When Jordan submodules are bimodules

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August 6, 2009

Abstract

Let $\mathcal{A}$ be an algebra and let X be an $\mathcal{A}$-bimodule. We call a linear subspace Y of X a Jordan $\mathcal{A}$-submodule of X if $Ay + yA \in Y$ for all $A \in \mathcal{A}$ and $y \in Y$ (if $X = \mathcal{A}$, then this coincides with the classical concept of a Jordan ideal). When is a Jordan $\mathcal{A}$-submodule a submodule? We give a thorough analysis of this question in both algebraic and analytic context. In the first part of the paper we consider general algebras and general Banach algebras. In the second part we treat some more specific topics, such as symmetrically normed Jordan $\mathcal{A}$-submodules. Some of our results are of interest also in the classical situation; in particular, we show that there exist C*-algebras having Jordan ideals that are not ideals.

1 Introduction

Let $\mathcal{A}$ be an associative algebra over a field F with $\mathrm{char}(F) \neq 2$. A linear subspace J of $\mathcal{A}$ is called a *Jordan ideal of* $\mathcal{A}$ if

$$A \circ B := AB + BA \in J, \text{ for all } A \in \mathcal{A} \text{ and } B \in J. \tag{1.1}$$

Under which conditions on $\mathcal{A}$ is each Jordan ideal of $\mathcal{A}$ in fact a two-sided ideal? This question was discussed in pure algebra and in functional analysis in a number of publications for more than fifty years. The mixture of algebra and analysis is also an essential feature of the present paper.

One of our main motivations for this work has been the aforementioned question for C*-algebras, that is, is every Jordan ideal J of a C*-algebra an ideal? For a long time it was known that the answer is "yes" if J is closed, and recently the positive answer was also obtained for W*-algebras (see below). The general case, however, was to the best of our knowledge so far unsettled. In the last section of this paper we give an example showing that the answer is in general "no". This is one of the main results of this paper, which has an advantage that it can be easily formulated. In order to explain the other principal results we have to introduce some additional terminology and notation, and also to give some more historical details.

Let us call an algebra $\mathcal{A}$ *Jordan ideal free* if each Jordan ideal of $\mathcal{A}$ is a two-sided ideal. So we are searching for conditions under which an algebra is Jordan ideal free. Fundamental results in this area are due to Jacobson and Rickart [15] and Herstein [14]; in [15] it was proved that the matrix algebra $M_n(\mathcal{B})$, where $n \geq 2$ and $\mathcal{B}$ is a unital algebra, is Jordan ideal free, and in [14] it was proved that simple algebras are Jordan ideal free. Later Fong, Miers and Sourour [11] gave a simple proof of the fact that the algebra $B(H)$ of all bounded operators on a separable Hilbert space H is Jordan ideal free ($B(H)$ is isomorphic to $M_n(\mathbb{C})$, if dim $H = n < \infty$, and to $M_2(\mathcal{B})$, if dim $H = \infty$). Fong and Murphy in [10] showed that all properly infinite W*-algebras are Jordan ideal free. Civin and Yood established in [6] that each closed Jordan ideal of a C*-algebra is a two-sided ideal. Recently Brešar, A. Fošner and M. Fošner proved in [4] that every algebra generated by its commutators is Jordan ideal free. Using this they showed that all W*-algebras are Jordan ideal free (this result was also obtained by L. J. Bunce (a private communication)).

Apart from an algebraic motivation for our work provided by the papers mentioned above, another stimulus came from the question about the structure of non-closed, symmetrically normed (s. n.) Jordan ideals of C*-algebras. This question arose in the study by Kissin and Shulman [16] of $\mathcal{B}$-Lipschitz functions on semisimple Hermitian Banach *-algebras $\mathcal{B}$ that continued an analytic investigation initiated by Daletskii and Krein [7] and Birman and Solomyak [1]. It was shown there that the class of non-unital algebras $\mathcal{B}$, for which all polynomials are $\mathcal{B}$-Lipschitz functions, consists of all s. n. Jordan ideals of C*-algebras. This means that $\mathcal{B}$ is a subalgebra of some C*-algebra $\mathcal{A}$, that (1.1) holds and $\|A \circ B\|_{\mathcal{B}} \leq D\|A\|\|B\|_{\mathcal{B}}$ for some $D > 0$. It was conjectured in [16] that s. n. Jordan ideals of C*-algebras are two-sided s. n. ideals and was proved to be true for $B(H)$ and for the ideal $C(H)$ of all compact operators on H.

We will now extend the concept of a Jordan ideal as follows. Let X be an $\mathcal{A}$-bimodule. A subspace Y of X is called a *Jordan $\mathcal{A}$-submodule of X* if

$$A \circ y = Ay + yA \in Y, \text{ for all } A \in \mathcal{A} \text{ and } y \in Y.$$

A Jordan $\mathcal{A}$-submodule of $\mathcal{A}$ itself is, clearly, a Jordan ideal of $\mathcal{A}$.

The bimodule approach is more flexible and has wide and fruitful applications. For example, information about the structure of Jordan $\mathcal{A}$-submodules can be used to consider Jordan ideals of a larger algebra $\mathcal{C}$ that contains $\mathcal{A}$: $\mathcal{C}$ is an $\mathcal{A}$-bimodule and its Jordan ideals are Jordan $\mathcal{A}$-submodules. Also, from the technical point of view the bimodule setting is more challenging. Let us mention an analogy: the question whether a Jordan derivation is a derivation is much more entangled if one considers Jordan derivations from an algebra into its bimodule than just those from an algebra into itself (see, for example, [3] and [20]).

Definition 1.1 *We say that an algebra $\mathcal{A}$ is Jordan free, if every Jordan $\mathcal{A}$-submodule of each $\mathcal{A}$-bimodule is an $\mathcal{A}$-bimodule.*

In Section 2 we consider the following general question: *When are algebras Jordan free*? This problem lends itself to a fuller investigation than the more "narrow" problem of finding conditions under which algebras are Jordan *ideal* free. Clearly, all Jordan free algebras are also Jordan ideal free. However, while all commutative algebras are Jordan ideal free, they are the main "culprits" as far as Jordan freeness is concerned (see Proposition 2.2).

The main result of Section 2 is Theorem 2.7 which states, in particular, that a *unital* algebra $\mathcal{A}$ is Jordan free if and only if it has no two-sided ideals $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative. Another necessary and sufficient condition is the equality

$$\mathcal{A} = \text{Alg}([\mathcal{A},\mathcal{A}]), \tag{1.2}$$

where Alg($[\mathcal{A},\mathcal{A}]$) is the subalgebra of $\mathcal{A}$ generated by all commutators $[A, B] = AB - BA$, for $A, B \in \mathcal{A}$. Condition (1.2) is still sufficient for *non-unital* algebras to be Jordan free but we do not know if it is necessary. On the other hand, the condition that a non-unital algebra $\mathcal{A}$ has no commutative quotients is not, in general, sufficient for $\mathcal{A}$ even to be Jordan ideal free (see

Proposition 2.10). Using condition (1.2), we show that all non-commutative simple algebras and all W*-algebras without non-zero commutative ideals are Jordan free.

Using (1.2), we also prove that the algebra $M_n(\mathcal{B})$, $n \geq 2$, is Jordan free for each algebra $\mathcal{B}$ with $\mathcal{B}^2 = \mathcal{B}$ and, in particular, for each C*-algebra $\mathcal{B}$. This gives rise to many examples of Jordan free algebras: the algebra $C(H)$ of all compact operators on H and the Calkin algebra $B(H)/C(H)$ are Jordan free; the algebras of all bounded operators on Schatten ideals C_p, $1 \leq p < \infty$, are Jordan free; the algebras $B(C(H))$, $B(l_p)$ and $B(c_0)$ of all bounded operators on $C(H)$, l_p and c_0 are Jordan free. (Förster and Nagy in [12] proved earlier that $B(l_p)$ and $B(c_0)$ are Jordan ideal free.) On the other hand, all Schatten ideals C_p, $1 \leq p < \infty$, and all quotient algebras C_p/C_q, $1 \leq q < p$, are not Jordan free.

Starting with Section 3 we study Jordan $\mathcal{A}$-submodules for Banach algebras $\mathcal{A}$. We say that a Banach algebra $\mathcal{A}$ is *topologically Jordan free* if each *closed* Jordan $\mathcal{A}$-submodule of a Banach $\mathcal{A}$-bimodule is an $\mathcal{A}$-bimodule. The class of topologically Jordan free Banach algebras contains the class of Jordan free Banach algebras and does not coincide with it: all Schatten ideals C_p, for example, are topologically Jordan free algebras while, as it was mentioned above, they are not Jordan free. The conditions in Theorem 3.8 for a Banach algebra to be topologically Jordan free are identical to the conditions in Theorem 2.7 for an algebra to be Jordan free, except for the additional requirement of the closure. In particular, one of the sufficient conditions for a Banach algebra $\mathcal{A}$ to be topologically Jordan free is an analogue of condition (1.2):

$$\mathcal{A} = \overline{\mathrm{Alg}([\mathcal{A},\mathcal{A}])}. \tag{1.3}$$

If $\mathcal{A}$ is unital or an arbitrary C*-algebra, then condition (1.3) is necessary and sufficient for $\mathcal{A}$ to be topologically Jordan free. It is also equivalent to the condition that $\mathcal{A}$ has no non-zero multiplicative functionals. In general, however, non-unital Banach algebras without multiplicative functionals may be not topologically Jordan free. The equivalence of these two properties is especially interesting in the class of all non-unital Banach algebras in which every closed ideal is contained in a maximal ideal. It is strongly related to the well-known problem whether all topologically simple commutative Banach algebras are one-dimensional (see Proposition 3.10).

In Section 4 we study a more subtle problem of finding conditions on Banach algebras $\mathcal{A}$ under which all symmetrically normed (s. n.) Jordan $\mathcal{A}$-submodules Y of Banach $\mathcal{A}$-bimodules X (Y are not necessarily closed in $\|\cdot\|_X$ but are complete in some norm $\|\cdot\|_Y$) are Banach $\mathcal{A}$-bimodules in $\|\cdot\|_Y$. We call such algebras *s. n. Jordan free.* They constitute a proper subset in the class of topologically Jordan free Banach algebras, so the sufficient condition for a Banach algebra $\mathcal{A}$ to be s. n. Jordan free must be stronger than (1.3). The condition we obtain is a quantative version of condition (1.3). Let $\mathbf{b}_1(\mathcal{A})$ be the unit ball of $\mathcal{A}$, let $\mathcal{K} = [\mathbf{b}_1(\mathcal{A}), \mathbf{b}_1(\mathcal{A})]$ and let $G(t\mathcal{K})$, for $t > 0$, be the absolutely convex semigroup in $\mathcal{A}$ generated by $t\mathcal{K}$. We show that $\mathcal{A}$ is s. n. Jordan free if, for each t, the closure $\overline{G(t\mathcal{K})}$ contains a ball of $\mathcal{A}$.

Using this result, we establish in Theorem 4.19 that the classes of topologically Jordan free C*-algebras and s. n. Jordan free C*-algebras coincide and consist of all C*-algebras that have no non-zero multiplicative functionals. Among other results of Section 4 we mention the following analogue of Jacobson-Rickart theorem obtained in [15]: all algebras $M_n(\mathcal{B})$, where $n \geq 2$ and $\mathcal{B}$ is a unital Banach algebra, are s. n. Jordan free.

In Section 5 we consider the problems of the existence of Jordan and s. n. Jordan ideals in C*-algebras. The fact that all commutative algebras and all C*-algebras without commutative quotients are Jordan ideal free, the results of [6] that all closed Jordan ideals of C*-algebras are two-sided ideals and of [4] that all W*-algebras are Jordan ideal free gave rise to the question whether all C*-algebras are Jordan ideal free. As already mentioned above, in Section 5 we give a negative answer: there exists a non-closed Jordan ideal J of a unital C*-algebra (which is a commutative extension of the algebra $C(H)$) that is not a two-sided ideal. Moreover, this Jordan ideal turns out to be an s. n. Jordan ideal that, in turn, gives a negative answer to the conjecture made in [16] that all s. n. Jordan ideals of C*-algebras are two-sided s. n. ideals. On the other hand, we show in Section 5 that all s. n. Jordan ideals of W*-algebras and all *reflexive* s. n. Jordan ideals of C*-algebras are two-sided s. n. ideals.

Another algebraic concept that extends the notion of Jordan ideals is the notion of *inner ideals.* A linear subspace U of an algebra $\mathcal{A}$ is an inner ideal if $u\mathcal{A}u \subseteq U$ for all $u \in U$. One-sided ideals and Jordan ideals are inner ideals and $a\mathcal{A}b$ is an inner ideal for all $a, b \in \mathcal{A}$. The structure of weak *-closed ideals in W*-algebras and norm closed ideals in C*-algebras was studied by Edwards and Rüttimann in [8],[9] and by Bunce in [5]. In particular, it was established that each norm closed inner ideal U in a C*-algebra $\mathcal{A}$ satisfies $U = \overline{\mathcal{A}U} \cap \overline{U\mathcal{A}}$. For any Jordan ideal J of a unital algebra $\mathcal{A}$, $\mathcal{A}J = J\mathcal{A}$ is a two-sided ideal, so it does not coincide with J, if J is not an ideal. The example of a non-closed Jordan ideal J of a unital C*-algebra constructed in Section 5 shows that unital C*-algebras may have non-closed inner ideals U such that $U \subsetneqq \mathcal{A}U = U\mathcal{A}$.

The authors are very grateful to the referee for the careful reading of the paper and for numerous perceptive and helpful comments.

2 Jordan free algebras

In this section we consider some necessary and sufficient conditions for an algebra $\mathcal{A}$ to be Jordan free. We also consider various classes of Jordan free algebras.

Lemma 2.1 (i) *If Jordan free subalgebras $\mathcal{A}$, $\mathcal{B}$ of an algebra $\mathcal{C}$ generate $\mathcal{C}$, then $\mathcal{C}$ is Jordan free.*

(ii) *For each two-sided ideal $\mathcal{I}$ of a Jordan free algebra $\mathcal{A}$, the quotient algebra $\widehat{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ is Jordan free.*

Proof. Let Y be a Jordan $\mathcal{C}$-submodule of an $\mathcal{C}$-bimodule X. Then X is an $\mathcal{A}$- and $\mathcal{B}$-bimodule and Y is a Jordan $\mathcal{A}$- and $\mathcal{B}$-submodule of X. Hence Y is an $\mathcal{A}$- and $\mathcal{B}$-bimodule, so it is an $\mathcal{C}$-bimodule. Part (i) is proved.

Let Y be a Jordan $\widehat{\mathcal{A}}$-submodule of an $\widehat{\mathcal{A}}$-bimodule X. Then X is an $\mathcal{A}$-bimodule with multiplications $Ax = \widehat{A}x$, $xA = x\widehat{A}$, for $A \in \mathcal{A}$ and $x \in X$, and Y is a Jordan $\mathcal{A}$-submodule. If $\mathcal{A}$ is Jordan free then Y is an $\mathcal{A}$-bimodule. Hence it is an $\widehat{\mathcal{A}}$-bimodule. ∎

For any subset $\mathcal{S}$ of $\mathcal{A}$, denote by $\mathrm{Alg}(\mathcal{S})$ the subalgebra and by $\mathrm{Id}(\mathcal{S})$ the ideal of $\mathcal{A}$ generated by $\mathcal{S}$. Clearly,

$$\mathcal{A}^2 := \left\{ \sum_{i=1}^{n} A_i B_i : A_i, B_i \in \mathcal{A},\ n \in \mathbb{N} \right\} = \mathrm{Id}(\mathcal{A}^2). \tag{2.1}$$

Let $\mathcal{A}$ be unital. An $\mathcal{A}$-bimodule X is called *unital* if $\mathbf{1}x = x\mathbf{1} = x$, for each $x \in X$.

Proposition 2.2 *Let $\mathcal{A}$ have a proper two-sided ideal $\mathcal{I}$ such that the quotient algebra $\mathcal{A}/\mathcal{I}$ is commutative* (*in particular, this holds if $\mathcal{A}^2 \neq \mathcal{A}$*). *Then*

(i) *$\mathcal{A}$ is not Jordan free.*

(ii) *If $\mathcal{A}$ is unital and* $\dim\,\mathcal{A} \geq 2$, *then there is a unital $\mathcal{A}$-bimodule containing a Jordan $\mathcal{A}$-submodule that is not an $\mathcal{A}$-bimodule.*

Proof. First assume that $\mathcal{A}^2 = \{0\}$. Let f be a non-zero linear functional on $\mathcal{A}$ and let Z be a linear space. It is easy to check that the direct sum $Z \dotplus Z$ is an $\mathcal{A}$-bimodule with multiplications

$$A(x \dotplus y) = f(A)y \dotplus 0 \text{ and } (x \dotplus y)A = -f(A)y \dotplus 0, \text{ for all } A \in \mathcal{A} \text{ and } x, y \in Z.$$

The subspace $Y = \{x \dotplus x : x \in Z\}$ is a Jordan $\mathcal{A}$-submodule of $Z \dotplus Z$ but not an $\mathcal{A}$-bimodule, so $\mathcal{A}$ is not Jordan free.

Assume now that $\mathcal{A}^2 \neq \{0\}$. The direct sum $X = \mathcal{A} \dotplus \mathcal{A}$ is an $\mathcal{A}$-bimodule with multiplications

$$A(B \dotplus C) = AB \dotplus 0 \text{ and } (B \dotplus C)A = 0 \dotplus CA, \text{ for } A, B, C \in \mathcal{A}.$$

If $\mathcal{A}$ is commutative, the subspace $Y = \{B \dotplus B : B \in \mathcal{A}\}$ of X is a Jordan $\mathcal{A}$-submodule, since

$$A \circ (B \dotplus B) = AB \dotplus BA = AB \dotplus AB \in Y, \text{ for } A, B \in \mathcal{A},$$

and it is not an $\mathcal{A}$-bimodule, since $A(B \dotplus B) = AB \dotplus 0 \notin Y$, if $AB \neq 0$. Thus $\mathcal{A}$ is not Jordan free.

If $\mathcal{A}$ has an ideal $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative then, combining the above with Lemma 2.1(ii), we complete the proof of (i).

Let $\mathcal{A}$ be commutative and unital. The $\mathcal{A}$-bimodule $X = \mathcal{A} \dotplus \mathcal{A}$ in the proof of (i) is not unital. The algebraic tensor product $X_1 = \mathcal{A} \otimes \mathcal{A}$ with multiplication generated by

$$C(A \otimes B) = CA \otimes B \text{ and } (A \otimes B)C = A \otimes BC, \text{ for } A, B, C \in \mathcal{A},$$

is a unital $\mathcal{A}$-bimodule. Let j be a linear operator on X_1 such that $j(A \otimes B) = B \otimes A$ and let $Y = \{x \in X_1 : j(x) = x\}$. Since $j(C \circ (A \otimes B)) = C \circ j(A \otimes B)$, for all $A, B, C \in \mathcal{A}$, we have $j(C \circ x) = C \circ j(x)$ for $x \in X_1$. Hence Y is a Jordan $\mathcal{A}$-submodule of X_1.

If Y is an $\mathcal{A}$-bimodule, $C(\mathbf{1} \otimes \mathbf{1}) = C \otimes \mathbf{1} \in Y$ for all $C \in \mathcal{A}$. Hence $C = \lambda\mathbf{1}$ and $\mathcal{A}$ is one-dimensional. Thus Y is not an $\mathcal{A}$-bimodule.

If $\mathcal{A}$ is not commutative, but has an ideal $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative, then, using the above argument and the argument in the proof of Lemma 2.1(ii), we construct a unital $\mathcal{A}$-bimodule that contains a Jordan $\mathcal{A}$-submodule which is not an $\mathcal{A}$-bimodule. ■

The recent papers [4] and [16] use an elementary argument which is also applicable in the present more general context. For $A, B \in \mathcal{A}$ and $x \in X$, let

$$[A, B] = AB - BA \text{ and } [A, x] = Ax - xA$$

be their commutators. The following easily checked identity connects the Jordan and Lie products:

$$[[A, B], x] = A \circ (B \circ x) - B \circ (A \circ x). \tag{2.2}$$

Therefore

$$\begin{aligned} [A, B]x &= \frac{1}{2}\left([A, B] \circ x + [[A, B], x]\right) = \frac{1}{2}\left([A, B] \circ x + A \circ (B \circ x) - B \circ (A \circ x)\right), \\ x[A, B] &= \frac{1}{2}\left([A, B] \circ x - [[A, B], x]\right) = \frac{1}{2}\left([A, B] \circ x - A \circ (B \circ x) + B \circ (A \circ x)\right). \end{aligned} \tag{2.3}$$

It is also easy to check that, for all $F, G, B, C, D \in \mathcal{A}$,

$$F[[B,C],D]G = [F[[B,C],D],G] + [GF[B,C],D] - [GF,D][B,C]. \tag{2.4}$$

For subsets $\mathcal{S}, \mathcal{R}$ of $\mathcal{A}$, let $[\mathcal{S},\mathcal{R}] = \{[A,B]\colon A \in \mathcal{S}, B \in \mathcal{R}\}$. It follows from (2.4) that

$$\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}]) \subseteq \mathrm{Alg}([\mathcal{A},\mathcal{A}]) \subseteq \mathrm{Id}([\mathcal{A},\mathcal{A}]) \subseteq \mathcal{A}. \tag{2.5}$$

¿From (2.3) we obtain immediately the following analogue of Theorem 2.1 [4].

Lemma 2.3 *If Y is a Jordan $\mathcal{A}$-submodule of X, then Y is an $\mathrm{Alg}([\mathcal{A},\mathcal{A}])$-bimodule. In particular, if $\mathcal{A} = \mathrm{Alg}([\mathcal{A},\mathcal{A}])$ then $\mathcal{A}$ is Jordan free.*

The condition

$$\mathcal{A} = \mathrm{Alg}([\mathcal{A},\mathcal{A}]). \tag{2.6}$$

plays an important role in this paper. Making use of Lemma 2.3 and the Zorn's Lemma, we have

Corollary 2.4 (i) *If an ideal $\mathcal{I}$ of $\mathcal{A}$ and the quotient algebra $\mathcal{A}/\mathcal{I}$ satisfy* (2.6), *then $\mathcal{A}$ also satisfies* (2.6), *so $\mathcal{A}$ is Jordan free.*

(ii) *Every algebra $\mathcal{A}$ has an ideal $\mathcal{I}$ satisfying* (2.6) *such that $\mathcal{A}/\mathcal{I}$ has no non-zero ideals satisfying* (2.6).

Denote by $Z(\mathcal{A})$ the center of $\mathcal{A}$. The following result was proved in [4, Lemma 2.4].

Lemma 2.5 (i) *If $C \in \mathcal{A}$ satisfies $[\mathcal{A}, C] \in Z(\mathcal{A})$, then $[A,C]^2 = 0$ for each $A \in \mathcal{A}$.*

(ii) *Let $Z(\mathcal{A})$ have no non-zero nilpotent elements. If $[\mathcal{A}, C] \in Z(\mathcal{A})$ then $C \in Z(\mathcal{A})$.*

Proof. If $[A,C] \in Z(\mathcal{A})$, for each $A \in \mathcal{A}$, then also $[AC, C] \in Z(\mathcal{A})$. Hence

$$[A,C]^2 = [A,[A,C]C] = [A,[AC,C]] = 0.$$

Part (i) is proved. Part (ii) follows immediately from (i). ∎

We say that $\mathcal{A}$ has *max-property* if every proper ideal of $\mathcal{A}$ is contained in a maximal proper ideal of $\mathcal{A}$. If, for example, $\mathcal{A}$ has a finite subset which is not contained in any proper ideal of $\mathcal{A}$, then it follows from the Zorn's lemma that $\mathcal{A}$ has max-property. Thus all unital algebras, simple algebras and finitely generated algebras have max-property.

The class of all algebras with max-property is closed under extension.

Proposition 2.6 *If an ideal $\mathcal{I}$ of $\mathcal{A}$ and the quotient $\mathcal{A}/\mathcal{I}$ have max-property, then $\mathcal{A}$ has max-property.*

Proof. Denote by π the canonical map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$ and let $\mathcal{J}$ be a proper ideal of $\mathcal{A}$. If the ideal $\pi(\mathcal{J})$ of $\mathcal{A}/\mathcal{I}$ is proper, there is a maximal ideal $\mathcal{M}$ of $\mathcal{A}/\mathcal{I}$ containing $\pi(\mathcal{J})$. Then the ideal $\mathcal{L} = \pi^{-1}(\mathcal{M})$ is proper, since $\pi(\mathcal{L}) = \mathcal{M} \neq \mathcal{A}/\mathcal{I}$. It is also maximal. Indeed, if an ideal $\mathcal{L}'$ of $\mathcal{A}$ is larger than $\mathcal{L}$, then $\pi(\mathcal{L}') \supsetneqq \pi(\mathcal{L}) = \mathcal{M}$. Hence $\pi(\mathcal{L}') = \mathcal{A}/\mathcal{I}$, therefore $\mathcal{L}' = \mathcal{A}$, as $\mathcal{I} \subseteq \mathcal{L}'$.

Assume now that $\pi(\mathcal{J}) = \mathcal{A}/\mathcal{I}$ whence $\mathcal{A} = \mathcal{I} + \mathcal{J}$. Let $\mathcal{K}$ be a maximal proper ideal of $\mathcal{I}$ containing $\mathcal{J} \cap \mathcal{I}$. Then $\mathcal{K} + \mathcal{J}$ is a maximal proper ideal of $\mathcal{A}$ containing $\mathcal{J}$. Indeed, $\mathcal{K} + \mathcal{J} \neq \mathcal{A}$ because $(\mathcal{K} + \mathcal{J}) \cap \mathcal{I} = \mathcal{K} + (\mathcal{J} \cap \mathcal{I}) = \mathcal{K} \neq \mathcal{I}$. On the other hand, if an ideal $\mathcal{N}$ of $\mathcal{A}$ contains $\mathcal{K} + \mathcal{J}$, then $\mathcal{N} = \mathcal{J} + (\mathcal{N} \cap \mathcal{I})$, because for any $A \in \mathcal{M}$, there is $B \in \mathcal{J}$ with $\pi(A) = \pi(B)$. As $\mathcal{N} \cap \mathcal{I} \supseteq \mathcal{K}$, either $\mathcal{N} \cap \mathcal{I} = \mathcal{K}$ and $\mathcal{N} = \mathcal{K} + \mathcal{J}$, or $\mathcal{N} \cap \mathcal{I} = \mathcal{I}$ and $\mathcal{N} = \mathcal{A}$. ■

We shall now prove the main theorem of this section.

Theorem 2.7 *Consider the following conditions for an algebra* $\mathcal{A}$:

(i) $\mathcal{A}$ *is Jordan free.*

(ii) $\mathrm{Id}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$.

(iii) $\mathrm{Alg}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$.

(iv) $\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}]) = \mathcal{A}$.

(v) $\mathcal{A}$ *has no proper two-sided ideals* $\mathcal{I}$ *such that the quotient algebra* $\mathcal{A}/\mathcal{I}$ *is commutative.*

(vi) $\mathcal{A}$ *has no non-zero multiplicative linear functionals.*

Then

$$\text{(iv)} \Longleftrightarrow \text{(iii)} \Longrightarrow \text{(i)} \Longrightarrow \text{(v)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(vi)}. \tag{2.7}$$

If $\mathcal{A}$ *has max-property, then conditions* (i)-(v) *are equivalent.*

Proof. (iv) ⇒ (iii) follows from (2.5).

(iii) ⇒ (iv). Assume that $\mathcal{I} = \mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}]) \neq \mathcal{A}$. Since $[[A,B],C] \in \mathcal{I}$, for all $A,B,C \in \mathcal{A}$, the quotient algebra $\mathcal{B} = \mathcal{A}/\mathcal{I}$ satisfies $[[\mathcal{B},\mathcal{B}],\mathcal{B}] = \{0\}$. Hence $[\mathcal{B},\mathcal{B}] \subseteq Z(\mathcal{B})$. If (iii) holds, we have

$$\{0\} \neq \mathcal{B} = \mathrm{Alg}([\mathcal{B},\mathcal{B}]) \subseteq Z(\mathcal{B}).$$

Therefore $[\mathcal{B},\mathcal{B}] = \{0\}$, so that $\mathcal{B} = \{0\}$, a contradiction.

(iii) ⇒ (i) follows from Lemma 2.3, and (i) ⇒ (v) follows from Proposition 2.2(i).

(v) $\Longleftrightarrow$ (ii) follows from the fact that $\mathrm{Id}([\mathcal{A},\mathcal{A}])$ is the smallest out of all the ideals $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is commutative.

(ii) ⇒ (vi) follows from the fact that $\mathrm{Id}([\mathcal{A},\mathcal{A}])$ lies in the kernel of any multiplicative functional on $\mathcal{A}$.

Suppose now that $\mathcal{A}$ has max-property. To show that conditions (i)-(v) are equivalent we only need to prove (ii) $\Longrightarrow$ (iv). Assume that $\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}]) \neq \mathcal{A}$ and let $\mathcal{I}$ be a maximal ideal of $\mathcal{A}$ containing $\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])$. Then $\mathcal{B} = \mathcal{A}/\mathcal{I} \neq \{0\}$ has only one proper ideal $\{0\}$ and, as in the proof of (iii) $\Longrightarrow$ (iv), satisfies $[[\mathcal{B},\mathcal{B}],\mathcal{B}] = \{0\}$. Hence $[\mathcal{B},\mathcal{B}] \subseteq Z(\mathcal{B})$ and, by Lemma 2.5(i),

$$R^2 = 0 \text{ for each } R \in [\mathcal{B},\mathcal{B}]. \tag{2.8}$$

We have from (ii) that $\mathrm{Id}([\mathcal{B},\mathcal{B}]) = \mathcal{B} \neq \{0\}$, so there is $0 \neq C \in [\mathcal{B},\mathcal{B}]$. Then $\mathcal{B}C$ is a two-sided ideal in $\mathcal{B}$. Hence either $\mathcal{B}C = \mathcal{B}$ or $\mathcal{B}C = \{0\}$. If $\mathcal{B}C = \mathcal{B}$ then, by (2.8), $\{0\} = \mathcal{B}C^2 = \mathcal{B}C = \mathcal{B}$. Thus $\mathcal{B}C = \{0\}$. Hence the one-dimensional space L generated by C is an ideal of $\mathcal{B}$. Since $\mathcal{B}$ has only one proper ideal $\{0\}$, $\mathcal{B} = L$. Therefore $\mathcal{B} = \mathrm{Id}([\mathcal{B},\mathcal{B}]) = \{0\}$. This contradiction shows that $\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}]) = \mathcal{A}$, so (ii) $\Longrightarrow$ (iv). ■

Since for non-commutative simple algebras condition (v) holds, Theorem 2.7 yields the following generalization of Herstein's result.

Corollary 2.8 *All non-commutative simple algebras are Jordan free.*

If $\mathcal{I}$ is a proper ideal of a unital Banach algebra $\mathcal{A}$ over $\mathbb{C}$, its closure is also a proper ideal of $\mathcal{A}$. Taking this into account and the fact that every unital commutative Banach algebra has a non-zero multiplicative linear functional, we obtain

Corollary 2.9 *For unital Banach algebras, all conditions* (i)-(vi) *of Theorem* 2.7 *are equivalent.*

For unital C*-algebras, the equivalence of conditions (ii), (iii), (vi) of Theorem 2.7 was established in [4]. As the example below shows, for non-unital algebras, conditions (i)-(v) of Theorem 2.7 are not, in general, equivalent.

Let $\mathcal{G}$ be the Grassman algebra (without an identity element) on the set $\{x_1, x_2, \ldots\}$ over a field F. Thus all generators x_i of $\mathcal{G}$ satisfy the following relations:

$$x_i x_j + x_j x_i = 0 \text{ and } x_i^2 = 0 \text{ for all } i \text{ and } j.$$

A typical element of $\mathcal{G}$ is therefore a linear combination of monomials $x_{i_1} x_{i_2} \ldots x_{i_n}$ with $i_1 < i_2 < \ldots < i_n$. Let us point out a few elementary properties of $\mathcal{G}$:

1) monomials of even degree lie in the center of $\mathcal{G}$;
2) any two monomials of odd degree anti-commute;
3) the commutator of any two elements is of even degree;
4) $[[\mathcal{G}, \mathcal{G}], \mathcal{G}] = 0$.

Let $\mathcal{I}_i$, for $i \in \mathbb{N}\backslash 0$, be the ideal of $\mathcal{G}$ generated by the element

$$x_i - x_{3i-1} x_{3i} x_{3i+1} = x_i - \frac{1}{2}[x_{3i-1}, x_{3i}] x_{3i+1},$$

and let $\mathcal{I}$ be the ideal of $\mathcal{G}$ generated by all elements of the form $x_i - x_{3i-1} x_{3i} x_{3i+1}$, for $i = 1, 2, \ldots$, that is, $\mathcal{G}$ is the linear span of $\left\{\cup_{i \in \mathbb{N}\backslash 0} \mathcal{I}_i\right\}$. Let us show that $\mathcal{I} \neq \mathcal{G}$. We claim that $x_1 \notin \mathcal{I}$. Suppose this was not true. Then $x_1 \in \mathcal{I}_1 + \ldots + \mathcal{I}_n$, for some n. Using the properties stated above, it is easily seen that all $\mathcal{I}_i^2 = 0$. Accordingly,

$$(\mathcal{I}_1 + \ldots + \mathcal{I}_n)(x_1 - x_2 x_3 x_4)(x_2 - x_5 x_6 x_7) \ldots (x_n - x_{3n-1} x_{3n} x_{3n+1}) = 0.$$

If $x_1 \in \mathcal{I}_1 + \ldots + \mathcal{I}_n$, then

$$\begin{aligned} & (-1)^n x_1 x_2 x_3 x_4 \ldots x_{3n} x_{3n+1} \\ = \; & x_1 (x_1 - x_2 x_3 x_4)(x_2 - x_5 x_6 x_7) \ldots (x_n - x_{3n-1} x_{3n} x_{3n+1}) = 0, \end{aligned}$$

a contradiction. Thus $\mathcal{I} \neq \mathcal{G}$, so the quotient algebra $\mathcal{A} = \mathcal{G}/\mathcal{I} \neq \{0\}$.

Proposition 2.10 *The algebra $\mathcal{A} = \mathcal{G}/\mathcal{I}$ has no proper ideals $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J}$ is commutative and*

$$0 = [[\mathcal{A}, \mathcal{A}], \mathcal{A}] \subsetneq \mathrm{Alg}([\mathcal{A}, \mathcal{A}]) \subsetneq \mathrm{Id}([\mathcal{A}, \mathcal{A}]) = \mathcal{A}.$$

The algebra $\mathcal{A}$ has a Jordan ideal which is not a two-sided ideal, so $\mathcal{A}$ is not Jordan ideal free.

Proof. The algebra $\mathcal{A}$ is generated (as an algebra) by elements $\widehat{x}_i = x_i + \mathcal{I}$ that satisfy $\widehat{x}_i = \frac{1}{2}[\widehat{x}_{3i-1}, \widehat{x}_{3i}]\widehat{x}_{3i+1} \in \mathrm{Id}([\mathcal{A},\mathcal{A}])$. Hence $\mathrm{Id}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$. By (2.7), this implies that $\mathcal{A}$ has no proper ideals $\mathcal{J}$ such that $\mathcal{A}/\mathcal{J}$ is commutative.

Clearly, $\mathcal{A}$ inherits the property 4): $[[\mathcal{A},\mathcal{A}],\mathcal{A}] = 0$.

Since $\mathrm{Id}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$, we have that $\mathcal{A}$ is not commutative. As $[\mathcal{A},\mathcal{A}] \subseteq Z(\mathcal{A})$, we have

$$\{0\} \neq [\mathcal{A},\mathcal{A}] \subseteq \mathrm{Alg}([\mathcal{A},\mathcal{A}]) \subseteq Z(\mathcal{A}) \neq \mathcal{A}.$$

Finally, consider a linear subspace J of $\mathcal{A}$ generated by all odd monomials. By property 1),

$$\begin{aligned}\widehat{x}_{j_1}...\widehat{x}_{j_{2m}} \circ \widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}} &= \widehat{x}_{j_1}...\widehat{x}_{j_{2m}}\widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}} + \widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}}\widehat{x}_{j_1}...\widehat{x}_{j_{2m}} \\ &= 2\widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}}\widehat{x}_{j_1}...\widehat{x}_{j_{2m}} \in J.\end{aligned}$$

By property 2),

$$\widehat{x}_{j_1}...\widehat{x}_{j_{2m+1}} \circ \widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}} = \widehat{x}_{j_1}...\widehat{x}_{j_{2m+1}}\widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}} + \widehat{x}_{i_1}...\widehat{x}_{i_{2n+1}}\widehat{x}_{j_1}...\widehat{x}_{j_{2m+1}} = 0 \in J.$$

Hence J is a Jordan ideal of $\mathcal{A}$. Since the product of odd monomials is an even monomial, J is not a two-sided ideal. ■

The above proposition shows that, in general, the equivalent properties (ii) and (v) of Theorem 2.7 do not imply (i). We do not know if (i) implies (iii).

Problem 2.11 *Does the equality* $\mathrm{Alg}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$ *hold for all Jordan free algebras*?

We will consider now some special classes of Jordan free algebras.

Corollary 2.12 *The algebra* $\mathcal{F}$ *of all finite rank operators on a linear space* X, $\dim X \neq 1$, *is Jordan free.*

Proof. As $\mathcal{F}$ is simple and non-commutative, the proof follows from Corollary 2.8. ■

Corollary 2.13 *Every* W^**-algebra* $\mathcal{A}$ *without commutative weakly closed ideals is Jordan free.*

Proof. It follows from the proof of Corollary 2.6 of [4] that $\mathcal{A} = \mathrm{Alg}([\mathcal{A},\mathcal{A}])$. By Lemma 2.3, $\mathcal{A}$ is Jordan free. ■

Proposition 2.14 (i) *If* $\mathcal{A}$, $\mathcal{B}$ *are unital and Jordan free, the tensor product* $\mathcal{A} \otimes \mathcal{B}$ *is Jordan free.*

(ii) *Let* $\mathcal{A} = \mathrm{Alg}([\mathcal{A},\mathcal{A}])$ *and let* $\mathcal{B}$ *coincide with the linear span of squares of its elements. Set* $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$. *Then* $\mathcal{C} = \mathrm{Alg}([\mathcal{C},\mathcal{C}])$, *so* $\mathcal{C}$ *is Jordan free.*

(iii) *If* $\mathcal{B}^2 \neq \mathcal{B}$ *then, for each algebra* $\mathcal{A}$, $\mathcal{A} \otimes \mathcal{B}$ *is not Jordan free.*

Proof. The algebra $\mathcal{A}\otimes\mathbf{1}_{\mathcal{B}}$ is isomorphic to $\mathcal{A}$, the algebra $\mathbf{1}_{\mathcal{A}}\otimes\mathcal{B}$ is isomorphic to $\mathcal{B}$ and they generate $\mathcal{A}\otimes\mathcal{B}$. Hence (i) follows from Lemma 2.1(i).

For $D\in\mathcal{B}$ and $A,B\in\mathcal{A}$,

$$[A,B]\otimes D^2=[A\otimes D,B\otimes D]\in \text{Alg}([\mathcal{C},\mathcal{C}]).$$

If $\mathcal{B}$ is the linear span of squares of all $D\in\mathcal{B}$, $[A,B]\otimes T\in \text{Alg}([\mathcal{C},\mathcal{C}])$, for $A,B\in\mathcal{A}$, $T\in\mathcal{B}$. Set

$$\mathcal{F}=\{A\in\mathcal{A}: A\otimes D\in \text{Alg}([\mathcal{C},\mathcal{C}]) \text{ for all } D\in\mathcal{B}\}.$$

Then $[\mathcal{A},\mathcal{A}]\subseteq\mathcal{F}$. If $A,B\in\mathcal{F}$, then $A+B\in\mathcal{F}$ and, for each $D\in\mathcal{B}$,

$$AB\otimes D^2=(A\otimes D)(B\otimes D)\in \text{Alg}([\mathcal{C},\mathcal{C}]).$$

Since $\mathcal{B}$ is the linear span of squares of all $D\in\mathcal{B}$, we have $AB\in\mathcal{F}$. Hence $\mathcal{F}$ is an algebra containing $[\mathcal{A},\mathcal{A}]$. Therefore $\mathcal{A}=\text{Alg}([\mathcal{A},\mathcal{A}])=\mathcal{F}$. Thus $\mathcal{C}=\text{Alg}([\mathcal{C},\mathcal{C}])$. Part (ii) is proved.

By (2.1), $(\mathcal{A}\otimes\mathcal{B})^2=\mathcal{A}^2\otimes\mathcal{B}^2\subseteq\mathcal{A}\otimes\mathcal{B}^2\neq\mathcal{A}\otimes\mathcal{B}$, so (iii) follows from Proposition 2.2(i). ■

Corollary 2.15 *Let $\mathcal{B}$ be an algebra over a field F. For each $n\geq 2$, the algebra $\mathcal{A}=M_n(\mathcal{B})$ is Jordan free if and only if $\mathcal{B}^2=\mathcal{B}$. In particular, $M_n(\mathcal{B})$ is Jordan free if $\mathcal{B}$ is unital.*

Proof. We have $\mathcal{A}=M_n(F)\otimes\mathcal{B}$. If $\mathcal{B}^2\neq\mathcal{B}$, by Proposition 2.14(iii), $\mathcal{A}$ is not Jordan free.

Let now $\mathcal{B}^2=\mathcal{B}$. Denote by $\{e_{ij}\}$ the matrix identity in $M_n(F)$. Let $i\neq j$. For all $A,B\in\mathcal{B}$, we have $e_{ij}\otimes AB=[e_{ii}\otimes A,e_{ij}\otimes B]\in[\mathcal{A},\mathcal{A}]$. Since $\mathcal{B}^2=\mathcal{B}$, it follows from (2.1) that $e_{ij}\otimes C\in \text{Alg}([\mathcal{A},\mathcal{A}])$ for all $C\in\mathcal{B}$.

For $A,B\in\mathcal{B}$, $e_{ii}\otimes AB=(e_{ij}\otimes A)(e_{ji}\otimes B)\in\text{Alg}([\mathcal{A},\mathcal{A}])$. As $\mathcal{B}^2=\mathcal{B}$, we have from (2.1) that $e_{ii}\otimes C\in\text{Alg}([\mathcal{A},\mathcal{A}])$, for all $C\in\mathcal{B}$. Hence $\mathcal{A}=\text{Alg}([\mathcal{A},\mathcal{A}])$. By Lemma 2.3, $\mathcal{A}$ is Jordan free. ■

The above result is a generalization of the theorem about Jordan ideals of the algebras $M_n(\mathcal{B})$ established by Jacobson and Rickart in [15, Theorem 11].

If $\mathcal{B}$ is a C*-algebra then each $0<A\in\mathcal{B}$ is represented as $A=B^2$ with $0<B\in\mathcal{B}$. Since each $G\in\mathcal{B}$ is represented as $G=G_1-G_2+i(G_3-G_4)$ with $0\leq G_i\in\mathcal{B}$, we have $\mathcal{B}^2=\mathcal{B}$.

Corollary 2.16 *Let $\mathcal{B}$ be a C*-algebra. For any $n\geq 2$, the matrix algebra $M_n(\mathcal{B})$ is Jordan free.*

Corollary 2.17 *Let H be a Hilbert space with $n=\dim H>1$.*

(i) *The algebra $B(H)$ and the ideal $C(H)$ of all compact operators on H are Jordan free.*

(ii) *The Calkin algebra $B(H)/C(H)$ is Jordan free.*

Proof. If $n<\infty$, then $B(H)=C(H)=M_n(\mathbb{C})$, so $B(H)$ and $C(H)$ are Jordan free.

If $\dim H=\infty$, then $H=K\oplus K$, so $B(H)=M_2(B(K))$ and $C(H)=M_2(C(K))$. By Corollary 2.16, $B(H)$ and $C(H)$ are Jordan free. Part (ii) follows from Lemma 2.1. ■

Corollary 2.17(i) generalizes the result of Fong, Miers and Sourour [11, Theorem 3] that $B(H)$ is Jordan ideal free. For Schatten ideals C_p, $1\leq p<\infty$, (see the definition of the ideals C_p in the next section) the situation is different.

Corollary 2.18 (i) *All Schatten ideals C_p, $1 \le p < \infty$, are not Jordan free.*

(ii) *The quotient algebras $B(H)/C_p$ and $C(H)/C_p$ are Jordan free.*

(iii) *The quotient algebras C_p/C_q, for $1 \le q < p$, are not Jordan free.*

Proof. It is well known that $\{0\} \neq (C_p)^2 \subseteq C_{\frac{p}{2}} \neq C_p$ for $1 \le p < \infty$. Therefore $\{0\} \neq (C_p/C_q)^2 \subseteq C_{\frac{p}{2}}/C_q \neq C_p/C_q$ for $q < p$. Hence, by Proposition 2.2(i), C_p and C_p/C_q are not Jordan free. This proves parts (i) and (iii). Part (ii) follows Lemma 2.1(ii) and Corollary 2.17. ■

We saw above that, for Hilbert spaces H with dim $H \neq 1$, the algebra $B(H)$ is Jordan free. We will consider now the algebra $B(X)$ of all bounded operators on a Banach space X.

Proposition 2.19 *The algebra $B(X)$ is Jordan free, if there are $E_{ij} \in B(X)$ such that*

$$E_{ij}E_{km} = \delta_{jk}E_{im} \text{ and } E_{11} + ... + E_{nn} = \mathbf{1},\ n \ge 2. \tag{2.9}$$

Proof. Since $B(X)$ is isomorphic to $M_n(E_{11}B(X)E_{11})$, apply Corollary 2.15. ■

Corollary 2.20 *If $X = C_p$ or $X = l_p$, for $1 \le p < \infty$, or if $X = C(H)$ or $X = c_0$, then $B(X) \approx M_2(\mathcal{B})$ for some unital algebra $\mathcal{B}$, so $B(X)$ is Jordan free.*

Proof. Let $H = K \oplus K$ and let U_{ij} be the matrix identity in $B(H) \approx M_2(B(K))$. Let $X = C_p$. For $A \in X$, set $E_{ij}(A) = U_{ij}A$. Then E_{ij} satisfy (2.9) and $E_{ij} \in B(X)$, since $E_{ij}(A) \in X$ and

$$\|E_{ij}(A)\|_p = \|U_{ij}A\|_p \le \|U_{ij}\|\|A\|_p = \|A\|_p.$$

Hence all $B(C_p)$ are Jordan free. Similarly, $B(C(H))$ is Jordan free.

Let $X = l_p$, $1 \le p \le \infty$, or $X = c_0$. Let E_{11} be the operator of multiplication on X by $(1, 0, 1, 0, ...)$ and $E_{22} = \mathbf{1} - E_{11}$. Let $E_{12}(\lambda_1, \lambda_2, ...) = (\lambda_2, 0, \lambda_4, 0, \lambda_6, 0, ...)$ and $E_{21}(\lambda_1, \lambda_2, ...) = (0, \lambda_1, 0, \lambda_3, 0, \lambda_5, ...)$. Then E_{ij} satisfy (2.9), so $B(l_p)$ and $B(c_0)$ are Jordan free. ■

Corollary 2.20(ii) extends the result of Förster and Nagy in [12] (see also [4]) where it was shown that the algebras $B(l_p)$ and $B(c_0)$ are Jordan ideal free.

3 Topologically Jordan free Banach algebras

In this section we always assume that $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$. The natural class of $\mathcal{A}$-bimodules to consider is the class of Banach $\mathcal{A}$-bimodules, that is, Banach spaces X which are $\mathcal{A}$-bimodules and

$$\|Ax\|_X \le \|A\|\|x\|_X \text{ and } \|xA\|_X \le \|A\|\|x\|_X, \text{ for all } A \in \mathcal{A} \text{ and } x \in X. \tag{3.1}$$

Let $\widetilde{\mathcal{A}} = \mathbb{C}\mathbf{1} + \mathcal{A}$ be the unitization of $\mathcal{A}$ with norm $\|\lambda\mathbf{1}+A\| = |\lambda| + \|A\|$, for $A \in \mathcal{A}$. Setting $\mathbf{1}x = x\mathbf{1} = x$, for all $x \in X$, we have that X is a Banach $\widetilde{\mathcal{A}}$-bimodule.

Lemma 3.1 *Let $(X, \|\cdot\|_X)$ be a Banach space and an $\mathcal{A}$-bimodule. Then X has an equivalent norm $\|\cdot\|'_X$ with respect to which it is a Banach $\mathcal{A}$-bimodule, if and only if, for some $C > 0$,*

$$\|Ax\|_X \le C\|A\|\|x\|_X \text{ and } \|xA\|_X \le C\|A\|\|x\|_X \text{ for all } A \in \mathcal{A} \text{ and } x \in X. \tag{3.2}$$

Proof. We only need to prove "if" part. Set $K = \max(1, C, C^2)$. Then

$$\|AxB\|_X \le K\|A\|\|B\|\|x\|_X, \text{ for all } A, B \in \widetilde{\mathcal{A}} \text{ and } x \in X.$$

The norm $\|x\|'_X = \sup\{\|UxV\|_X\colon U, V \in \widetilde{\mathcal{A}}, \|U\|_{\widetilde{\mathcal{A}}} \le 1, \|V\|_{\widetilde{\mathcal{A}}} \le 1\}$ on X is equivalent to $\|\cdot\|_X$, as

$$\begin{aligned}\|x\|_X &= \|\mathbf{1}x\mathbf{1}\|_X \le \|x\|'_X = \sup\{\|UxV\|_X\colon U, V \in \widetilde{\mathcal{A}},\ \|U\|_{\widetilde{\mathcal{A}}} \le 1,\ \|V\|_{\widetilde{\mathcal{A}}} \le 1\} \\ &\le K\sup\{\|U\|\|x\|_X\|V\|\colon U, V \in \widetilde{\mathcal{A}},\ \|U\|_{\widetilde{\mathcal{A}}} \le 1,\ \|V\|_{\widetilde{\mathcal{A}}} \le 1\} = K\|x\|_X,\end{aligned}$$

and $\|AxB\|'_X \le \|A\|\|B\|\|x\|'_X$ for $A, B \in \widetilde{\mathcal{A}}$ and $x \in X$. Thus $(X, \|\cdot\|'_X)$ is a Banach $\widetilde{\mathcal{A}}$-bimodule. ■

Definition 3.2 *We say that a Banach algebra $\mathcal{A}$ is topologically Jordan free if each closed Jordan $\mathcal{A}$-submodule of a Banach $\mathcal{A}$-bimodule is an $\mathcal{A}$-bimodule.*

All Jordan free Banach algebras are topologically Jordan free. The opposite inclusion is not true (this follows from Corollaries 2.18 and 3.7), so the class of topologically Jordan free Banach algebras is larger than the class of Jordan free Banach algebras.

For a subalgebra $\mathcal{B}$ of a Banach algebra $\mathcal{A}$, denote by $\overline{\mathcal{B}}$ its norm closure in $\mathcal{A}$. The following result is an analogue of Lemma 2.3 for Banach algebras.

Lemma 3.3 *If Y is a closed Jordan $\mathcal{A}$-submodule of a Banach $\mathcal{A}$-bimodule, then Y is a Banach $\overline{\mathrm{Alg}([\mathcal{A},\mathcal{A}])}$-bimodule. In particular, if $\mathcal{A} = \overline{\mathrm{Alg}([\mathcal{A},\mathcal{A}])}$ then $\mathcal{A}$ is topologically Jordan free.*

Recall that a Banach algebra $\mathcal{A}$ is *topologically simple* if it has no *closed* two-sided ideals except $\{0\}$ and itself. Topologically simple Banach algebras are often not simple algebras. For example, the Schatten ideals C_p, $1 \le p < \infty$, are topologically simple Banach algebras but not simple algebras.

Proposition 3.4 *Each topologically simple non-commutative Banach algebra $\mathcal{A}$ is topologically Jordan free.*

Proof. First let us show that the center $Z(\mathcal{A})$ of any topologically simple Banach algebra $\mathcal{A}$, $\dim \mathcal{A} \neq 1$, has no non-zero nilpotent elements. Let $C \in Z(\mathcal{A})$ be such that $C^n = 0$ and $C^{n-1} \neq 0$, $n > 1$. If $\mathcal{A}C^{n-1} = \{0\}$, the one-dimensional space generated by C^{n-1} is a closed ideal of $\mathcal{A}$. Hence $\mathcal{A}C^{n-1} \neq \{0\}$. Since $\mathcal{A}$ is topologically simple and $\overline{\mathcal{A}C^{n-1}}$ is a closed ideal in $\mathcal{A}$, $\overline{\mathcal{A}C^{n-1}} = \mathcal{A}$. Then $\overline{\mathcal{A}C} = \overline{\mathcal{A}C^n} = \{0\}$. Therefore the one-dimensional space generated by C is a closed ideal of $\mathcal{A}$. Thus $Z(\mathcal{A})$ has no non-zero nilpotent elements.

By Lemma 2.5, $[[\mathcal{A}, C], \mathcal{A}] = 0$ implies that $C \in Z(\mathcal{A})$. Hence, since $\mathcal{A}$ is simple and non-commutative, there are $B, C, D \in \mathcal{A}$ such that $[[B, C], D] \neq 0$. It follows from (2.4) that the two-sided ideal $\mathcal{I} = \mathcal{A}[[B, C], D]\mathcal{A}$ is contained in $\mathrm{Alg}([\mathcal{A},\mathcal{A}])$.

Let Y be a closed Jordan $\mathcal{A}$-submodule of X. By Lemma 2.3, Y is an $\mathcal{I}$-bimodule. Since $\mathcal{A}$ is simple, $\overline{\mathcal{I}} = \mathcal{A}$. Since Y is a closed subspace of X, it is a Banach $\mathcal{A}$-bimodule. ■

The next result is an analogue of Lemma 2.1(ii) for Banach algebras.

Proposition 3.5 *Let a Banach algebra $\mathcal{A}$ be topologically Jordan free. Then, for each closed two-sided ideal $\mathcal{I}$, the quotient Banach algebra $\widehat{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ is topologically Jordan free.*

Proof. Let Y be a closed Jordan $\widehat{\mathcal{A}}$-submodule of a Banach $\widehat{\mathcal{A}}$-bimodule X. Then X is an $\mathcal{A}$-bimodule with multiplications $Ax = \widehat{A}x$, $xA = x\widehat{A}$, for $A \in \mathcal{A}$ and $x \in X$, and Y is a Jordan $\mathcal{A}$-submodule. Moreover,

$$\|Ax\|_X = \|\widehat{A}x\|_X \le \|\widehat{A}\|\|x\|_X \le \|A\|\|x\|_X,$$

Thus X is a Banach $\mathcal{A}$-bimodule. Since $\mathcal{A}$ is topologically Jordan free, Y is an $\mathcal{A}$-bimodule. Since $Ax = \widehat{A}x$, $xA = x\widehat{A}$, it is an $\widehat{\mathcal{A}}$-bimodule. ■

Making use of Proposition 3.5 and replacing in the beginning of the proof of Proposition 2.2(i) the linear space Z by a Banach space and the linear functional f on $\mathcal{A}$ by a bounded functional f with $\|f\| = 1$, we obtain the following analogue of Proposition 2.2(i).

Proposition 3.6 *If a Banach algebra $\mathcal{A}$ has a closed two-sided ideal $\mathcal{I}$ such that the quotient algebra $\mathcal{A}/\mathcal{I}$ is commutative, then $\mathcal{A}$ is not topologically Jordan free.*

Let H be a separable Hilbert space. A two-sided ideal $\mathcal{J}$ of $B(H)$ is a *symmetrically normed* (s. n.) ideal if it is a Banach space with respect to a norm $\|\cdot\|_{\mathcal{J}}$ and

$$\|AXB\|_{\mathcal{J}} \le \|A\|\|X\|_{\mathcal{J}}\|B\|, \text{ for all } A, B \in B(H) \text{ and } X \in \mathcal{J}.$$

S. n. ideals of $B(H)$ are related to symmetric norming functions on the space of all sequences $\xi = \{\xi_i\}$ of real numbers converging to 0. Denote by Φ the set of all such functions. Each $\phi \in \Phi$ defines an s. n. ideal $(J^\phi, \|\cdot\|_\phi)$ of $B(H)$ (for the detailed discussion, see [13]). For example, the functions

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}, \text{ for } 1 \le p < \infty, \text{ and } \phi_\infty(\xi) = \sup |\xi_i|$$

define, respectively, the Schatten ideals C_p and the ideal $C(H) = J^{\phi_\infty}$ of all compact operators on H. Let $\mathcal{F}$ be the set of all finite rank operators on H. Then $\mathcal{F} \subseteq J^\phi$, the closure of $\mathcal{F}$ in $\|\cdot\|_\phi$ is a separable s. n. ideal J_0^ϕ and $J_0^\phi \subseteq J^\phi$. Each separable s. n. ideal of $B(H)$ is isomorphic to J_0^ϕ for some $\phi \in \Phi$. In many cases (for example, for all ϕ_p above) the ideals J_0^ϕ and J^ϕ coincide. For each s. n. ideal $\mathcal{J}$ of $B(H)$, there is $\phi \in \Phi$ such that $J_0^\phi \subseteq \mathcal{J} \subseteq J^\phi$; the first inclusion is isometric and the second one is continuous.

Theorem 3.7 (i) *All separable s. n. ideals of $B(H)$ (in particular, $C(H)$ and all Schatten ideals C_p, $1 \le p < \infty$) are topologically Jordan free.*

(ii) *All non-separable s. n. ideals of $B(H)$ are not topologically Jordan free.*

Proof. If $\mathcal{I}$ is a closed ideal of a separable ideal J_0^ϕ, it follows easily that $\mathcal{F} \subseteq \mathcal{I}$. As J_0^ϕ is the closure of $\mathcal{F}$, $\mathcal{I} = J_0^\phi$. Thus J_0^ϕ is topologically simple and part (i) follows from Proposition 3.4.

If $\mathcal{J}$ is a non-separable s. n. ideal of $B(H)$, then $J_0^\phi \subsetneqq \mathcal{J} \subseteq J^\phi$ for some $\phi \in \Phi$ (see [17, Proposition 2.1]). Hence $(J_0^\phi)^2 \subseteq \mathcal{J}^2 \subseteq J^\phi C(H)$. From Theorem III.6.3 [13] we obtain that $J_0^\phi = \overline{(J_0^\phi)^2}$, so $J_0^\phi \subseteq \overline{\mathcal{J}^2} \subseteq \overline{J^\phi C(H)}$. Let finite rank projections P_n strongly converge to $\mathbf{1}_H$. By Theorem III.6.3 [13], for $A \in J^\phi$, $B \in C(H)$,

$$\|ABP_n - AB\|_\phi \le \|A\|_\phi\|BP_n - B\| = \|A\|_\phi\|P_nB^* - B^*\| \to 0, \text{ as } n \to \infty.$$

Since $ABP_n \in \mathcal{F}$, it follows from the definition of J_0^ϕ that $AB \in J_0^\phi$. Thus

$$J^\phi C(H) \subseteq J_0^\phi. \tag{3.3}$$

Hence $\overline{\mathcal{J}^2} = J_0^\phi \neq \mathcal{J}$, so the quotient algebra $\mathcal{J}/J_0^\phi$ is commutative. By Proposition 3.6, $\mathcal{J}$ is not topologically Jordan free. ■

Note that it was shown in Corollary 2.18 that all C_p are not Jordan free.

We say that a Banach algebra $\mathcal{A}$ has property (max-t), if every closed proper two-sided ideal of $\mathcal{A}$ lies in a maximal closed proper two-sided ideal of $\mathcal{A}$. For example, all topologically simple and all unital algebras have property (max-t).

In Section 2 we considered some conditions for algebras to be Jordan free. Below we consider similar conditions for Banach algebras to be topologically Jordan free. They are weaker than the conditions of Section 2.

Theorem 3.8 *Consider the following conditions for a Banach algebra $\mathcal{A}$.*

(i) *$\mathcal{A}$ is topologically Jordan free.*

(ii) $\overline{\mathrm{Id}([\mathcal{A},\mathcal{A}])} = \mathcal{A}$.

(iii) $\overline{\mathrm{Alg}([\mathcal{A},\mathcal{A}])} = \mathcal{A}$.

(iv) $\overline{\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])} = \mathcal{A}$.

(v) *$\mathcal{A}$ has no closed two-sided ideals $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative.*

(vi) *$\mathcal{A}$ has no non-zero multiplicative linear functionals.*

Then (iv) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (v) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (vi).
If $\mathcal{A}$ has property (max-t) *then conditions* (i)-(v) *are equivalent.*
If $\mathcal{A}$ is a unital algebra or a C^*-algebra then condition (i)-(vi) *are equivalent.*

Proof. The proof of (iv) $\Longleftrightarrow$ (iii) for Banach algebras is the same as in Theorem 2.7.

(iii) $\Rightarrow$ (i) follows from Lemma 3.3 and (i) $\Rightarrow$ (v) follows from Proposition 3.6.

(v) $\Longleftrightarrow$ (ii) follows from the fact that $\overline{\mathrm{Id}([\mathcal{A},\mathcal{A}])}$ is the smallest out of all closed ideals $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is commutative and it lies in all of them.

(ii) $\Rightarrow$ (vi) follows from the fact that $\overline{\mathrm{Id}([\mathcal{A},\mathcal{A}])}$ lies in the kernel of every multiplicative functional on $\mathcal{A}$, since multiplicative functionals on Banach algebras are automatically continuous.

Equivalence of conditions (i)-(v) *for algebras with property* (max-t). We only need to prove (ii) $\Longrightarrow$ (iv). Assume that (ii) holds and $\mathcal{I} = \overline{\mathrm{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])} \neq \mathcal{A}$. Since every closed proper ideal of $\mathcal{A}$ lies in a maximal closed proper ideal of $\mathcal{A}$, there is a maximal closed proper ideal $\mathcal{J}$ of $\mathcal{A}$ containing $\mathcal{I}$. The quotient Banach algebra $\mathcal{B} = \mathcal{A}/\mathcal{J} \neq \{0\}$ is topologically simple and non-commutative, since $\overline{\mathrm{Id}([\mathcal{B},\mathcal{B}])} = \mathcal{B}$. Since $[[A, B], C] \in \mathcal{I} \subseteq \mathcal{J}$, for all $A, B, C \in \mathcal{A}$, $[[\mathcal{B},\mathcal{B}],\mathcal{B}] = \{0\}$. Therefore $[\mathcal{B},\mathcal{B}] \subseteq Z(\mathcal{B})$. It follows from the proof of Proposition 3.4 that $Z(\mathcal{B})$ does not have non-zero nilpotent elements. Hence, by Lemma 2.5, $\mathcal{B} \subseteq Z(\mathcal{B})$, so $\mathcal{B}$ is commutative. This contradiction proves (iv).

Equivalence of conditions (i)-(vi) *for unital and for C^*-algebras.* Since unital Banach algebras have property (max-t), conditions (i)-(v) are equivalent for them. To show that they are equivalent for all C*-algebras $\mathcal{A}$, we only need to prove (ii) $\Longrightarrow$ (iv). Assume that $\mathcal{I} = \overline{\text{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])} \neq \mathcal{A}$. Then the quotient C*-algebra $\mathcal{B} = \mathcal{A}/\mathcal{I} \neq \{0\}$ satisfies $\text{Id}([[\mathcal{B},\mathcal{B}],\mathcal{B}]) = \{0\}$. Therefore $[\mathcal{B},\mathcal{B}] \subseteq Z(\mathcal{B})$. Hence, by Lemma 2.5, $[A,B]^2 = 0$ for all $A, B \in \mathcal{B}$. If A, B are selfadjoint, $T = i[A,B]$ is selfadjoint and $T^2 = 0$. Hence $T = 0$. Thus $\mathcal{B}$ is commutative. Therefore, by (ii), $\mathcal{B} = \overline{\text{Id}([\mathcal{B},\mathcal{B}])} = \{0\}$. This contradiction proves (iv).

Finally, let us prove (vi) $\Rightarrow$ (v) for a unital Banach algebra or a C*-algebra $\mathcal{A}$. If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is commutative, then $\mathcal{A}/\mathcal{I}$ is a unital algebra or a C*-algebra. Hence it has a non-zero multiplicative functional. Its "extension" to $\mathcal{A}$ is a multiplicative functional. ∎

We showed above that, for Banach algebras *with* property (max-t), condition (ii) implies (iv) and, therefore, implies (i). For Banach algebras *without* property (max-t), we will prove below that an algebraic analogue of condition (ii) implies (iv) and, hence, implies (i).

Corollary 3.9 *If* $\text{Id}([\mathcal{A},\mathcal{A}]) = \mathcal{A}$, *for a Banach algebra* $\mathcal{A}$, *then* $\mathcal{A}$ *is topologically Jordan free.*

Proof. Let $\mathcal{I} = \overline{\text{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])} \neq \mathcal{A}$. Then $\mathcal{B} = \mathcal{A}/\mathcal{I} \neq \{0\}$ and $[[\mathcal{B},\mathcal{B}],\mathcal{B}] = \{0\}$, since $[[A,B],C] \in \mathcal{I}$ for $A, B, C \in \mathcal{A}$. Hence $[\mathcal{B},\mathcal{B}] \subseteq Z(\mathcal{B})$. Since $\text{Id}([\mathcal{B},\mathcal{B}]) = \mathcal{B}$, each $R \in \mathcal{B}$ can be written as

$$R = \sum_{i=1}^{m}[D_i, F_i] + \sum_{i=1}^{n} C_i[A_i, B_i] \text{ for some } n, m \in \mathbb{N} \text{ and some } A_i, B_i, C_i, D_i, F_i \in \mathcal{B}.$$

By Lemma 2.5, $[A,B]^2 = 0$ for all $A, B \in \mathcal{B}$. Therefore $R^{n+m+1} = 0$, so that $\mathcal{B}$ consists of nilpotent elements. It follows from the Grabiner theorem [2, Theorem 46.3] that $\mathcal{B}$ is nilpotent: $\mathcal{B}^k = \{0\}$ for some $k \in \mathbb{N}$. Since $\mathcal{B}^2 = \mathcal{B}$ implies $\mathcal{B}^k = \mathcal{B}$, we have $\mathcal{B}^2 \neq \mathcal{B}$. Hence $\text{Id}([\mathcal{B},\mathcal{B}]) \subseteq \mathcal{B}^2 \neq \mathcal{B}$. This contradiction shows that $\overline{\text{Id}([[\mathcal{A},\mathcal{A}],\mathcal{A}])} = \mathcal{A}$. By Theorem 3.8, $\mathcal{A}$ is topologically Jordan free. ∎

For Banach algebras without identity, (vi) does not necessary implies (v). For example, any nilpotent Banach algebra $\mathcal{A}$ has no non-zero multiplicative functionals, while the algebra $\mathcal{A}/\overline{\mathcal{A}^2} \neq \{0\}$ is commutative.

Denote by $\mathbb{A}$ the class of all Banach algebras $\mathcal{A}$ with property (max-t) satisfying $\overline{\mathcal{A}^2} = \mathcal{A}$. The implication (vi) $\Longrightarrow$ (v) of Theorem 3.8 for such algebras appears to be a difficult problem: it is equivalent to the classical problem whether all topologically simple commutative Banach algebras are one-dimensional.

Proposition 3.10 *The following statements are equivalent.*

(i) *The implication* (vi) $\Longrightarrow$ (v) *of Theorem* 3.8 *holds for all* $\mathcal{A} \in \mathbb{A}$.

(ii) *Each topologically simple commutative Banach algebra is one-dimensional.*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathcal{A}$ be a topologically simple commutative Banach algebra. Then either $\overline{\mathcal{A}^2} = \{0\}$ or $\overline{\mathcal{A}^2} = \mathcal{A}$. If $\overline{\mathcal{A}^2} = \{0\}$, $\mathcal{A}$ has no non-zero multiplicative functionals. For $0 \neq R \in \mathcal{A}$, $L = \mathbb{C}R$ is a closed ideal of $\mathcal{A}$. As part (vi) of Theorem 3.8 implies (v), $\mathcal{A} = L$, so $\dim \mathcal{A} = 1$.

Let now $\overline{\mathcal{A}^2} = \mathcal{A}$. Since $\mathcal{A}$ is topologically simple, it has property (max-t), so $\mathcal{A} \in \mathbb{A}$ and part (vi) of Theorem 3.8 implies (v). If $\dim \mathcal{A} \neq 1$ then, since $\mathcal{A}$ is commutative, condition (v) does not hold, so condition (vi) does not hold either and $\mathcal{A}$ has a non-zero bounded multiplicative functional

f. Its kernel $\mathrm{Ker}(f) \neq \{0\}$ is a closed ideal of $\mathcal{A}$ of codimension 1 which contradicts the assumpion that $\mathcal{A}$ is topologically simple. Hence dim $\mathcal{A} = 1$.

(ii) $\Longrightarrow$ (i). Assume that the property of Theorem 3.8(v) does not hold for some $\mathcal{A} \in \mathbb{A}$. Then $\mathcal{A}$ has a closed ideal $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative. Let $\mathcal{J}$ be a maximal closed ideal of $\mathcal{A}$ that contains $\mathcal{I}$. Then $\mathcal{B} = \mathcal{A}/\mathcal{J}$ is commutative and topologically simple. By (ii), dim $\mathcal{B} = 1$. Hence either $\mathcal{B}^2 = \{0\}$ or $\mathcal{B}^2 = \mathcal{B}$. If $\mathcal{B}^2 = \{0\}$ then $\overline{\mathcal{A}^2} \subseteq \mathcal{J}$, so $\mathcal{A} \notin \mathbb{A}$. Therefore $\mathcal{B}^2 = \mathcal{B}$ and $\mathcal{J}$ is the kernel of a non-zero bounded multiplicative functional on $\mathcal{A}$. Thus the property of Theorem 3.8(vi) does not hold for $\mathcal{A}$. ■

The Grassman algebra $\mathcal{G}$ over $\mathbb{C}$ considered in Section 2 can be turned into a normed algebra. For distinct monomials $\{w_i\}_{i=1}^n$ in $\mathcal{G}$ and complex numbers $\{\alpha_i\}_{i=1}^n$, set

$$\|\sum_{i=1}^{n} \alpha_i w_i\| = \sum_{i=1}^{n} |\alpha_i| \,.$$

Then $\mathcal{G}$ is a normed algebra. Denote by $\overline{\mathcal{G}}$ its completion.

Proposition 3.11 (i) $[[\overline{\mathcal{G}}, \overline{\mathcal{G}}], \overline{\mathcal{G}}] = \{0\}$.

(ii) *The Banach algebra* $\overline{\mathcal{G}}$ *has a closed Jordan ideal which is not a two-sided ideal, so it is not topologically Jordan free.*

Proof. Part (i) follows from property 4) of $\mathcal{G}$.

Let J be the linear space of $\mathcal{G}$ generated by all odd monomials. It follows from properties 1) and 2) of $\mathcal{G}$ that J is a Jordan but not a two-sided ideal of $\mathcal{G}$. Its closure $\overline{J}$ is a closed Jordan ideal and the distance from each even monomial in $\mathcal{G}$ to J and, hence, to $\overline{J}$ equals 1. Therefore $x_1 \in \overline{J}$ and $x_1 x_2 \notin \overline{J}$. Thus $\overline{J}$ is not a two-sided ideal of $\overline{\mathcal{G}}$. ■

Let $\mathcal{I}$ be the ideal of $\mathcal{G}$ considered in Proposition 2.10. The question arises as to whether the closure of $\mathcal{I}$ coincides with $\overline{\mathcal{G}}$.

4 Symmetrically normed Jordan free Banach algebras

In this section we consider Banach algebras $\mathcal{A}$ and Banach $\mathcal{A}$-bimodules but widen the class of Jordan $\mathcal{A}$-submodules. Let Y be a Jordan $\mathcal{A}$-submodule of an Banach $\mathcal{A}$-bimodule $(X, \|\cdot\|_X)$. It is called *symmetrically normed* (s. n.), if it is a Banach space in a norm $\|\cdot\|_Y$,

$$\|y\|_X \leq \|y\|_Y \text{ for all } y \in Y, \tag{4.1}$$

and there exists $D = D(Y) > 0$ such that

$$\|A \circ y\|_Y \leq D\|A\|\|y\|_Y, \text{ for all } A \in \mathcal{A} \text{ and } y \in Y, \tag{4.2}$$

Clearly, all Jordan $\mathcal{A}$-submodules of X closed in $\|\cdot\|_X$ are s. n. Jordan $\mathcal{A}$-submodules. However, the converse is not true. For example, $B(H)$ as a Banach $B(H)$-bimodule has only one non-trivial closed ideal $C(H)$, while it has a huge variety of s. n. ideals with rich analytic and algebraic structure.

In this section we study conditions on $\mathcal{A}$ under which each s. n. Jordan $\mathcal{A}$-submodule $(Y, \|\cdot\|_Y)$ of every $\mathcal{A}$-bimodule is a Banach $\mathcal{A}$-bimodule in an equivalent norm. By Lemma 3.1, Y is a Banach $\mathcal{A}$-bimodule if and only if Y is an $\mathcal{A}$-bimodule and (3.2) holds.

Definition 4.1 *We say that a Banach algebra $\mathcal{A}$ is s. n. Jordan free if each s. n. Jordan $\mathcal{A}$-submodule $(Y, \|\cdot\|_Y)$ of every Banach $\mathcal{A}$-bimodule is a Banach $\mathcal{A}$-bimodule in a norm equivalent to $\|\cdot\|_Y$.*

Clearly, all s. n. Jordan free Banach algebras and all Jordan free Banach algebras are topologically Jordan free. All Schatten ideals C_p, $1 \leq p < \infty$, are not Jordan free (see Corollary 2.18), while they are topologically Jordan free (see Theorem 3.7). The Schatten ideal C_1 is s. n. Jordan free (see Theorem 4.13), while it is not Jordan free. Thus we have the following relations for these sets:

$$\begin{aligned} \{\text{Jordan free Banach algebras}\} &\subsetneqq \{\text{topologically Jordan free Banach algebras}\}, \\ \{\text{s. n. Jordan free algebras}\} &\subseteq \{\text{topologically Jordan free Banach algebras}\}, \\ \{\text{s. n. Jordan free Banach algebras}\} &\nsubseteq \{\text{Jordan free Banach algebras}\}. \end{aligned}$$

Problem 4.2 (i) *Do there exist Jordan free Banach algebras that are not s. n. Jordan free?*
(ii) *Do there exist topologically Jordan free Banach algebras that are not s. n. Jordan free?*

The next result is an analogue of Lemma 2.1.

Proposition 4.3 (i) *Let a Banach algebra $\mathcal{C}$ be the sum of closed subalgebras $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are s. n. Jordan free then $\mathcal{C}$ is s. n. Jordan free.*
(ii) *Let a Banach algebra $\mathcal{A}$ be s. n. Jordan free. Then, for each closed two-sided ideal $\mathcal{I}$, the quotient algebra $\widehat{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ is s. n. Jordan free.*

Proof. Let Y be an s. n. Jordan $\mathcal{C}$-submodule of a Banach $\mathcal{C}$-bimodule X. Then X is a Banach $\mathcal{A}$- and $\mathcal{B}$-bimodule and Y is an s. n. Jordan $\mathcal{A}$- and $\mathcal{B}$-submodule of X. Since $\mathcal{A}$ and $\mathcal{B}$ are s. n. Jordan free, Y is a Banach $\mathcal{A}$- and $\mathcal{B}$-bimodule, so Y is a $\mathcal{C}$-bimodule.

Let $\mathcal{A} \dotplus \mathcal{B}$ be the direct sum of $\mathcal{A}$ and $\mathcal{B}$ with norm $\|A + B\| = \|A\| + \|B\|$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$. Since the map $A \dotplus B \to A + B$ from $\mathcal{A} \dotplus \mathcal{B}$ onto $\mathcal{C}$ is continuous, it follows from the open mapping theorem that there is $K > 0$ such that, for each $R \in \mathcal{C}$, there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with

$$R = A + B, \ \|A\| \leq K\|R\| \text{ and } \|B\| \leq K\|R\|.$$

Hence, by (3.2), for all $y \in Y$,

$$\begin{aligned} \|Ry\|_Y &\leq \|Ay\|_Y + \|By\|_Y \leq C_{\mathcal{A}}\|A\|\|y\|_Y + C_{\mathcal{B}}\|B\|\|y\|_Y \\ &\leq K(C_{\mathcal{A}} + C_{\mathcal{B}})\|R\|\|y\|_Y. \end{aligned}$$

Similarly, $\|yR\|_Y \leq K(C_{\mathcal{A}} + C_{\mathcal{B}})\|R\|\|y\|_Y$. By (3.2), Y is a Banach $\mathcal{C}$-bimodule. Part (i) is proved.

Let Y be an s. n. Jordan $\widehat{\mathcal{A}}$-submodule of a Banach $\widehat{\mathcal{A}}$-bimodule X. Then X is an $\mathcal{A}$-bimodule with multiplication $Ax = \widehat{A}x$, $xA = x\widehat{A}$, for $A \in \mathcal{A}$ and $x \in X$, and Y is a Jordan $\mathcal{A}$-submodule. Moreover,

$$\|Ax\|_X = \|\widehat{A}x\|_X \leq \|\widehat{A}\|\|x\|_X \leq \|A\|\|x\|_X,$$

$$\|A \circ y\|_Y = \|\widehat{A} \circ y\|_Y \leq D\|\widehat{A}\|\|y\|_Y \leq D\|A\|\|y\|_Y.$$

Thus Y is an s. n. Jordan $\mathcal{A}$-submodule of the Banach $\mathcal{A}$-bimodule X. Since $\mathcal{A}$ is s. n. Jordan free, Y is a Banach $\mathcal{A}$-bimodule and there is $C > 0$ such that (3.2) holds.

Since $Ax = \widehat{A}x$ and $xA = x\widehat{A}$, Y is an $\widehat{\mathcal{A}}$-bimodule. For all $T \in \mathcal{I}$, we have from (3.2)

$$\|\widehat{A}y\|_Y = \|(\widehat{A+T})y\|_Y = \|(A+T)y\|_Y \le C\|A+T\|\|y\|_Y.$$

Hence $\|\widehat{A}y\|_Y \le C \inf_{T\in\mathcal{I}} \|A+T\|\|y\|_Y = C\|\widehat{A}\|\|y\|_Y$. Similarly, $\|y\widehat{A}\|_Y \le C\|\widehat{A}\|\|y\|_Y$. Thus Y is a Banach $\widehat{\mathcal{A}}$-bimodule. ■

Since s. n. Jordan free algebras are topologically Jordan free, Proposition 3.6 yields

Proposition 4.4 *Any s. n. Jordan free Banach algebra $\mathcal{A}$ has no closed two-sided ideal $\mathcal{I}$ such that $\mathcal{A}/\mathcal{I}$ is commutative.*

For a Banach space X, denote by $\mathbf{b}_r(X)$ the ball of radius r: $\mathbf{b}_r(X) = \{x \in X\colon \|x\|_X \le r\}$.

For a subset S of $\mathcal{A}$, denote by S_m, $m \ge 1$, the set of all products $A_1...A_n$, $1 \le n \le m$, of elements from S. Denote by $\overline{\mathrm{co}(S_m)}$ the closure in $\mathcal{A}$ of the absolutely convex set $\mathrm{co}(S_m)$ of all linear combinations $\sum_{k=1}^p \lambda_k B_k$ with $B_k \in S_m$, $\lambda_k \in \mathbb{C}$ and $\sum_{k=1}^p |\lambda_k| \le 1$.

Denote by $\overline{G(S)}$ the closure in $\mathcal{A}$ of the absolutely convex semigroup $G(S)$ generated by S. Then $S_1 \subseteq S_2 \subseteq ....$ and $G(S) = \cup\, \mathrm{co}(S_m)$. We also have

$$\begin{aligned} t^m \mathrm{co}(S_m) &\subseteq \mathrm{co}(tS)_m \subseteq G(tS), \text{ for } 0 < t \le 1, \\ \mathrm{co}(S_m) &\subseteq \mathrm{co}(tS)_m \subseteq G(tS) \text{ and } G(S) \subseteq G(tS), \text{ for } 1 < t. \end{aligned} \tag{4.3}$$

Consider the set $\mathcal{K} = \mathcal{K}(\mathcal{A}) = [\mathbf{b}_1(\mathcal{A}), \mathbf{b}_1(\mathcal{A})] = \{[A,B]\colon A, B \in \mathbf{b}_1(\mathcal{A})\}$.

Definition 4.5 (i) *We say that a Banach algebra $\mathcal{A}$ belongs to the class $\mathfrak{L}_{m,r}$, for some $m \in \mathbb{N}$ and $r > 0$, if $\overline{\mathrm{co}(\mathcal{K}_m)}$ contains $\mathbf{b}_r(\mathcal{A})$.*

(ii) *We say that a Banach algebra $\mathcal{A}$ belongs to the class $\mathfrak{L}$ if, for each $t > 0$, the closed semigroup $\overline{G(t\mathcal{K})}$ contains some ball $\mathbf{b}_r(\mathcal{A})$. We write $r(t, \mathcal{A})$ for the maximal r with this property.*

It follows from (4.3) that $\mathfrak{L}_{m,r} \subseteq \mathfrak{L}_{m+1,r} \subseteq \mathfrak{L}$ and $\mathfrak{L}_{m,r} \subseteq \mathfrak{L}_{m,\rho}$, for all m, r, ρ such that $\rho < r$. We have $\mathcal{K}(\mathcal{A}) \subseteq \mathcal{A}^2$, so $G(\mathcal{K}(\mathcal{A})) \subseteq \mathcal{A}^2$. If

$$\overline{\mathcal{A}^2} \ne \mathcal{A} \text{ then } \overline{G(\mathcal{K}(\mathcal{A}))} \subseteq \overline{\mathcal{A}^2} \ne \mathcal{A}. \text{ Thus } \mathcal{A} \notin \mathfrak{L}. \tag{4.4}$$

Lemma 4.6 *Let $\mathcal{A}$ be the linear span of closed subalgebras $\{\mathcal{A}_i\}_{i=1}^n$ and let all $\mathcal{A}_i \in \mathfrak{L}_{m,r}$ (respectively, all $\mathcal{A}_i \in \mathfrak{L}$). Then there is $K > 0$ such that $\mathcal{A} \in \mathfrak{L}_{m,\frac{r}{nK}}$ (respectively, $\mathcal{A} \in \mathfrak{L}$). If $\mathcal{A}$ is the direct sum of $\mathcal{A}_i \in \mathfrak{L}_{m,r}$ and $\|A\| = \sup \|A_i\|$, for $A = A_1 \dotplus ... \dotplus A_n$, then $\mathcal{A} \in \mathfrak{L}_{m,r}$.*

Proof. Let $\mathcal{B} = \mathcal{A}_1 \dotplus ... \dotplus \mathcal{A}_n$ be the direct sum of $\{\mathcal{A}_i\}_{i=1}^n$ with norm $\|A_1 \dotplus ... \dotplus A_n\|_{\mathcal{B}} = \sum_{i=1}^n \|A_i\|$ for $A_i \in \mathcal{A}_i$. Then $\psi\colon A_1 \dotplus ... \dotplus A_n \to \sum_{i=1}^n A_i$ is a linear map from $\mathcal{B}$ onto $\mathcal{A}$ and $\|\psi\| = 1$. It follows from the open mapping theorem that there is $K > 0$ such that, for each $A \in \mathcal{A}$, there is $B \in \psi^{-1}(A)$ satisfying $\|B\|_{\mathcal{B}} \le K\|A\|$ .Therefore, for each $A \in \mathcal{A}$, there are $A_i \in \mathcal{A}_i$ with

$$A = \sum_{i=1}^n A_i \text{ and } \|A_i\| \le K\|A\|.$$

We will consider the case $\mathcal{A}_i \in \mathfrak{L}_{m,r}$. For all i and m, $\mathcal{K}_m(\mathcal{A}_i) \subseteq \mathcal{K}_m(\mathcal{A})$, so $\mathrm{co}(\mathcal{K}_m(\mathcal{A}_i)) \subseteq \mathrm{co}(\mathcal{K}_m(\mathcal{A}))$. Let $A \in \mathbf{b}_{\frac{r}{nK}}(\mathcal{A})$. Then $A = A_1 + ... + A_n$ and all $A_i \in \mathbf{b}_{\frac{r}{n}}(\mathcal{A}_i)$. Hence

$$nA_i \in \mathbf{b}_r(\mathcal{A}_i) \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}_i))} \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}.$$

As $\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$ is absolutely convex, $A = \frac{1}{n}(nA_1) + ... + \frac{1}{n}(nA_n) \in \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$, so $\mathbf{b}_{\frac{r}{nK}}(\mathcal{A}) \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$.

If $\mathcal{A}$ is the direct sum of $\mathcal{A}_i$ and $\|A\| = \sup \|A_i\|$, then $\mathbf{b}_r(\mathcal{A})$ is the direct sum of $\mathbf{b}_r(\mathcal{A}_i)$, $\mathcal{K}_m(\mathcal{A})$ is the direct sum of $\mathcal{K}_m(\mathcal{A}_i)$ and $\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$ is the direct sum of $\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}_i))}$. Hence $\mathcal{A} \in \mathfrak{L}_{m,r}$. ■

Lemma 4.7 *Let $\{\mathcal{A}_\lambda\}$ be an increasing net of closed unital subalgebras of a Banach unital algebra $\mathcal{A}$, let $\mathcal{A} = \overline{\cup \mathcal{A}_\lambda}$ and let all $\mathcal{A}_\lambda \in \mathfrak{L}_{m,r}$. Then $\mathcal{A} \in \mathfrak{L}_{m,r}$.*

Proof. Since the ball $\mathbf{b}_r(\mathcal{A}_\lambda)$ of radius r of $\mathcal{A}_\lambda$ lies in $\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}_\lambda))} \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$,

$$\mathbf{b}_r(\mathcal{A}) = \overline{\cup \mathbf{b}_r(\mathcal{A}_\lambda)} \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}. \quad ■$$

Theorem 4.8 *Let $\mathcal{A} \in \mathfrak{L}_{m,r}$ (respectively, $\mathcal{A} \in \mathfrak{L}$). Then, for every unital Banach algebra $\mathcal{B}$, the projective tensor product $\mathcal{C} = \mathcal{A}\widehat{\otimes}\mathcal{B}$ belongs to $\mathfrak{L}_{m,r}$ (respectively, $\mathcal{C} \in \mathfrak{L}$).*

Proof. For subsets $U \subseteq \mathcal{A}$ and $V \subseteq \mathcal{B}$, set $U \otimes V = \{A \otimes B : A \in U \text{ and } B \in V\}$. For all $A, C \in \mathbf{b}_1(\mathcal{A})$ and $B \in \mathbf{b}_1(\mathcal{B})$,

$$[A, C] \otimes B = [A \otimes B, C \otimes \mathbf{1}_\mathcal{B}] \in [\mathbf{b}_1(\mathcal{C}), \mathbf{b}_1(\mathcal{C})] = \mathcal{K}(\mathcal{C}).$$

Hence $\mathcal{K}(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \mathcal{K}(\mathcal{C})$. For all $A_1, ..., A_m \in \mathcal{K}(\mathcal{A})$ and $B \in \mathbf{b}_1(\mathcal{B})$,

$$A_1....A_m \otimes B = (A_1 \otimes B)(A_2 \otimes \mathbf{1}_\mathcal{B})...(A_m \otimes \mathbf{1}_\mathcal{B}) \in \mathcal{K}_m(\mathcal{C}).$$

Therefore $\mathcal{K}_m(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \mathcal{K}_m(\mathcal{C})$. Thus $\mathrm{co}(\mathcal{K}_m(\mathcal{A})) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \mathrm{co}(\mathcal{K}_m(\mathcal{C}))$. Hence

$$\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))} \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{C}))}.$$

If $\mathcal{A} \in \mathfrak{L}_{m,r}$, $\mathbf{b}_r(\mathcal{A})$ lies in $\overline{\mathrm{co}(\mathcal{K}_m(\mathcal{A}))}$, so $\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{C}))}$. Hence

$$\overline{\mathrm{co}(\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}))} \subseteq \overline{\mathrm{co}(\mathcal{K}_m(\mathcal{C}))}.$$

By the definition of the projective tensor product, $\overline{\mathrm{co}(\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}))} = \mathbf{b}_r(\mathcal{C})$. Thus $\mathcal{C} \in \mathfrak{L}_{m,r}$.

Let $\mathcal{A} \in \mathfrak{L}$. As above, $\mathrm{co}(t\mathcal{K}(\mathcal{A}))_m \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \mathrm{co}(t\mathcal{K}(\mathcal{C}))_m$ for each m and $t > 0$. Hence

$$G(t\mathcal{K}(\mathcal{A})) \otimes \mathbf{b}_1(\mathcal{B}) = (\cup \mathrm{co}(t\mathcal{K}(\mathcal{A}))_m) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq (\cup \mathrm{co}(t\mathcal{K}(\mathcal{C}))_m) = G(t\mathcal{K}(\mathcal{A})).$$

Since $\mathbf{b}_r(\mathcal{A}) \subseteq \overline{G(t\mathcal{K}(\mathcal{A}))}$ for some r, $\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}) \subseteq \overline{G(t\mathcal{K}(\mathcal{A}))}$. Hence $\overline{\mathrm{co}(\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}))} \subseteq \overline{G(t\mathcal{K}(\mathcal{A}))}$. As $\overline{\mathrm{co}(\mathbf{b}_r(\mathcal{A}) \otimes \mathbf{b}_1(\mathcal{B}))} = \mathbf{b}_r(\mathcal{C})$, for projective tensor products, we have $\mathcal{C} \in \mathfrak{L}$. ■

Lemma 4.9 *Let $\mathcal{A} \in \mathfrak{L}$, let X be a Banach $\mathcal{A}$-module and let Y be a Jordan $\mathcal{A}$-submodule of X and a Banach space in $\|\cdot\|_Y$. If $\mathbf{b}_1(Y)$ is closed in $\|\cdot\|_X$, then*

(i) *Y is an $\mathcal{A}$-bimodule and*

$$\|Ay\|_Y \le C\|A\|\|y\|_Y \text{ and } \|yA\|_Y \le C\|A\|\|y\|_Y, \text{ for all } A \in \mathcal{A} \text{ and } y \in Y, \tag{4.5}$$

with $C = \frac{1}{r(t,\mathcal{A})}$, for $t = (D + D^2)^{-1}$, where $r(t, \mathcal{A})$ is defined in Definition 4.5 *and D in* (4.2);

(ii) *Y is a Banach $\mathcal{A}$-bimodule in some equivalent norm $\|\cdot\|'_Y$.*

Proof. For $A, B \in \mathbf{b}_1(\mathcal{A})$ and $y \in \mathbf{b}_1(Y)$, we have from (2.3)

$$[A,B]y = \frac{1}{2}([A,B] \circ y + A \circ (B \circ y) - B \circ (A \circ y)) \in Y,$$

so that $\|[A,B]y\|_Y \le D + D^2 = t^{-1}$. Similarly $y[A,B] \in Y$ and $\|y[A,B]\|_Y \le t^{-1}$. Hence

$$Ty, yT \in Y,\ \|Ty\|_Y \le t^{-1} \text{ and } \|yT\|_Y \le t^{-1},\ \text{for all } T \in \mathcal{K} = \mathcal{K}(\mathcal{A}).$$

Then $\|Ty\|_Y \le 1$, $\|yT\|_Y \le 1$ for all $T \in t\mathcal{K}$. Therefore $\|Ty\|_Y \le 1$, $\|yT\|_Y \le 1$ for all $T \in G(t\mathcal{K})$ and $y \in \mathbf{b}_1(Y)$.

Let $T_n \in G(t\mathcal{K})$ and $T_n \to T \in \overline{G(t\mathcal{K})}$. All $T_n y$ belong to $\mathbf{b}_1(Y)$ and converge to Ty in $\|\cdot\|_X$. As $\mathbf{b}_1(Y)$ is closed in $\|\cdot\|_X$, $Ty \in \mathbf{b}_1(Y)$. Similarly, $yT \in \mathbf{b}_1(Y)$. Since the ball $\mathbf{b}_r(\mathcal{A})$, with $r = r(t, \mathcal{A})$, lies in $\overline{G(t\mathcal{K})}$ (see Definition 4.5), we have that $Ty, yT \in \mathbf{b}_1(Y)$, for all $y \in \mathbf{b}_1(Y)$ and $T \in \mathbf{b}_r(\mathcal{A})$. Hence Y is an $\mathcal{A}$-bimodule and (4.5) holds with $C = \frac{1}{r}$. Part (i) is proved. Part (ii) follows from (i) and Lemma 3.1. ■

Sometimes the proof of the fact that $\mathbf{b}_1(Y)$ is closed in X requires substantial efforts and techniques, as in the case when $X = B(H)$ and $Y = J^{\phi}$ is a symmetrically normed ideal of $B(H)$ (see Theorem III.5.1 [13]). In other cases it can be comparatively easily checked. For example, $\mathbf{b}_1(Y)$ is closed in X when the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ coincide on Y. Another class of examples is given by the following result.

Proposition 4.10 *Let Y be a linear subspace of a Banach space X and a Banach space in a norm $\|\cdot\|_Y$ and let* (4.1) *hold. If Y is a reflexive Banach space then $\mathbf{b}_1(Y)$ is closed in X.*

Proof. Since Y is reflexive, the ball $\mathbf{b}_1(Y)$ is compact in the $\sigma(Y, Y^*)$-topology. By (4.1), for each functional $f \in X^*$, its restriction $f|_Y$ belongs to Y^*. Hence $\mathbf{b}_1(Y)$ is compact in the $\sigma(X, X^*)$-topology. Hence it is closed in this topology, so it is closed in $\|\cdot\|_X$. ■

For every s. n. Jordan $\mathcal{A}$-submodule of a Banach $\mathcal{A}$-bimodule X, we shall now construct a special s. n. Jordan $\mathcal{A}$-submodule of X with closed unit ball larger than Y.

Lemma 4.11 *Let X be an $\mathcal{A}$-module and a Banach space that satisfies* (4.5)*. Let $Y \neq X$, be an s. n. Jordan $\mathcal{A}$-submodule of X, that is,* (4.1) *and* (4.2) *hold. If $\mathbf{b}_1(Y)$ is not closed in $\|\cdot\|_X$, there exists an s. n. Jordan $\mathcal{A}$-submodule $\widetilde{Y}$ of X such that $Y \subsetneqq \widetilde{Y} \neq X$ and $\|y\|_{\widetilde{Y}} \le \|y\|_Y$ for $y \in Y$, that $\mathbf{b}_1(\widetilde{Y})$ is closed in $\|\cdot\|_X$ and inequality* (4.2) *for $\widetilde{Y}$ holds with the same constant D, as for Y.*

Proof. We will use the following construction considered in [18]. Let $\mathbf{b}_1(Y)$ be not closed in $\|\cdot\|_X$ and let $\overline{\mathbf{b}_r(Y)}$ be the closure of $\mathbf{b}_r(Y)$ in $\|\cdot\|_X$. Set

$$\widetilde{Y} = \cup_{r>0}\overline{\mathbf{b}_r(Y)} \text{ with } \|y\|_{\widetilde{Y}} = \inf\{r\colon\ y \in \overline{\mathbf{b}_r(Y)}\}.$$

Then (see Section I.1.2 of [18]) $(\widetilde{Y}, \|\cdot\|_{\widetilde{Y}})$ is a Banach space, $Y \subsetneqq \widetilde{Y} \neq X$,

$$\|y\|_{\widetilde{Y}} \le \|y\|_Y \text{ for } y \in Y,\ \|x\|_X \le \|x\|_{\widetilde{Y}} \text{ for } x \in \widetilde{Y},$$

and $\mathbf{b}_r(\widetilde{Y}) = \overline{\mathbf{b}_r(Y)}$. Hence all balls $\mathbf{b}_r(\widetilde{Y})$ are closed in $\|\cdot\|_X$.

It is only left to show that $\widetilde{Y}$ is a Jordan $\mathcal{A}$-submodule of X and that the inequality (4.2) for $\widetilde{Y}$ holds with the same constant D, as for Y. Let $y \in \widetilde{Y}$ and $r = \|y\|_{\widetilde{Y}}$. Then there are $y_n \in \mathbf{b}_r(Y)$ such that $\|y - y_n\|_X \to 0$. For each $A \in \mathcal{A}$,

$$\begin{aligned} \|A \circ y - A \circ y_n\|_X &\leq \|Ay - Ay_n\|_X + \|yA - y_nA\|_X \leq 2C\|A\|\|y - y_n\|_X \to 0, \\ \text{and } \|A \circ y_n\|_Y &\leq D\|A\|\|y_n\|_Y \leq D\|A\|r. \end{aligned}$$

Set $R = D\|A\|r$. Then $A \circ y_n \in \mathbf{b}_R(Y)$. Therefore $A \circ y \in \overline{\mathbf{b}_R(Y)} = \mathbf{b}_R(\widetilde{Y})$. Hence $\|A \circ y\|_{\widetilde{Y}} \leq R = D\|A\|r = D\|A\|\|y\|_{\widetilde{Y}}$. The proof is complete. ■

Now we will show that the class $\mathfrak{L}$ consists of s. n. Jordan free algebras.

Theorem 4.12 *Every Banach algebra $\mathcal{A}$ in $\mathfrak{L}$ is s. n. Jordan free.*

Proof. Let $(Y, \|\cdot\|_Y)$ be an s. n. Jordan $\mathcal{A}$-submodule of a Banach $\mathcal{A}$-bimodule X and let (4.2) hold for Y with constant D. Set $t = (D + D^2)^{-1}$. As $\mathcal{A} \in \mathfrak{L}$, consider $r(t, \mathcal{A})$ defined in Definition 4.5 and set $C = \frac{1}{r(t,\mathcal{A})}$. Denote by $\mathcal{W}$ the set of all $\mathcal{A}$-subbimodules Z of X containing Y such that each of them is a Banach space in norm $\|\cdot\|_Z$,

$$\|z\|_X \leq \|z\|_Z \text{ for all } z \in Z, \text{ and } \|y\|_Z \leq \|y\|_Y \text{ for all } y \in Y, \tag{4.6}$$

and (4.5) is satisfied with constant C:

$$\|Az\|_Z \leq C\|A\|\|z\|_Z \text{ and } \|zA\|_Z \leq C\|A\|\|z\|_Z \text{ for all } A \in \mathcal{A} \text{ and } z \in Z. \tag{4.7}$$

We say that $Z_1 < Z_2$, for $Z_1, Z_2 \in \mathcal{W}$, if $Z_2 \subset Z_1$ and $\|z\|_{Z_1} \leq \|z\|_{Z_2}$ for all $z \in Z_2$. Then $\mathcal{W}$ is a partially ordered set. Let $\{Z_\lambda\}_{\lambda \in \Lambda}$ be a linearly ordered subset of $\mathcal{W}$. Set

$$Z = \{z \in \cap_{\lambda \in \Lambda} Z_\lambda : \|z\|_Z = \sup_\lambda \|z\|_{Z_\lambda} < \infty\}.$$

Then $Y \subseteq Z$, $\|y\|_Z \leq \|y\|_Y$ for all $y \in Y$, $\|z\|_X \leq \|z\|_Z$ for all $z \in Z$, and (see [18, Section I.3.4]) Z is a Banach space.

For each $z \in Z$, we have $Az, zA \in \cap_{\lambda \in \Lambda} Z_\lambda$ and, by (4.7),

$$\|Az\|_Z = \sup_\lambda \|Az\|_{Z_\lambda} \leq C \sup_\lambda \|A\|\|z\|_{Z_\lambda} = C\|A\|\|z\|_Z.$$

Similarly, $\|zA\|_Z \leq C\|A\|\|z\|_Z$. Hence Z is an $\mathcal{A}$-bimodule and it belongs to the set $\mathcal{W}$. Clearly, $Z_\lambda \leq Z$ for all $\lambda \in \Lambda$. It is easy to check that $Z \leq Z'$ for each majorant Z' of $\{Z_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{W}$. Thus each linearly ordered subset of $\mathcal{W}$ has a supremum in $\mathcal{W}$. By Zorn's Lemma, $\mathcal{W}$ has a maximal element - a minimal $\mathcal{A}$-bimodule Z_0 in X containing Y such that Z_0 is a Banach space in $\|\cdot\|_{Z_0}$ that satisfies

$$\|y\|_{Z_0} \leq \|y\|_Y \text{ for all } y \in Y, \ \|z\|_X \leq \|z\|_{Z_0} \text{ for all } z \in Z_0,$$
$$\|Az\|_{Z_0} \leq C\|A\|\|z\|_{Z_0} \text{ and } \|zA\|_{Z_0} \leq C\|A\|\|z\|_{Z_0} \text{ for all } A \in \mathcal{A} \text{ and } z \in Z_0.$$

Let us show that $\mathbf{b}_1(Y)$ is closed in Z_0 in $\|\cdot\|_{Z_0}$. If not, then, by Lemma 4.11, there is an s. n. Jordan $\mathcal{A}$-submodule $(\widetilde{Y}, \|\cdot\|_{\widetilde{Y}})$ of Z_0, $Y \subsetneqq \widetilde{Y} \neq Z_0$, such that $\mathbf{b}_1(\widetilde{Y})$ is closed in Z_0 in $\|\cdot\|_{Z_0}$, inequality (4.2) for $\widetilde{Y}$ holds with the same constant D and

$$\|y\|_{\widetilde{Y}} \leq \|y\|_Y \text{ for } y \in Y, \text{ and } \|z\|_{Z_0} \leq \|z\|_{\widetilde{Y}} \text{ for } z \in \widetilde{Y},$$

Hence

$$\|z\|_X \le \|z\|_{Z_0} \le \|z\|_{\widetilde{Y}} \text{ for } z \in \widetilde{Y}.$$

Thus (4.6) holds for $\widetilde{Y}$. Moreover, by Lemma 3.1, Z_0 is a Banach $\mathcal{A}$-bimodule in some equivalent norm $\|\cdot\|'_{Z_0}$. Hence $\mathbf{b}_1(\widetilde{Y})$ is closed in Z_0 in $\|\cdot\|'_{Z_0}$. As $\mathcal{A} \in \mathfrak{L}$, it follows from Lemma 4.9(i) that $\widetilde{Y}$ is an $\mathcal{A}$-bimodule and inequalities (4.5) hold for $\mathcal{A}$ and $\widetilde{Y}$ with $C = \frac{1}{r(t,\mathcal{A})}$ where $t = (D + D^2)^{-1}$. Therefore inequalities (4.7) hold for $\widetilde{Y}$. Hence $\widetilde{Y} \in \mathcal{W}$ and $Z_0 < \widetilde{Y}$ ($\widetilde{Y} \subsetneqq Z_0$) which contradicts the fact that Z_0 is a maximal element in $\mathcal{W}$. This contradiction shows that $\mathbf{b}_1(Y)$ is closed in Z_0 in $\|\cdot\|_{Z_0}$ and, therefore, in $\|\cdot\|'_{Z_0}$.

As $(Z_0, \|\cdot\|'_{Z_0})$ is a Banach $\mathcal{A}$-bimodule and since $\mathcal{A} \in \mathfrak{L}$ and $\mathbf{b}_1(Y)$ is closed in $\|\cdot\|'_{Z_0}$, it follows from Lemma 4.9(ii) that Y is a Banach $\mathcal{A}$-bimodule in some equivalent norm $\|\cdot\|'_Y$. ■

Let $\mathcal{A}$ be a Banach *-algebra. Denote by $\mathcal{P}(\mathcal{A})$ the set of all selfadjoint projections in $\mathcal{A}$ such that, for each $P \in \mathcal{P}(\mathcal{A})$, there is $U \in \mathbf{b}_1(\mathcal{A})$ satisfying

$$P = UU^*, \ Q = U^*U \text{ is a projection and } PQ = \mathbf{0}.$$

If $P \in \mathcal{P}(\mathcal{A})$, $P - Q = [U, U^*] \in \mathcal{K}(\mathcal{A})$ and $P + Q = (P - Q)^2 \in \mathcal{K}_2(\mathcal{A})$. Hence $P \in \mathrm{co}(\mathcal{K}_2(\mathcal{A}))$.

Using Theorem 4.12, we will show that $C(H)$ and the Schatten ideal C_1 are s. n. Jordan free.

Theorem 4.13 *The Schatten ideal C_1 belongs to $\mathfrak{L}_{2,\frac{1}{2}}$, the ideal $C(H)$ belongs to $\mathfrak{L}_{2,\frac{1}{4}}$, every full matrix algebra $M_n(\mathbb{C})$ belongs to $\mathfrak{L}_{2,\frac{1}{8}}$, for even n, and to $\mathfrak{L}_{2,\frac{1}{12}}$, for odd $n \ge 3$. Thus they are all s. n. Jordan free.*

Proof. All one-dimensional projections belong to $\mathcal{P}(C_1)$. Indeed, let $x \in H$ and P be the projection on $\mathbb{C}x$. For some y in H such that $y \perp x$, let Q be the projection on $\mathbb{C}y$. Then there is a partial isometry U such that $P = UU^*$ and $Q = U^*U$. By definition of $\|\cdot\|_1$, $\|U\|_1 = \|(U^*U)^{\frac{1}{2}}\|_1 = \|Q\|_1 = 1$. Hence $P \in \mathcal{P}(C_1)$.

For $A = A^* \in \mathbf{b}_1(C_1)$, let $\{\lambda_n\}$ be all eigenvalues of A repeated according to their multiplicity, $|\lambda_1| \ge .. \ge |\lambda_n| ...$, and let P_n be the one-dimensional mutually orthogonal projections on the corresponding eigenspaces. Then all $P_n \in \mathbf{b}_1(C_1)$ and

$$\|A\|_1 = \sum_n |\lambda_n| \le 1. \tag{4.8}$$

All $P_n \in \mathrm{co}(\mathcal{K}_2(C_1))$ and, by (4.8), $\sum_{n=1}^N |\lambda_n| \le 1$, for each $N \in \mathbb{N}$, so

$$A_N = \sum_{n=1}^N \lambda_n P_n \in \ \mathrm{co}(\mathcal{K}_2(C_1)).$$

Since $\|A - A_N\|_1 \to 0$, as $N \to \infty$, $A \in \overline{\mathrm{co}(\mathcal{K}_2(C_1))}$.

Each B in $\mathbf{b}_1(C_1)$ is represented as $B = B_1 + iB_2$, with $B_i = B_i^* \in \mathbf{b}_1(C_1)$. Hence $B \in 2\overline{\mathrm{co}(\mathcal{K}_2(C_1))}$, so that $\mathbf{b}_{\frac{1}{2}}(C_1)$ lies in $\overline{\mathrm{co}(\mathcal{K}_2(C_1))}$. Thus $C_1 \in \mathfrak{L}_{2,\frac{1}{2}}$.

Let P_k be finite-dimensional mutually orthogonal projections and let $A = \sum_{k=1}^m \lambda_k P_k$ be a finite rank positive operator in $\mathbf{b}_1(C(H))$, where $1 \ge \lambda_1 > .. > \lambda_m > 0$. Set $R_k = P_1 + ... + P_k$. Then R_k are finite-dimensional orthogonal projections and

$$A = (\lambda_1 - \lambda_2)R_1 + (\lambda_2 - \lambda_3)R_2 + .. + (\lambda_{m-1} - \lambda_m)R_{m-1} + \lambda_m R_m. \tag{4.9}$$

Each projection in $C(H)$ lies in $\mathcal{P}(C(H))$, so all $R_k \in \text{co}(\mathcal{K}_2(C(H)))$. Hence $A \in \text{co}(\mathcal{K}_2(C(H)))$. Since all positive operators in $\mathbf{b}_1(C(H))$ are norm limits of the above type operators, they lie in $\overline{\text{co}(\mathcal{K}_2(C(H)))}$. Each $A \in \mathbf{b}_1(C(H))$ has form $A = A_1 - A_2 + i(A_3 - A_4)$, where all A_i are positive and lie in $\mathbf{b}_1(C(H))$. Hence $A \in 4\overline{\text{co}(\mathcal{K}_2(C(H)))}$, so $\mathbf{b}_{\frac{1}{4}}(C(H)) \subseteq \overline{\text{co}(\mathcal{K}_2(C(H)))}$. Thus $C(H) \in \mathfrak{L}_{2,\frac{1}{4}}$.

Let $[\frac{n}{2}]$ be the integral part of $\frac{n}{2}$. Each orthogonal projection P in $M_n(\mathbb{C})$ with dim $P \leq [\frac{n}{2}]$ lies in $\mathcal{P}(M_n(\mathbb{C}))$, so $P \in \text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. If $[\frac{n}{2}] <$ dim $P \leq 2[\frac{n}{2}]$, then $P = Q + T$, where Q, T are mutually orthogonal projections and max(dim Q,dim T) $\leq [\frac{n}{2}]$. Hence $P \in 2\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. If n is odd, then $\mathbf{1} = P + Q + T$, where P, Q, T are mutually orthogonal projections and max(dim P,dim Q,dim T) $\leq [\frac{n}{2}]$. Hence $\mathbf{1} \in 3\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$.

Each positive operator A in $\mathbf{b}_1(M_n(\mathbb{C}))$ has form $A = \sum_{k=1}^{m} \lambda_k P_k$, where $1 \geq \lambda_1 > .. > \lambda_m > 0$ and P_k are mutually orthogonal projections. Hence A has form (4.9), where $R_k = P_1 + ... + P_k$ are orthogonal projections. If n is even, then all R_k lie in $2\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$, so that $A \in 2\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. If n is odd, then all R_k lie in $3\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$, so that $A \in 3\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$.

Each $B \in \mathbf{b}_1(M_n(\mathbb{C}))$ has form $B = B_1 - B_2 + i(B_3 - B_4)$, where all B_i are positive and lie in $\mathbf{b}_1(M_n(\mathbb{C}))$. If n is even, then $B \in 8\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. Thus $\mathbf{b}_{\frac{1}{8}}(M_n(\mathbb{C})) \subseteq \text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. Hence $M_n(\mathbb{C}) \in \mathfrak{L}_{2,\frac{1}{8}}$. If n is odd, then $B \in 12\text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. Thus $\mathbf{b}_{\frac{1}{12}}(M_n(\mathbb{C})) \subseteq \text{co}(\mathcal{K}_2(M_n(\mathbb{C})))$. Hence $M_n(\mathbb{C}) \in \mathfrak{L}_{2,\frac{1}{12}}$. ■

By Theorem 3.7, all non-separable s. n. ideals are not topologically Jordan free and, therefore, not s. n. Jordan free. On the other hand, all separable s. n. ideals are topologically Jordan free and, by Theorem 4.13, their "extreme" points - the smallest s. n. ideal C_1 and the largest s. n. ideal $C(H)$ - are s. n. Jordan free.

Problem 4.14 *Are all separable s. n. ideals (in particular, all Schatten ideals C_p, for $1 < p < \infty$) s. n. Jordan free?*

Corollary 4.15 *Let $\mathcal{A} = \overline{\cup \mathcal{A}_n}$ be an AF-algebra, where $\{\mathcal{A}_n\}$ is an increasing by inclusion sequence of finite-dimensional C*-algebras:*

$$\mathcal{A}_n = M_{n(1)}(\mathbb{C}) \oplus M_{n(2)}(\mathbb{C}) \oplus ... \oplus M_{n(m_n)}(\mathbb{C}).$$

If some algebras $\mathcal{A}_{n_j}$, with $n_j \to \infty$, have no one-dimensional summands ($n_j(k) \neq 1$, for all $k = 1, ..., m_{n_j}$), then $\mathcal{A} \in \mathfrak{L}_{2,\frac{1}{12}}$. Thus $\mathcal{A}$ is s. n. Jordan free.

Proof. By Lemma 4.6 and Theorem 4.13, all $\mathcal{A}_{n_j} \in \mathfrak{L}_{2,\frac{1}{12}}$. Applying Lemma 4.7 and Theorem 4.12, we conclude the proof. ■

In Theorem 4.19 we will obtain necessary and sufficient conditions for C*-algebras to be s. n. Jordan free. They imply that $C(H)$ and all AF-algebras satisfying the conditions of Corollary 4.15 are s. n. Jordan free.

Let $\mathcal{B}$ be a unital Banach algebra and let $\{E_{ij}\}$ be the matrix identity in $M_n(\mathcal{B})$, for $n \geq 2$. For $B \in \mathcal{B}$, denote by BE_{ij} the matrix in $M_n(\mathcal{B})$ with (i,j)-entry B and all other entries 0. We always assume that the norm in $M_n(\mathcal{B})$ is a cross-norm of the tensor product $M_n \otimes \mathcal{B}$. Hence $\|E_{ij}\| = 1$ and $\|BE_{ij}\| = \|B\|$, for all $B \in \mathcal{B}$.

Theorem 4.16 *Let $\mathcal{A} = M_n(\mathcal{B})$. If n is even then $\mathcal{A} \in \mathfrak{L}_{3,\frac{1}{4}}$, if n is odd then $\mathcal{A} \in \mathfrak{L}_{3,\frac{1}{300}}$. Thus in both cases $\mathcal{A}$ is s. n. Jordan free.*

Proof. Let $B \in \mathcal{B}$ and $\|B\| \le 1$. Then, for $i \neq j$,

$$\begin{aligned} BE_{ij} &= [E_{ii}, BE_{ij}] \in [\mathbf{b}_1(\mathcal{A}), \mathbf{b}_1(\mathcal{A})] = \mathcal{K}_1, \\ BE_{ii} - BE_{jj} &= [E_{ij}, BE_{ji}] \in [\mathbf{b}_1(\mathcal{A}), \mathbf{b}_1(\mathcal{A})] = \mathcal{K}_1, \\ T_{ij} &= [E_{ij}, E_{ji}] = E_{ii} - E_{jj} \in [\mathbf{b}_1(\mathcal{A}), \mathbf{b}_1(\mathcal{A})] = \mathcal{K}_1. \end{aligned} \tag{4.10}$$

Hence $E_{ii} = \frac{1}{2}T_{ij} + \frac{1}{2}T_{ij}^2 \in \mathrm{co}(\mathcal{K}_2)$ and

$$BE_{ii} = E_{ii}(BE_{ii} - BE_{jj}) = \frac{1}{2}T_{ij}[E_{ij}, BE_{ji}] + \frac{1}{2}T_{ij}^2[E_{ij}, BE_{ji}] \in \mathrm{co}(\mathcal{K}_3). \tag{4.11}$$

Let $n = 2$. For $A = (B_{ij}) \in \mathbf{b}_{\frac{1}{4}}(\mathcal{A})$, we have

$$\|B_{ij}E_{ij}\| = \|E_{ii}AE_{jj}\| \le \|E_{ii}\|\|A\|\|E_{jj}\| \le \frac{1}{4}, \text{ for all } i, j,$$

Hence $\|B_{ij}\| \le \frac{1}{4}$. By (4.10) and (4.11), $4B_{ij}E_{ij} \in \mathrm{co}(\mathcal{K}_3)$. Since $\mathrm{co}(\mathcal{K}_3)$ is absolutely convex,

$$A = \sum_{i,j=1}^{2} \frac{1}{4}(4B_{ij}E_{ij}) \in \mathrm{co}(\mathcal{K}_3).$$

Thus the ball $\mathbf{b}_{\frac{1}{4}}(\mathcal{A})$ lies in $\mathrm{co}(\mathcal{K}_3)$. Therefore $M_2(\mathcal{B}) \in \mathfrak{L}_{3,\frac{1}{4}}$.

Since $M_{2k}(\mathcal{B}) = M_2(M_k(\mathcal{B}))$, we also have that $\mathcal{A} = M_{2k}(\mathcal{B}) \in \mathfrak{L}_{3,\frac{1}{4}}$.

Let now $n = 2p+1 \ge 3$. Consider the projections $P = \mathbf{1} - E_{nn}$, $Q = \mathbf{1} - E_{11}$ and $R = E_{11} + E_{nn}$. Then $\|P\| \le 2$, $\|Q\| \le 2$, $\|R\| \le 2$. The subalgebras $P\mathcal{A}P$ and $Q\mathcal{A}Q$ of $\mathcal{A}$ are isomorphic to $M_{2p}(\mathcal{B})$. Hence they belong to $\mathfrak{L}_{3,\frac{1}{4}}$. The subalgebra $R\mathcal{A}R$ of $\mathcal{A}$ is isomorphic $M_2(\mathcal{B})$. Hence it also lies in $\mathfrak{L}_{3,\frac{1}{4}}$. It is easy to check that, for each $A \in \mathcal{A}$,

$$\begin{aligned} A_1 &= PAP \in P\mathcal{A}P \text{ and } \|A_1\| = \|PAP\| \le 4\|A\|, \\ A_2 &= Q(A - A_1)Q \in Q\mathcal{A}Q \text{ and } \|A_2\| = \|Q(A - A_1)Q\| \le 4\|A - A_1\| \le 20\|A\|, \\ A_3 &= A - A_1 - A_2 \in R\mathcal{A}R \text{ and } \|A_3\| = \|A - A_1 - A_2\| \le 25\|A\|, \\ A &= A_1 + A_2 + A_3. \end{aligned}$$

We have from Lemma 4.6(i) that $\mathcal{A} \in \mathfrak{L}_{3,\frac{\frac{1}{4}}{3\times 25}} = \mathfrak{L}_{3,\frac{1}{300}}$. ■

It follows from Theorem 4.16 that $B(H)$ is s. n. Jordan free. This extends the result of [16, Corollary 4.9] that every s. n. Jordan ideal of $B(H)$ is an s. n. two-sided ideal of $B(H)$.

The Cuntz C*-algebra $\mathcal{O}_n$ is generated by isometries $\{U_i\}_{i=1}^n$ on an infinite dimensional Hilbert space H such that $P_i = U_iU_i^*$ are mutually orthogonal projections,

$$U_1^*U_1 = U_2^*U_2 = ... = U_n^*U_n = \mathbf{1} \text{ and } P_1 + P_2 + ... + P_n = \mathbf{1}.$$

The operators $E_{ij} = U_iU_j^*$ satisfy $E_{ij}E_{ij}^* = P_i$ and $E_{ij}^*E_{ij} = P_j$, so all P_i are equivalent. Hence $\mathcal{O}_n$ is isomorphic to $M_n(\mathcal{B})$, where $\mathcal{B} = P_1\mathcal{O}_nP_1|_{P_1H}$.

Corollary 4.17 *All Cuntz algebra $\mathcal{O}_n$, $n \ge 2$, are s. n. Jordan free.*

Theorem 4.18 *A W*-algebra $\mathcal{A}$ is s. n. Jordan free if and only if it has no commutative weakly closed ideals. In this case it belongs to the class $\mathfrak{L}_{3,\frac{1}{300}}$.*

Proof. If $\mathcal{A}$ is s. n. Jordan free, it follows from Proposition 4.4 that $\mathcal{A}$ does not have commutative weakly closed ideals.

Conversely, $\mathcal{A}$ has decomposition $\mathcal{A} = \mathcal{A}_{\rm I} \oplus \mathcal{A}_{\rm II} \oplus \mathcal{A}_{\rm III}$ where $\mathcal{A}_{\rm I}, \mathcal{A}_{\rm II}, \mathcal{A}_{\rm III}$ are W*-algebras of type I,II,III respectively. The algebra $\mathcal{B} = \mathcal{A}_{\rm II} \oplus \mathcal{A}_{\rm III}$ has no abelian projections. By Proposition 2.2.13 [19], there are orthogonal equivalent projections P and Q such that $\mathbf{1}_{\mathcal{B}} = P + Q$. Hence $\mathcal{B}$ is isomorphic to $M_2(P\mathcal{B}P)$ and, by Theorem 4.16, $\mathcal{B} \in \mathfrak{L}_{3,\frac{1}{4}}$. Thus $\mathcal{B}$ is s. n. Jordan free.

The W*-algebra $\mathcal{A}_{\rm I}$ decomposes uniquely into the direct sum of W*-algebras $\mathcal{A}_n$ of type I_n:

$$\mathcal{A}_{\rm I} = \oplus_{n\geq 1}\mathcal{A}_n,$$

where n is the number of mutually orthogonal abelian equivalent projections P_α in $\mathcal{A}_n$ such that $\mathbf{1}_{\mathcal{A}_n} = \sum P_\alpha$. Since $\mathcal{A}$ has no commutative weakly closed ideals, the commutative component $\mathcal{A}_1$ in the above decomposition vanishes. If n is even or infinite, then there are orthogonal equivalent projections P and Q such that $\mathbf{1}_{\mathcal{A}_n} = P + Q$. Hence $\mathcal{A}_n$ is isomorphic to $M_2(P\mathcal{A}_nP)$ and, by Theorem 4.16, $\mathcal{A}_n \in \mathfrak{L}_{3,\frac{1}{4}}$. If n is odd, then $\mathcal{A}_n$ is isomorphic to $M_n(\mathcal{B})$, where $\mathcal{B}$ is a W*-algebra, and, by Theorem 4.16, $\mathcal{A}_n \in \mathfrak{L}_{3,\frac{1}{300}}$. Since $\mathfrak{L}_{3,\frac{1}{4}} \subseteq \mathfrak{L}_{3,\frac{1}{300}}$, all components $\mathcal{A}_n$ belong to $\mathfrak{L}_{3,\frac{1}{300}}$. By Lemma 4.6(*ii*), $\mathcal{A}_{\rm I} \in \mathfrak{L}_{3,\frac{1}{300}}$ and, therefore, $\mathcal{A} = \mathcal{A}_{\rm I} \oplus \mathcal{B} \in \mathfrak{L}_{3,\frac{1}{300}}$. Hence $\mathcal{A}$ is s. n. Jordan free. ∎

Now we will give a description of s. n. Jordan free C*-algebras.

Theorem 4.19 *Let $\mathcal{A}$ be a C*-algebra. The following conditions are equivalent.*

(i) *$\mathcal{A}$ is s. n. Jordan free.*

(ii) *$\mathcal{A}$ is topologically Jordan free.*

(iii) *$\mathcal{A}$ has no non-zero multiplicative functionals.*

Proof. (i) $\Longrightarrow$ (ii) is evident. (ii) $\Longleftrightarrow$ (iii) follows from Theorem 3.8.

(iii) $\Longrightarrow$ (i). Suppose that $\mathcal{A}$ has no multiplicative functionals. Then $\mathcal{A}$ has no commutative *-representations. The second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is the *-weak closure of $\pi(\mathcal{A})$ where π is the universal representation π of $\mathcal{A}$. If $\mathcal{A}^{**}$ has a weakly closed commutative ideal $\mathcal{J} \neq \{0\}$, then there is a projection P in the centre of $\mathcal{A}^{**}$ such that $\mathcal{J} = P\mathcal{A}^{**}$ and $\mathcal{J}$ is the weak closure of the commutative C*-algebra $P\pi(\mathcal{A})$. Thus $P\pi(\mathcal{A}) \neq \{0\}$, so $A \in \mathcal{A} \to P\pi(A)$ is a commutative *-representation of $\mathcal{A}$. This contradiction shows that $\mathcal{A}^{**}$ has no weakly closed commutative ideals. By Theorem 4.18, $\mathbf{b}_{\frac{1}{300}}(\mathcal{A}^{**}) \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}^{**}))}$.

As before, for a subset S of $\mathcal{A}$ and $m \in \mathbb{N}$, let S_m be the set of all products $A_1...A_n$, $n \leq m$, with $A_i \in S$. Let $\overline{S}^\sigma$ be the closure of S in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology. Then

$$\left(\overline{S}^\sigma\right)_m \subseteq \overline{S_m}^\sigma. \tag{4.12}$$

By the bipolar theorem, $\mathbf{b}_r(\mathcal{A}^{**}) = \overline{\mathbf{b}_r(\mathcal{A})}^\sigma$ for each r. Hence

$$\mathcal{K}(\mathcal{A}^{**}) = [\mathbf{b}_1(\mathcal{A}^{**}),\mathbf{b}_1(\mathcal{A}^{**})] = [\overline{\mathbf{b}_1(\mathcal{A})}^\sigma,\overline{\mathbf{b}_1(\mathcal{A})}^\sigma] \subseteq \overline{[\mathbf{b}_1(\mathcal{A}),\mathbf{b}_1(\mathcal{A})]}^\sigma = \overline{\mathcal{K}(\mathcal{A})}^\sigma$$

whence, by (4.12), $\mathcal{K}_3(\mathcal{A}^{**}) \subseteq (\overline{\mathcal{K}(\mathcal{A})}^\sigma)_3 \subseteq \overline{\mathcal{K}_3(\mathcal{A})}^\sigma$. Therefore $\mathrm{co}(\mathcal{K}_3(\mathcal{A}^{**})) \subseteq \mathrm{co}\left(\overline{\mathcal{K}_3(\mathcal{A})}^\sigma\right) \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}^\sigma$. Thus

$$\mathbf{b}_{\frac{1}{300}}(\mathcal{A}^{**}) \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}^{**}))} \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}^\sigma. \tag{4.13}$$

Let us show that $\mathbf{b}_{\frac{1}{300}}(\mathcal{A}) \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}$. Suppose that there is $B \in \mathbf{b}_{\frac{1}{300}}(\mathcal{A})$ such that $B \notin \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}$. By Hahn-Banach theorem, there is $f \in \mathcal{A}^*$ such that $|f(A)| \leq 1$, for all $A \in \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}$, and $f(B) > 1$. Hence B does not belong to the closure of $\overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology which contradicts (4.13). Therefore $\mathbf{b}_{\frac{1}{300}}(\mathcal{A}) \subseteq \overline{\mathrm{co}(\mathcal{K}_3(\mathcal{A}))}$, so $\mathcal{A} \in \mathfrak{L}_{3,\frac{1}{300}}$. Thus $\mathcal{A}$ is s. n. Jordan free. ∎

It follows from Theorem 4.19 that all simple C*-algebras (in particular, $C(H)$ and Cuntz algebras $\mathcal{O}_n$) and all AF-algebras satisfying the conditions of Corollary 4.15 are s. n. Jordan free.

5 Jordan ideals of C*-algebras

Recall that an algebra is called Jordan ideal free if it does not have proper Jordan ideals, that is, Jordan ideals that are not two-sided ideals. Since each Jordan free algebra is Jordan ideal free, our Theorem 2.7, for C*-algebras with max-property, and Lemma 2.5 of [4], for the unital case, show that they are Jordan ideal free if they have no commutative quotients. Since all commutative algebras (the opposite case) are also Jordan ideal free, and since there are many other results which establish that under some restrictions Jordan ideals of C*-algebras are two-sided ideals (see, for example, [6, 4] and the bibliography there), it is natural to conjecture that all C*-algebras are Jordan ideal free. We will produce below an example which shows that this conjecture is false: there are C*-algebras which are not Jordan ideal free. Moreover, the same example also shows that C*-algebras can have s. n. Jordan ideals that are not two-sided s. n. ideals.

We will start this section by constructing Jordan ideals in some unital algebras. Let $\mathcal{B}$ be a non-unital algebra and let $\widetilde{\mathcal{B}} = \mathbb{C}\mathbf{1} + \mathcal{B}$ be its unitization. By Corollary 2.15, the algebra $M_n(\widetilde{\mathcal{B}}) = M_n(\mathbb{C}) \otimes \widetilde{\mathcal{B}}$, for $n \geq 2$, has no Jordan ideals that are not two-sided ideals. Let $\mathcal{D}_n = \{D = (D_{ij})\colon D_{ij} = 0 \text{ if } i \neq j,\ D_{ii} = \lambda_i \mathbf{1} \text{ with } \lambda_i \in \mathbb{C}\}$ be the subalgebra of all diagonal scalar matrices in $M_n(\widetilde{\mathcal{B}})$. We will show that under some condition on $\mathcal{B}$ the subalgebra $\mathcal{A} = M_n(\mathcal{B}) + \mathcal{D}_n$ of $M_n(\widetilde{\mathcal{B}})$ has non-trivial Jordan ideals. Denote by $\{e_{ij}\}$ the matrix identity in $M_n(\mathbb{C})$.

Theorem 5.1 *Let $\mathcal{J}$ be a two-sided ideal of a non-unital algebra $\mathcal{B}$. For $n \geq 2$, let Λ be a set of pairs (i, j) with $i < j \leq n$. Suppose that there is a set $\mathcal{X} = \{X_{ij}\}_{(i,j)\in\Lambda}$ of elements in $\mathcal{B}\backslash\mathcal{J}$ such that*

$$\mathcal{B}X_{ij} \subseteq \mathcal{J} \text{ and } X_{ij}\mathcal{B} \subseteq \mathcal{J} \text{ for all } (i,j) \in \Lambda. \tag{5.1}$$

Then

$$I_\Lambda(\mathcal{X}) = M_n(\mathcal{J}) + \sum_{(i,j)\in\Lambda} \mathbb{C}(e_{ij} + e_{ji}) \otimes X_{ij}$$

is a Jordan ideal of the unital algebra $\mathcal{A} = M_n(\mathcal{B}) + \mathcal{D}_n$ and not a two-sided ideal of $\mathcal{A}$.

If $\mathcal{B}$ is a *-algebra, $\mathcal{J}$ is a selfadjoint ideal of $\mathcal{B}$ and all $X_{ij}^* = X_{ij}$, then $I_\Lambda(\mathcal{X})$ is a selfadjoint Jordan ideal of the *-algebra $\mathcal{A}$.

Proof. Clearly, $M_n(\mathcal{J})$ is a two-sided ideal of $\mathcal{A}$. Set $C_{ij} = \mathbb{C}(e_{ij}+e_{ji}) \otimes X_{ij}$. If $D = (D_{km}) \in \mathcal{D}_n$ with $D_{kk} = \lambda_k \mathbf{1}$, then $((e_{ij}+e_{ji}) \otimes X_{ij}) \circ D = (\lambda_i + \lambda_j)(e_{ij}+e_{ji}) \otimes X_{ij} \in C_{ij}$. Hence $C_{ij} \circ \mathcal{D}_n \subseteq C_{ij}$. It follows from (5.1) that $C_{ij} \circ M_n(\mathcal{B}) \subseteq M_n(\mathcal{J})$. Therefore $C_{ij} \circ \mathcal{A} \subseteq M_n(\mathcal{J}) + C_{ij} \subseteq I_\Lambda(\mathcal{X})$. Hence $I_\Lambda(\mathcal{X})$ is a Jordan ideal.

However, $I_\Lambda(\mathcal{X})$ is not a two-sided ideal of $\mathcal{A}$. Indeed, let $(i, j) \in \Lambda$. Let $D = (D_{km}) \in \mathcal{D}_n$ with $D_{kk} = \lambda_k \mathbf{1}$, be such that $\lambda_i = 1$ and $\lambda_k = 0$ for all $k \neq i$. Then $DC_{ij} = \mathbb{C}(e_{ij} \otimes X_{ij}) \notin I_\Lambda(\mathcal{X})$. ∎

Let H be a separable Hilbert space and let $\mathcal{B} = C(H)$ be the algebra of all compact operators on H. Let J^ϕ be a non-separable s. n. ideal of compact operators in $B(H)$, that is, $J_0^\phi \neq J^\phi$ (see Section 3). Set $\mathcal{J} = J_0^\phi$. Then $\mathcal{J} \subseteq \mathcal{B}$. It follows from (3.3) that $X\mathcal{B} \subseteq \mathcal{J}$, for each selfadjoint $X \in J^\phi \diagdown J_0^\phi$. As $X^* = X$ and $\mathcal{J}, \mathcal{B}$ are selfadjoint algebras, $\mathcal{B}X \subseteq \mathcal{J}$. For $n \geq 2$, let $\mathcal{D}_n$ be the algebra of all diagonal operators on $H \oplus ... \oplus H$ (repeated n times) such that all their entries belong to $\mathbb{C}\mathbf{1}_H$. Theorem 5.1 yields

Corollary 5.2 *Let J^ϕ be a non-separable s. n. ideal of $B(H)$. For $n \geq 2$, let Λ be a set of pairs (i,j) with $i < j \leq n$, and $\mathcal{X} = \{X_{ij}\}_{(i,j)\in\Lambda}$ be a set of selfadjoint elements in $J^\phi \diagdown J_0^\phi$. Then*

$$I_\Lambda(\mathcal{X}) = M_n(J_0^\phi) + \sum_{(i,j)\in\Lambda} \mathbb{C}(e_{ij} + e_{ji}) \otimes X_{ij}$$

is a selfadjoint Jordan ideal, but not a two-sided ideal of the unital C-algebra $\mathcal{A} = M_n(C(H)) + \mathcal{D}_n$.*

In particular, for each selfadjoint $X \in J^\phi \diagdown J_0^\phi$,

$$I(X) = \left\{ \begin{pmatrix} A & B + \lambda X \\ C + \lambda X & D \end{pmatrix} : \ A, B, C, D \in J_0^\phi, \ \lambda \in \mathbb{C} \right\} \tag{5.2}$$

is a selfadjoint Jordan ideal, but not a two-sided ideal of the unital C-algebra $M_2(C(H)) + \mathcal{D}_2$.*

As stated in the introduction, a linear subspace U of an algebra $\mathcal{A}$ is an inner ideal if $u\mathcal{A}u \subseteq U$ for all $u \in U$. The structure of weakly *-closed inner ideals in W*-algebras and norm closed inner ideals in C*-algebras was studied in [8], [9], [5]. In particular, it was established that each norm closed inner ideal U in a C*-algebra $\mathcal{A}$ satisfies $U = \overline{\mathcal{A}U} \cap \overline{U\mathcal{A}}$.

Jordan ideals J are inner ideals, as $j^2 = \frac{1}{2} j \circ j \in J$, for $j \in J$, so $jaj = \frac{1}{2}(j \circ (j \circ a) - j^2 \circ a) \in J$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ is unital then $J \subseteq \mathcal{A}J \cap J\mathcal{A}$, so that, for all $j \in J$ and $a \in \mathcal{A}$, we have $aj = a \circ j - ja \in J\mathcal{A}$. Thus $\mathcal{A}J \subseteq J\mathcal{A}$. Similarly, $J\mathcal{A} \subseteq \mathcal{A}J$, so $\mathcal{A}J = J\mathcal{A}$ is a two-sided ideal that does not coincide with J, if J is not an ideal. The example of a non-closed Jordan ideal of a unital C*-algebra considered in Corollary 5.2 shows that unital C*-algebras may have non-closed inner ideals U such that $U \subsetneqq \mathcal{A}U = U\mathcal{A}$.

Although all closed Jordan ideals of C*-algebras are two-sided ideals (see [6]), the approach of Theorem 5.2 can be used to construct Hermitian semisimple Banach *-algebras that have closed proper selfadjoint Jordan ideals.

Example 5.3 Let $J_0^\phi \neq J^\phi$. The ideal $M_2(J^\phi)$ of $B(H \oplus H)$ is isometrically isomorphic to J^ϕ. It is easy to see that for each selfadjoint $X \in J^\phi \diagdown J_0^\phi$, $I(X)$ given in (5.2) is a selfadjoint Jordan ideal, but not a two-sided ideal of the unital Banach *-algebra

$$\mathcal{A}_1 = M_2(J^\phi) + \mathcal{D}_2 = \left\{ T = A + D\colon \ A \in M_2(J^\phi), \ D = \begin{pmatrix} \lambda\mathbf{1} & 0 \\ 0 & \mu\mathbf{1} \end{pmatrix}, \ \lambda, \mu \in \mathbb{C} \right\}$$

with norm $\|A + D\|_{\mathcal{A}_1} = \|A\|_{J^\phi} + |\lambda| + |\mu|$. As the separable s. n. ideal J_0^ϕ is closed in J^ϕ in $\|\cdot\|_{J^\phi}$, I(X) is closed in $\mathcal{A}_1$.

To show that $\mathcal{A}_1$ is Hermitian, that is, $\mathrm{Sp}_{\mathcal{A}_1}(T) \subseteq \mathbb{R}$ for each $T = T^* \in \mathcal{A}_1$, we only need to prove that T is invertible in $\mathcal{A}_1$ if it is invertible in $B(H \oplus H)$. The inverse S of $T = A + D \in \mathcal{A}_1$ in $B(H \oplus H)$ belongs to C*-algebra $C(H \oplus H) + \mathcal{D}_2$, so that it must be of the form $S = X + D_1$ with

$X \in C(H \oplus H)$ and $D_1 \in \mathcal{D}_2$. We only have to prove that $X \in M_2(J^\phi)$. The condition $ST = \mathbf{1}_{H\oplus H}$ gives

$$XA + D_1A + XD + D_1D = \mathbf{1}_{H\oplus H}.$$

As A and X are compact, $D_1D = \mathbf{1}_{H\oplus H}$. Hence D and D_1 are invertible and

$$X = -XAD^{-1} - D_1AD^{-1} \in M_2(J^\phi),$$

as $A \in M_2(J^\phi)$. ■

Another example of a semisimple Banach (but non-Hermitian) *-algebra with a proper closed Jordan ideal can be obtained by modifying the example from [4] of a proper Jordan ideal in the free algebra $\mathcal{F}_2$ with two generates x and y. Denote by W the set $\{x, y, x^2, xy, yx, y^2, x^3, x^2y,\ xyx, ...\}$ of all monomials in $\mathcal{F}_2$. Set

$$\mathcal{A} = \left\{ u = \sum_{w\in W} \lambda_w(u)w : \lambda_w(u) \in \mathbb{C} \text{ and } \|u\| = \sum_{w\in W} |\lambda_w(u)| < \infty \right\}.$$

Then $\mathcal{A}$ is a Banach algebra. It is easy to show that it does not contain non-zero quasinilpotents (see [2, p. 254]), so, in particular, it is semisimple. Further, $\mathcal{A}$ has an involution which is uniquely determined by $x^* = x$ and $y^* = y$. Let

$$\mathcal{B} = \{u \in \mathcal{A} : \lambda_{xy}(u) = \lambda_{yx}(u)\}.$$

It is easy to see that $\mathcal{B}$ is a closed selfadjoint Jordan ideal of $\mathcal{A}$, but not a two-sided ideal.

We have already mentioned that W*-algebras have no proper Jordan ideals (see [4]). We shall prove now that in W*-algebras all s. n. Jordan ideals are two-sided s. n. ideals.

Corollary 5.4 *Each s. n. Jordan ideal of a W*-algebra $\mathcal{A}$ is a two-sided s. n. ideal of $\mathcal{A}$.*

Proof. Let J be an s. n. Jordan ideal of $\mathcal{A}$. By Theorem 4.18, $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$, where $\mathcal{B}$ is an s. n. Jordan free W*-algebra and $\mathcal{C}$ is a commutative W*-algebra. Hence J is a Banach $\mathcal{B}$-bimodule: $RB, BR \in J$, for all $R \in J$ and $B \in \mathcal{B}$, and there is $K > 0$ such that

$$\|RB\|_J \le K\|B\|\|R\|_J \text{ and } \|BR\|_J \le K\|B\|\|R\|_J.$$

Let $R = R_\mathcal{B} + R_\mathcal{C} \in J$ where $R_\mathcal{B} \in \mathcal{B}$ and $R_\mathcal{C} \in \mathcal{C}$. Then $R_\mathcal{C} = \mathbf{1}_\mathcal{C}R = \frac{1}{2}(\mathbf{1}_\mathcal{C} \circ R) \in J$. By (4.2),

$$\|R_\mathcal{C}\|_J = \frac{1}{2}\|\mathbf{1}_\mathcal{C} \circ R\|_J \le \frac{1}{2}D\|\mathbf{1}_\mathcal{C}\|\|R\|_J = \frac{1}{2}D\|R\|_J.$$

Hence, for all $C \in \mathcal{C}$ and $B \in \mathcal{B}$,

$$R(B + C) = RB + (R_\mathcal{B} + R_\mathcal{C})C = RB + R_\mathcal{C}C = RB + \frac{1}{2}C \circ R_\mathcal{C} \in J$$

Therefore we have from (4.2) and the above inequalities

$$\begin{aligned} \|R(B + C)\|_J &\le \|RB\|_J + \frac{1}{2}\|C \circ R_\mathcal{C}\|_J \le K\|B\|\|R\|_J + \frac{1}{2}D\|C\|\|R_\mathcal{C}\|_J \\ &\le K\|B\|\|R\|_J + \frac{1}{4}D^2\|C\|\|R\|_J \le (K + \frac{1}{4}D^2)\|B + C\|\|R\|_J. \end{aligned}$$

Similarly, $\|(B+C)R\|_J \leq (K+\frac{1}{4}D^2)\|B+C\|\|R\|_J$. It follows from Lemma 3.1 that J has an equivalent norm $\|\cdot\|'_J$ with respect to which it is an $\mathcal{A}$-bimodule:

$$\|AR\|'_J \leq \|A\|\|R\|'_J \text{ and } \|RA\|'_J \leq \|A\|\|R\|'_J, \text{ for all } A \in \mathcal{A} \text{ and } R \in J,$$

that is J is an s. n. two-sided ideal of $\mathcal{A}$. ■

An s. n. Jordan ideals J of a Banach algebra is reflexive if its second dual space $J^{**} = J$. As the example of s. n. ideals of $B(H)$ shows, reflexive s. n. ideals constitute a wide class of s. n. ideals.

Theorem 5.5 *Every reflexive s. n. Jordan ideal of a C*-algebra is a two-sided s. n. ideal.*

Proof. Let J be a reflexive s. n. Jordan ideal of a C*-algebra $\mathcal{A}$ and let π be the universal representation of $\mathcal{A}$ on H_π. Identify $\mathcal{A}$ with $\pi(\mathcal{A})$ and J with $\pi(J)$. Then the second dual $\mathcal{A}^{**}$ is isomorphic to the closure of $\mathcal{A}$ in the weak operator topology (*wot*) on H_π. We will show that J is a s. n. Jordan ideal of $\mathcal{A}^{**}$.

For $B \in J$, the operator $P_B(A) = B \circ A$ from $\mathcal{A}$ into J is bounded and

$$\|P_B(A)\|_J = \|B \circ A\|_J \leq D\|B\|_J\|A\|, \text{ so } \|P_B\| \leq D\|B\|_J.$$

Let P_B^* be the adjoint operator from J^* into $\mathcal{A}^*$. Then P_B^{**} is a bounded operator from $\mathcal{A}^{**}$ into $J^{**} = J$ and $\|P_B^{**}\| = \|P_B\| \leq D\|B\|_J$.

Let $A \in \mathcal{A}^{**}$ and let $A_\lambda \in \mathcal{A}$, $\lambda \in \Lambda$, be such that $A_\lambda \overset{wot}{\to} A$. Since $P_B^{**}(A) \in J^{**} = J$, we have for each $g \in J^*$,

$$g(P_B^{**}(A)) = P_B^{**}(A)(g) = A(P_B^*(g)) = P_B^*(g)(A),$$

where $P_B^*(g) \in A^*$. Since all functionals from $\mathcal{A}^*$ are continuous on $\mathcal{A}^{**}$ in *wot* on H_π,

$$P_B^*(g)(A) = \lim_\lambda P_B^*(g)(A_\lambda) = \lim_\lambda g(P_B(A_\lambda)) = \lim_\lambda g(B \circ A_\lambda).$$

Hence $g(P_B^{**}(A)) = \lim\limits_\lambda g(B \circ A_\lambda)$, for all $g \in J^*$, so $B \circ A_\lambda \to P_B^{**}(A)$ in the $\sigma(J, J^*)$-topology. Hence, for each $\mu \in \Lambda$, there are linear convex combinations of $B \circ A_\lambda$, with $\lambda \geq \mu$, that converge to $P_B^{**}(A)$ in $\|\cdot\|_J$. As $\|C\| \leq \|C\|_J$, for all $C \in J$, they also converge to $P_B^{**}(A)$ in $\|\cdot\|$ and, therefore, in *wot* on H_π.

As $A_\lambda \overset{wot}{\to} A$, we have $B \circ A_\lambda \overset{wot}{\to} B \circ A$. For $x, y \in H_\pi$ and $\varepsilon > 0$, let $\lambda_\varepsilon \in \Lambda$ be such that, for $\lambda > \lambda_\varepsilon$, $|((B \circ A - B \circ A_\lambda)x, y)| < \varepsilon$. By the above argument,

$$\left|((P_B^{**}(A) - \sum_i \tau_i(B \circ A_{\lambda_i}))x, y)\right| < \varepsilon,$$

for some $\lambda_i > \lambda_\varepsilon$ with $\tau_i > 0$ and $\sum_i \tau_i = 1$. Hence

$$\begin{aligned} |((B \circ A - P_B^{**}(A))x, y)| &\leq \left|((B \circ A - \sum_i \tau_i(B \circ A_{\lambda_i}))x, y)\right| + \left|((\sum_i \tau_i(B \circ A_{\lambda_i}) - P_B^{**}(A))x, y)\right| \\ &< \sum_i \tau_i \left|((B \circ A - B \circ A_{\lambda_i})x, y)\right| + \varepsilon < 2\varepsilon. \end{aligned}$$

Therefore $P_B^{**}(A) = B \circ A$, so $B \circ A \in J$ and

$$\|B \circ A\|_J = \|P_B^{**}(A)\|_J \leq \|P_B^{**}\|\|A\| \leq D\|B\|_J\|A\|.$$

Thus J is an s. n. Jordan ideal of $\mathcal{A}^{**}$. By Corollary 5.4, J is a two-sided s. n. ideal of $\mathcal{A}^{**}$. Hence it is a two-sided s. n. ideal of $\mathcal{A}$. ■

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