JORDAN HOMOMORPHISMS REVISITED

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Abstract. Let θ be a Jordan homomorphism from an algebra A into an algebra B. We find various conditions under which the restriction of θ to the commutator ideal of A is the sum of a homomorphism and an antihomomorphism. Algebraic results, obtained in the first part of the paper, are applied to the second part dealing with the case where A and B are C^*-algebras.

1. Introduction

Jordan homomorphisms of associative rings and algebras play a significant role in various mathematical areas, in particular in ring theory and in the theory of operator algebras. The standard problem in this context is whether Jordan homomorphisms can be expressed through homomorphisms and antihomomorphisms. This problem has a long and rich history. Let us mention, just to give the flavor of the subject, a few classical results from the 1950's.

In [12] Jacobson and Rickart proved that a Jordan homomorphism from an arbitrary ring into a domain is either a homomorphism or an antihomomorphism. The same conclusion holds for Jordan homomorphisms onto prime rings, as shown by Herstein [10] and Smiley [17]. Jacobson and Rickart also proved in the same paper [12] that a Jordan homomorphism from the ring of $n \times n$ matrices over an arbitrary unital ring (where $n \geq 2$) into another ring is the sum of a homomorphism and an antihomomorphism. This result was used by Kadison [13] for proving that a Jordan $*$-homomorphism from a W^*-algebra onto a C^*-algebra is the sum of a $*$-homomorphism and a $*$-antihomomorphism.

In the present paper we will show that under suitable conditions a Jordan homomorphism defined on an algebra A is the sum of a homomorphism and an antihomomorphism, however, not necessarily on A but only on the commutator ideal of A. As we shall see, there are good reasons for the restriction to the commutator ideal.

The paper is organized as follows. Section 2 is of an introductory nature, it contains definitions and remarks on the concepts that are studied. Section 3 considers the action of Jordan homomorphisms on tetrads. It is known that this action can be decisive, so the idea to treat this is not new. What seems to be new, and what is one of the principal ideas upon which this paper is based, is to consider those tetrads that one of the four elements involved is an idempotent. Section 4 considers Jordan homomorphisms that also preserve tetrads (which we call reversal homomorphisms). A rough summary of both Sections 3 and 4 is given in Theorem 4.4; it tells us that under a mild assumption on the range, Jordan homomorphisms can be described on the commutator ideal provided that the algebra on which they are defined is linearly spanned by its idempotents. We believe that the algebraic results from these first sections are interesting in their own right. However, their

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main purpose is to apply them, in Section 5, to the study of Jordan homomorphisms on C^*-algebras that are either surjective or adjoint preserving (i. e. Jordan $*$-homomorphisms). The main result is Theorem 5.2. In its corollaries we in particular give generalizations of Kadison's theorem.

The present paper continues the series [6, 7, 8]. In these papers it was showed that various problems about the Jordan structure of an associative algebra A (about Jordan ideals, Jordan modules, Jordan derivations, etc.) can be solved not on the entire algebra A, but on some of its ideals; in particular the commutator ideal of A often came into play. From the technical point of view these papers are quite different from the present one. Namely, it is not clear to us how to use the ideas upon which these related papers are based in order to handle a similar, but apparently a more entangled problem of describing Jordan homomorphisms on A. In the present paper we use a different approach which remarkably leads to analogous conclusion: Jordan homomorhisms can be "controlled" on the commutator ideal of A. All these indicate that something deeper, not yet fully understood might be hidden behind the results from [6, 7, 8] and this paper.

2. Preliminaries

Throughout, F will denote a field with characteristic not 2, and we will consider algebras over F. Except in the last section on C^*-algebras (where F is therefore $\mathbb{C}$), we shall not impose any further assumptions on F. It should be mentioned that the algebraic part of the paper could be presented in the setting of 2-torsionfree rings, but for simplicity of the exposition we shall deal with algebras over F.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite set and let $F\langle X\rangle$ be the free (associative) algebra on X, i. e. the algebra of all polynomials in noncommuting indeterminates x_i. The reversal involution on $F\langle X\rangle$ is defined by $x_i^* = x_i$ for every $x_i \in X$ (and consequently $(x_{i_1}x_{i_2}\ldots x_{i_n})^* = x_{i_n}\ldots x_{i_2}x_{i_1}$ for all $x_{i_k} \in X$). Let us denote by $H\langle X\rangle$ the set of all symmetric elements with respect to this involution. Clearly, as a linear space $H\langle X\rangle$ is generated by elements of the form

$$\{x_{i_1}x_{i_2}\ldots x_{i_n}\} = x_{i_1}x_{i_2}\ldots x_{i_n} + x_{i_n}\ldots x_{i_2}x_{i_1},$$

$x_{i_k} \in X$. The case when $n = 4$ and all x_{i_k} are different is of special interest. Such elements, that is $\{x_{i_1}x_{i_2}x_{i_3}x_{i_4}\} = x_{i_1}x_{i_2}x_{i_3}x_{i_4} + x_{i_4}x_{i_3}x_{i_2}x_{i_1}$, $x_{i_k} \neq x_{i_l}$ if $k \neq l$, are called *tetrads*.

Further, let $J\langle X\rangle$ be the free special Jordan algebra, i. e. $J\langle X\rangle$ is the Jordan subalgebra of $F\langle X\rangle$ generated by 1 and all $x_i \in X$. Its elements are the so-called *Jordan polynomials.* Let us state a few concrete examples:

$$\begin{aligned}
x_i \circ x_j &= x_i x_j + x_j x_i,\\
x_i^2 &= \frac{1}{2} x_i \circ x_i,\\
x_i x_j x_i &= \frac{1}{2}\big(x_i \circ (x_i \circ x_j) - x_i^2 \circ x_j\big),\\
\{x_i x_j x_k\} &= (x_i + x_k)x_j(x_i + x_k) - x_i x_j x_i - x_k x_j x_k,\\
[[x_i, x_j], x_k] &= (x_j \circ x_k) \circ x_i - (x_i \circ x_k) \circ x_j;
\end{aligned}$$

here, as usual, $[u, v]$ denotes $uv - vu$. It is clear that $J\langle X\rangle \subseteq H\langle X\rangle$. It is easy to see that tetrads do not lie in $J\langle X\rangle$ (see e. g. [11, p. 8]), so that $J\langle X\rangle \neq H\langle X\rangle$. A well-known theorem by Cohn states that $H\langle X\rangle$ is generated as a Jordan algebra by X and all tetrads [11, p. 8].

Now let A and B be (associative) algebras over F, and let $\theta : A \to B$, $a \mapsto a^\theta$, be a linear map. We say that θ is a *Jordan homomorphism* if

$$(a_1 \circ a_2)^\theta = a_1^\theta \circ a_2^\theta$$

for all $a_1, a_2 \in A$; equivalently, θ preserves all Jordan polynomials, that is

$$f(a_1, \dots, a_n)^\theta = f(a_1^\theta, \dots, a_n^\theta) \tag{1}$$

for all $f(x_1, \dots, x_n) \in J\langle X\rangle$ and all $a_i \in A$. So, in particular, a Jordan homomorphism satisfies

$$(a_1 a_2 a_1)^\theta = a_1^\theta a_2^\theta a_1^\theta, \tag{2}$$
$$[[a_1, a_2], a_3]^\theta = [[a_1^\theta, a_2^\theta], a_3^\theta] \tag{3}$$

for all $a_1, a_2, a_3 \in A$.

Let us call θ a *reversal homomorphism* if (1) holds for all $f(x_1, \dots, x_n) \in H\langle X\rangle$ and all $a_i \in A$. In view of Cohn's theorem, θ is a reversal homomorphism if and only if θ is a Jordan homomorphism and it preserves tetrads:

$$\{a_1 a_2 a_3 a_4\}^\theta = \{a_1^\theta a_2^\theta a_3^\theta a_4^\theta\}$$

for all $a_1, a_2, a_3, a_4 \in A$.

The ultimate goal when considering Jordan homomorphisms is to show that they arise from homomorphisms and antihomomorphisms. In this regard we introduce the following concept: we shall say that $\theta : A \to B$ is the *sum of a homomorphism and an antihomomorphism* (SHA for brevity) if there exists ideals I and J of $\mathrm{Alg}(A^\theta)$, the (associative) subalgebra of B generated by the range of θ, a homomorphism $\varphi : A \to I$ and an antihomomorphism $\psi : A \to J$ such that $\theta = \varphi + \psi$ and $IJ = JI = 0$ (so, in particular, $A^\varphi A^\psi = A^\psi A^\varphi = 0$). For example, $A \mapsto (A, A^t)$, where A^t denotes the transpose of the matrix A, is an SHA from $M_n(F)$ to $M_n(F) \times M_n(F)$. Note that the range of this SHA is not an associative algebra. SHAs from one algebra *onto* a semiprime algebra can be described in greater detail.

Lemma 2.1. *Let θ be an SHA from an algebra A onto a semiprime algebra B. Then there exist ideals I_0 and J_0 of A and ideals I and J of B such that*

(a) *$I_0 + J_0 = A$ and $I_0 \cap J_0 = \ker\theta$;*
(b) *$I \oplus J = B$;*
(c) *$\theta|I_0$ is a homomorphism from I_0 onto I;*
(d) *$\theta|J_0$ is an antihomomorphism from J_0 onto J.*

Proof. Let φ, ψ, I and J be as in the above definition, and define $I_0 = \ker\psi$ and $J_0 = \ker\varphi$. Since $IJ = 0$ we have $(I \cap J)^2 = 0$, and hence $I \cap J = 0$ as B is semiprime. Moreover, since $x^\varphi + x^\psi = x^\theta$ for every $x \in A$, we have $I + J = B$ and so (b) holds. Further, given $x \in A$ we have $x^\varphi \in B$ and so there exists $x_0 \in A$ such that $x_0^\theta = x^\varphi$. Thus $I \ni (x - x_0)^\varphi = x_0^\psi \in J$ and hence $x - x_0 \in J_0$ and $x_0 \in I_0$. This proves that $I_0 + J_0 = A$. It is obvious that $I_0 \cap J_0 = \ker\theta$, so (a) holds. Further, it is clear that $\theta|I_0 = \varphi|I_0$ and $\theta|J_0 = \psi|J_0$. It remains to show that $I_0^\theta = I$ and $J_0^\theta = J$. Pick $y \in I$. Then there is $x \in A$ such that $y = x^\theta = x^\varphi + x^\psi$. As $I \ni y - x^\varphi = x^\psi \in J$ it follows that $x^\psi = 0$, i.e. $x \in I_0$. Thus $I_0^\theta = I$, and similarly we see that $J_0^\theta = J$. $\square$

Let us add a few additional remarks to Lemma 2.1 If θ is bijective, then (a) can be read as

(a') $I_0 \oplus J_0 = A$,

and we have the decomposition of θ that one could call a "direct sum" of an isomorphism and an antiisomorphism. Next, if B is a unital algebra, then (b) is equivalent to

(b') $I = eB$ and $J = (1 - e)B$ for some central idempotent $e \in B$, and in this case $x \mapsto ex^\theta$ is a homomorphism and $x \mapsto (1-e)x^\theta$ is an antihomomorphism.

We remark that the conditions (a)–(d) appeared in the study of Jordan homomorphisms onto semiprime rings in [3, 4]. Lemma 2.1 shows that in this context a seemingly weaker condition that θ is an SHA is actually equivalent to (a)–(d). This lemma will not be used in the sequel, we just thought it is appropriate it to record it in order to help the reader to understand better the concept of an SHA.

The following example shows the nontriviality of the concepts that were introduced.

Example 2.2. Let A be the exterior algebra with generators $e_1, e_2, \dots, e_n$. Thus $e_i^2 = e_ie_j + e_je_i = 0$ for all i and j, and as a linear space A is generated by elements $e_{i_1}e_{i_2}\dots e_{i_k}$ with $i_1 < i_2 < \dots < i_k$, $1 \le k \le n$. Define a linear map $\theta : A \to A$ by $e_i^\theta = e_i$ and $(e_{i_1}e_{i_2}\dots e_{i_k})^\theta = 0$ if $k \ge 2$.

We claim that θ is a Jordan homomorphism. It is enough to show that

$$(e_{i_1}e_{i_2}\dots e_{i_k} \circ e_{j_1}e_{j_2}\dots e_{j_l})^\theta = (e_{i_1}e_{i_2}\dots e_{i_k})^\theta \circ (e_{j_1}e_{j_2}\dots e_{j_l})^\theta.$$

If $k \ge 2$ or $l \ge 2$, then both sides are obviously 0. But actually the same is true if $k = l = 1$. Indeed, $(e_i \circ e_j)^\theta = 0$ and $e_i^\theta \circ e_j^\theta = e_i \circ e_j = 0$. Our claim is thus proved.

Suppose that θ is the sum of a homomorphism φ and an antihomomorphism ψ. Then $e_ie_i^\varphi = (e_i^\varphi + e_i^\psi)e_i^\varphi = (e_i^2)^\varphi + e_i^\psi e_i^\varphi = 0$. Therefore $e_i^\varphi = \lambda_ie_i + h_i$, where $\lambda_i \in F$ and h_i lies in the linear span of all elements $e_{i_1}e_{i_2}\dots e_{i_k}$ where $k \ge 2$. Accordingly, $e_i^\psi = e_i^\theta - e_i^\varphi = (1-\lambda_i)e_i - h_i$. Suppose that $\lambda_j \ne 0$ for some j, say $\lambda_1 \ne 0$. Since $e_1^\varphi e_i^\psi = 0$ for every $i \ge 2$, that is $(\lambda_1e_1 + h_1)((1-\lambda_i)e_i - h_i) = 0$, it follows that

$$\lambda_1(1-\lambda_i)e_1e_i = \lambda_1e_1h_i - (1-\lambda_i)h_1e_i + h_1h_i.$$

This readily yields $\lambda_1(1-\lambda_i) = 0$ and so $\lambda_i = 1$ for all $i \ge 2$. But then $e_2^\varphi e_1^\psi = 0$ implies that $\lambda_1 = 1$ as well. Hence we have

$$0 = (e_1e_2)^\theta = e_1^\varphi e_2^\varphi + e_2^\psi e_1^\psi = (e_1 + h_1)(e_2 + h_2) + (-h_2)(-h_1),$$

and so $e_1e_2 = -h_1e_2 - e_1h_2 - h_1h_2 - h_2h_1$, which is obviously a contradiction. Therefore $\lambda_j = 0$ for every j, so that $e_i^\varphi = h_i$ and $e_i^\psi = e_i - h_i$. However, again considering $e_1^\varphi e_2^\varphi + e_2^\psi e_1^\psi = (e_1e_2)^\theta = 0$, we see that this is also impossible. Thus θ is not an SHA.

If $n < 4$, then $A^4 = 0$ and hence $\{a_1a_2a_3a_4\} = 0$ for all $a_i \in A$, so that θ is trivially a reversal homomorphism. If, however, $n \ge 4$, then $\{e_1e_2e_3e_4\}^\theta = 0$ while $\{e_1^\theta e_2^\theta e_3^\theta e_4^\theta\} = \{e_1e_2e_3e_4\} = 2e_1e_2e_3e_4 \ne 0$. Thus θ is not a reversal homomorphism in this case.

Note that the algebra A from this example has plenty of nilpotent elements, including those that lie in the center of A. In the sequel, when considering a Jordan homomorphism $\theta : A \to B$, we shall be frequently forced to assume that the center of $\mathrm{Alg}(A^\theta)$ does not contain nonzero nilpotents.

It is easy to check that every SHA is also a reversal homomorphism. So we have

$$\theta \text{ is an SHA} \Longrightarrow \theta \text{ is a reversal homomorphism} \Longrightarrow \theta \text{ is a Jordan homomorphism}$$

and, as Example 2.2 shows, none of these implications can be reversed in general. When can they be reversed?

3. When Jordan homomorphisms are reversal homomorphisms

Throughout this section A and B will be arbitrary algebras and $\theta : A \to B$ will be a Jordan homomorphism.

The next lemmas are simple and elementary; anyhow, when gathered together they yield Theorem 3.5, which describes a useful property of Jordan homomorphisms. The first lemma is actually known [12, Corollary 2], and the second one could be derived from [12, Corollary 1]; anyway we give the proofs since they are very short.

Lemma 3.1. *If $e = e^2 \in A$ commutes with $a \in A$, then $(ea)^\theta = e^\theta a^\theta = a^\theta e^\theta$.*

Proof. Since e commutes with a we have $e \circ a = ea + ae = 2eae$, hence $e^\theta a^\theta + a^\theta e^\theta = 2e^\theta a^\theta e^\theta$ by (2). Since e^θ is also an idempotent it follows easily that $e^\theta a^\theta = a^\theta e^\theta$. But then $(ea)^\theta = (eae)^\theta = e^\theta a^\theta e^\theta = e^\theta a^\theta$. □

Lemma 3.1 in particular shows that if e is a central idempotent in A, then e^θ is a central idempotent in $\mathrm{Alg}(A^\theta)$, and if A is unital, then 1^θ is the unity of $\mathrm{Alg}(A^\theta)$.

Lemma 3.2. *Suppose that the center of $\mathrm{Alg}(A^\theta)$ does not contain nonzero nilpotent elements. Then for all $a, b \in A$, $ab = ba = 0$ implies $a^\theta b^\theta = b^\theta a^\theta = 0$.*

Proof. First note that $a^\theta b^\theta = -b^\theta a^\theta$ since $a \circ b = 0$. Next, for every $x \in \mathcal{A}$ we have $[[a, b], x] = 0$ which, in view of (3), yields $[[a^\theta, b^\theta], x^\theta] = 0$. That is, $2[a^\theta b^\theta, x^\theta] = 0$, so that $a^\theta b^\theta$ lies in the center of $\mathrm{Alg}(A^\theta)$. Since $(a^\theta b^\theta)^2 = (aba)^\theta b^\theta = 0$ it follows from our assumption that $a^\theta b^\theta = 0$. □

Let $a_1, a_2, a_3, a_4 \in A$. Our main concern in this section is the question whether θ satisfies

$$\{a_1a_2a_3a_4\}^\theta = \{a_1^\theta a_2^\theta a_3^\theta a_4^\theta\}. \tag{4}$$

Lemma 3.3. *Suppose that $a_1a_2 = a_2a_1$ and $(a_1a_2)^\theta = a_1^\theta a_2^\theta$. Then (4) holds for all $a_3, a_4 \in A$.*

Proof. Note that our assumptions imply that $(a_1a_2)^\theta = a_1^\theta a_2^\theta = a_2^\theta a_1^\theta$. Accordingly, we have $\{a_1a_2a_3a_4\}^\theta = \{(a_1a_2)\, a_3\, a_4\}^\theta = \{(a_1a_2)^\theta\, a_3^\theta\, a_4^\theta\} = \{a_1^\theta a_2^\theta a_3^\theta a_4^\theta\}$. □

Let us consider a modified version of (4):

$$\{a_{\pi(1)}a_{\pi(2)}a_{\pi(3)}a_{\pi(4)}\}^\theta = \{a_{\pi(1)}^\theta a_{\pi(2)}^\theta a_{\pi(3)}^\theta a_{\pi(4)}^\theta\} \tag{5}$$

where $\pi \in S_4$ is a permutation.

Lemma 3.4. *If (5) holds for some $\pi \in S_4$, then (5) holds for every $\pi \in S_4$.*

Proof. Without loss of generality we may assume that (4) holds, and we have to show (5) holds for every permutation π. From the identities

$$\begin{aligned}
\{a_2a_1a_3a_4\} &= \{(a_1 \circ a_2)a_3a_4\} - \{a_1a_2a_3a_4\},\\
\{a_3a_2a_1a_4\} &= \{a_1a_2a_3\} \circ a_4 - \{a_1a_2a_3a_4\},\\
\{a_4a_2a_3a_1\} = \{a_1a_3a_2a_4\} &= \{a_1(a_2 \circ a_3)a_4\} - \{a_1a_2a_3a_4\},\\
\{a_1a_4a_3a_2\} &= a_1 \circ \{a_2a_3a_4\} - \{a_1a_2a_3a_4\},\\
\{a_1a_2a_4a_3\} &= \{a_1a_2(a_3 \circ a_4)\} - \{a_1a_2a_3a_4\}
\end{aligned}$$

it follows that (5) holds whenever π is a transposition. But then it holds for every permutation π. □

We now have enough information to prove the main result of this section.

Theorem 3.5. *Let $\theta : A \to B$ be a Jordan homomorphism such that the center of $\mathrm{Alg}(A^\theta)$ does not contain nonzero nilpotent elements. Then $\{a_1a_2a_3e\}^\theta = \{a_1^\theta a_2^\theta a_3^\theta e^\theta\}$ for all $a_1, a_2, a_3 \in A$ and all idempotents $e \in A$.*

Proof. For every $a \in A$ we write $ae^{\perp} = a - ae$ and $e^{\perp}a = a - ea$ (if A was unital then we would simply define $e^{\perp}$ as $1 - e$). In view of the Peirce decomposition,

$$A = eAe \oplus e^{\perp}Ae \oplus eAe^{\perp} \oplus e^{\perp}Ae^{\perp},$$

we may assume without loss of generality that

$$a_1, a_2, a_3 \in eAe \cup e^{\perp}Ae \cup eAe^{\perp} \cup e^{\perp}Ae^{\perp}.$$

Suppose first that at least one a_i lies in $eAe \cup e^{\perp}Ae^{\perp}$. Then e commutes with a_i and so $(ea_i)^{\theta} = e^{\theta}a_i^{\theta} = a_i^{\theta}e^{\theta}$ by Lemma 3.1. Consequently, Lemma 3.3 implies that $\{ea_ia_ja_k\}^{\theta} = \{e^{\theta}a_i^{\theta}a_j^{\theta}a_k^{\theta}\}$, where $\{i, j, k\} = \{1, 2, 3\}$. The desired conclusion then follows from Lemma 3.4. Thus we may assume that $a_1, a_2, a_3 \in e^{\perp}Ae \cup eAe^{\perp}$. But then at least two among these three elements lie in one of the sets on the right hand side, i. e. either $a_i, a_j \in e^{\perp}Ae$ or $a_i, a_j \in eAe^{\perp}$, $i \neq j$. Therefore $a_ia_j = a_ja_i = 0$. The desired result now follows from Lemmas 3.2, 3.3 and 3.4. □

Corollary 3.6. *Suppose that the algebra A is equal to the linear span of its idempotents. If $\theta : A \to B$ is a Jordan homomorphism such that the center of* $\mathrm{Alg}(A^{\theta})$ *does not contain nonzero nilpotent elements, then θ is a reversal homomorphism.*

In Section 5 we will need the following anayltic version of Corollary 3.6.

Corollary 3.7. *Let A and B be normed algebras. Suppose that A is equal to the closed linear span of its idempotents. If $\theta : A \to B$ is a continuous Jordan homomorphism such that the center of* $\mathrm{Alg}(A^{\theta})$ *does not contain nonzero nilpotent elements, then θ is a reversal homomorphism.*

4. When reversal homomorphisms are SHAs

Let us mention at the beginning that some ideas from McCrimmon's paper [15] (and partially also from Martindale's paper [14]) will be hidden in our arguments in this section. However, unlike the approach in [15] (and [14]), which is based on Zelmanov's theory of Jordan algebras, our approach is rather elementary and direct.

In what follows the commutator ideal, i. e. the ideal generated by all commutators in the algebra in question, will play an important role.

Lemma 4.1. *Let A be an arbitrary algebra and let $k \in A$. Then k lies in the commutator ideal K of A if and only if there exist $a_{i_j} \in A$ such that*

$$k = \sum a_{i_1}a_{i_2}\ldots a_{i_n} \quad \textit{and} \quad \sum a_{i_n}\ldots a_{i_2}a_{i_1} = 0. \tag{6}$$

Proof. Consider the element of the form $k = a[b, c]d \in K$ with $a, b, c, d \in A$. We have

$$k = a \cdot b \cdot c \cdot d - a \cdot cb \cdot d \quad \text{and} \quad d \cdot c \cdot b \cdot a - d \cdot cb \cdot a = 0.$$

Similarly we consider elements of the form $a[b, c]$, $[b, c]d$ and $[b, c]$. Since K is linearly spanned with such elements, this proves the "only if" part. Conversely, if (6) holds, then

$$k = \sum a_{i_1}a_{i_2}\ldots a_{i_n} - \sum a_{i_n}\ldots a_{i_2}a_{i_1},$$

so it suffices to show that every element of the form $a_1a_2\ldots a_n - a_n\ldots a_2a_1$ lies in K. Using the identity

$$\begin{aligned} & a_1a_2\ldots a_n - a_n\ldots a_2a_1 \\ = \quad & [a_1a_2\ldots a_{n-1}, a_n] + a_n(a_1a_2\ldots a_{n-1} - a_{n-1}\ldots a_2a_1) \end{aligned}$$

this follows immediately by induction on n. □

Theorem 4.2. *Let A and B be arbitrary algebras and let $\theta : A \to B$ be a reversal homomorphism. Suppose that $\mathrm{Alg}(A^\theta)$ does not contain nonzero nilpotent central ideals. Then the restriction of θ to the commutator ideal K of A is an SHA.*

Proof. Set $T = \mathrm{Alg}(A^\theta)$, and define $\varphi, \psi : K \to T$ as follows: for every k of the form (6) let

$$k^\varphi = \sum a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta \quad \text{and} \quad k^\psi = \sum a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta.$$

In order to prove that φ and ψ are well-defined, we have to show that the conditions

$$\sum a_{i_1} a_{i_2} \ldots a_{i_n} = 0 \quad \text{and} \quad \sum a_{i_n} \ldots a_{i_2} a_{i_1} = 0$$

imply that

$$u = \sum a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta \quad \text{and} \quad v = \sum a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta$$

are both zero.

Note first of all that

$$\sum a_{i_1} a_{i_2} \ldots a_{i_n} + \sum a_{i_n} \ldots a_{i_2} a_{i_1} = 0,$$

and hence, since θ is a reversal homomorphism, $u + v = 0$. Secondly, we claim that $u^2 = 0$, i. e. $uv = 0$. We have

$$\begin{aligned} uv &= \Big(\sum a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta\Big)\Big(\sum a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta\Big) \\ &= \sum\Big(a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta a_{j_n}^\theta \ldots a_{j_2}^\theta a_{j_1}^\theta + a_{j_1}^\theta a_{j_2}^\theta \ldots a_{j_n}^\theta a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta\Big) \\ &+ \sum a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta. \end{aligned}$$

Since θ is a reversal homomorphism it follows that

$$\begin{aligned} uv &= \sum(a_{i_1} a_{i_2} \ldots a_{i_n} a_{j_n} \ldots a_{j_2} a_{j_1} + a_{j_1} a_{j_2} \ldots a_{j_n} a_{i_n} \ldots a_{i_2} a_{i_1})^\theta \\ &\quad + \sum(a_{i_1} a_{i_2} \ldots a_{i_n} a_{i_n} \ldots a_{i_2} a_{i_1})^\theta \\ &= \Big(\big(\sum a_{i_1} a_{i_2} \ldots a_{i_n}\big)\big(\sum a_{i_n} \ldots a_{i_2} a_{i_1}\big)\Big)^\theta = 0, \end{aligned}$$

establishing our claim.

Further, we have

$$\sum a_{i_1} a_{i_2} \ldots a_{i_n} y_1 y_2 \ldots y_m + \sum y_m \ldots y_2 y_1 a_{i_n} \ldots a_{i_2} a_{i_1} = 0$$

for all $y_i \in A$, implying that

$$\sum a_{i_1}^\theta a_{i_2}^\theta \ldots a_{i_n}^\theta y_1^\theta y_2^\theta \ldots y_m^\theta + \sum y_m^\theta \ldots y_2^\theta y_1^\theta a_{i_n}^\theta \ldots a_{i_2}^\theta a_{i_1}^\theta = 0.$$

That is,

$$u y_1^\theta y_2^\theta \ldots y_m^\theta + y_m^\theta \ldots y_2^\theta y_1^\theta v = 0,$$

and so

$$u y_1^\theta y_2^\theta \ldots y_m^\theta = y_m^\theta \ldots y_2^\theta y_1^\theta u$$

for all $y_i \in A$. Taking $m = 1$ in the last identity we see that $[u, A^\theta] = 0$, and so u lies in the center of T. Next, taking $m = 2$ it follows that $u[y_1^\theta, y_2^\theta] = 0$, which further yields

$$\begin{aligned} &[u y_1^\theta \ldots y_s^\theta, y^\theta] = u[y_1^\theta \ldots y_s^\theta, y^\theta] \\ = \ &u[y_1^\theta, y^\theta] y_2^\theta \ldots y_s^\theta + u y_1^\theta [y_2^\theta, y^\theta] y_3^\theta \ldots y_s^\theta + \ldots + u y_1^\theta \ldots y_{s-1}^\theta [y_s^\theta, y^\theta] = 0 \end{aligned}$$

for all $y_i, y \in A$. This shows that the ideal N of T generated by u lies in the center of T. Since $u^2 = 0$ it is clear that also $N^2 = 0$. But then $N = 0$ according to our assumption. That is, $u = 0$, and hence of course also $v = 0$. We have thereby proved that φ and ψ are well-defined.

From the definition of φ and ψ it is clear that

$$(kx)^\varphi = k^\varphi x^\theta,\ (xk)^\varphi = x^\theta k^\varphi,\ (kx)^\psi = x^\theta k^\psi,\ (xk)^\psi = k^\psi x^\theta \tag{7}$$

holds for all $x \in A$ and all $k \in K$. Further, it is easy to show that

$$(kk')^\varphi = k^\varphi k'^\varphi \quad \text{and} \quad (kk')^\psi = k'^\psi k^\psi \tag{8}$$

for all $k, k' \in K$. Indeed, just write k as in (6), and similarly, let $k' = \sum b_{j_1} b_{j_2} \dots b_{j_m}$ where $\sum b_{j_m} \dots b_{j_2} b_{j_1} = 0$. Then we have

$$kk' = \sum a_{i_1} a_{i_2} \dots a_{i_n} b_{j_1} b_{j_2} \dots b_{j_m} \quad \text{and} \quad \sum b_{j_m} \dots b_{j_2} b_{j_1} a_{i_n} \dots a_{i_2} a_{i_1} = 0,$$

and now (8) follows directly from the definition of φ and ψ. Thus φ is a homomorphism and ψ is an antihomomorphism.

Next, picking $k \in K$ and representing it as in (6), we have

$$\begin{aligned} k^\theta &= \Big(\sum a_{i_1} a_{i_2} \dots a_{i_n}\Big)^\theta = \Big(\sum a_{i_1} a_{i_2} \dots a_{i_n} + a_{i_n} \dots a_{i_2} a_{i_1}\Big)^\theta \\ &= \sum a_{i_1}^\theta a_{i_2}^\theta \dots a_{i_n}^\theta + \sum a_{i_n}^\theta \dots a_{i_2}^\theta a_{i_1}^\theta = k^\varphi + k^\psi. \end{aligned}$$

Thus, the restriction of θ to K is the sum of φ and ψ.

Let $k, k' \in K$. By (8) we have $(kk')^\varphi = k^\varphi k'^\varphi$, and on the other hand, in view of (7), $(kk')^\varphi = k^\varphi k'^\theta = k^\varphi(k'^\varphi + k'^\psi)$. Comparing we get $K^\varphi K^\psi = 0$. By induction on n it can be easily deduced from (7) that $(kx_1 \dots x_{n-1} x_n)^\varphi = k^\varphi x_1^\theta \dots x_{n-1}^\theta x_n^\theta$ for all $k \in K$ and $x_i \in A$. Therefore

$$k^\varphi x_1^\theta \dots x_{n-1}^\theta x_n^\theta k'^\psi = (kx_1 \dots x_{n-1} x_n)^\varphi k'^\psi \in K^\varphi K^\psi = 0,$$

which shows that $K^\varphi T K^\psi = 0$. Similarly one proves that $K^\psi K^\varphi = 0$ and $K^\psi T K^\varphi = 0$. Denoting by I the ideal of T generated by K^φ, and by J the ideal of T generated by K^ψ, we thus have $IJ = JI = 0$. Thus, the restriction of θ to K is indeed an SHA. $\square$

Remark 4.3. Suppose additionally that A (from Theorem 4.2) is an algebra with involution $*$ and θ is a Jordan $*$-homomorphism, i. e. a Jordan homomorphism that also preserves adjoints: $(x^*)^\theta = (x^\theta)^*$. From the definition of φ and ψ it is clear that they also preserve adjoints, so they are a $*$-homomorphism and a $*$-antihomomorphism, respectively.

Consider Example 2.2 for $n < 4$: θ from this example is a reversal homomorphism, which is not an SHA on A. But on the commutator ideal K of A it certainly is an SHA, in fact $K^\theta = 0$. So, the restriction to the commutator ideal in Theorem 4.2 is really necessary. See also Example 5.1 below.

Combining Corollary 3.6 and Theorem 4.2 we get the following result.

Theorem 4.4. *Let A and B be algebras, and suppose that A is equal to the linear span of its idempotents. If $\theta : A \to B$ is a Jordan homomorphism such that the center of $\mathrm{Alg}(A^\theta)$ does not contain nonzero nilpotent elements, then the restriction of θ to the commutator ideal K of A is an SHA.*

5. Jordan homomorphisms on C^*-algebras

Jordan homomorphisms are of special importance in the theory of C^*-algebras. In his classical 1951 work [13] Kadison proved that a linear bijective map from one C^*-algebra onto another one is an isometry if and only if it is a Jordan $*$-homomorphism followed by left multiplication by a unitary element. In the same paper he also proved that a Jordan $*$-homomorphism from a W^*-algebra onto a C^*-algebra is an SHA [13, Theorem 10]. In 1965 Størmer generalized Kadison's theorem as follows: If θ is a (not necessarily surjective) Jordan $*$-homomorphism from a C^*-algebra A into another C^*-algebra $B \subseteq B(H)$, then the weak closure of the C^*-algebra generated by A^θ contains a central projection e such that $x \mapsto ex^\theta$ is a $*$-homomorphism and $x \mapsto (1-e)x^\theta$ is a $*$-antihomomorphism.

Our aim in this section is to extend these results by Kadison and Størmer in different directions. In our proofs we will rely heavily on algebraic results from the previous sections. Let us mention that both Kadison and Størmer also used an entirely algebraic result as the basic tool in their proofs, namely the theorem by Jacobson and Rickart on Jordan homomorphisms on matrix rings mentioned in the introduction. In our approach we will be able to avoid using this result.

Of course, Størmer's result is in some sense definitive of its kind. In particular it shows that a Jordan $*$-homomorphism from a C^*-algebra A into a C^*-algebra B is an SHA; however, the ranges of the homomorphism and the antihomomorphism from this decomposition may not lie in B but in a bigger algebra (incidentally, results showing that Jordan homomorphisms can be expressed as SHA's only when extending the target algebra also appear in pure algebra, see e. g. [1] and [2]). Our goal is to find an intrinsic decomposition of θ, that is, we wish to express θ as the sum of a homomorphism and an antihomomorphism that both also map in B. The next example shows, however, that in this context we have to face some limitations. This example is based on the same idea as the one given in [1, p. 458]. The authors of [1] attributed that example to Kaplansky.

Example 5.1. Let $K(H)$ be the C^*-algebra of all compact operators on an infinite dimensional, separable Hilbert space H. Let A be the unitization of the C^*-algebra $K(H) \times K(H)$. Define $\theta : A \to A$ by $((S,T)+\lambda)^\theta = (S,T^t)+\lambda$; here T^t denotes the transpose of T relative to a fixed orthonormal basis. Note that θ is a Jordan $*$-automorphism, and moreover a reversal homomorphism. However, θ is not an SHA, at least not on the entire A. Indeed, suppose that θ was equal to $\varphi+\psi$ as in the above definition. Then $1 = 1^\theta = e+f$ where $e = 1^\varphi$ and $f = 1^\psi$. Clearly e and f are orthogonal idempotents. Moreover, e commutes with all elements from A^φ (since φ is a homomorphism) and it also commutes with all elements from A^ψ (since $eA^\psi = A^\psi e = 0$). Thus e commutes with $A^\theta = A$, that is to say, e lies in the center of A. But clearly the only central idempotents in A are 0 and 1, which yields that $\varphi = 0$ or $\psi = 0$ – a contradiction.

On the other hand, the restriction of θ to $K = K(H) \times K(H)$ is clearly the (direct) sum of a homomorphism $(S,T) \mapsto (S,0)$ and an antihomomorphism $(S,T) \mapsto (0,T^t)$. Note that K is the commutator ideal of A (actually, every element in K is the sum of commutators in A, see [16, Theorem 1]).

Let us finally point out that θ actually *is an SHA on A* if we allow that a homomorphism and an antihomomorphism from this decomposition have their ranges in an algebra containing A. Specifically, let $A_1 = K(H)_1 \times K(H)_1$ where $K(H)_1$ is the unitization of $K(H)$. We may consider A as a subalgebra of A_1 via the embedding $(S,T)+\lambda \mapsto (S+\lambda, T+\lambda)$. In this setting θ is the sum of a homomorphism $\varphi : A \to A_1$, $((S,T)+\lambda)^\varphi = (S+\lambda, 0)$, and an antihomomorphism $\psi : A \to A_1$, $((S,T)+\lambda)^\psi = (0, T^t+\lambda)$.

Example 5.1 justifies the restriction to the commutator ideal in the next theorem.

Theorem 5.2. *Let θ be a Jordan homomorphism from a C^*-algebra A into a C^*-algebra B. If either* (i) *θ is surjective or* (ii) *θ is a Jordan $*$-homomorphism, then the restriction of θ to the commutator ideal K of A is an SHA.*

Proof. First we note that each of the assumptions (i) or (ii) implies that θ is continuous. If (i) holds, this follows from [9, Theorem 5.8]; in case of (ii) this is more elementary, see e. g. [18, p. 439].

We claim that if (i) holds, then we may assume without loss of generality that θ is bijective. Indeed, $J = \ker\theta$ is a closed Jordan ideal of A and as such it is an ideal [9, Theorem 5.3] (see also [7] for an alternative proof). Thus A/J is a C^*-algebra, and θ induces a Jordan isomorphism $\bar{\theta}$ from A/J onto B. Now if the restriction of $\bar{\theta}$ to $\bar{K}$, the commutator ideal of A/J, was the sum of a homomorphism $\bar{\varphi} : \bar{K} \to B$ and an antihomomorphism $\bar{\psi} : \bar{K} \to B$, then the restriction of θ to K, the commutator ideal of A, would be the sum of a homomorphism $\varphi : k \mapsto (k+J)^{\bar{\varphi}}$ and an antihomomorphism $\psi : k \mapsto (k+J)^{\bar{\psi}}$. This establishes our claim.

Suppose that the center of $\mathrm{Alg}(A^\theta)$ contains an element u such that $u^2 = 0$. If θ is bijective then $u = t^\theta$ for some $t \in A$, and we have $(tt^*t)^\theta = t^\theta (t^*)^\theta t^\theta = u(t^*)^\theta u = 0$, hence $tt^*t = 0$ and so $t = 0$, yielding $u = 0$. If θ is a Jordan $*$-homomorphism, then $u^* \in \mathrm{Alg}(A^\theta)$, so that u commutes with u^*; therefore $uu^*u = 0$ which forces $u = 0$. Thus, in any case, the center of $\mathrm{Alg}(A^\theta)$ does not contain nonzero central nilpotent elements.

Since θ is continuous we may consider $\theta^{**} : A^{**} \to B^{**}$. One can check that θ^{**} is also a Jordan homomorphism (cf. [18, Lemma 3.1]). Moreover, θ^{**} is either bijective or a Jordan $*$-homomorphism. Therefore, as above we see that $\mathrm{Alg}({A^{**}}^{\theta^{**}})$ does not have nonzero central nilpotent elements. Since A^{**} is a W^*-algebra, it is equal to the closed linear span of its idempotents. Corollary 3.7 therefore tells us that θ^{**} is a reversal homomorphism. But then θ is also a reversal homomorphism. Therefore the desired conclusion follows from Theorem 4.2. □

One can view Theorem 5.2 (ii) as a supplement to the aforementioned results by Kadison and Størmer. Let us also mention that if θ is a Jordan $*$-homomorhism, then the corresponding homomorphism and an antihomomorphism also preserve adjoints (see Remark 4.3).

Corollary 5.3. *Let A be a unital C^*-algebra having no nonzero multiplicative functionals, and let θ be a Jordan homomorphism from A into a C^*-algebra B. If either* (i) *θ is surjective or* (ii) *θ is a Jordan $*$-homomorphism, then θ is an SHA.*

Proof. Since A is unital and has no nonzero multiplicative functionals, it is easy to see that A coincides with its commutator ideal [7, Lemma 2.5]. □

The assumption about the non-existence of nonzero multiplicative functionals is really necessary, as Example 5.1 shows.

In our last corollary we show that Kadison's theorem [13, Theorem 10] holds without assuming that a Jordan homomorphism θ preserves adjoints.

Corollary 5.4. *Let θ be a Jordan homomorphism from a W^*-algebra A onto a C^*-algebra B. Then θ is an SHA.*

Proof. Let e be a central projection in A such that eA is a commutative algebra and $(1-e)A$ has no abelian central summands. By Lemma 3.1 we have $(fx)^\theta = f^\theta x^\theta$ for every idempotent $f \in e\mathcal{A}$. Since the linear span of idempotents in eA is dense in eA and since θ is continuous (cf. the proof of Theorem 5.2) it follows that the restriction of θ to eA is a homomorphism. We have $1^\theta = 1'$, the identity element of B, $e' = e^\theta$ is a central idempotent in B, and moreover $(ex)^\theta = e'x^\theta$, $((1-e)x)^\theta = (1-e')x^\theta$ for every $x \in A$ (see Lemma 3.1 and the comment after its proof). Therefore $((1-e)A)^\theta = (1'-e')B$ is an algebra. Since the algebra $(1-e)A$ is equal to its commutator ideal (see e.g. [5, Lemma 2.6]) it follows from Theorem 5.2 that the restriction of θ to $(1-e)A$ is the sum of a homomorphism $\varphi_0 : (1-e)A \to (1-e')B$ and an antihomomorphism $\psi_0 : (1-e)A \to (1-e')B$. But then θ is the sum of the homomorphism $\varphi : x \mapsto (ex)^\theta + ((1-e)x)^{\varphi_0}$ and the antihomomorphism $\psi : x \mapsto ((1-e)x)^{\psi_0}$. □

Incidentally we mention that the same conclusion holds for a Jordan homomorphism from any algebra onto a W^*-algebra. This follows from [4, Theorem 1]. The proof, however, is entirely different.

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