mathbb{F}))$ denote the set of algebra homomorphisms from $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ to $M_n(\mathbb{F})$ that act as the identity on $M_n(\mathbb{F})$. Identifying $M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ with $M_n(A)$ one easily sees that there is a canonical isomorphism

$$\mathrm{Hom}_{M_n}(M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle, M_n(\mathbb{F})) \cong \mathrm{Hom}(A, \mathbb{F}).$$

Now take a GPI f of $M_n(\mathbb{F})$. This means that

$$f \in \cap_{\phi \in \mathrm{Hom}_{M_n}(M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle, M_n(\mathbb{F}))} \ker \phi,$$

or equivalently,

$$e_{1i}fe_{j1} \in \cap_{\phi \in \mathrm{Hom}(A, \mathbb{F})} \ker \phi$$

for every $1 \leq i, j \leq n$. Since A is the free algebra on $\overline{x}$ it can be easily shown that $\cap_{\phi \in \mathrm{Hom}(A, \mathbb{F})} \ker \phi$ is generated by

$$[x_{ij}^{(k)}, x_{pq}^{(l)}] = [ex_{ij}^{(k)}e, ex_{pq}^{(l)}e]$$

and additionally by

$$\left(x_{ij}^{(k)}\right)^q - x_{ij}^{(k)} = \left(ex_{ij}^{(k)}e\right)^q - ex_{ij}^{(k)}e$$

if $|\mathbb{F}| = q$. Hence $f = \sum_{i,j} e_{i1}(e_{1i}fe_{j1})e_{1j}$ lies in the T-ideal generated by $[ex_1e, ex_2e]$, and additionally by $(ex_1e)^q - ex_1e$ if $|\mathbb{F}| = q$. ∎

Let
$$q_n(x_1) = x_1^n + \tau_1(x_1)x_1^{n-1} + \cdots + \tau_n(x_1)$$
denote the Cayley-Hamilton polynomial. Thus, $\tau(x_1) = -\mathrm{tr}(x_1)$ and $\tau_n(x_1) = (-1)^n \det(x_1)$. Until the end of this section we assume that $\mathrm{char}(\mathbb{F}) = 0$. Then each $\tau_i(x_1)$ can be expressed as a $\mathbb{Q}$-linear combination of the products of $\mathrm{tr}(x_1^j)$. Let
$$Q_n = Q_n(x_1, \ldots, x_n)$$
denote the multilinear version of $q_n(x_1)$ obtained by full polarization. For example,
$$Q_2(x_1, x_2) = x_1x_2 + x_2x_1 - \mathrm{tr}(x_1)x_2 - \mathrm{tr}(x_2)x_1 + \mathrm{tr}(x_1)\mathrm{tr}(x_2) - \mathrm{tr}(x_1x_2).$$
As $Q_n(b_1, \ldots, b_n) = 0$ for all $b_i \in M_n(\mathbb{F})$, we call Q_n the *Cayley-Hamilton identity*. Recall that Q_n can be written as

$$(2.1) \qquad\qquad Q_n = \sum_{\sigma \in S_{n+1}} (-1)^\sigma \phi_\sigma(x_1, \ldots, x_n)$$

where $(-1)^\sigma = \pm 1$ denotes the sign of the permutation σ and ϕ_σ is defined using the cycle decomposition of
$$\sigma = (i_1, \ldots, i_{k_1})(j_1, \ldots, j_{k_2}) \ldots (u_1, \ldots, u_{k_t})(s_1, \ldots s_k, n+1)$$
as
$$\phi_\sigma(x_1, \ldots, x_n) = \mathrm{tr}(x_{i_1} \cdots x_{i_{k_1}})\mathrm{tr}(x_{j_1} \cdots x_{j_{k_2}}) \cdots \mathrm{tr}(x_{u_1} \cdots x_{u_{k_t}})x_{s_1} \cdots x_{s_k}.$$
Regarding $\mathrm{tr}(y)$ as $\sum_{i,j} e_{ij}ye_{ji}$, and consequently $\mathrm{tr}(y_1 \ldots y_k)$ as $\sum_{i,j} e_{ij}y_1 \ldots y_k e_{ji}$, we see that we may consider Q_n as a GPI of $M_n(\mathbb{F})$.

Corollary 2.2. *If* $\mathrm{char}(\mathbb{F}) = 0$, *then the T-ideal of all GPI's of* $M_n(\mathbb{F})$ *is generated by the Cayley-Hamilton identity* Q_n.

Proof. It suffices to show that the basic identity
$$[ex_1e, ex_2e] = ex_1ex_2e - ex_2ex_1e \in M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle,$$
where e is a rank one idempotent, follows from the Cayley-Hamilton identity. To this end we insert $[ex_1e, ex_2e]$ for one variable and e for the others in Q_n. Note that 1 needs to be in the last cycle of σ for $\phi_\sigma([ex_1e, ex_2e], e, \ldots, e)$ to be nonzero. In this case $\phi_\sigma([ex_1e, ex_2e], e, \ldots, e) = [ex_1e, ex_2e]$. Thus, we need to count the number of such permutations with the corresponding signs.

Take a cycle τ. Note that permutations with the corresponding signs having the last cycle τ do not sum to zero only if τ is of length n or $n+1$. For τ of length n we have $(n-1)(n-1)!$ permutations of sign $(-1)^{n-1}$, while for τ of length $n+1$ the number of permutation is $n!$ and all have sign $(-1)^n$. Hence,
$$Q_n([ex_1e, ex_2e], e, \ldots, e) = (-1)^n(n-1)![ex_1e, ex_2e].$$
∎

3. One-sided functional identities and syzygies

One-sided functional identities are intimately connected with syzygies on generic matrices. We first recall a result on syzygies that will be used in the sequel.

3.1. **Syzygies on a generic matrix.** First we introduce some auxiliary notation. Let $r, s \geq 1$, let $C_y = \mathbb{F}[y_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq s]$ be a polynomial algebra, and let $u_1, \ldots, u_r$ be generators of the free module C_y^n. The matrix of independent commutative variables $Y = (y_{ij})_{1 \leq i \leq r, 1 \leq j \leq s}$ is called a generic $r \times s$-matrix. With a syzygy on Y we mean a syzygy on the rows of Y; i.e., an element $\sum_{i=1}^r f_i u_i \in C_y^r$ with the property $\sum_{i=1}^r f_i y_{ij} = 0$ for $j = 1, \ldots, s$. If $s \leq r$ we denote by $[j_1, \ldots, j_s]$, where $1 \leq j_1 < \cdots < j_s \leq r$, the determinant of the $s \times s$-submatrix of Y, obtained by restricting Y to the rows indexed by $j_1, \ldots, j_s$.

Theorem 3.1 ([Onn94, Theorem 7.2]). *Let $1 \leq s < r$ and let Y be a generic $s \times r$-matrix. The set of determinantal relations*

$$G = \left\{ \sum_{\ell=0}^s (-1)^\ell [j_0, \ldots, \hat{j}_\ell, \ldots, j_s] u_{j_\ell} \mid 1 \leq j_0 < j_1 < \cdots < j_s \leq n \right\}$$

generates the module of syzygies on Y and is a Gröbner basis for it with respect to any lexicographic monomial order on C_y^r.

3.2. **Solving left-sided functional identities.** We restrict ourselves to the *left-sided functional identities*; i.e., functional identities of the form $\sum_{k \in K} F_k(\overline{x}_m^k) x_k = 0$. The right-sided functional identities $\sum_{l \in L} x_l G_l(\overline{x}_m^l) = 0$ can be of course treated in the same way. Besides, by applying the transpose operation to a right-sided functional identity we obtain a left-sided functional identity, so that the results that we will obtain are more or less directly applicable to right-sided functional identities.

The standard solution of $\sum_{k \in K} F_k(\overline{x}_m^k) x_k = 0$ is defined as $F_k = 0$ for each $k \in K$. If $|K| \leq n$, then there are no other solutions on $M_n(\mathbb{F})$ – this is an easy consequence of the general theory of functional identities (see e.g. [BCM07, Corollary 2.23]). To obtain a nonstandard solution we have to modify the Cayley-Hamilton identity. We take only its noncentral part and commute it with the product of a fixed matrix and a new variable. In this way we arrive at a basic functional identity on $M_n(\mathbb{F})$ in $n+1$ variables

(3.1) $$\left[\tilde{Q}_n(a_1 x_1, \ldots, a_n x_n), a_{n+1} x_{n+1}\right] = 0,$$

where $\tilde{Q}_n$ denotes the noncentral part of Q_n; i.e.,

$$\tilde{Q}_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_{n+1} \setminus S_n} \epsilon_\sigma \phi_\sigma(x_1, \ldots, x_n),$$

and $a_i \in M_n(\mathbb{F})$. Note that (3.1) can be indeed interpreted as a left-sided functional identity. For example, in the case $n = 2$ we have

$$\tilde{Q}_2(x_1, x_2) = x_1 x_2 + x_2 x_1 - \text{tr}(x_1) x_2 - \text{tr}(x_2) x_1,$$

so that

$$\left[\tilde{Q}_2(a_1 x_1, a_2 x_2), a_3 x_3\right] = \left(a_3 x_3(-a_2 x_2 + \text{tr}(a_2 x_2)) a_1\right) x_1 + \left(a_3 x_3(-a_1 x_1 + \text{tr}(a_1 x_1)) a_2\right) x_2$$

$$+ \left(\left(a_1 x_1 a_2 x_2 + a_2 x_2 a_1 x_1 - \text{tr}(a_1 x_1) a_2 x_2 - \text{tr}(a_2 x_2) a_1 x_1\right) a_3\right) x_3 = 0.$$

If we multiply (3.1) by a scalar-valued function $\lambda(x_{n+2}, \ldots, x_m)$, we get a left-sided functional identity of $M_n(\mathbb{F})$ in m variables for an arbitrary $m > n$. Of course, by permuting the variables

x_i we get further examples. We will see that in fact every left-sided functional identity of $M_n(\mathbb{F})$ is a sum of left-sided functional identities of such a type.

Before giving a full description of the unknown functions F_k satisfying $\sum_{k \in K} F_k(\overline{x}_m^k) x_k = 0$ we need to introduce some more notation. Let

$$C = \mathbb{F}\big[x_{ij}^{(k)} \mid 1 \le i, j \le n, 1 \le k \le m\big].$$

We denote by

$$[(i_1, j_1), \dots, (i_n, j_n)]$$

the determinant of the matrix with the ℓ-th row equal to the j_ℓ-th row of the generic matrix $X_{i_\ell} = (x_{ij}^{(i_\ell)})$. For example, in the case $n = 2$, $[(1,1),(2,1)]$ denotes the determinant of the matrix

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{11}^{(2)} & x_{12}^{(2)} \end{pmatrix},$$

and thus equals $x_{11}^{(1)} x_{12}^{(2)} - x_{12}^{(1)} x_{11}^{(2)}$. Further, $\mathbb{N}_d$ denotes the set $\{1, \dots, d\}$. If $I = \{i_1, \dots, i_p\} \subseteq \mathbb{N}_m$ with $i_1 < \cdots < i_p$, then $\overline{x}_m^I$ stands for $(x_1, \dots, \hat{x}_{i_1}, \dots, \hat{x}_{i_p}, \dots, x_m)$. Finally, as above by e_{ij} we denote the matrix units.

Without loss of generality we may assume that $K = \mathbb{N}_m$. When dealing with two-sided functional identities (1.1) it is often convenient to deal with the case where K and L are arbitrary subsets of $\mathbb{N}_m$, but for one-sided identities this would only cause notational complications.

Proposition 3.2. *Let $\sum_{k=1}^m F_k(\overline{x}_m^k) x_k = 0$ be a functional identity of $M_n(\mathbb{F})$, where $F_k : M_n(\mathbb{F})^{m-1} \to M_n(\mathbb{F})$ are multilinear functions. Then there exist multilinear scalar-valued functions $\lambda_{\ell I J}$ such that*

$$F_k(\overline{x}_m^k) = \sum_{\ell, I, J} (-1)^s \lambda_{\ell I J}(\overline{x}_m^I)\big[(i_1, j_1), \dots, \widehat{(i_s, j_s)}, \dots, (i_{n+1}, j_{n+1})\big] e_{\ell j_s},$$

where the sum runs over all $\ell \in \mathbb{N}_n$, all $I = \{i_1, \dots, i_{n+1}\} \subseteq \mathbb{N}_m$ such that $i_s = k$ for some s, $i_1 < \cdots < i_{n+1}$, and all $J = (j_1, \dots, j_{n+1}) \in \mathbb{N}_n^{n+1}$.

Proof. We can view $\sum_{k=1}^m F_k(\overline{x}_m^k) x_k = 0$ as the identity in $M_n(C)$, since F_k are multilinear and hence polynomial maps. We then have an identity of the form

$$\sum_{k=1}^m H_k X_k = 0,$$

where $X_k = (x_{ij}^{(k)})$, $1 \le k \le m$, are generic matrices, and $H_k \in M_n(C)$ correspond to F_k. Note that this identity is equivalent to n identities

$$\sum_{k=1}^m e_{\ell\ell} H_k X_k = 0, \quad 1 \le \ell \le n.$$

We thus first find solutions of each of them. Without loss of generality we assume that $\ell = 1$ and $H_k = e_{11} H_k$, $1 \le k \le m$. We can further rewrite this identity as

$$\sum_{k=1}^m \sum_{j=1}^n H_{1j}^{(k)} x_{jr}^{(k)} = 0$$

for $1 \le r \le n$, which can be viewed as

$$\left(H_{11}^{(1)}, \ldots, H_{1n}^{(1)}, \ \ldots, H_{11}^{(m)}, \ldots, H_{1n}^{(m)} \right) \begin{pmatrix} x_{11}^{(1)} & \cdots & x_{1n}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{n1}^{(1)} & \cdots & x_{nn}^{(1)} \\ & \vdots & \\ x_{11}^{(m)} & \cdots & x_{1n}^{(m)} \\ \vdots & \ddots & \vdots \\ x_{n1}^{(m)} & \cdots & x_{nn}^{(m)} \end{pmatrix} = (0 \ldots 0).$$

We write $X_{n,m}$ for the matrix on the right. By definition, $(H_{11}^{(1)}, \ldots, H_{1n}^{(1)}, \ \ldots, H_{11}^{(m)}, \ldots, H_{1n}^{(m)})$ is a syzygy on the rows of the generic matrix $X_{n,m}$. The Gröbner basis of the module of syzygies is described in Theorem 3.1. Since in our case $H_{ij}^{(k)}$ are multilinear as functions of $X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_m$, we look for the elements in the Gröbner basis with the same property. Note that

$$\left[(i_1, j_1), \ldots, (i_n, j_n) \right]$$

denotes the determinant of the submatrix of $X_{n,m}$ with the k-th row equal to j_k-th row of the generic matrix X_{i_k}. By Theorem 3.1 the desired generators are

$$\sum_{k=1}^{n+1} (-1)^k \left[(i_1, j_1), \ldots, \widehat{(i_k, j_k)}, \ldots, (i_{n+1}, j_{n+1}) \right] u_{(i_k-1)n+j_k}$$

for $1 \le i_1 < i_2 < \cdots < i_{n+1} \le m$, $1 \le j_1, \ldots, j_{n+1} \le n$, and the proposition follows. ∎

3.3. **Left-sided functional identity as a GPI.** Each function F_k is multilinear so it corresponds to a multilinear generalized polynomial $f_k \in M_n(\mathbb{F}) * \mathbb{F}\langle x \rangle$ such that the evaluation of f_k on $M_n(\mathbb{F})$ coincides with F_k. Note that this correspondence is uniquely determined up to the generalized polynomial identities. At any rate, the problem of describing functional identities $\sum_{k \in K} F_k(\overline{x}_m^k) x_k = 0$ is basically equivalent to the problem of describing GPI's of the form $\sum_{k \in K} f_k(\overline{x}_m^k) x_k$. In this section we will deal with the latter since the GPI setting seems to be more convenient for formulating the main result.

We start with a technical lemma. By $D_{j_1,\ldots,j_n}$ we denote the determinant of the matrix whose k-th row is equal to the j_k-th row of the generic matrix X_k.

Lemma 3.3. Let $i_\ell, j_\ell \in \mathbb{N}_n$, $1 \le \ell \le n+1$. If $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ then

$$e_{i_{n+1},j_n} x_n Q_{n-1}(e_{i_1 j_1} x_1, \ldots, e_{i_{n-1},j_{n-1}} x_{n-1}) e_{i_n j_{n+1}}$$

$$= (-1)^\tau D_{j_1,\ldots,j_n} e_{i_{n+1},j_{n+1}}$$

$$= - e_{i_{n+1},j_{n+1}} \tilde{Q}_n(e_{i_1 j_1} x_1, \ldots, e_{i_n j_n} x_n),$$

where $\tau \in S_n$ is given by $\tau(k) = i_k$, otherwise $Q_{n-1}(e_{i_1,j_1} x_1, \ldots, e_{i_{n-1},j_{n-1}} x_{n-1}) e_{i_n,j_n} = 0$.

Proof. The last assertion follows from the fact that Q_{n-1} is an identity of $M_{n-1}(\mathbb{F})$. Indeed, we may assume without loss of generality that $i_k = k$ for $1 \le k \le s < n$, $i_s = \cdots = i_n = s$, and hence

$$Q_{n-1}(e_{1j_1}x_1, \ldots, e_{sj_s}x_s, \ldots, e_{sj_{n-1}}x_{n-1})e_{sj_n} \in \big(M_n(\mathbb{F})e_{nn}\big)e_{sj_n} = \{0\}.$$

The first equality clearly follows by using the identity

$$e_{i_{n+1},j_n}x_n\phi_\sigma(e_{i_1j_1}x_1, \ldots, e_{i_{n-1},j_{n-1}}x_{n-1})e_{i_n,j_{n+1}} = x^{(1)}_{j_1i_{\sigma(1)}} \cdots x^{(n)}_{j_ni_{\sigma(n)}}e_{i_{n+1},j_{n+1}}$$

for $\sigma \in S_n$ in the expression (2.1) of Q_{n-1}, and the second one follows from

$$(-1)^\tau D_{j_1,\ldots,j_n} = -\tilde{Q}_n(e_{i_1j_1}x_1, \ldots, e_{i_nj_n}x_n),$$

which can be deduced in a similar way after applying the identity

$$\tilde{Q}_n(y_1, \ldots, y_n) = - \sum_{\sigma \in S_n \subseteq S_{n+1}} (-1)^\sigma \phi_\sigma(y_1, \ldots, y_n).$$

∎

Let us call $g \in M_n(\mathbb{F}) * \mathbb{F}\langle x\rangle$ a *central generalized polynomial* if all its evaluations on $M_n(\mathbb{F})$ are scalar matrices. For instance, $\tilde{Q}_n$ is a central generalized polynomial.

Theorem 3.4. *Let $f_1, \ldots, f_m \in M_n(\mathbb{F}) * \mathbb{F}\langle x\rangle$ be multilinear generalized polynomials such that $P = \sum_{k=1}^m f_k(\overline{x}_m^k)x_k$ is a GPI of $M_n(\mathbb{F})$. Then P can be written as a sum of GPI's of the form*

$$g\big[\tilde{Q}_n(a_1x_{k_1}, \ldots, a_nx_{k_n}), a_{n+1}x_{k_{n+1}}\big],$$

where $k_i \ne k_j$ if $i \ne j$, $a_i \in M_n(\mathbb{F})$, and g is a multilinear central generalized polynomial (in all variables except x_{k_i}).

Proof. Proposition 3.2 implies that P can be written as a sum of GPI's of the form

$$\sum_{k=1}^{n+1}(-1)^k\big[(i_1, j_1), \ldots, \widehat{(i_k, j_k)}, \ldots, (i_{n+1}, j_{n+1})\big]e_{\ell j_k}x_{i_k},$$

multiplied by central generalized polynomials; the determinants appearing in the identity can also be viewed as central generalized polynomials. It is thus enough to prove that this identity can be written in the desired way. We can assume without loss of generality that $\ell = 1$, $i_k = k$, $1 \le k \le n+1$. Hence we can write the identity as

$$\sum_{k=1}^{n+1}(-1)^k D_{j_1,\ldots,\hat{j}_k,\ldots,j_{n+1}}e_{1j_k}x_k,$$

where $D_{j_1,\ldots,\hat{j}_k,\ldots,j_{n+1}}$ stands for the determinant of the submatrix of the matrix that has the ℓ-th row equal to the j_ℓ-th row of the generic matrix X_ℓ, in which we remove the k-th row. Note that

$$\tilde{Q}_n(x_1, \ldots, x_n) = \sum_{k=1}^n Q_{n-1}(x_1, \ldots, \hat{x}_k, \ldots, x_n)x_k.$$

Applying Lemma 3.3 we thus obtain

$$e_{1j_{n+1}}x_{n+1}\tilde{Q}_n(e_{1j_1}x_1,\ldots,e_{nj_n}x_n)$$

$$=e_{1j_{n+1}}x_{n+1}\sum_{k=1}^{n}Q_{n-1}(e_{1j_1}x_1,\ldots,\widehat{e_{kj_k}x_k},\ldots,e_{nj_n}x_n)e_{kj_k}x_k$$

$$=\sum_{k=1}^{n}(-1)^{\tau_k}D_{j_1,\ldots,\hat{j}_k,\ldots,j_{n+1}}e_{1j_k}x_k,$$

where $\tau_k = (k, k+1, \ldots, n)$, and thus $(-1)^{\tau_k} = (-1)^{n-k}$. Applying Lemma 3.3 we obtain

$$\sum_{k=1}^{n+1}(-1)^k D_{j_1,\ldots,\hat{j}_k,\ldots,j_{n+1}}e_{1j_k}x_k$$

$$= (-1)^n e_{1j_{n+1}}x_{n+1}\tilde{Q}_n(e_{1j_1}x_1,\ldots,e_{nj_n}x_n) - (-1)^n Q_n(e_{1j_1}x_1,\ldots,e_{nj_n}x_n)e_{1j_{n+1}}x_{n+1},$$

which yields the desired conclusion. ∎

Remark 3.5. Let us remark that the identity (3.1) can be written as

$$\sum_{k=1}^{n+1}\tilde{Q}_n(a_1x_1,\ldots,\widehat{a_kx_k},\ldots,a_nx_n,\tilde{a}_{n+1,k}x_{n+1})\tilde{a}_kx_k = 0$$

for some $\tilde{a}_k, \tilde{a}_{n+1,k} \in M_n(\mathbb{F})$. It is enough to establish the statement in the case when $a_k = e_{i_k,j_k}$, $1 \leq k \leq n+1$, are matrix units. Applying Lemma 3.3 we obtain, similarly as in the proof of Corollary 3.4, the identity

$$e_{i_{n+1}j_{n+1}}x_{n+1}\tilde{Q}_n(e_{i_1j_1}x_1,\ldots,e_{i_nj_n}x_n)$$

$$=e_{i_{n+1}j_{n+1}}x_{n+1}\sum_{k}Q_{n-1}(e_{i_1j_1}x_1,\ldots,\widehat{e_{i_kj_k}x_k},\ldots,e_{i_nj_n}x_n)e_{i_kj_k}x_k$$

$$=\sum_{k}(-1)^{\tau}D_{j_1,\ldots,\hat{j}_k,\ldots,j_{n+1}}e_{i_{n+1},j_k}x_k$$

$$=\sum_{k}(-1)^n\tilde{Q}_n(e_{i_1j_1}x_1,\ldots,\widehat{e_{i_k,j_k}x_k},\ldots,e_{i_kj_{n+1}}x_{n+1})e_{i_{n+1}j_k}x_k,$$

where $\tau \in S_n$, $\tau(k) = i_k$.

3.4. **An application: Characterization of the determinant.** The determinants have appeared throughout our discussion on one-sided identities. To point out their role more clearly, we record two simple corollaries at the end of the section.

Let A be an algebra over a field $\mathbb{F}$. A function $T : A \to A$ is said to be the *trace of an r-linear function* $F : A^r \to A$ if $T(x) = F(x, \ldots, x)$ for all $x \in A$ (if $r = 0$ then T is a constant function). We remark that if F is symmetric and $\text{char}(F)$ is 0 or greater than r, then F is uniquely determined by its trace T. This can be shown by applying the linearization process.

Corollary 3.6. *Let $m > n$ and let $T_k : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be the trace of an $(m-1)$-linear function $F_k : M_n(\mathbb{F})^{m-1} \to M_n(\mathbb{F})$, $1 \leq k \leq m$. If $\sum_{k=1}^{m} F_k(\overline{x}_m^k)x_k = 0$ is a functional identity*

of $M_n(\mathbb{F})$, then each T_k can be written as $T_k(x) = \det(x)S_k(x)$ where S_k is the trace of an $(m-1-n)$-linear function.

Proof. Proposition 3.2 shows that $T_k(x_1)$ is a sum of the terms of the form

$$(-1)^s \lambda_{\ell IJ}(x_1, \ldots, x_1)\big[(1,j_1), \ldots, \widehat{(1,j_s)}, \ldots, (1,j_{n+1})\big] e_{\ell j_s}.$$

By definition, $\big[(1,j_1), \ldots, \widehat{(1,j_s)}, \ldots, (1,j_{n+1})\big] = \pm \det(x_1)$ if $j_1, \ldots, \hat{j}_s, \ldots, j_{n+1} \in \mathbb{N}_n$ are distinct, and 0 otherwise. This proves the corollary. ∎

In the final corollary we add more assumptions in order to obtain an abstract characterization of the determinant.

Corollary 3.7. *Let $F : M_n(\mathbb{F})^n \to M_n(\mathbb{F})$ be an n-linear function satisfying $F(1, \ldots, 1) = 1$, and let T be the trace of F. If $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > n$, then the following statements are equivalent:*

(i) *There exist multilinear functions $F_1, \ldots, F_n : M_n(\mathbb{F})^n \to M_n(\mathbb{F})$ such that*

$$(3.2) \qquad \sum_{k=1}^{n} F_k(\overline{x}_{n+1}^k)x_k + F(\overline{x}_{n+1}^{n+1})x_{n+1} = 0$$

is a functional identity of $M_n(\mathbb{F})$.

(ii) *$T(x) = \det(x) \cdot 1$.*

Proof. Corollary 3.6 shows that (i) implies (ii). To establish the converse, just note that $\det(x) \cdot 1$ is the trace of the function $\frac{1}{n!}\tilde{Q}_n(x_1, \ldots, x_n)$, and that $\frac{1}{n!}[\tilde{Q}_n(x_1, \ldots, x_n), x_{n+1}] = 0$ can be written in the form (3.2) with $F(\overline{x}_{n+1}^{n+1}) = \frac{1}{n!}\tilde{Q}_n(\overline{x}_{n+1}^{n+1})$. ∎

4. Two-sided functional identities

In this section we consider the general two-sided functional identities

$$(4.1) \qquad \sum_{k \in K} F_k(\overline{x}_m^k)x_k = \sum_{l \in L} x_l G_l(\overline{x}_m^l).$$

Let us first examine what result can be expected.

4.1. Solutions of two-sided functional identities.
Let us first consider (4.1) in an arbitrary algebra A with center Z. Suppose there exist multilinear functions

$$p_{kl} : A^{m-2} \to A, \ k \in K, \ l \in L, \ k \neq l,$$
$$\lambda_i : A^{m-1} \to Z, \ i \in K \cup L,$$

such that

$$F_k(\overline{x}_m^k) = \sum_{l \in L, \, l \neq k} x_l p_{kl}(\overline{x}_m^{kl}) + \lambda_k(\overline{x}_m^k), \quad k \in K,$$

$$(4.2) \qquad G_l(\overline{x}_m^l) = \sum_{k \in K, \, k \neq l} p_{kl}(\overline{x}_m^{kl})x_k + \lambda_l(\overline{x}_m^l), \quad l \in L,$$

$$\lambda_i = 0 \quad \text{if} \quad i \notin K \cap L.$$

Note that then (4.1) is fulfilled. We call (4.2) a *standard solution* of (4.1). In a large class of algebras a standard solution is also the only possible solution of (4.1) [BCM07]. Especially in infinite dimensional algebras this often turns out to be the case. For $A = M_n(\mathbb{F})$, this holds provided only if $|K| \leq n$ and $|L| \leq n$ (see e.g. [BCM07, Corollary 2.23]). Indeed, if $|K| > n$ or $|L| > n$ then there exist nonstandard solutions of one-sided identities. Do all nonstandard solutions of (4.1) in $M_n(\mathbb{F})$ arise from the one-sided identities? More specifically, let us call a solution of (4.1) a *standard solution modulo one-sided identities* if there exist multilinear functions

$$\varphi_k, \psi_l : A^{m-1} \to A, \ k \in K, \ l \in L,$$

$$p_{kl} : A^{m-2} \to A, \ k \in K, \ l \in L, \ k \neq l,$$

$$\lambda_i : A^{m-1} \to Z, \ i \in K \cup L,$$

such that

$$F_k(\overline{x}_m^k) = \sum_{l \in L, \, l \neq k} x_l p_{kl}(\overline{x}_m^{kl}) + \lambda_k(\overline{x}_m^k) + \varphi_k(\overline{x}_m^k), \quad k \in K,$$

(4.3)
$$G_l(\overline{x}_m^l) = \sum_{k \in K, \, k \neq l} p_{kl}(\overline{x}_m^{kl}) x_k + \lambda_l(\overline{x}_m^l) + \psi_l(\overline{x}_m^l), \quad l \in L,$$

$$\lambda_i = 0 \quad \text{if} \quad i \notin K \cap L,$$

$$\sum_{k \in K} \varphi_k(\overline{x}_m^k) x_k = \sum_{l \in L} x_l \psi_l(\overline{x}_m^l) = 0.$$

This notion is not vacuous. Namely, there do exist algebras admitting functional identities (4.1) having solutions that are not standard modulo one-sided identities. One can actually find algebras with this property in which all solutions of one-sided identities are standard; see e.g. [BCM07, Example 5.29].

Our goal in the rest of the paper is to show that every solution of (4.1) on $A = M_n(\mathbb{F})$ is standard modulo one-sided identities. As solutions of one-sided identities have been described in the preceding section, this will give a complete solution of the problem to which we have addressed ourselves in this paper.

4.2. **Gröbner basis of a module representing functional identities.** In this subsection we put functional identities aside, and establish auxiliary results needed for the proof of the main result, Theorem 4.7.

First we introduce the necessary framework. We take $m, n \in \mathbb{N}$ and define $n^2 \times n^2$-matrices

$$X_k' = \begin{pmatrix} x_{11}^{(k)} & \cdots & x_{1n}^{(k)} & & & & \\ & \vdots & & & & & \\ x_{n1}^{(k)} & \cdots & x_{nn}^{(k)} & & & & \\ & & & \ddots & & & \\ & & & & x_{11}^{(k)} & \cdots & x_{1n}^{(k)} \\ & & & & & \vdots & \\ & & & & x_{n1}^{(k)} & \cdots & x_{nn}^{(k)} \end{pmatrix} = X_k \otimes 1 \in M_n(\mathbb{F}) \otimes M_n(\mathbb{F}),$$

$$X_k'' = \begin{pmatrix} x_{11}^{(k)} & & & x_{n1}^{(k)} & \\ & \ddots & & & \cdots & & \ddots \\ & & x_{11}^{(k)} & & & x_{n1}^{(k)} \\ & & & \vdots & & \\ x_{1n}^{(k)} & & & x_{nn}^{(k)} & \\ & \ddots & & & \cdots & & \ddots \\ & & x_{1n}^{(k)} & & & x_{nn}^{(k)} \end{pmatrix} = 1 \otimes X_k^t \in M_n(\mathbb{F}) \otimes M_n(\mathbb{F}),$$

and write

$$\Xi = (X_1', \ldots, X_m', X_1'', \ldots, X_m'')^t.$$

Let M be the submodule of the module C^{n^2} generated by the rows of Ξ. We aim to find a Gröbner basis of M.

We need some more notation. Let $K = \{k_1, \ldots, k_a\}$ and $L = \{l_1, \ldots, l_b\}$ be subsets of $\mathbb{N}_m$, and let us write $Q = K \cap L = \{q_1, \ldots, q_c\}$ (this set may be empty). Let $(d_1, f_1), \ldots, (d_c, f_c)$ be such that $q_\ell = k_{d_\ell} = l_{f_\ell}$. We attach the tuples $V = (v_1, \ldots, v_a) \in \mathbb{N}_n^a$ and $S = (s_1, \ldots, s_b) \in \mathbb{N}_n^b$ to K and L, resp. For a subset $U \subseteq \mathbb{N}_a$, $|U| = a-c$, we write $U^{\mathsf{c}} = \{u_1', \ldots, u_c'\}$, $u_1' < \cdots < u_c'$, for the complement of U in $\mathbb{N}_a$. Let σ belong to $\operatorname{Sym} U^{\mathsf{c}}$, the permutation group of U^{c}. We choose $\lambda \in \mathbb{N}_n$ and write

$$D_\lambda^{\mathsf{c}}(Q_\sigma, L_S \setminus Q)$$

for the determinant of the $b \times b$-matrix $Y = (y_{ij})$, where

$$y_{i\ell} = \begin{cases} x_{is_\ell}^{(l_\ell)} & \text{for } 1 \le i \le b-1, \ \ell \in \mathbb{N}_b \setminus \{f_1, \ldots, f_c\}, \\ x_{\lambda s_\ell}^{(l_\ell)} & \text{for } i = b, \ \ell \in \mathbb{N}_b \setminus \{f_1, \ldots, f_c\}, \end{cases}$$

$$y_{if_\ell} = \begin{cases} x_{i\sigma(u_\ell')}^{(q_\ell)} & \text{for } 1 \le i \le b-1, \ 1 \le \ell \le c, \\ x_{\lambda\sigma(u_\ell')}^{(q_\ell)} & \text{for } i = b, \ 1 \le \ell \le c. \end{cases}$$

Analogously, let $\tau \in \operatorname{Sym} W^{\mathsf{c}}$, where $W^{\mathsf{c}} = \{w_1', \ldots, w_c'\} \subseteq \mathbb{N}_b$, and write

$$D_\lambda^{\mathsf{r}}(Q_\tau, K_V \setminus Q)$$

for the determinant of the $a \times a$-matrix $Z = (z_{ij})$, where

$$z_{\ell j} = \begin{cases} x_{v_\ell j}^{(k_\ell)} & \text{for } 1 \le j \le a-1, \ \ell \in \mathbb{N}_a \setminus \{d_1, \ldots, d_c\}, \\ x_{v_\ell \lambda}^{(k_\ell)} & \text{for } j = a, \ \ell \in \mathbb{N}_a \setminus \{d_1, \ldots, d_c\}, \end{cases}$$

$$z_{d_\ell j} = \begin{cases} x_{\tau(w_\ell')j}^{(q_\ell)} & \text{for } 1 \le j \le a-1, \ 1 \le \ell \le c, \\ x_{\tau(w_\ell')\lambda}^{(q_\ell)} & \text{for } j = a, \ 1 \le \ell \le c. \end{cases}$$

In particular, we write $D_\lambda^{\mathsf{c}}(L_S)$, $D_\lambda^{\mathsf{r}}(K_V)$ for $D_\lambda^{\mathsf{c}}(\varnothing, L_S)$, $D_\lambda^{\mathsf{r}}(\varnothing, K_V)$, resp. (Note that Y and Z are formed from the columns (resp. rows) of certain matrices, which is the reason for using $^{\mathsf{c}}$ (resp. $^{\mathsf{r}}$) in the above notation.)

We further denote by

$$d_{\lambda, W^{\mathsf{c}}}^{\mathsf{c}}(Q_\sigma), \quad d_{\lambda, W}^{\mathsf{c}}(L_S \setminus Q)$$

the determinant of the submatrix of Y containing the columns labeled by $f_1, \ldots, f_\ell$ (resp. by $\ell \in \mathbb{N}_b \setminus \{f_1, \ldots, f_\ell\}$) and rows labeled by $i_\ell \in W^{\mathbf{c}}$ (resp. $i_\ell \in W$), and by

$$d^{\mathbf{r}}_{\lambda, U^{\mathbf{c}}}(Q_\tau), \quad d^{\mathbf{r}}_{\lambda, U}(K_V \setminus Q)$$

the determinant of the submatrix of Z containing the rows labeled by $d_1, \ldots, d_\ell$ (resp. by $\ell \in \mathbb{N}_a \setminus \{d_1, \ldots, d_\ell\}$) and columns labeled by $j_\ell \in U^{\mathbf{c}}$ (resp. $j_\ell \in U$). We let the determinant of the empty matrix be 1.

Example 4.1. Let $n = 4$, $K = \{1, 2, 3\}$, $L = \{2, 3, 4, 5\}$, $V = (4, 1, 2)$, $S = (3, 4, 2, 1)$, $U = \{2\}$, $W = \{1, 3\}$, $\sigma = \mathrm{id}$, $\tau = (24)$, $\lambda = 4$. Then $Q = \{2, 3\}$,

$$D^{\mathbf{c}}_\lambda(Q_\sigma, L_S \setminus Q) = \begin{vmatrix} x^{(2)}_{11} & x^{(3)}_{13} & x^{(4)}_{12} & x^{(5)}_{11} \\ x^{(2)}_{21} & x^{(3)}_{23} & x^{(4)}_{22} & x^{(5)}_{21} \\ x^{(2)}_{31} & x^{(3)}_{33} & x^{(4)}_{32} & x^{(5)}_{31} \\ x^{(2)}_{41} & x^{(3)}_{43} & x^{(4)}_{42} & x^{(5)}_{41} \end{vmatrix}, \quad D^{\mathbf{r}}_\lambda(Q_\tau, K_V \setminus Q) = \begin{vmatrix} x^{(1)}_{41} & x^{(1)}_{42} & x^{(1)}_{44} \\ x^{(2)}_{41} & x^{(2)}_{42} & x^{(2)}_{44} \\ x^{(3)}_{21} & x^{(3)}_{22} & x^{(3)}_{24} \end{vmatrix},$$

$$d^{\mathbf{c}}_{\lambda, W^{\mathbf{c}}}(Q_\sigma) = \begin{vmatrix} x^{(2)}_{21} & x^{(3)}_{23} \\ x^{(2)}_{41} & x^{(3)}_{43} \end{vmatrix}, \quad d^{\mathbf{c}}_{\lambda, W}(L_S \setminus Q) = \begin{vmatrix} x^{(4)}_{12} & x^{(5)}_{11} \\ x^{(4)}_{32} & x^{(5)}_{31} \end{vmatrix},$$

$$d^{\mathbf{r}}_{\lambda, U^{\mathbf{c}}}(Q_\tau) = \begin{vmatrix} x^{(2)}_{41} & x^{(2)}_{44} \\ x^{(3)}_{21} & x^{(3)}_{24} \end{vmatrix}, \quad d^{\mathbf{r}}_{\lambda, U}(K_V \setminus Q) = \begin{vmatrix} x^{(1)}_{42} \end{vmatrix}.$$

Let $u_{\gamma, \delta}$ denote the $n(\gamma - 1) + \delta$-th basis element in the C-module C^{n^2}. We write

$$G' = \left\{ \sum_{\alpha = |K|}^{n} D^{\mathbf{r}}_\alpha(K_V) u_{\gamma, \alpha} \mid \gamma \in \mathbb{N}_n, K \subseteq \mathbb{N}_m, V \subseteq \mathbb{N}_n, |V| = |K| \right\},$$

$$G'' = \left\{ \sum_{\beta = |L|}^{n} D^{\mathbf{c}}_\beta(L_S) u_{\beta, \delta} \mid \delta \in \mathbb{N}_n, L \subseteq \mathbb{N}_m, S \subseteq \mathbb{N}_n, |S| = |L| \right\}.$$

Note that every polynomial in C can be treated as a function on $M_n(\mathbb{F})^m$. Let C_{mult} denote the elements in C that are multilinear in some set of variables $x_{k_1}, \ldots, x_{k_\ell}$, $1 \le k_i \ne k_j \le m$. We will say that G is a *multilinear Gröbner basis* of a C-module $N \subseteq C^r$ if there exists a Gröbner basis $\tilde{G}$ of N such that $G = \tilde{G} \cap C_{mult}$.

Proposition 4.2 ([Onn94, Theorem 8.4])**.** *The set G' is a multilinear Gröbner basis of the submodule M' of M generated by the first mn^2 rows of Ξ, and the set G'' is a multilinear Gröbner basis of the submodule M'' of M generated by the last mn^2 rows of Ξ.*

We will use this proposition in order to show that one can obtain a multilinear Gröbner basis of M by simply joining G' and G''. To establish this result in Lemma 4.5 we need some preliminary lemmas.

For a subset $U \subseteq \mathbb{N}_a$ we set $U_\alpha = U$ if $a \notin U$, and $U_\alpha = (U \setminus \{a\}) \cup \{\alpha\}$ if $a \in U$.

Lemma 4.3. Let $a, b, c \in \mathbb{N}$, $c \le a, b \le n$, $U \subseteq \mathbb{N}_a$, $W \subseteq \mathbb{N}_b$, $|U| = |W| = c$, $Q = \{q_1, \ldots, q_c\}$. For $a \le \alpha$, $b \le \beta$ we have

$$(4.4) \qquad \sum_{\sigma \in \mathrm{Sym}\, U_\alpha} (-1)^\sigma d^{\mathbf{c}}_{\beta, W}(Q_\sigma) = \sum_{\tau \in \mathrm{Sym}\, W_\beta} (-1)^\tau d^{\mathbf{r}}_{\alpha, U}(Q_\tau).$$

Proof. Notice that we can assume without loss of generality that $U = W = Q = \{1, \ldots, c\}$, $\alpha = \beta = c$. We compute

$$\sum_{\sigma \in \mathrm{Sym}\, U} (-1)^\sigma d^{\mathbf{c}}_{\beta, W}(Q_\sigma) = \sum_{\sigma \in \mathrm{Sym}\, c} (-1)^\sigma \sum_{\rho \in \mathrm{Sym}\, c} (-1)^\rho\, x^{(\rho(1))}_{1, \sigma\rho(1)} \cdots x^{(\rho(c))}_{c, \sigma\rho(c)}$$

$$= \sum_{\rho \in \mathrm{Sym}\, c} (-1)^{\rho^{-1}} \sum_{\sigma \in \mathrm{Sym}\, c} (-1)^{\sigma^{-1}}\, x^{(\sigma^{-1}(1))}_{\rho^{-1}\sigma^{-1}(1), 1} \cdots x^{(\sigma^{-1}(c))}_{\rho^{-1}\sigma^{-1}(c), c}$$

$$= \sum_{\tau \in \mathrm{Sym}\, W} (-1)^\tau d^{\mathbf{r}}_{\alpha, U}(Q_\tau).$$

∎

Lemma 4.4. Let $K = \{k_1, \ldots, k_a\}$, $L = \{l_1, \ldots, l_b\}$, $Q = K \cap L = \{q_1, \ldots, q_c\}$. Let $(d_1, f_1), \ldots, (d_c, f_c)$ be such that $q_\ell = k_{d_\ell} = l_{f_\ell}$, and let $V = (v_1, \ldots, v_a) \in \mathbb{N}_n^a$, $S = (s_1, \ldots, s_b) \in \mathbb{N}_n^b$. For $a \le \alpha \le n$, $b \le \beta \le n$ we have

$$(4.5)\qquad \begin{aligned} (-1)^{\sum d_\ell} &\sum_{U \subseteq \mathbb{N}_a, |U| = a-c} (-1)^{\sum_{U^c} u}\, d^{\mathbf{r}}_{\alpha, U}(K_V \setminus Q) \sum_{\sigma \in \mathrm{Sym}\, (U^c)_\alpha} (-1)^\sigma D^{\mathbf{c}}_\beta(Q_\sigma, L_S \setminus Q) \\ =(-1)^{\sum f_\ell} &\sum_{W \subseteq \mathbb{N}_b, |W| = b-c} (-1)^{\sum_{W^c} w}\, d^{\mathbf{c}}_{\beta, W}(L_S \setminus Q) \sum_{\tau \in \mathrm{Sym}\, (W^c)_\beta} (-1)^\tau D^{\mathbf{r}}_\alpha(Q_\tau, K_V \setminus Q). \end{aligned}$$

Proof. Using the Laplace expansion by the columns $f_1, \ldots, f_c$ we obtain

$$D^{\mathbf{c}}_\beta(Q_\sigma, L_S \setminus Q) = \sum_{W \subseteq \mathbb{N}_b, |W| = b-c} (-1)^{\sum f_\ell}(-1)^{\sum_{W^c} w} d^{\mathbf{c}}_{\beta, W}(L_S \setminus Q) d^{\mathbf{c}}_{\beta, W^c}(Q_\sigma),$$

and analogously using the Laplace expansion by the rows $d_1, \ldots, d_c$ we have

$$D^{\mathbf{r}}_\alpha(Q_\tau, K_V \setminus Q) = \sum_{U \subseteq \mathbb{N}_a, |U| = a-c} (-1)^{\sum d_\ell}(-1)^{\sum_{U^c} u} d^{\mathbf{r}}_{\alpha, U}(K_V \setminus Q) d^{\mathbf{r}}_{\alpha, U^c}(Q_\tau).$$

By applying Lemma 4.3 we arrive at the desired conclusion. ∎

Before proceeding to the proof of the next lemma we make a little digression and recall some facts concerning Gröbner bases and syzygies (see e.g. [Eis95]). Let A be a polynomial algebra. By $u_1, \ldots, u_r$ we denote the generators of the free module A^r. Let N be a module over A and $\{g_1, \ldots, g_t\}$ its Gröbner basis with respect to any monomial order on A^r. Let $\mathrm{in}(g_i)$ stand for the initial term of g_i. If $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of A^r, set

$$m_{ij} = \frac{\mathrm{in}(g_i)}{\mathrm{GCD}(\mathrm{in}(g_i), \mathrm{in}(g_j))} \in A.$$

For each such pair i, j choose an expression

$$\sigma_{ij} := m_{ji} g_i - m_{ij} g_j = \sum_{\ell} h_{\ell}^{(ij)} g_{\ell},$$

such that $\mathrm{in}(\sigma_{ij}) \geq \mathrm{in}(h_{\ell}^{(ij)} g_{\ell})$ for every ℓ; it is called a standard expression of σ_{ij} in terms of the g_{ℓ}, and its existence is guaranteed by the fact that $\{g_1, \ldots, g_t\}$ is a Gröbner basis of N. For other pairs i, j set $m_{ij} = 0$, $h_{\ell}^{(ij)} = 0$. By Shreyer's theorem (see e.g. [Eis95, Theorem 15.10]) the module of syzygies on the Gröbner basis $\{g_1, \ldots, g_t\}$ of a module N is generated by

$$\tau_{ij} := m_{ji} u_i - m_{ij} u_j - \sum_{\ell} h_{\ell}^{(ij)} u_{\ell},$$

$1 \leq i, j \leq t$.

Let us define a monomial order $>$ on the module C^{n^2}. On C we set $x_{i_1 j_1}^{(k_1)} > x_{i_2 j_2}^{(k_2)}$ if $(k_1, i_1, j_1) < (k_2, i_2, j_2)$ lexicographically (i.e. $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$, or $k_1 = k_2$, $i_1 = i_2$ and $j_1 < j_2$). We define $p u_{\alpha, \beta} > q u_{\gamma, \delta}$ if $(\alpha, \beta, q) < (\gamma, \delta, p)$ lexicographically (i.e. $\alpha < \gamma$, or $\alpha = \gamma$ and $\beta < \delta$, or $\alpha = \gamma$, $\beta = \delta$ and $p > q$).

Lemma 4.5. The set $G' \cup G''$ is a multilinear Gröbner basis of M with respect to the order $>$ on C^{n^2}.

Proof. Using the Buchberger's criterion (see e.g. [Eis95, Theorem 15.8]) together with Proposition 4.2 we see that it suffices to verify that σ_{ij} has a standard expression in terms of $g_{\ell} \in G' \cup G''$ for $g_i \in G'$, $g_j \in G''$, initial terms of which involve the same basis element in C^{n^2} and for which σ_{ij} is multilinear. Choose $g_i \in G'$, $g_j \in G''$ such that $\mathrm{in}(g_i)$ and $\mathrm{in}(g_j)$ involve the same basis element of C^{n^2}. Define sets $K, L \subseteq \mathbb{N}_m$ such that g_i depends on the variables $x_{k_{\ell}}$ for $k_{\ell} \in K$, g_j on $x_{l_{\ell}}$ for $l_{\ell} \in L$. Then the initial term involves the variables in $Q = K \cap L$. For σ_{ij} to be multilinear the factors in the initial terms of g_i and g_j dependent on the variables in Q need to coincide. Hence,

$$g_i = \sum_{\alpha = a}^{n} D_{\alpha}^{\mathbf{r}}(Q_{\tau}, K_V \setminus Q) u_{b, \alpha},$$

$$g_j = \sum_{\beta = b}^{n} D_{\beta}^{\mathbf{c}}(Q_{\sigma}, L_S \setminus Q) u_{\beta, a},$$

where $|K| = a$, $|L| = b$, $|Q| = c$, $V \in \mathbb{N}_n^a$, $S \in \mathbb{N}_n^b$, and $W = \{d_1, \ldots, d_c\}^{\mathbf{c}}$, $\tau = \mathrm{id}$, $U = \{f_1, \ldots, f_c\}^{\mathbf{c}}$, $\sigma = \mathrm{id}$, and we have

$$\mathrm{in}(g_i) = \prod_{\{\ell \,|\, k_{\ell} \in Q\}} x_{f_{\ell}, d_{\ell}}^{(q_{\ell})} \prod_{\{\ell \,|\, k_{\ell} \in K \setminus Q\}} x_{v_{\ell}, \ell}^{(k_{\ell})}, \qquad \mathrm{in}(g_j) = \prod_{\{\ell \,|\, l_{\ell} \in Q\}} x_{f_{\ell}, d_{\ell}}^{(q_{\ell})} \prod_{\{\ell \,|\, l_{\ell} \in L \setminus Q\}} x_{\ell, s_{\ell}}^{(l_{\ell})}.$$

By Lemma 4.4 we deduce

$$\sum_{\lambda=b}^{n}(-1)^{\sum f_\ell}\sum_{W\subseteq\mathbb{N}_b,|W|=b-c}(-1)^{\sum_{W^{\mathsf{c}}}w}\,d_{\lambda,W}^{\mathbf{c}}(L_S\setminus Q)\sum_{\tau\in\mathrm{Sym}\,(W^{\mathsf{c}})_\lambda}(-1)^\tau\sum_{\alpha=a}^{n}D_\alpha^{\mathbf{r}}(Q_\tau,K_V\setminus Q)u_{\lambda,\alpha}$$

$$=\sum_{\lambda=a}^{n}(-1)^{\sum d_\ell}\sum_{U\subseteq\mathbb{N}_a,|U|=a-c}(-1)^{\sum_{U^{\mathsf{c}}}u}\,d_{\lambda,U}^{\mathbf{r}}(K_V\setminus Q)\sum_{\sigma\in\mathrm{Sym}\,(U^{\mathsf{c}})_\lambda}(-1)^\sigma\sum_{\beta=b}^{n}D_\beta^{\mathbf{c}}(Q_\sigma,L_S\setminus Q)u_{\beta,\lambda}.$$

Indeed, restricting to the basis element $u_{\gamma,\delta}\in C^{n^2}$ on both sides of this identity we get the identity (4.5) for $\alpha=\delta,\beta=\gamma$. It remains to prove that this identity induces a standard expression for σ_{ij}. It is enough to check that the initial terms of the elements in the Gröbner basis (except for g_i and g_j) that appear in this identity and involve the same basis element of C^{n^2} are smaller or equal to $\mathrm{in}(\sigma_{ij})$. Those are

$$(4.6)\quad \mathrm{in}\Big(d_{\lambda,W}^{\mathbf{c}}(L_S\setminus Q)D_a^{\mathbf{r}}(Q_\tau,K_V\setminus Q)\Big)\leq\prod_{\{\ell|\,l_\ell\in L\setminus Q\}}x_{w_l,s_l}^{(l_\ell)}\prod_{\{\ell|\,k_\ell\in K\setminus Q\}}x_{v_\ell,\ell}^{(k_\ell)}\prod_{\{\ell|\,k_\ell\in Q\}}x_{\tau(w_\ell'),d_\ell}^{(q_\ell)},$$

$$(4.7)\quad \mathrm{in}\Big(d_{\lambda,U}^{\mathbf{r}}(K_V\setminus Q)D_b^{\mathbf{c}}(Q_\sigma,L_S\setminus Q)\Big)\leq\prod_{\{\ell|\,k_\ell\in K\setminus Q\}}x_{v_l,u_l}^{(k_\ell)}\prod_{\{\ell|\,l_\ell\in L\setminus Q\}}x_{\ell,s_\ell}^{(l_\ell)}\prod_{\{\ell|\,l_\ell\in Q\}}x_{f_\ell,\sigma(u_\ell')}^{(q_\ell)},$$

where $W=\{w_1,\ldots,w_{b-c}\}\subseteq\mathbb{N}_b$, $W^{\mathsf{c}}=\{w_1',\ldots,w_c'\}$, $U=\{u_1,\ldots,u_{a-c}\}\subseteq\mathbb{N}_a$, $U^{\mathsf{c}}=\{u_1',\ldots,u_c'\}$, $\tau\in\mathrm{Sym}\,W^{\mathsf{c}}$, $\sigma\in\mathrm{Sym}\,U^{\mathsf{c}}$, and we need to exclude $W=\{d_1,\ldots,d_c\}^c$, $\tau=\mathrm{id}$, and $U=\{f_1,\ldots,f_c\}^c$, $\sigma=\mathrm{id}$. Note that the equalities hold for $\lambda=b$ (resp. $\lambda=a$).

One easily infers that $\mathrm{in}(\sigma_{ij})$ equals the product

$$(4.8)\qquad\qquad \prod_{\{\ell|\,k_\ell\in Q\}}x_{f_\ell,d_\ell}^{(q_\ell)}\prod_{\{\ell|\,k_\ell\in K\setminus Q\}}x_{v_\ell,\ell}^{(k_\ell)}\prod_{\{\ell|\,l_\ell\in L\setminus Q\}}x_{\ell,s_\ell}^{(l_\ell)},$$

in which we replace the factor $x_{v_{a-1},a-1}^{(k_{a-1})}x_{v_a,a}^{(k_a)}$ (resp. $x_{b-1,s_{b-1}}^{(l_{b-1})}x_{b,s_b}^{(l_b)}$) by $x_{v_a,a-1}^{(k_{a-1})}x_{v_{a-1},a}^{(k_a)}$ (resp. $x_{b-1,s_b}^{(l_{b-1})}x_{b,s_{b-1}}^{(l_b)}$) if $k_{a-1}\geq l_{b-1}$ (resp. if $k_{a-1}<l_{b-1}$). The terms in (4.6) are obtained by permuting the indices corresponding to the rows of the elements $x_{\ell,s_\ell}^{(l_\ell)}$ appearing in (4.8) (notice that the elements $x_{f_\ell,d_\ell}^{(q_\ell)}$ are also of that form), while the terms in (4.7) are obtained by permuting the indices corresponding to the columns of the elements $x_{v_\ell,\ell}^{(k_\ell)}$ appearing in (4.8) (notice that the elements $x_{f_\ell,d_\ell}^{(q_\ell)}$ are also of that form). Since the permutation described in order to obtain the initial term of σ_{ij} leads to the biggest monomial among the monomials in (4.6), (4.7) with respect to the given order $>$ on C, σ_{ij} has a standard expression, which concludes the proof. ∎

We will need a slight generalization of Lemma 4.5. Let $K=\{k_1,\ldots,k_a\}\subseteq\mathbb{N}_m$, $L=\{l_1,\ldots,l_b\}\subseteq\mathbb{N}_m$. We denote

$$\Xi^{(KL)}=(X_{k_1}',\ldots,X_{k_a}',X_{l_1}'',\ldots,X_{l_b}''),$$

and write $M^{(KL)}$ for the module generated by the rows of $\Xi^{(KL)}$. Let $G'^{(K)}\subseteq G'$ be a multilinear Gröbner basis on the rows of $(X_{k_1}',\ldots,X_{k_a}')$, and $G''^{(L)}\subseteq G''$ be a multilinear Gröbner basis on the rows of $(X_{l_1}'',\ldots,X_{l_b}'')$. We state the next lemma without proof since

one only needs to inspect the proof of Lemma 4.5, and notice that it carries over to a more general situation of the following lemma.

Lemma 4.6. *The set $G'^{(K)} \cup G''^{(L)}$ is a multilinear Gröbner basis of $M^{(KL)}$.*

4.3. **Reduction to standard and one-sided identities.** We are now in a position to establish our main result on two-sided functional identities, which together with Theorem 3.4 gives a full description of functional identities on $M_n(\mathbb{F})$.

Theorem 4.7. *Let $m \geq 2$ and $K, L \subseteq \mathbb{N}_m$. Every solution of the functional identity*

$$\sum_{k \in K} F_k(\overline{x}_m^k) x_k = \sum_{l \in L} x_l G_l(\overline{x}_m^l)$$

on $M_n(\mathbb{F})$ is standard modulo one-sided identities.

Proof. The functional identity in question can be written coordinate-wise as a system of equations. We will treat the situation where $K = L = \mathbb{N}_m$. Finding all solutions of our functional identity in this case, we will obtain the desired ones among those with $F_k = 0$ for $k \in \mathbb{N}_m \setminus K$, $G_l = 0$ for $l \in \mathbb{N}_m \setminus L$. For this let us first denote

$$F'_k = (F_{11}^{(k)}, \ldots, F_{1n}^{(k)}, \ldots, F_{n1}^{(k)}, \ldots, F_{nn}^{(k)}), \quad G''_l = (G_{11}^{(l)}, \ldots, G_{1n}^{(l)}, \ldots, G_{n1}^{(l)}, \ldots, G_{nn}^{(l)}),$$

$$H = \big(F'_1, \ldots, F'_m, G''_1, \ldots, G''_m\big).$$

Then the system of equations reads as

$$(4.9) \qquad H\Xi = \big(F'_1, \ldots, F'_m, G''_1, \ldots, G''_m\big) \begin{pmatrix} X'_1 \\ \vdots \\ X'_m \\ X''_1 \\ \vdots \\ X''_m \end{pmatrix} = 0$$

This system can thus be interpreted as a syzygy on the rows of the matrix Ξ.

In our case F_k, G_l are multilinear so we can restrict ourselves to the syzygies on the rows of Ξ which yield multilinear functions in $x_1, \ldots, x_m$. These are generated by τ_{ij} for i, j such that g_i, g_j belong to a multilinear Gröbner basis G of M. By Lemma 4.5 we have $G = G' \cup G''$. If we take elements $g_i, g_j \in G'$ (resp. $g_i, g_j \in G''$), then $\sigma_{ij} = m_{ji} g_i - m_{ij} g_j$ can be expressed in terms of $g_\ell \in G'$ (resp. $g_\ell \in G''$), since G' (resp. G'') is a (multilinear) Gröbner basis of the module generated by the first (resp. last) mn^2 rows of Ξ. These τ_{ij} thus yield the one-sided functional identities. It remains to consider τ_{ij} for $g_i \in G'$, $g_j \in G''$. Both elements g_i, g_j are multilinear of degree at most n. Let g_i involve the variables appearing in X_k for $k \in K' \subseteq \mathbb{N}_m$, $|K'| \leq n$, and g_j those appearing in X_l for $l \in L' \subseteq \mathbb{N}_m$, $|L'| \leq n$. We can treat g_i, g_j as the elements of the Gröbner basis on the rows of the matrix $\Xi^{(K'L')}$. By Lemma 4.6, σ_{ij} can be expressed in terms of those g_ℓ that form a Gröbner basis on the $\Xi^{(K'L')}$, which implies that the functional identity of the form

$$\sum_{k \in K'} F_k(\overline{x}_m^k) x_k = \sum_{l \in L'} x_l G_l(\overline{x}_m^l)$$

corresponds to the syzygy τ_{ij}. Since $|K'|, |L'| \leq n$, this identity has standard solutions by [BCM07, Corollary 2.23]. ∎

4.4. **An application: Commuting traces.** Let A be an algebra over a field $\mathbb{F}$, and let $T : A \to A$ be the trace of an r-linear function $t : A^r \to A$ (see Subsection 3.4). We say that T is *commuting* if $[T(x), x] = 0$ for all $x \in A$. Such a map is said to be of a *standard form* if there exist traces of $(r - i)$-linear functions $\mu_i : A \to \mathbb{F}$, $0 \leq i \leq r$, such that $T(x) = \sum_{i=0}^{r} \mu_i(x) x^i$ for all $x \in A$. The question whether every commuting trace of an r-linear function on A is of a standard form has been studied extensively for different classes of algebras A (see [Bre04, BCM07] for surveys), but, paradoxically, for the basic case where $A = M_n(\mathbb{F})$ this question had been opened for a long time and has been answered in affirmative only very recently [BS14], and only under the assumption that $\text{char}(\mathbb{F}) = 0$. Our goal now is to show that Theorem 4.7 can be used for obtaining an alternative, independent proof, and moreover, a slight generalization with respect to restrictions on $\text{char}(F)$.

Corollary 4.8. *Let* $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ *be a commuting trace of an r-linear function t. If* $\text{char}(\mathbb{F}) = 0$ *or* $\text{char}(\mathbb{F}) > r + 1$, *then T is of a standard form.*

Proof. The proof is by induction on r. The $r = 0$ case is trivial, so we may assume that $r > 0$ and that the result holds for $r - 1$.

Note that the complete linearization of $[T(x), x] = 0$ yields the functional identity

$$\sum_{k=1}^{m} F(\overline{x}_m^k) x_k = \sum_{l=1}^{m} x_l F(\overline{x}_m^l) \quad \text{for all } x_1 \ldots, x_m \in M_n(\mathbb{F}),$$

where $m = r + 1$ and

$$F(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} t(x_{\sigma(1)}, \ldots, x_{\sigma(r)}).$$

Applying Theorem 4.7 we see that for each $k = 1, \ldots, m = r + 1$ there exist functions

$$\varphi_k : M_n(\mathbb{F})^{m-1} \to M_n(\mathbb{F}), \ k \in \mathbb{N}_m,$$
$$p_{kl} : M_n(\mathbb{F})^{m-2} \to M_n(\mathbb{F}), \ k, l \in \mathbb{N}_m, \ k \neq l,$$
$$\lambda_k : M_n(\mathbb{F})^{m-1} \to \mathbb{F}, \ k \in \mathbb{N}_m,$$

such that

(4.10) $$\sum_{k=1}^{m} \varphi_k(\overline{x}_m^k) x_k = 0$$

and

$$F(\overline{x}_m^k) = \sum_{l \in \mathbb{N}_m, l \neq k} x_l p_{kl}(\overline{x}_m^{kl}) + \lambda_k(\overline{x}_m^k) + \varphi_k(\overline{x}_m^k).$$

Setting $x_1 = \cdots = x_m = x$ and using $F(x, \ldots, x) = r! T(x)$ we arrive at

(4.11) $$T(x) = x P_k(x) + \Lambda_k(x) + \Phi_k(x), \quad 1 \leq k \leq m,$$

where P_k is the trace of $\frac{1}{r!} \sum_{l \in \mathbb{N}_m, l \neq k} p_{kl}$, Λ_k is the trace of $\frac{1}{r!} \lambda_k$, and Φ_k is the trace of $\frac{1}{r!} \varphi_k$. Note that (4.10) yields $\sum_{k=1}^{m} \Phi_k(x) x = 0$ for all $x \in M_n(\mathbb{F})$. Interpreting this identity as

$AX = 0$ where X is a generic matrix and A is the matrix corresponding to $\sum_{k=1}^{m} \Phi_k$, it follows (since X is invertible) that $A = 0$, and hence that

$$\text{(4.12)} \qquad \sum_{k=1}^{m} \Phi_k = 0.$$

Further, (4.11) shows that

$$\Phi_1(x) - \Phi_\ell(x) = x\big(P_\ell(x) - P_1(x)\big) + \Lambda_\ell(x) - \Lambda_1(x), \quad 2 \leq \ell \leq m.$$

Summing up these identities and using (4.12) we get

$$\text{(4.13)} \qquad m\Phi_1(x) = x\Big(\sum_{\ell=2}^{m} P_\ell(x) - (m-1)P_1(x)\Big) + \sum_{\ell=2}^{m} \Lambda_\ell(x) - (m-1)\Lambda_1(x).$$

Accordingly,

$$\Phi_1(x) - x\tilde{P}_1(x) \in \mathbb{F} \quad \text{for all } x \in M_n(\mathbb{F}),$$

where $\tilde{P}_1 = \frac{1}{m}\big(\sum_{\ell=2}^{m} P_\ell - (m-1)P_1\big)$. Setting $P = P_1 + \tilde{P}_1$ we thus see from (4.11) that

$$\text{(4.14)} \qquad T(x) - xP(x) \in \mathbb{F} \quad \text{for all } x \in M_n(\mathbb{F}).$$

Since T is commuting it follows that $[xP(x), x] = 0$ for every $x \in M_n(\mathbb{F})$, which can be written as $x[P(x), x] = 0$. Interpreting this identity through a generic matrix we see, similarly as above, that $[P(x), x] = 0$. Thus, P is a commuting trace of an $(r-1)$-linear function. By induction assumption, P is of a standard form. From (4.14) we thus see that T is of a standard form, too. ∎

A more careful analysis is needed if one wishes to further weaken the assumption on $\mathrm{char}(\mathbb{F})$. Let us examine only the case where $r = 2$, which is the one that plays the most prominent role in applications of functional identities. It naturally appears in the study of Lie isomorphisms, commutativity preserving maps, Lie-admissible maps, Poisson algebras, and several other topics (see [Bre04] and [BCM07, Section 1.4]). Any new information on commuting traces of bilinear functions is therefore of interest.

We will thus consider the case where T is a commuting trace of a bilinear map F. We have to add the assumption that F satisfies the functional identity

$$\text{(4.15)} \qquad F(x, y)z + F(z, x)y + F(y, z)x = zF(x, y) + yF(z, x) + xF(y, z).$$

One way of looking at (4.15) is that $(x, y) \mapsto F(x, y)$ is a nonassociative commutative product on $M_n(F)$ which is connected with the ordinary product through a version of the Jacobi identity: $[F(x, y), z] + [F(z, x), y] + [F(y, z), x] = 0$.

Note that (4.15) is actually equivalent to T being commuting provided that F is symmetric and $\mathrm{char}(F) \neq 2, 3$. However, we do not wish to impose these assumptions. The point of the next corollary is that it has no restrictions on $\mathrm{char}(F)$.

Corollary 4.9. *Let T be the trace of a bilinear function $F : M_n(\mathbb{F})^2 \to M_n(\mathbb{F})$. If T is commuting and F satisfies (4.15), then T is of a standard form.*

Proof. If char($\mathbb{F}$) $\neq$ 2, then the result follows from [BCM07, Theorem 5.32 and Remark 5.33]. Assume, therefore, that char($\mathbb{F}$) = 2. From now on we simply follow the proof of Corollary 4.8. Thus, we first derive (4.11) with $m = 3$, and after that (4.12) and (4.13). Note that $m\Phi_1(x) = \Phi_1(x)$ since $m = 3$. Therefore, (4.14) follows with P being a linear function. This implies that P is commuting, and hence is of a standard form [BCM07, Corollary 5.28]. Consequently, T is of a standard form as well. ∎

These corollaries can be extended, by using scalar extensions and other results on functional identities, to considerably more general algebras than $M_n(\mathbb{F})$ (cf. [BS14]). Also, some other special identities that have proved to be useful for applications could now be examined in greater detail. However, we do not wish to make this section lengthy and technical. Our aim has been just to give an indication of the applicability of Theorem 4.7.

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