JORDAN SUPERHOMOMORPHISMS

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Abstract. Herstein's theorem on Jordan homomorphisms onto prime associative algebras is extended to prime associative superalgebras.

1. Introduction

Throughout the paper, by an algebra we shall mean an algebra over a fixed unital commutative ring Φ, and we assume (without further mentioning) that $\frac{1}{2} \in \Phi$. However, Φ will play an insignificant role in this paper, and our arguments also work in rings of characteristic not 2. Nevertheless, we decided to follow several recent related papers and work in the setting of algebras.

Let $\mathcal{A}$ be an associative algebra. Introducing a new product in $\mathcal{A}$, the so-called Jordan product, by $x \circ y = \frac{1}{2}(xy + yx)$, $\mathcal{A}$ becomes a Jordan algebra. Similarly we can make of $\mathcal{A}$ a Lie algebra by defining the Lie product $[x, y] = \frac{1}{2}(xy - yx)$. The study of the relationship between the associative and the Jordan and Lie structure of an associative algebra was initiated in the 1950's by Herstein, who, together with some of his students, obtained most of the classical results of this theory (see e.g. [12, 13]). Its considerable part, the one which concerns the structure of Jordan and Lie ideals, has been recently extended to superalgebras by several authors [6, 7, 8, 9, 10, 16, 17]. However, as far as we know such extensions have not yet been done for results on Jordan and Lie homomorphisms. In this paper we make the first step in this direction, proving a superalgebra version of Herstein's theorem on Jordan homomorphisms. Recall that a *Jordan homomorphism* between associative algebras $\mathcal{B}$ and $\mathcal{A}$ is a Φ-module homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ such that $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in \mathcal{B}$.

Herstein's theorem. *A Jordan homomorphism from an arbitrary associative algebra onto a prime associative algebra is either a homomorphism or an antihomomorphism.*

Actually, Herstein proved this result under the additional assumption that the characteristic of algebras is different from 3. Smiley [19] removed this assumption and also simplified the proof. In fact, his proof is very short and involves only elementary combinatorial argument, while this can not be said for other related results on Jordan and Lie homomorphisms. Only very

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recently all Herstein's conjectures [12] on Lie homomorpisms of associative rings (with and without involution) were solved (see e.g. [1, 2, 3]), and their proofs depend heavily on the theory of functional identities (see e.g. [5]). It is our goal, in the future, to apply functional identities also to the study of Lie and Jordan maps of superalgebras. The present paper, however, is entirely self-contained and the arguments are elementary.

2. Superalgebra preliminaries

An *associative superalgebra* $\mathcal{A}$ is a $\mathbf{Z}_2$-graded associative algebra; that is, $\mathcal{A}$ is an associative algebra and there exist Φ-submodules $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_0\mathcal{A}_0 \subseteq \mathcal{A}_0$ (i.e. $\mathcal{A}_0$ is a subalgebra), $\mathcal{A}_0\mathcal{A}_1 \subseteq \mathcal{A}_1$, $\mathcal{A}_1\mathcal{A}_0 \subseteq \mathcal{A}_1$, and $\mathcal{A}_1\mathcal{A}_1 \subseteq \mathcal{A}_0$. We say that $\mathcal{A}_0$ is the even, and $\mathcal{A}_1$ is the odd part of $\mathcal{A}$. If $x \in \mathcal{A}_i$, $i = 0, 1$, then we say that x is *homogeneous* of degree i and we write $|x| = i$. We now list a few illustrative examples (cf. [9, 10, 16]). For simplicity we assume that in these examples Φ is a field.

Examples. 1. A *trivial* superalgebra: $\mathcal{A}_1 = 0$ and $\mathcal{A}_0 = \mathcal{A}$.

2.. Let $\mathcal{A}$ be a unital associative algebra containing an element t such that $t^2 = 1$. Setting $\mathcal{A}_0 = \{x \in \mathcal{A} \mid [x, t] = 0\}$ and $\mathcal{A}_1 = \{x \in \mathcal{A} \mid x \circ t = 0\}$, $\mathcal{A}$ becomes an associative superalgebra. Indeed, every element $x \in \mathcal{A}$ can be written as $x = \frac{1}{2}(x+txt)+\frac{1}{2}(x-txt)$, and $\frac{1}{2}(x+txt) \in \mathcal{A}_0$, $\frac{1}{2}(x-txt) \in \mathcal{A}_1$. In the special case, when $\mathcal{A} = M_{r+s}$ (the $(r+s) \times (r+s)$ matrices over Φ with $r, s \geq 1$) and $t = \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix}$ (here, I_p is the identity in M_p), the corresponding superalgebra is denoted by $M(r|s)$. Note that $M(r|s)_0$ consists of matrices of the form $\begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix}$, $R \in M_r$, $S \in M_s$, and $M(r|s)_1$ consists of matrices of the form $\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}$, where U is an $r \times s$ matrix and V is an $s \times r$ matrix. When $r = s$ we write $M(r) = M(r|r)$.

3. Given an associative algebra $\mathcal{R}$, the algebra $\mathcal{A} = \mathcal{R} \times \mathcal{R}$ then becomes an associative superalgebra by setting $\mathcal{A}_0 = \{(x, x) \mid x \in \mathcal{R}\}$ and $\mathcal{A}_1 = \{(x, -x) \mid x \in \mathcal{R}\}$. In the case when $\mathcal{R} = M_n$, this superalgebra is denoted by $\mathcal{Q}(n)$.

4. Let $\mathcal{A} = \mathcal{Q}(\alpha, \beta)$ be the quaternion algebra, i.e. $\mathcal{A}$ is a 4-dimensional algebra with Φ-basis $1, u, v, uv$ and multiplication given by $uv = -vu$, $u^2 = \alpha \in \Phi$, $v^2 = \beta \in \Phi$. Letting $\mathcal{A}_0 = \Phi 1 + \Phi uv$, $\mathcal{A}_1 = \Phi u + \Phi v$, $\mathcal{A}$ becomes an associative superalgebra, called a *quaternion superalgebra.* Note that the superalgebra $M(1)$ is in fact the quaternion superalgebra $\mathcal{Q}(1, -1)$.

Superalgebras of types $M(r|s)$, $Q(n)$, and trivial superalgebras are the only examples of finite-dimensional simple associative superalgebras over an algebraically closed field [20] (see also [14]). By simplicitly of an associative superalgebra $\mathcal{A}$ we mean that $\mathcal{A}^2 \neq 0$ and $\mathcal{A}$ has no proper graded ideals, i.e. ideals $\mathcal{I}$ of an algebra $\mathcal{A}$ such that $\mathcal{I} = \mathcal{I} \cap \mathcal{A}_0 + \mathcal{I} \cap \mathcal{A}_1$.

An associative superalgebra $\mathcal{A}$ is said to be *prime* if the product of any two nonzero graded ideals in $\mathcal{A}$ is nonzero. It is well-known and easy to see that the primeness of an algebra (not superalgebra!) $\mathcal{A}$ can be characterized by the condition that $a\mathcal{A}b = 0$, where $a, b \in \mathcal{A}$, implies $a = 0$ or $b = 0$. Making some obvious modifications in the argument one gets an analogous result for superalgebras:

Lemma 2.1. *An associative superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is prime if and only if for any homogeneous elements a and b (i.e. $a, b \in \mathcal{A}_0 \cup \mathcal{A}_1$) $a\mathcal{A}b = 0$ implies $a = 0$ or $b = 0$.*

If $\mathcal{A}$ is prime as a superalgebra, this does not necessarily mean that $\mathcal{A}$ is prime as an algebra (consider $\mathcal{Q}(n)$) nor that its even part $\mathcal{A}_0$ is a prime algebra (consider $M(r|s)$). However, at least one of the algebras $\mathcal{A}$ and $\mathcal{A}_0$ must be prime [16, Lemma 1.3], and both are semiprime [16, Lemma 1.2]. Let us just show that $\mathcal{A}_0$ must be semiprime since this is the only among these facts that shall be needed. Noting that for each $a_0 \in \mathcal{A}_0$ we have $a_0\mathcal{A}a_0\mathcal{A}a_0 \subseteq (a_0\mathcal{A}_0a_0)\mathcal{A}a_0 + a_0\mathcal{A}(a_0\mathcal{A}_0a_0) + a_0\mathcal{A}_0a_0$, we see that $a_0\mathcal{A}_0a_0 = 0$ implies $a_0\mathcal{A}a_0\mathcal{A}a_0 = 0$ and hence $a_0 = 0$. We continue with an observation of a similar nature:

Lemma 2.2. *Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a prime associative superalgebra. If $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$ are such that $a_0\mathcal{A}_1a_1 = a_1\mathcal{A}_1a_0 = 0$, then either $a_0 = 0$ or $a_1 = 0$.*

Proof. Note that $(a_0\mathcal{A}_0a_1\mathcal{A}_1)\mathcal{A}_0(a_0\mathcal{A}_0a_1\mathcal{A}_1) \subseteq a_0\mathcal{A}_0(a_1\mathcal{A}_1a_0)\mathcal{A}_0a_1\mathcal{A}_1 = 0$, and so $a_0\mathcal{A}_0a_1\mathcal{A}_1 = 0$ by the semiprimeness of $\mathcal{A}_0$. Since $\mathcal{A}a_1 \subseteq \mathcal{A}_1a_1 + \mathcal{A}_1$, this yields $(a_0\mathcal{A}_0a_1)\mathcal{A}a_1 = 0$ which in turn implies $a_0\mathcal{A}_0a_1 = 0$ by Lemma 2.1. Since, according to our assumption, also $a_0\mathcal{A}_1a_1 = 0$, we have $a_0\mathcal{A}a_1 = 0$ and so $a_0 = 0$ or $a_1 = 0$ by Lemma 2.1.

We remark that assuming only one relation, say $a_1\mathcal{A}_1a_0 = 0$, is not enough for concluding $a_0 = 0$ or $a_1 = 0$. Consider, for example, $\mathcal{A} = M(r|s)$ and note that $a_0 = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} \in \mathcal{A}_0$, $a_1 = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix} \in \mathcal{A}_1$ satisfy $a_1\mathcal{A}_1a_0 = 0$.

Now let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be any associative superalgebra. Introducing a new product in $\mathcal{A}$ by

$$x \circ_s y = \frac{1}{2}(xy + (-1)^{|x||y|}yx),$$

where x, y are any homogeneous elements, $\mathcal{A}$ becomes a Jordan superalgebra. Jordan superalgebras of this kind also appear in Kac's classification [15] of finite-dimensional simple Jordan superalgebras over an algebraically closed field (see also [18] for a concise survey). Given homogeneous elements $x, y \in \mathcal{A}$, note that

$$x \circ_s y = x \circ y \quad \text{whenever } |x| = 0 \text{ or } |y| = 0, \tag{1}$$

and also

$$x \circ_s y = [x, y] \quad \text{whenever } |x| = 1 \text{ and } |y| = 1.$$

Let $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ be another associative superalgebra. We shall say that a Φ-module homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ is a *Jordan superhomomorphism* if it preserves the $\mathbf{Z}_2$-gradation (that is, $\varphi(\mathcal{B}_i) \subseteq \mathcal{A}_i$, $i = 0, 1$) and satisfies

$$\varphi(x \circ_s y) = \varphi(x) \circ_s \varphi(y)$$

for all homogeneous elements $x, y \in \mathcal{B}$. A Jordan superhomomorphism is, of course, the analogue of a Jordan homomorphism in the superalgebra setting (moreover, in trivial superalgebras these two concepts coincide). Let us now define analogues of homomorphisms and antihomomorphisms. A $\mathbf{Z}_2$-gradation preserving Φ-module homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ will be called a *superhomomorphism* (resp. *superantihomomorphism*) if $\varphi(xy) = \varphi(x)\varphi(y)$ (resp. $\varphi(xy) = (-1)^{|x||y|}\varphi(y)\varphi(x)$) for all homogeneous elements $x, y \in \mathcal{B}$. Of course, a superhomomorphism is nothing but a homomorphism of superalgebras. Clearly, superhomomorphisms and superantihomomorphisms are examples of Jordan superhomomorphisms. The natural question that appears is under which conditions these are also the only possible examples.

It seems rather obvious how to find concrete examples of superhomomorphisms. Let us list a few examples of superantihomomorphisms and nontrivial Jordan superhomomorphisms, i.e. those that are different from superhomomorphisms and superantihomomorphisms.

Examples. 1. Of course, a superantihomomorphism between associative superalgebras $\mathcal{B}$ and $\mathcal{A}$ is not, in general, an antihomomorphism between algebras $\mathcal{B}$ and $\mathcal{A}$. However, the following is true: If Φ contains an element i such that $i^2 = -1$ and we define $\iota : \mathcal{A} \to \mathcal{A}$ by $\iota(x_0 + x_1) = x_0 + ix_1$, $x_i \in \mathcal{A}_i$, then a $\mathbf{Z}_2$-gradation preserving Φ-module homomorphism $\varphi : \mathcal{B} \to \mathcal{A}$ is a superantihomomorphism between superalgebras if and only if $\iota \circ \varphi$ is an antihomomorphism between algebras.

2. Let $\psi : M_r \to M_r$ be an antihomomorphism. Then $\varphi : M(r) \to M(r)$,

$$\varphi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} \psi(D) & -\psi(B) \\ \psi(C) & \psi(A) \end{bmatrix}$$

is a superantihomomorphism.

3. *Superinvolutions* are examples of superantiautomorphisms (see e.g. [10, Section 3] and examples therein). In particular, if ψ in example above is the matrix transposition, then φ is the so-called transposition superinvolution.

4. Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra which is commutative as an algebra (for example, $\mathcal{A} = \mathcal{Q}(1)$). Pick $a_0 \in \mathcal{A}_0$ and define $\varphi : \mathcal{A} \to \mathcal{A}$ by $\varphi(x_0 + x_1) = x_0 + a_0 x_1$. Then φ is a Jordan superhomomorphism which is not necessarily a superhomomorphism nor a superantihomomorphism.

5. Pick an invertible $\gamma \in \Phi$ such that $\gamma \neq \pm 1$ and define $\varphi : \mathcal{Q}(\alpha,\beta) \to \mathcal{Q}(\alpha,\beta)$ by

$$\varphi(\lambda_0 + \lambda_1 uv + \lambda_2 u + \lambda_3 v) = \lambda_0 + \lambda_1 uv + \lambda_2 \gamma u + \lambda_3 \gamma^{-1} v$$

Then φ is a Jordan superhomomorphism which is neither a superhomomorphism nor a superantihomomorphism.

The last two examples show that nontrivial Jordan superhomomorphisms exist even in some basic examples of associative superalgebras. However, both algebras appearing in Examples 4 and 5 are examples of associative superalgebras whose even part is commutative (and in fact, they are rather general examples of such algebras, cf. [16, Lemma 1.9]). Such algebras have turned out to be rather exceptional in the study of Jordan and Lie ideals (see e.g. [9, 10, 16]) so it is not so surprising that they also admit nontrivial examples in the study of homomorphisms. We shall see that these are the only prime associative superalgebras where nontrivial examples can exist.

Finally we fix the notation that will be used in the next sections. For a map φ between (super)algebras $\mathcal{B}$ and $\mathcal{A}$, we set

$$\begin{aligned} \tau(x,y) &= \varphi(xy) - \varphi(x)\varphi(y), \\ \omega(x,y) &= \varphi(xy) - \varphi(y)\varphi(x), \\ \rho(x,y) &= \varphi(xy) + \varphi(y)\varphi(x) \quad \text{for all } x, y \in \mathcal{B}. \end{aligned}$$

3. Jordan homomorphisms

In this section we make the first step towards the proof of the main result. The auxiliary result which we are going to prove is already a slight generalization of Herstein's theorem, and we shall state in the setting of algebras (rather than superalgebras). Basically we will just follow the argument given in [4], but nevertheless we shall give a complete proof for the sake of completness.

Proposition 3.1. *Let φ be a Φ-module homomorphism from an algebra $\mathcal{B}$ onto an algebra $\mathcal{A}$. Suppose there exist subalgebras $\mathcal{B}_0$ of $\mathcal{B}$ and $\mathcal{A}_0$ of $\mathcal{A}$ such that the following conditions are fulfilled:*

(i) $\varphi(x \circ x_0) = \varphi(x) \circ \varphi(x_0)$ for all $x \in \mathcal{B}$, $x_0 \in \mathcal{B}_0$;

(ii) $\varphi(\mathcal{B}_0) \subseteq \mathcal{A}_0$;

(iii) $a_0\mathcal{A}b_0\mathcal{A}a_0 = 0$, where $a_0, b_0 \in \mathcal{A}_0$, implies $a_0 = 0$ or $b_0 = 0$.

Then $\varphi|_{\mathcal{B}_0}$, the restriction of φ to $\mathcal{B}_0$, is either a homomorphism or an antihomomorphism.

Proof. Using $x_0 x x_0 = 2x_0 \circ (x_0 \circ x) - x_0^2 \circ x$ it follows at once that

$$\varphi(x_0 x x_0) = \varphi(x_0)\varphi(x)\varphi(x_0) \quad \text{for all } x \in \mathcal{B},\ x_0 \in \mathcal{B}_0. \tag{2}$$

Linearizing (2) we get

$$\varphi(x_0 x y_0 + y_0 x x_0) = \varphi(x_0)\varphi(x)\varphi(y_0) + \varphi(y_0)\varphi(x)\varphi(x_0) \tag{3}$$

for all $x \in \mathcal{B}$, $x_0, y_0 \in \mathcal{B}_0$. Now pick any $x \in \mathcal{B}$, $x_0, y_0 \in \mathcal{B}_0$ and set $W = \varphi(x_0y_0xy_0x_0 + y_0x_0xx_0y_0)$. Using (2) we get

$$\begin{aligned} W &= \varphi(x_0(y_0xy_0)x_0) + \varphi(y_0(x_0xx_0)y_0) \\ &= \varphi(x_0)\varphi(y_0xy_0)\varphi(x_0) + \varphi(y_0)\varphi(x_0xx_0)\varphi(y_0) \\ &= \varphi(x_0)\varphi(y_0)\varphi(x)\varphi(y_0)\varphi(x_0) + \varphi(y_0)\varphi(x_0)\varphi(x)\varphi(x_0)\varphi(y_0). \end{aligned}$$

On the other hand, making use of (3) we get

$$\begin{aligned} W &= \varphi((x_0y_0)x(y_0x_0) + (y_0x_0)x(x_0y_0)) \\ &= \varphi(x_0y_0)\varphi(x)\varphi(y_0x_0) + \varphi(y_0x_0)\varphi(x)\varphi(x_0y_0). \end{aligned}$$

Comparing both expressions and using $\varphi(x_0y_0) + \varphi(y_0x_0) = \varphi(x_0)\varphi(y_0) + \varphi(y_0)\varphi(x_0)$ we arrive at

$$\tau(x_0, y_0)\varphi(x)\omega(x_0, y_0) + \omega(x_0, y_0)\varphi(x)\tau(x_0, y_0) = 0 \tag{4}$$

for all $x \in \mathcal{B}$, $x_0, y_0 \in \mathcal{B}_0$ (cf. [19, p. 427], [4, Lemma 2.1]). We have to show that either $\tau(\mathcal{B}_0, \mathcal{B}_0) = 0$ or $\omega(\mathcal{B}_0, \mathcal{B}_0) = 0$.

Pick $x_0, y_0 \in \mathcal{B}_0$ and write $\tau = \tau(x_0, y_0)$, $\omega = \omega(x_0, y_0)$ for brevity. Since φ is onto, (4) can be written as $\tau a\omega + \omega a\tau = 0$ for all $a \in \mathcal{A}$. Accordingly, $\tau a(\omega b\tau) = -\tau(a\tau b)\omega = (\omega a\tau)b\tau = -\tau a\omega b\tau$ for all $a, b \in \mathcal{A}$, so that $\tau\mathcal{A}\omega\mathcal{A}\tau = 0$. Since $\tau, \omega \in \mathcal{A}_0$, it follows from our assumption that $\tau = 0$ or $\omega = 0$. That is, for each pair $x_0, y_0 \in \mathcal{B}_0$ we have either $\tau(x_0, y_0) = 0$ or $\omega(x_0, y_0) = 0$. It remains to show (in a standard way) that one of these two conditions is fulfilled for all $x_0, y_0 \in \mathcal{B}_0$. For any fixed $x_0 \in \mathcal{B}_0$, the sets $\{y_0 \in \mathcal{B}_0 \,|\, \tau(x_0, y_0) = 0\}$ and $\{y_0 \in \mathcal{B}_0 \,|\, \omega(x_0, y_0) = 0\}$ are additive subgroups of $\mathcal{B}_0$ whose union is, by what we proved, equal to $\mathcal{B}_0$. However, a group cannot be the union of its proper subgroups, so it follows that either $\tau(x_0, \mathcal{B}_0) = 0$ or $\omega(x_0, \mathcal{B}_0) = 0$. But this means that $\mathcal{B}_0$ is the union of its additive subgroups $\{x_0 \in \mathcal{B}_0 \,|\, \tau(x_0, \mathcal{B}_0) = 0\}$ and $\{x_0 \in \mathcal{B}_0 \,|\, \omega(x_0, \mathcal{B}_0) = 0\}$, and so one of them equals $\mathcal{B}_0$. This completes the proof.

4. The main result

We first fix the notation for this last section. By φ we denote a Jordan superhomomorphism from an arbitrary associative superalgebra $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ onto a prime associative superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. Further, by $\mathcal{Z}$ and $\mathcal{C}_0$ we denote the center of the algebra $\mathcal{A}$ and $\mathcal{A}_0$, respectively. We assume that the algebra $\mathcal{A}_0$ is noncommutative, that is, $\mathcal{C}_0 \neq \mathcal{A}_0$.

Let us record a few useful identities. First of all, it is clear that

$$\tau(x, x_0) = -\tau(x_0, x),\ \ \omega(x, x_0) = -\omega(x_0, x)\ \text{ for all } x \in \mathcal{B},\ x_0 \in \mathcal{B}_0, \tag{5}$$

$$\tau(x_1, y_1) = \tau(y_1, x_1),\ \ \rho(x_1, y_1) = \rho(y_1, x_1)\ \text{ for all } x_1, y_1 \in \mathcal{B}_1. \tag{6}$$

Let $y = y_0 + y_1 \in \mathcal{B}_0 \oplus \mathcal{B}_1$ and $x_1 \in \mathcal{B}_1$. Using

$$[x_1^2, y] = 2[x_1, x_1 \circ y_0] + 2x_1 \circ [x_1, y_1]$$

it follows immediately from the definition of a Jordan superhomomorphism together with (1) that

$$\varphi([x_1^2, y]) = [\varphi(x_1)^2, \varphi(y)] \quad \text{for all } x_1 \in \mathcal{B}_1,\ y \in \mathcal{B}. \tag{7}$$

Using both Lemma 2.1 and (1) we see that φ satisfies all conditions of Proposition 3.1. Thus we have

Lemma 4.1. *$\varphi|_{\mathcal{B}_0}$ is either a homomorphism or an antihomomorphism of $\mathcal{B}_0$ onto $\mathcal{A}_0$.*

Lemma 4.2. *If $\varphi|_{\mathcal{B}_0}$ is a homomorphism, then φ is a superhomomorphism.*

Proof. Our assumption implies that $\varphi([x_1^2, x_0]) = [\varphi(x_1^2), \varphi(x_0)]$ for all $x_0 \in \mathcal{B}_0$, $x_1 \in \mathcal{B}_1$. Comparing this relation with (7) it follows that $[\tau(x_1, x_1), \varphi(x_0)] = 0$. That is to say, $\tau(x_1, x_1) \in \mathcal{C}_0$ for all $x_1 \in \mathcal{B}_1$. Linearizing and using (6) we get

$$\tau(\mathcal{B}_1, \mathcal{B}_1) \subseteq \mathcal{C}_0. \tag{8}$$

Now consider the expression $\varphi(x_0x_1y_1)$ with $x_0 \in \mathcal{B}_0$, $x_1, y_1 \in \mathcal{B}_1$. On the one hand,

$$\varphi(x_0(x_1y_1)) = \varphi(x_0)\varphi(x_1y_1) = \varphi(x_0)\varphi(x_1)\varphi(y_1) + \varphi(x_0)\tau(x_1, y_1),$$

and on the other hand,

$$\begin{aligned} \varphi((x_0x_1)y_1) &= \varphi(x_0x_1)\varphi(y_1) + \tau(x_0x_1, y_1) \\ &= \varphi(x_0)\varphi(x_1)\varphi(y_1) + \tau(x_0, x_1)\varphi(y_1) + \tau(x_0x_1, y_1). \end{aligned}$$

Comparing we obtain

$$\tau(x_0, x_1)\varphi(y_1) = \varphi(x_0)\tau(x_1, y_1) - \tau(x_0x_1, y_1) \tag{9}$$

for all $x_0 \in \mathcal{B}_0$, $x_1, y_1 \in \mathcal{B}_1$. In a similar fashion, by computing $\varphi(y_1x_1x_0)$ in two different ways and then using (5) and (6), we get

$$\varphi(y_1)\tau(x_0, x_1) = -\tau(x_1, y_1)\varphi(x_0) + \tau(x_1x_0, y_1) \tag{10}$$

for all $x_0 \in \mathcal{B}_0$, $x_1, y_1 \in \mathcal{B}_1$.

From (8) and (9) it follows that $\tau(x_0, x_1)\varphi(y_1)$ commutes with $\varphi(x_0)$, so that $[\tau(x_0, \mathcal{B}_1)\mathcal{A}_1, \varphi(x_0)] = 0$ for all $x_0 \in \mathcal{B}_0$. Accordingly, for any $a_0 \in \mathcal{A}_0$, $a_1 \in \mathcal{A}_1$ we have $a_1a_0 \in \mathcal{A}_1$ and so

$$\tau(x_0, x_1)a_1[a_0, \varphi(x_0)] = [\tau(x_0, x_1)a_1a_0, \varphi(x_0)] - [\tau(x_0, x_1)a_1, \varphi(x_0)]a_0 = 0,$$

that is,

$$\tau(x_0, \mathcal{B}_1)\mathcal{A}_1[\mathcal{A}_0, \varphi(x_0)] = 0 \quad \text{for all } x_0 \in \mathcal{B}_0. \tag{11}$$

Analogously, (8) and (10) imply $[\mathcal{A}_1\tau(x_0, \mathcal{B}_1), \varphi(x_0)] = 0$ and considering an element $a_0a_1 \in \mathcal{A}_1$ we arrive at

$$[\mathcal{A}_0, \varphi(x_0)]\mathcal{A}_1\tau(x_0, \mathcal{B}_1) = 0 \quad \text{for all } x_0 \in \mathcal{B}_0. \tag{12}$$

Now compare (11) and (12) and note that Lemma 2.2 can be used. Hence it follows that for each $x_0 \in \mathcal{B}_0$, either $\tau(x_0, \mathcal{B}_1) = 0$ or $\varphi(x_0) \in \mathcal{C}_0$. Again using the fact a group cannot be the union of its proper subgroups, as well as our assumption that $\mathcal{C}_0 \neq \mathcal{A}_0$, it follows that $\tau(\mathcal{B}_0, \mathcal{B}_1) = 0$. Whence also $\tau(\mathcal{B}_1, \mathcal{B}_0) = 0$ by (5).

Therefore, given $x_1, y_1 \in \mathcal{B}_1$ we have $x_1^2 \in \mathcal{B}_0$ and hence $\varphi([x_1^2, y_1]) = [\varphi(x_1^2), \varphi(y_1)]$. On the other hand, $\varphi([x_1^2, y_1]) = [\varphi(x_1)^2, \varphi(y_1)]$ by (7), so that $[\tau(x_1, x_1), \varphi(y_1)] = 0$. That is, $[\tau(x_1, x_1), \mathcal{A}_1] = 0$ which together with (8) gives $\tau(x_1, x_1) \in \mathcal{Z}$ for every $x_1 \in \mathcal{B}_1$. Linearizing and using (6) we get $\tau(\mathcal{B}_1, \mathcal{B}_1) \subseteq \mathcal{Z}$. Consequently, (10) shows that $\tau(\mathcal{B}_1, \mathcal{B}_1)\mathcal{A}_0 \subseteq \mathcal{Z}$ which in turn implies $\tau(\mathcal{B}_1, \mathcal{B}_1)[\mathcal{A}_0, \mathcal{A}_0] = 0$. However, since $\tau(\mathcal{B}_1, \mathcal{B}_1) \subseteq \mathcal{A}_0 \cap \mathcal{Z}$ and $[\mathcal{A}_0, \mathcal{A}_0] \neq 0$, we infer from Lemma 2.1 that $\tau(\mathcal{B}_1, \mathcal{B}_1) = 0$.

Combining all our conclusions together with our assumption (which can be written as $\tau(\mathcal{B}_0, \mathcal{B}_0) = 0$) it follows that $\tau(\mathcal{B}, \mathcal{B}) = 0$. That is, φ is a superhomomorphism.

Lemma 4.3. *If $\varphi|_{\mathcal{B}_0}$ is an antihomomorphism, then φ is a superantihomomorphism.*

The proof of Lemma 4.3 is a simple modification of that of Lemma 4.2, so we give only an outline. First we observe using (7) that $\rho(\mathcal{B}_1, \mathcal{B}_1) \subseteq \mathcal{C}_0$. Then we compute $\varphi(x_0 x_1 y_1)$ and $\varphi(y_1 x_1 x_0)$ in two different ways to obtain

$$\begin{aligned} \varphi(y_1)\omega(x_0, x_1) &= -\rho(x_1, y_1)\varphi(x_0) + \rho(x_0 x_1, y_1), \\ \omega(x_0, x_1)\varphi(y_1) &= \varphi(x_0)\rho(x_1, y_1) - \rho(x_1 x_0, y_1). \end{aligned}$$

Using these relations we derive

$$\omega(x_0, \mathcal{B}_1)\mathcal{A}_1[\mathcal{A}_0, \varphi(x_0)] = [\mathcal{A}_0, \varphi(x_0)]\mathcal{A}_1\omega(x_0, \mathcal{B}_1) = 0$$

from which $\omega(\mathcal{B}_0, \mathcal{B}_1) = \omega(\mathcal{B}_1, \mathcal{B}_0) = 0$ follows. Finally, using this together with (7) we show that $\rho(\mathcal{B}_1, \mathcal{B}_1) \subseteq \mathcal{Z}$ from which it can be deduced that $\rho(\mathcal{B}_1, \mathcal{B}_1) = 0$, completing the proof.

From Lemmas 4.1, 4.2 and 4.3 we infer the main result.

Theorem 4.4. *A Jordan superhomomorphism from an arbitrary associative superalgebra onto a prime associative superalgebra whose even part is noncommutative is either a superhomomorphism or a superantihomomorphism.*

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