FUNCTIONAL IDENTITIES AND RINGS OF QUOTIENTS

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Abstract. The fundamental theorem on functional identities states that a prime ring R with $\deg(R) \geq d$ is a d-free subset of its maximal left ring of quotients $Q_{ml}(R)$. We consider the question whether the same conclusion holds for symmetric rings of quotients. This indeed turns out to be the case for the maximal symmetric ring of quotients $Q_{ms}(R)$, but not for the symmetric Martindale ring of quotients $Q_s(R)$. We show, however, that if the maps from the basic functional identities have their ranges in R, then the maps from their standard solutions have their ranges in $Q_s(R)$. We actually prove a more general theorem which implies both aforementioned results. Its proof is somewhat shorter and more compact than the standard proof used for establishing d-freeness in various situations.

1. Introduction

The theory of functional identities [4] deals with identical relations on rings which, besides arbitrary elements from rings (or from their subsets), involve arbitrary maps that are considered as unknowns. The basic concept of the theory is that of a d-free set where d is a positive integer. The exact definition will be given in Section 3; speaking roughly, a subset of a ring is d-free if certain functional identities in an appropriate number of variables connected to d have only standard solutions, i.e., solutions that are "obvious" and independent of the structure of the ring in question. From the definition it is not at all clear that d-free sets actually exist in any ring. However, Beidar [2] proved that every prime ring R is a d-free subset of its maximal left ring of quotients $Q_{ml}(R)$, as long as the (by all means necessary) condition that $\deg(R)$, the degree of algebraicity of R over the extended centroid, is greater or equal to d is fulfilled (see also [4, Theorem 5.11]). This fundamental theorem has been used to find further examples of d-free sets, and moreover, has turned out to be applicable to problems in various areas; see [4]. In spite of its indisputable importance, the theorem still has some room for improvement. One of its downsides is that $Q_{ml}(R)$ may be much larger than R. Although some larger rings are unavoidable at this level of generality, one might wish to replace $Q_{ml}(R)$ by a ring that is more tightly connected to R. Further, as the notion of a functional identity is left-right symmetric, it would be more natural to deal with the symmetric rings of quotients instead of the left ones. Finally, the

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theorem does not say much about the situation which is, especially from the point of view of applications, most desirable, i.e., when R is a d-free subset of itself (in such a case we simply call R a d-free ring). To the best of our knowledge, so far it has not even been known if the free algebra in at most two variables, which one can view as a model of a prime ring R with $\deg(R) = \infty$, has this property. The work on this paper actually begun with an attempt to solve this question. As we will see, the answer is positive. In fact, every symmetrically closed prime ring R with $\deg(R) \geq d$ is d-free. More generally, we will show for an arbitrary prime ring R with $\deg(R) \geq d$ that the maps from standard solutions of the basic functional identities on R have their ranges in the symmetric Martindale ring of quotients $Q_s(R)$, provided that the maps from these identities have their ranges in R. We remark that this is analogous to the results on identities involving automorphisms and derivations [3]. Further, we will show that a prime ring R with $\deg(R) \geq d$ is a d-free subset of $Q_{ms}(R)$, the maximal symmetric ring of quotients of R. All the results just stated will be derived as corollaries to our main theorem establishing d-freeness in connection with what we call the symmetric fractional degree, a variation of the notion of the fractional degree (see [4]). On the other hand, we will also establish some negative results: not every prime ring R with $\deg(R) \geq d$ is a d-free subset of $Q_s(R)$, and a centrally closed prime ring R with $\deg(R) \geq d$ may not be d-free.

Besides "shrinking" the rings of quotients, the main contribution of the article is a significant modification of the method used to establish d-freeness. The standard method, used everywhere in [4], depends on the auxiliary notion of $(t;d)$-freeness which is technically and notationally heavy. We will be able to avoid it completely. To be more precise, we will still have to deal with an element t satisfying the same requirements as one must impose when treating $(t;d)$-freeness, but our treatment will be more direct and transparent.

The paper is organized as follows. In the second section we consider the connection between symmetric rings of quotients and pairs of maps φ, ψ satisfying the identity

$$\varphi(x)ya = bx\psi(y),$$

where a and b are fixed elements. The reason for this is that general functional identities from the definition of d-freeness can be reduced to this very special one (with $a = b$). This will be shown in the third section, which contains the main theorem and its corollaries. The final, fourth section provides examples showing that the corollaries are, in some sense, optimal.

2. Symmetric rings of quotients

Throughout this section we assume that R is a prime ring. Recall that a ring $Q = Q_s(R)$ is called a *symmetric Martindale ring of quotients* of R if it satisfies the following conditions:

(a) R is a subring of Q.
(b) For every $q \in Q$ there exists a nonzero ideal I of R such that $Iq \subseteq R$ and $qI \subseteq R$.

(c) For every $q \in Q$ and every nonzero ideal J of R, each of the conditions $Jq = 0$ and $qJ = 0$ implies $q = 0$.

(d) If I, J are nonzero ideals of R and $f : I \to R$, $g : J \to R$ are maps satisfying

$$f(u)v = ug(v) \quad \text{for all } u \in I,\, v \in J, \tag{2.1}$$

then there exists $q \in Q$ such that $f(u) = uq$ and $g(v) = qv$ for all $u \in I$, $v \in J$.

It is well-known that $Q_s(R)$ exists for every prime ring R and is unique up to isomorphism (see [7, Propositions 1.4 and 1.6] or [3, Proposition 2.2.3]).

Remark 2.1. One usually assumes that the map f (resp. g) from (d) is a left (resp. right) R-module homomorphism. But this actually follows from (2.1). Namely, taking xu for u with $x \in R$ we have $f(xu)v = xug(v) = xf(u)v$, so that

$$\big(f(xu) - xf(u)\big)J = 0$$

which yields $f(xu) = xf(u)$ by (c). Similarly, substituting $u + u'$ for u we obtain

$$\big(f(u + u') - f(u) - f(u')\big)J = 0$$

and hence $f(u+u') = f(u)+f(u')$. Analogously we show that g is a right R-module homomorphism.

The next theorem gives an alternative description of the symmetric Martindale ring of quotients.

Theorem 2.2. *Let R be a prime ring. A ring Q is a symmetric Martindale ring of quotients of R if and only if it satisfies the following conditions:*

(a') *R is a subring of Q.*

(b') *For every $q \in Q$ there exists $a \neq 0$ in R such that $aRq \subseteq R$ and $qRa \subseteq R$.*

(c') *For every $q \in Q$ and every $b \neq 0$ in R, each of the conditions $bRq = 0$ and $qRb = 0$ implies $q = 0$.*

(d') *If a, b are nonzero elements in R and $\varphi, \psi : R \to R$ are maps satisfying*

$$\varphi(x)ya = bx\psi(y) \quad \text{for all } x, y \in R, \tag{2.2}$$

then there exists $q \in Q$ such that $\varphi(x) = bxq$ and $\psi(y) = qya$ for all $x, y \in R$.

Proof. It is clear that (b') is equivalent to (b), and (c') is equivalent to (c). Since (a) and (a') are identical, it suffices to show that the conditions (a')–(d') imply (d) are that (a)–(d) imply (d').

(a')–(d') $\Longrightarrow$ (d). Let $f : I \to R$, $g : J \to R$ satisfy (2.1). Given $u \in I \setminus \{0\}$ and $v \in J \setminus \{0\}$ we then have $f(ux)yv = uxg(yv)$ for all $x, y \in R$. Using (d') it follows that there exists $q_{u,v} \in Q$ such that $f(ux) = uxq_{u,v}$ for all $x \in R$ and $g(yv) = q_{u,v}yv$ for all $y \in R$. Taking another $v' \in J \setminus \{0\}$ we have, by the same reason, $f(ux) = uxq_{u,v'}$ for all $x \in R$. Accordingly, $uR(q_{u,v} - q_{u,v'}) = 0$ and so $q_{u,v} = q_{u,v'}$ by (c'). Similarly we see that $q_{u,v} = q_{u',v}$ for all $u, u' \in I \setminus \{0\}$ and $v \in J \setminus \{0\}$. Consequently, $q_{u,v} = q_{u,v'} = q_{u',v'}$ holds for arbitrary $u, u' \in I \setminus \{0\}$ and $v, v' \in J \setminus \{0\}$. Setting $q = q_{u,v}$ we thus have $f(ux) = uxq$ and $g(yv) = qyv$

for all $x \in R$ and all $u \in I$, $v \in J$ (trivially also for $u = 0$ and $v = 0$). Hence we have

$$(f(u) - uq)yv = ug(yv) - uqyv = 0$$

for all $y \in R$, $u \in I$, $v \in J$, yielding $f(u) = uq$ by (c'). Similarly we see that $g(v) = qv$ for all $v \in J$.

(a)–(d) $\Longrightarrow$ (d'). Let $\varphi, \psi : R \to R$ satisfy (2.2). Setting $x + x'$ for x one immediately derives

$$\big(\varphi(x + x') - \varphi(x) - \varphi(x')\big)Ra = 0.$$

Since R is prime it follows that $\varphi(x+x') = \varphi(x)+\varphi(x')$ for all $x, x' \in R$. Similarly we see that ψ is additive. Take $x, y, z \in R$ and consider the element $bxby\psi(z)$. On the one hand, it is equal to $bx\big(by\psi(z)\big) = bx\varphi(y)za$, and on the other hand is equal to $b(xby)\psi(z) = \varphi(xby)za$. Comparing both expressions we obtain

$$\big(\varphi(xby) - bx\varphi(y)\big)Ra = 0,$$

and so

$$\varphi(xby) = bx\varphi(y) \quad \text{for all } x, y \in R \tag{2.3}$$

by the primeness of R. Similarly, $bx\psi(y)za = \varphi(x)yaza = bx\psi(yaz)$, which yields

$$\psi(yaz) = \psi(y)za \quad \text{for all } y, z \in R. \tag{2.4}$$

Set $I = RbR$, $J = RaR$, and define $f : I \to R$, $g : J \to R$ by

$$f\Big(\sum_i x_i b y_i\Big) = \sum_i x_i\varphi(y_i), \quad g\Big(\sum_j z_j a w_j\Big) = \sum_j \psi(z_j)w_j.$$

We must show that f and g are well-defined. Suppose that $\sum_i x_i b y_i = 0$. Using (2.3) and the additivity of φ we obtain

$$by\Big(\sum_i x_i\varphi(y_i)\Big) = \sum_i \varphi(yx_iby_i) = \varphi\Big(y \cdot \sum_i x_iby_i\Big) = 0$$

for every $y \in R$. In view of (c) this gives $\sum_i x_i\varphi(y_i) = 0$. This means that f is well-defined. Similarly we see that so is g. For all $x_i, y_i, z_j, w_j \in R$ we have

$$\begin{aligned} f\Big(\sum_i x_iby_i\Big)\Big(\sum_j z_jaw_j\Big) &= \sum_{i,j} x_i\varphi(y_i)z_jaw_j \\ &= \sum_{i,j} x_iby_i\psi(z_j)w_j = \Big(\sum_i x_iby_i\Big)g\Big(\sum_j z_jaw_j\Big), \end{aligned}$$

that is, $f(u)v = ug(v)$ for all $u \in I$, $v \in J$. According to (d) there exists $q \in Q$ such that $f(u) = uq$ and $g(v) = qv$ for all $u \in I$, $v \in J$. Consequently, $x\varphi(y) = f(xby) = xbyq$ for all $x, y \in R$. This clearly yields $\varphi(y) = byq$. Similarly we see that $\psi(z) = qza$. □

We remark that the standard treatment of functional identities involves maps φ satisfying (2.3) [4]. As we will see, the approach we take in the next section leads to pairs of maps satisfying (2.2). This is new. As a matter of fact, we will arrive at (2.2) with a and b equal. We could in fact assume in (d') that $a = b$ – this is because in (d) we could assume without loss of generality that $I = J$ (by first replacing both I and J by $I \cap J$). We have decided to work with possibly different a and b simply because the traditional statement of condition (d) involves two ideals.

We now proceed to the maximal symmetric ring of quotients. First recall that a left ideal U of R is said to be *dense* if given $r_1, r_2 \in R$ with $r_1 \neq 0$, there exists $r \in R$ such that $rr_1 \neq 0$ and $rr_2 \in U$. A dense right ideal is defined analogously. Every nonzero two-sided ideal is clearly dense as a left (or right) ideal. A ring $Q = Q_{ms}(R)$ is called a *maximal symmetric ring of quotients of* R if it satisfies the following conditions:

(a") R is a subring of Q.
(b") For every $q \in Q$ there exist a dense left ideal U of R and a dense right ideal V of R such that $Uq \subseteq R$ and $qV \subseteq R$.
(c") For every $q \in Q$, every dense left ideal U of R and every dense right ideal V of R, each of the conditions $Uq = 0$ and $qV = 0$ implies $q = 0$.
(d") If U is a dense left ideal of R, V is dense right ideal of R, and $f : U \to R$, $g : V \to R$ are maps satisfying

$$f(u)v = ug(v) \quad \text{for all } u \in U,\ v \in V,$$

then there exists $q \in Q$ such that $f(u) = uq$ and $g(v) = qv$ for all $u \in I$, $v \in J$.

Remark 2.3. Repeating the argument from Remark 2.1 we see that f (resp. g) is a left (resp. right) R-module homomorphism.

For the existence and uniqueness (up to isomorphism) of $Q_{ms}(R)$ we refer the reader to [6]. We also remark that $Q_s(R) \subseteq Q_{ms}(R) \subseteq Q_{ml}(R)$ and that $Q_{ms}(Q_{ms}(R)) = Q_{ms}(R)$. The latter will be important for us.

The next lemma considers a pair of maps φ, ψ satisfying the same identity as in the previous theorem, but having their ranges in $Q_{ms}(R)$ rather than in R. Its proof will be a modification of that of the theorem.

Lemma 2.4. *Let R be a prime ring. If a, b are nonzero elements in R and $\varphi, \psi : R \to Q_{ms}(R)$ are maps satisfying*

$$\varphi(x)ya = bx\psi(y) \quad \textit{for all } x, y \in R,$$

then there exists $q \in Q_{ms}(R)$ such that $\varphi(x) = bxq$ and $\psi(y) = qya$ for all $x, y \in R$.

Proof. Since nonzero two-sided ideals are dense as left or right ideals, the same proof as above, just that instead of to the primeness we refer to (c"), shows that φ and ψ are additive and that $\varphi(xby) = bx\varphi(y)$, $\psi(yaz) = \psi(y)za$ for all $x, y, z \in R$. Set $U = QbR$, where $Q = Q_{ms}(R)$. Let us show that U is a dense left ideal of Q. Take $q_1, q_2 \in Q$ with $q_1 \neq 0$. By (b") there exists a dense left ideal W of R such that $Wq_2 \subseteq R$. In view of (c"), there exists $w \in W$ such that $wq_1 \neq 0$. Since U contains a nonzero two-sided ideal of R, there is $u \in U$ such that $u(wq_1) \neq 0$. The element $q = uw$ therefore satisfies $qq_1 \neq 0$ and $qq_2 \in U$. Thus U is indeed dense.

Similarly we see that $V = RaQ$ is a dense right ideal of Q. Now define $f : U \to Q$, $g : V \to Q$ by

$$f\Big(\sum_i q_i b y_i\Big) = \sum_i q_i \varphi(y_i), \quad g\Big(\sum_j z_j a s_j\Big) = \sum_j \psi(z_j) s_j.$$

To show that f is well-defined, suppose that $\sum_i q_i b y_i = 0$. By (b"), for every i there exists a dense left ideal W_i of R such that $W_i q_i \subseteq R$. It is easy to see that the intersection of dense left ideals is dense. Therefore $W = \cap_i W_i$ is a dense left ideal satisfying $W q_i \subseteq R$ for every i. Note that for every $y \in W$ we have

$$by\Big(\sum_i q_i \varphi(y_i)\Big) = \sum_i \varphi(y q_i b y_i) = \varphi\Big(y \cdot \sum_i q_i b y_i\Big) = 0.$$

Thus, $bW\Big(\sum_i q_i \varphi(y_i)\Big) = 0$, and hence $(RbR)W\Big(\sum_i q_i \varphi(y_i)\Big) = 0$. Since both RbR and W are dense left ideals of R, using (c") twice we obtain $\sum_i q_i \varphi(y_i) = 0$. Thus f is well-defined. Of course, similarly we see that so is g. Since

$$\begin{aligned} f\Big(\sum_i q_i b y_i\Big)\Big(\sum_j z_j a s_j\Big) &= \sum_{i,j} q_i \varphi(yi) z_j a s_j \\ &= \sum_{i,j} q_i b y_i \psi(z_j) s_j = \Big(\sum_i q_i b y_i\Big) g\Big(\sum_j z_j a s_j\Big), \end{aligned}$$

we may use (d") to obtain the existence of $q \in Q_{ms}(Q) = Q$ such that $f(u) = uq$ and $g(v) = qv$ for all $u \in U$, $v \in V$. Consequently, $\varphi(y) = f(by) = byq$ for all $y \in R$, and $\psi(z) = g(za) = qza$ for all $z \in R$. $\square$

3. Functional identities

Let Q be a unital ring with center C, and let R be a nonempty subset of Q. We are actually interested in the case where R is a (not necessarily unital) subring of Q, but definitions that follow make sense for any subset. Assume further that S is a subring of Q such that

$$R \subseteq S \subseteq Q.$$

The involvement of this additional subring is a novelty. In the usual definitions the role of S is played by Q. The main reason for involving S is that we would also like to cover the case where $S = R$.

Let m be a positive integer. For elements $x_i \in R$, $i = 1, 2, \dots, m$, we set

$$\begin{aligned} \overline{x}_m &= (x_1, \dots, x_m) \in R^m, \\ \overline{x}_m^i &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in R^{m-1}, \\ \overline{x}_m^{ij} = \overline{x}_m^{ji} &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1} \dots, x_m) \in R^{m-2}. \end{aligned}$$

Let I, J be subsets of $\{1, 2, \dots, m\}$, and for each $i \in I$ and $j \in J$ let

$$E_i : R^{m-1} \to S \quad \text{and} \quad F_j : R^{m-1} \to S$$

be arbitrary maps. If $m=1$, then we can regard E_i's and F_j's as elements in S. The basic functional identities, on which the general theory is based, are

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i + \sum_{j\in J} x_jF_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m \in R^m, \tag{3.1}$$

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i + \sum_{j\in J} x_jF_j(\overline{x}_m^j) \in C \quad \text{for all } \overline{x}_m \in R^m. \tag{3.2}$$

Note that (3.1) trivially implies (3.2), so one should not understand that (3.1) and (3.2) are satisfied simultaneously by the same maps E_i and F_j. Each of the two identities should be treated separately.

The *standard solution* of functional identities (3.1) and (3.2) is defined as

$$\begin{aligned} E_i(\overline{x}_m^i) &= \sum_{\substack{j\in J,\\ j\neq i}} x_jp_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad i\in I,\\ F_j(\overline{x}_m^j) &= -\sum_{\substack{i\in I,\\ i\neq j}} p_{ij}(\overline{x}_m^{ij})x_i - \lambda_j(\overline{x}_m^j), \quad j\in J,\\ &\lambda_k = 0 \quad \text{if} \quad k\notin I\cap J, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &p_{ij}: R^{m-2}\to Q,\ \ i\in I,\ j\in J,\ i\neq j,\\ &\lambda_k: R^{m-1}\to C,\ \ k\in I\cup J, \end{aligned}$$

are arbitrary maps (for $m=1$ one should understand this as that $p_{ij}=0$ and λ_k is an element in C). Note that (3.3) indeed implies (3.1), and hence also (3.2). Let us emphasize that we have assumed that the ranges of the maps E_i, F_j lie in S, while the ranges of the p_{ij}'s may be contained in Q. This can easily happen, as we will see in Section 4. We also remark that the cases when one of the sets I and J is empty are not excluded. We will follow the convention that the sum over $\emptyset$ is 0. Thus, if $J=\emptyset$, (3.1) reads as

$$\sum_{i\in I} E_i(\overline{x}_m^i)x_i = 0 \quad \text{for all } \overline{x}_m\in R^m,$$

and the standard solution of this functional identity is $E_i=0$ for all $i\in I$. Similarly, the standard solution of

$$\sum_{j\in J} x_jF_j(\overline{x}_m^j) = 0 \quad \text{for all } \overline{x}_m\in R^m$$

is $F_j=0$ for each j.

We are now in a position to introduce the basic definition.

Definition 3.1. Let d be a positive integer. We will say that R is *d-free relative to the pair* (S,Q) if the following two conditions hold for all $m\geq 1$ and all $I,J\subseteq \{1,2,\ldots,m\}$:

(a) If $\max\{|I|,|J|\}\leq d$, then (3.1) implies (3.3).
(b) If $\max\{|I|,|J|\}\leq d-1$, then (3.2) implies (3.3).

Roughly speaking, (a) and (b) state that the functional identities (3.1) and (3.2) have only standard solutions (for arbitrary maps E_i and F_j!), provided that the index sets $|I|$ and $|J|$ are small enough. Note that the definition implies that these standard solutions are unique.

In the classical case we have $S = Q$. If R is d-free relative to (Q, Q), then we say that R is a *d-free subset of Q*. This is the central notion of the book [4]. The most important case is when $R = S = Q$; if the ring R is a d-free subset of itself, then we simply say that *R is d-free*.

Assume now that R is a prime ring. If Q is either $Q_s(R)$ or $Q_{ms}(R)$ (or the left or right Martindale or maximal ring of quotients), then its center C is a field, called the *extended centroid* of R. Given $t \in R$, we denote by $\deg(t)$ the degree of algebraicity of t over C (if t is algebraic over C) or ∞ (if it is not algebraic). We set $\deg(R) = \deg\{\deg(t) \,|\, t \in R\}$. It is well-known that $\deg(R) \leq n < \infty$ if and only if R satisfies the standard polynomial identity of degree $2n$, or equivalently, R can be embedded into the ring of $n \times n$ matrices over a field. As we have already written in the introduction, the fundamental theorem on functional identities states R is a d-free subset of $Q_{ml}(R)$, the maximal left ring of quotients of R, if (and only if) $\deg(R) \geq d$ [4, Theorem 5.11]. Our goal is to refine this result. To this end, we give another definition.

Definition 3.2. Let R, S and Q be as at the beginning of the section, and assume additionally that R is a ring. Let $t \in R$. We will say that the *symmetric fractionable degree of t relative to* (S, Q) is greater than n, and write this as

$$\text{sf-deg}_{(S,Q)}(t) > n,$$

if there exist $a_k, b_k \in R$ such that

$$\sum_k a_k t^i b_k = 0, \;\; 0 \leq i \leq n-1, \;\; \text{and} \;\; a = \sum_k a_k t^n b_k$$

satisfies the following conditions:

(∗) If $q \in Q$ is such that $aRq = 0$ or $qRa = 0$, then $q = 0$.

(∗∗) If $\varphi, \psi : R \to S$ are maps satisfying

$$\varphi(x)ya = ax\psi(y) \quad \text{for all } x, y \in R,$$

then there exists $q \in Q$ such that

$$\varphi(x) = axq, \;\; \psi(y) = qya \quad \text{for all } x, y \in R.$$

Further, we define $\text{sf-deg}_{(S,Q)}(t) = n$ if $\text{sf-deg}_{(S,Q)}(t) > n-1$ but $\text{sf-deg}_{(S,Q)}(t) \not> n$, and $\text{sf-deg}_{(S,Q)}(t) = \infty$ if $\text{sf-deg}_{(S,Q)}(t) > n$ for every positive integer n.

Remark 3.3. We have assumed at the beginning that Q must be unital. This, however, follows from (∗∗), as one can easily check by considering the maps $\varphi(x) = ax$ and $\psi(y) = ya$.

Example 3.4. Let R be a prime ring. For every $t \in R$ we have

$$\deg(t) = \text{sf-}\deg_{(R,Q_s(R))}(t) = \text{sf-}\deg_{(Q_{ms}(R),Q_{ms}(R))}(t).$$

Namely, the elements $1, t, \ldots, t^n$ are linearly independent over the extended centroid C of R if and only there exist $a_k, b_k \in R$ such that $\sum_k a_k t^i b_k = 0$ if $0 \leq i \leq n-1$ and $\sum_k a_k t^n b_k \neq 0$ [3, Theorem 2.3.3]. The desired equalities therefore follow from Theorem 2.2 and Lemma 2.4.

This example shows that the definition of $\text{sf-deg}_{(S,Q)}(\,\cdot\,)$ makes sense, yet it may still strike the reader as lengthy and maybe even artificial. However, $\text{sf-deg}_{(S,Q)}(\,\cdot\,)$ is a symmetric, refined (and also slightly simplified) version of the concept of the fractional degree, which has turned out to be useful, sometimes unexpectedly, in various situations (see, e.g., [1]). We have therefore decided to work in the abstract setting, although we are primarily interested in the two cases specified in Example 3.4.

We continue with some auxiliary definitions. We will say that $H : R^p \to S$ is a *left i-map* if there is a map $E : R^{p-1} \to S$ such that

$$H(\overline{x}_p) = E(\overline{x}_p^i)x_i \quad \text{for all } \overline{x}_m \in R^p.$$

A sum of left i-maps will be called a *left map*. Thus, H is a left map if it can be written as

$$H(\overline{x}_p) = \sum_{i=1}^{p} E_i(\overline{x}_p^i)x_i$$

for some $E_i : R^{p-1} \to S$. Similarly we define *right j-maps* and *right maps*. The functional identity (3.1) (resp. (3.2)) can thus be described as that the sum of a left map and a right map is zero (resp. central). Our approach to these identities is based on certain transformations of maps of several variables, which we now describe.

Let $t \in R$ and let $H : R^p \to S$ be an arbitrary map. We will write

$$\begin{aligned} H(x_it) &\ \text{ for } \ H(x_1, \ldots, x_{i-1}, x_it, x_{i+1}, \ldots, x_p), \\ H(x_it, x_jt) &\ \text{ for } \ H(x_1, \ldots, x_{i-1}, x_it, x_{i+1}, \ldots, x_{j-1}, x_jt, x_{j+1}, \ldots, x_p), \ \text{ etc.} \end{aligned}$$

For any $1 \leq \ell \leq p$ we define $\mathcal{R}_{\ell,t}(H) : R^p \to S$ by

$$\begin{aligned} \mathcal{R}_{\ell,t}(H)(\overline{x}_p) = H(\overline{x}_p)t^{p-1} &- \sum_{\substack{1\leq i\leq p,\\ i\neq \ell}} H(x_it)t^{p-2} \\ &+ \sum_{\substack{1\leq i<j\leq p,\\ i,j\neq \ell}} H(x_it, x_jt)t^{p-3} - \sum_{\substack{1\leq i<j<k\leq p,\\ i,j,k\neq \ell}} H(x_it, x_jt, x_kt)t^{p-4} \\ &+ \cdots + (-1)^{p-1}H(x_1t, \ldots, x_{\ell-1}t, x_{\ell+1}t, \ldots, x_pt). \end{aligned}$$

For example, if $\ell = 1$ and $p = 4$,

$$\begin{aligned}
&\mathcal{R}_{1,t}(H)(x_1, x_2, x_3, x_4)\\
&\qquad = H(x_1, x_2, x_3, x_4)t^3\\
&\qquad - H(x_1, x_2t, x_3, x_4)t^2 - H(x_1, x_2, x_3t, x_4)t^2 - H(x_1, x_2, x_3, x_4t)t^2\\
&\qquad + H(x_1, x_2t, x_3t, x_4)t + H(x_1, x_2t, x_3, x_4t)t + H(x_1, x_2, x_3t, x_4t)t\\
&\qquad - H(x_1, x_2t, x_3t, x_4t).
\end{aligned}$$

It is obvious that

$$\mathcal{R}_{\ell,t}\Big(\sum_i H_i\Big) = \sum_i \mathcal{R}_{\ell,t}(H_i). \tag{3.4}$$

Lemma 3.5. *If H is a right j-map, then $\mathcal{R}_{\ell,t}(H)$ is also a right j-map. Accordingly, if H is a right map, then so is $\mathcal{R}_{\ell,t}(H)$.*

Proof. The first assertion is clear, and the second one follows from (3.4). □

Lemma 3.6. *If H is a left map, i.e., $H(\overline{x}_p) = \sum_{i=1}^p E_i(\overline{x}_p^i)x_i$, then there exist maps $G_i : R^{p-1} \to S$, $i = 0, 1, \dots, p-2$, such that*

$$\mathcal{R}_{\ell,t}(H)(\overline{x}_p) = \sum_{i=0}^{p-2} G_i(\overline{x}_p^\ell)x_\ell t^i + E_\ell(\overline{x}_p^\ell)x_\ell t^{p-1}$$

for all $\overline{x}_p \in R^p$.

Proof. The lemma is obvious if H is a left ℓ-map, i.e., if $H(\overline{x}_p) = E_\ell(\overline{x}_p^\ell)x_\ell$. In view of (3.4) it is therefore enough to show that $\mathcal{R}_{\ell,t}(H) = 0$ for every left i-map H with $i \neq \ell$. Let us prove this. For notational simplicity, assume that $\ell = 1$ and $i = 2$; thus, $H(\overline{x}_p) = E(\overline{x}_p^2)x_2$. Note that the first two terms from the definition of $\mathcal{R}_{1,t}(H)(\overline{x}_p)$ cancel out. The next terms, $-H(x_it)t^{p-2}$ with $i \geq 3$, cancel out with the terms $H(x_2t, x_it)t^{p-3}$ from the next summation. Further, the terms $H(x_it, x_jt)t^{p-3}$ with $3 \leq i < j \leq p$ cancel out with the terms $-H(x_2t, x_it, x_jt)t^{p-4}$. Etc. Finally, the terms $(-1)^{p-2}H(x_3t, \dots, x_pt)t$ and $(-1)^{p-1}H(x_2t, x_3t, \dots, x_pt)$ cancel out. □

We also need the "left" version of $\mathcal{R}_{\ell,t}(H)$, which we denote by $\mathcal{L}_{r,t}(H)$. It is defined in the same way as $\mathcal{R}_{\ell,t}(H)$, just that t and its powers are moved from the right-hand to the left-hand side, i.e.,

$$\mathcal{L}_{r,t}(H)(\overline{x}_p) = t^{p-1}H(\overline{x}_p) - \sum_{\substack{1\leq i\leq p,\\ i\neq r}} t^{p-2}H(tx_i) + \dots$$

Similar lemmas of course hold for $\mathcal{L}_{r,t}(H)$.

Lemma 3.7. *If H is a left i-map, then $\mathcal{L}_{r,t}(H)$ is also a left i-map. Accordingly, if H is a left map, then so is $\mathcal{L}_{r,t}(H)$.*

Lemma 3.8. *If H is a right map, i.e., $H(\overline{x}_p) = \sum_{j=1}^{p} x_j F_j(\overline{x}_p^j)$, then there exist maps $K_j : R^{p-1} \to S$, $j = 0, 1, \dots, p-2$, such that*

$$\mathcal{L}_{r,t}(H)(\overline{x}_p) = \sum_{j=0}^{p-2} t^j x_r K_j(\overline{x}_p^r) + t^{p-1} x_r F_r(\overline{x}_p^r)$$

for all $\overline{x}_p \in R^p$.

We are now in a position to establish our main theorem.

Theorem 3.9. *Let $R \subseteq S \subseteq Q$ be rings. Suppose that the centralizer of R in Q is equal to the center C of Q. Let d be a positive integer. If there exists $t \in R$ such that* sf-deg$_{(S,Q)}(t) \geq d$*, then R is d-free relative to (S, Q).*

Proof. Our assumption is that there exist $a_k, b_k \in R$ such that $\sum_k a_k t^i b_k = 0$, $0 \leq i \leq d-2$, and $a = \sum_k a_k t^{d-1} b_k$ satisfies $(*)$ and $(**)$ from Definition 3.2. We have to prove that this implies the validity of conditions (a) and (b) from Definition 3.1. A general remark before we start: we will consider the transformation $\mathcal{R}_{\ell,t}(\,\cdot\,)$ with respect to the variables x_i with $i \in I$ (so that $|I|$ plays the role of p). Of course, the maps E_i and F_j may also depend on other variables, but we treat them as fixed when dealing with $\mathcal{R}_{\ell,t}(\,\cdot\,)$. Analogously we will consider $\mathcal{L}_{r,t}(\,\cdot\,)$ with respect to the variables x_j with $j \in J$.

(a) Assume that (3.1) holds with $\max\{|I|, |J|\} \leq d$. Our goal is to show E_i and F_j are of the form (3.3). Assume that $I \neq \emptyset$, and pick $\ell \in I$. Apply $\mathcal{R}_{\ell,t}(\,\cdot\,)$ to (3.1); by making use of Lemmas 3.5 and 3.6 we obtain

$$\sum_{i=0}^{|I|-2} G_i(\overline{x}_m^\ell) x_\ell t^i + E_\ell(\overline{x}_m^\ell) x_\ell t^{|I|-1} + \sum_{j \in J} x_j H_j(\overline{x}_m^j) = 0$$

for some $G_i, H_j : R^{m-1} \to S$ (and all $\overline{x}_m \in R^m$). Replace x_ℓ by $x_\ell a_k$ and multiply the identity so obtained from the right by $t^{d-|I|} b_k$ (here we use that $|I| \leq d$). Summing up over all k we obtain

$$E_\ell(\overline{x}_m^\ell) x_\ell a + \sum_{j \in j} x_j L_j(\overline{x}_m^j) = 0 \tag{3.5}$$

for some maps $L_j : R^{m-1} \to S$ (that arise from H_j).

If $J = \emptyset$, then, in view of $(*)$, (3.5) implies that $E_\ell = 0$. Thus, (3.3) holds is $J = \emptyset$, and, analogously, if $I = \emptyset$. We may therefore assume that $I \neq \emptyset$ and $J \neq \emptyset$. Moreover, we now see that it is enough to show that the E_i's are of the desired form (3.3). Namely, if this holds, then (3.1) can be written as

$$\sum_{j \in J} x_j \Big(F_j(\overline{x}_m^j) + \sum_{\substack{i \in I, \\ i \neq j}} p_{ij}(\overline{x}_m^{ij}) x_i + \lambda_j(\overline{x}_m^j) \Big) = 0;$$

this is an identity of the type (3.1) with $I = \emptyset$, so the corresponding maps are all zero, meaning that the F_j's are of the desired form.

Pick $r \in J$. Applying $\mathcal{L}_{r,t}(\,\cdot\,)$ to (3.5) we obtain, in light of Lemmas 3.7 and 3.8,

$$U(\overline{x}_m^\ell)x_\ell a + \sum_{j=0}^{|J|-2} t^j x_r K_j(\overline{x}_m^r) + t^{|J|-1}x_r L_r(\overline{x}_m^r) = 0$$

for some $U, K_j : R^{m-1} \to S$. Replacing x_r by $b_k x_r$, multiplying from the left by $a_k t^{d-|J|}$ (recall that $|J| \le d$), and summing up over all k we get

$$V(\overline{x}_m^\ell)x_\ell a = -a x_r L_r(\overline{x}_m^r) \tag{3.6}$$

for some $V : R^{m-1} \to S$. If $r \neq \ell$, then, by fixing all variables except x_ℓ and x_r, we can use $(**)$. Hence there exists $p_{\ell r}(\overline{x}_m^{\ell r}) \in Q$ such that $-L_r(\overline{x}_m^r) = p_{\ell r}(\overline{x}_m^{\ell r})x_\ell a$. Suppose that $\ell \in J$ and take $r = \ell$. Replacing x_ℓ by $y_\ell x_\ell$ in (3.6) we get $\big(V(\overline{x}_m^\ell)y_\ell\big)x_\ell a = a y_\ell\big(-x_\ell L_\ell(\overline{x}_m^\ell)\big)$, which makes it possible for us to apply $(**)$ in this case, too. Hence there is $\lambda_\ell(\overline{x}_m^\ell) \in Q$ such that $-x_\ell L_\ell(\overline{x}_m^\ell) = \lambda_\ell(\overline{x}_m^\ell)x_\ell a$ for all $x_i \in R$. Consequently,

$$y_\ell \lambda_\ell(\overline{x}_m^\ell)x_\ell a = -y_\ell x_\ell L_\ell(\overline{x}_m^\ell) = \lambda_\ell(\overline{x}_m^\ell)(y_\ell x_\ell)a,$$

so that $[R, \lambda_\ell(\overline{x}_m^\ell)]Ra = 0$. From $(*)$ it follows that $\lambda_\ell(\overline{x}_m^\ell)$ lies in the centralizer of R in Q, which is equal to C by assumption. Set $\lambda_\ell(\overline{x}_m^\ell) = 0$ if $\ell \notin J$. Note that (3.5) can be now rewritten as

$$\Big(E_\ell(\overline{x}_m^\ell) - \sum_{\substack{j\in J,\\ j\neq \ell}} x_j p_{\ell j}(\overline{x}_m^{\ell j}) - \lambda_\ell(\overline{x}_m^\ell)\Big)Ra = 0.$$

Applying $(*)$ we get that E_ℓ is of the form (3.3), as desired.

(b) Assume now that

$$\mu(\overline{x}_m) = \sum_{i\in I} E_i(\overline{x}_m^i)x_i + \sum_{j\in J} x_j F_j(\overline{x}_m^j) \in C \tag{3.7}$$

and $\max\{|I|, |J|\} \le d-1$. Since we now know that (a) holds, it suffices to prove that $\mu(\overline{x}_m) = 0$. Take $\ell \in I$ and apply $\mathcal{R}_{\ell,t}(\,\cdot\,)$ to (3.7). Using Lemmas 3.5 and 3.6 we get

$$\mu(\overline{x}_m)t^{|I|-1} + \sum_{i=0}^{|I|-2} \mu_i(\overline{x}_m)t^i = \sum_{i=0}^{|I|-1} G_i(\overline{x}_m^\ell)x_\ell t^i + \sum_{j\in J} x_j H_j(\overline{x}_m^j) \tag{3.8}$$

for some $G_i, H_j : R^{m-1} \to S$ and $\mu_i : R^m \to C$. Substituting $x_\ell t$ for x_ℓ gives

$$\sum_{i=0}^{|I|-1} \mu_i'(x_\ell t)t^i = \Big(\sum_{i=0}^{|I|-1} G_i(\overline{x}_m^\ell)x_\ell t^i\Big)t + \sum_{j\in J} x_j H_j'(\overline{x}_m^j),$$

where $H_j' : R^{m-1} \to S$ and $\mu_i' : R^m \to C$. Rewriting the first summation on the right-hand side in terms of (3.8) we arrive at

$$\mu(\overline{x}_m)t^{|I|} + \sum_{i=0}^{|I|-1} \nu_i(\overline{x}_m)t^i = \sum_{j\in J} x_j L_j(\overline{x}_m^j) \tag{3.9}$$

where $L_j : R^{m-1} \to S$ and $\nu_i : R^m \to C$. Take $r \in J$ and apply $\mathcal{L}_{r,t}(\,\cdot\,)$ to (3.9). On the left-hand side we obtain an expression lying in $\sum_{i\geq 0} Ct^i$, and on the right hand-side we obtain

$$\sum_{j=0}^{|J|-2} t^j x_r K_j(\overline{x}_m^r) + t^{|J|-1} x_r L_r(\overline{x}_m^r)$$

for some $K_j : R^{m-1} \to S$. The latter expression therefore commutes with t. Note that this can be written as

$$t^{|J|} x_r L_r(\overline{x}_m^r) + \sum_{j=0}^{|J|-1} t^j x_r M_j(\overline{x}_m^r) = 0$$

for some $M_j : R^{m-1} \to S$. Replacing x_r by $b_k x_r$, multiplying from the left by $a_k t^{d-|J|-1}$ (note that $d-|J|-1$ is nonnegative by our assumption), and summing up over all k we obtain $aRL_r(\overline{x}_m^r) = 0$. Consequently, $L_r(\overline{x}_m^r) = 0$ for every $r \in J$. This means that the right-hand side, and therefore also the left-hand side of (3.9) is zero. Multiplying this identity from the left by $a_k t^{d-|I|-1}$, from the right by b_k, and summing up over all k thus results in $\mu(\overline{x}_m)a = 0$. From $(*)$ we infer that $\mu(\overline{x}_m) = 0$. □

Combining Theorem 3.9 with Theorem 2.2 (cf. Example 3.4) we get the following corollary.

Corollary 3.10. *Let R be a prime ring and let $d \geq 1$. If $\deg(R) \geq d$, then R is d-free relative to $(R, Q_s(R))$.*

Recall that R is said to be *symmetrically closed* if $R = Q_s(R)$. As an immediate consequence of Corollary 3.10 we have

Corollary 3.11. *Let R be a symmetrically closed prime ring and let $d \geq 1$. If $\deg(R) \geq d$, then R is a d-free ring.*

Let F be a field and X a set with $|X| \geq 2$. It is well-known that the free algebra $F\langle X\rangle$ is symmetrically closed [5] and it is trivial that $\deg(F\langle X\rangle) = \infty$. Hence we have

Corollary 3.12. *The free algebra $F\langle X\rangle$, where $|X| \geq 2$, is a d-free ring for every $d \geq 1$.*

Another corollary to Theorem 3.9 follows from Lemma 2.4 (cf. Example 3.4).

Corollary 3.13. *Let R be a prime ring and let $d \geq 1$. If $\deg(R) \geq d$, then R is a d-free subset of $Q_{ms}(R)$.*

4. Examples

The goal of this section is to show that in Corollary 3.11 we cannot replace "symmetrically closed" by "centrally closed", and that in Corollary 3.13 we cannot replace $Q_{ms}(R)$ by $Q_s(R)$. Both counterexamples will be derived from the following proposition.

Proposition 4.1. *Let $R \subseteq S \subset Q$ be rings with the same unity. Assume that for every $p \in Q$, $[R,R]p = 0$ implies $p = 0$. If there exists $q \in Q \setminus S$ such that $q[R,R], [R,R]q \subseteq S$, then R is not a 2-free subset of S.*

Proof. Define maps E_1, E_2, F_3, F_4 on R^3 by

$$\begin{aligned} E_1(x_2,x_3,x_4) &= [x_4,x_3]qx_2, \\ E_2(x_1,x_3,x_4) &= -[x_4,x_3]qx_1, \\ F_3(x_1,x_2,x_4) &= x_4q[x_2,x_1], \\ F_4(x_1,x_2,x_3) &= -x_3q[x_2,x_1] \end{aligned}$$

According to our assumption, their ranges lie in S. As one can immediately check, these maps satisfy the functional identity

$$E_1(x_2,x_3,x_4)x_1 + E_2(x_1,x_3,x_4)x_2 + x_3F_3(x_1,x_2,x_4) + x_4F_4(x_1,x_2,x_3) = 0.$$

If R was a 2-free subset of S, there would exist maps $p_{ij} : R^2 \to S$ such that, in particular,

$$E_1(x_2,x_3,x_4) = x_3p_{13}(x_2,x_4) + x_4p_{14}(x_2,x_3) \tag{4.1}$$

for all $x_i \in R$. It is actually clear from the definition that we can write E_1 in such a form, namely,

$$E_1(x_2,x_3,x_4) = x_3(-x_4qx_2) + x_4(x_3qx_2). \tag{4.2}$$

However, the maps $(x_2,x_4) \mapsto -x_4qx_2$ and $(x_2,x_3) \mapsto x_3qx_2$ have their ranges in Q but not in S, as we see by taking $x_2 = x_3 = x_4 = 1$. Comparing (4.1) and (4.2) we get

$$x_3f(x_2,x_4) + x_4g(x_2,x_3) = 0, \tag{4.3}$$

where

$$f(x_2,x_4) = p_{13}(x_2,x_4) + x_4qx_2, \quad g(x_2,x_3) = p_{14}(x_2,x_3) - x_3qx_2.$$

for all $x_i \in R$. Setting $x_3 = 1$ in (4.3) we get $f(x_2,x_4) = -x_4g(x_2,1)$, setting $x_4 = 1$ in (4.3) we get $g(x_2,x_3) = -x_3f(x_2,1)$ and setting $x_3 = x_4 = 1$ in (4.3) we get $f(x_2,1) = -g(x_2,1)$. We can therefore rewrite (4.3) as $[x_3,x_4]f(x_2,1) = 0$. According to our assumption this implies that $f(x_2,1) = 0$ for every $x_2 \in R$. But then $f(x_2,x_4) = g(x_2,x_3) = 0$ for all $x_i \in R$. That is, $p_{13}(x_2,x_4) = -x_4qx_2$ and $p_{14}(x_2,x_3) = x_3qx_2$, which is a contradiction. □

If a prime ring R is non-PI (i.e., it does not satisfy a nonzero polynomial identity), then $\deg(R) = \infty$ and so R is a d-free subset of $Q_{ms}(R)$ for every $d \geq 1$ by Corollary 3.13. The next corollaries in particular show that such a ring may not be 2-free, and may even not be a 2-free subset of $Q_s(R)$.

Recall that a prime ring R is said to be *centrally closed* if it coincides with its central closure; for a unital ring this is the same as saying that R contains its extended centroid.

Corollary 4.2. *There exists a non-PI centrally closed prime ring R, even a primitive ring with nonzero socle, which is not 2-free.*

Proof. Take an infinite dimensional vector space V over a field F. Let R be the ring of linear operators from V into V of the form $\lambda 1 + f$, where $\lambda \in F$, 1 is the identity operator, and f is an operator of finite rank. It is well-known that R is a primitive (and hence prime) non-PI ring with nonzero socle. Its extended centroid consists of scalar multiples of 1 (cf. [3, Theorem 4.3.7]), so R is centrally closed. Since $[R,R]$ consists of finite rank operators, which form an ideal of $Q = \mathrm{End}_F(V)$, we have $q[R,R], [R,R]q \subseteq R$ for all $q \in Q$, including those that are not contained in R. It is also clear that $[R,R]p = 0$ implies $p = 0$ for every $p \in Q$. Proposition 4.1 thus implies that R is not a 2-free subset of itself. $\square$

Corollary 4.3. *There exists a non-PI prime ring R, even an Ore domain, which is not a 2-free subset of $Q_s(R)$.*

Proof. We take the ring
$$R = F[t][u, v \mid uv = tvu]$$
used by Passman [7, Section 4] for showing that $Q_s(\,\cdot\,)$ is not the closure operation. More specifically, he showed that
$$Q_s(R) = F(t)[u, v \mid uv = tvu]$$
and
$$Q_s(Q_s(R)) = F(t)[u, u^{-1}, v, v^{-1} \mid uv = tvu].$$
We also mention that R is an Ore domain with center $Z = F[t]$, and that
$$R = \bigoplus_{n,m \geq 0} F[t]v^n u^m.$$

Note that $u^k v^\ell = t^{kl} v^\ell u^k$. Hence
$$[v^n u^m, v^s u^r] = (t^{ms} - t^{nr})v^{n+s}u^{m+r}$$
for all $n, m, s, r \geq 0$. This readily implies that
$$[R,R] \subset \bigoplus_{n,m \geq 1} F[t]v^n u^m.$$
We claim that this yields
$$v^{-1}[R,R] \subset R \subset Q_s(R) \quad \text{and} \quad [R,R]v^{-1} \subset Q_s(R).$$
The first inclusion is clear, and the second follows from $u^m v^{-1} = t^{-m}v^{-1}u^m$. Since $v^{-1} \in Q_s(Q_s(R)) \setminus Q_s(R)$, Proposition 4.1 tells us that R is not a 2-free subset of $Q_s(R)$. $\square$

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