Lie ideals: from pure algebra to C*-algebras.

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1 Introduction

Throughout this paper $\mathcal{A}$ will be an algebra over a field $\mathbb{F}$. We shall tacitly assume that $\mathrm{char}(\mathbb{F}) \neq 2$, although this is not always necessary. It is well-known that $\mathcal{A}$ becomes a Lie algebra if we replace the original product by the new product, the so-called Lie product, given by the commutator

$$[x, y] = xy - yx \text{ for } x, y \in \mathcal{A}.$$

Ideals of $\mathcal{A}$ with respect to the Lie product are called Lie ideals of $\mathcal{A}$. Thus, a *Lie ideal of* $\mathcal{A}$ is a linear subspace L of $\mathcal{A}$ such that

$$[a, x] \in L \text{ for all } a \in \mathcal{A} \text{ and } x \in L.$$

How does the associative structure of the algebra $\mathcal{A}$ effect the Lie structure of $\mathcal{A}$? In particular, is it possible to describe Lie ideals of $\mathcal{A}$ through associative ideals of $\mathcal{A}$? This and related questions such as the link between Lie ideals and conjugate-invariant subspace of unital algebras, have been an active area of research for more than 50 years. They have been studied in pure algebra (see e. g. [A, BFM, H1, H2, H3, JR, LM, MM1, MM2, Mu]) and, more or less independently, also in functional analysis, particularly in operator algebras (see e. g. [BM, CY, FM, FR, FMS, FN, HMS, HP, Ma, MaMu, MS, Mi, To]). One of the goals of this paper is to "glue" these two areas; that is, we will apply purely algebraic results, derived in the first part of the paper, to the second part dealing with Banach algebras, especially with W*-algebras and C*-algebras.

By $\mathfrak{Z}_{\mathcal{A}}$ we denote the center of $\mathcal{A}$. For linear subspaces K and M of $\mathcal{A}$, we denote

$$KM = \text{ linear span}\{ab\colon a \in K,\, b \in M\} \text{ and } [K, M] = \text{ linear span}\{[a, b]\colon a \in K,\, b \in M\}.$$

Ideals of $\mathcal{A}$ are obvious, but rather special examples of Lie ideals. Another and, as we shall see, a more important example of a Lie ideal that arises from an ideal J is $[J, \mathcal{A}]$. But even this is not sufficiently general, more complicated examples related to this one can be easily constructed. So what could be a satisfactory description of Lie ideals in terms of ideals? Before giving what we believe is an adequate answer to this question, we mention two rather old and well-known results.

The first one is a classical theorem by Herstein [H1]:

T0.0 **Theorem 1.1** *If $\mathcal{A}$ is a simple algebra, then a linear subspace L of $\mathcal{A}$ is a Lie ideal of $\mathcal{A}$ if and only if either $L \supseteq [\mathcal{A}, \mathcal{A}]$, or $L \subseteq \mathfrak{Z}_{\mathcal{A}}$.*

The second result is due to Fong, Miers and Sourour [FMS]:

T0.1 **Theorem 1.2** *A linear subspace L of the algebra $\mathcal{A} = B(H)$ of all bounded linear operators on a separable Hilbert space H is a Lie ideal of $\mathcal{A}$ if and only if there exists an ideal J of $\mathcal{A}$ such that*

$$[J, \mathcal{A}] \subseteq L \subseteq J + \mathbb{C}\mathbf{1}_H. \tag{1.1}$$ 0.3

These two characterizations of Lie ideals might appear rather different, but they can be viewed inside the same framework, as we shall now see. For any subspace K of $\mathcal{A}$, we set

$$N(K) = \{x \in \mathcal{A}\colon\ [x, \mathcal{A}] \subseteq K\}.$$

For example, $N(\{0\}) = \mathfrak{Z}_{\mathcal{A}}$ and $N([\mathcal{A}, \mathcal{A}]) = \mathcal{A}$. The condition that K is a Lie ideal of $\mathcal{A}$ can be expressed as that $K \subseteq N(K)$. Further, note that in this case *every* linear space lying between K and $N(K)$ is a Lie ideal. In particular, if J is an ideal of $\mathcal{A}$, then every subspace L such that

$$[J, \mathcal{A}] \subseteq L \subseteq N([J, \mathcal{A}]) \tag{1.2}$$ ena

is a Lie ideal. We will say that such a Lie ideal L is *embraced by an ideal* J. We regard Lie ideals that are embraced by ideals as "trivial" ones in the sense that they arise from associative ideals, so they reflect just the structure of $\mathcal{A}$ and are not concerned with some peculiar properties of $\mathcal{A}^-$. Note that the aforementioned results imply the following:

T1.1 **Theorem 1.3** (Herstein) *Every Lie ideal of a simple algebra is embraced by an ideal.*

T1.2 **Theorem 1.4** (Fong-Miers-Sourour) *Every Lie ideal of $B(H)$ is embraced by an ideal.*

Theorem 1.3 is clearly equivalent to Theorem 1.1, while Theorem 1.4 seemingly tells us less than Theorem 1.2. Nevertheless, it is equivalent to Theorem 1.2 due to Calkin's result [C] saying that $N(J) = J + \mathbb{C}\mathbf{1}$, for each ideal J of $\mathcal{A} = B(H)$, so that $J + \mathbb{C}\mathbf{1} \subseteq N([J, \mathcal{A}]) \subseteq N(J) = J + \mathbb{C}\mathbf{1}$ and, therefore, $N([J, \mathcal{A}]) = J + \mathbb{C}\mathbf{1}$.

Motivated by Theorems 1.3 and 1.4, we now propose the following general problem. Given an algebra $\mathcal{A}$, *is every Lie ideal of $\mathcal{A}$ embraced by an ideal*? If the answer is "yes", then all Lie ideals of $\mathcal{A}$ arise from ideals of $\mathcal{A}$, that is, they are completely determined by associative ideals of $\mathcal{A}$. In general this is the optimal description of Lie ideals one can hope for; in a particular algebra, one can of course also try to determine $[J, A]$ and $N([J, A])$ for all ideals J, and hence obtain a more accurate description. But in the present paper this problem will not be in the center of our attention. Nevertheless, as a consequence of special properties of algebras that we shall consider, we will usually obtain a sharper conclusion than just (1.2).

To the best of our knowledge, the idea of considering condition (1.2) is new. There is, however, a similar condition that has gained some interest in the past. We shall say that a subspace L of $\mathcal{A}$ is *related to an ideal* J of $\mathcal{A}$ if

$$[J, \mathcal{A}] \subseteq L \subseteq N(J). \tag{1.3}$$ dva

As $N([J, \mathcal{A}]) \subseteq N(J)$, (1.2) implies (1.3), so if L is embraced by J, then it is also related to J. The converse is not true in general. Moreover, unlike (1.2), (1.3) does not imply that L is a Lie ideal. So the condition of being related to an ideal is not entirely satisfactory for describing Lie ideals. Anyhow, this condition naturally appears in the study of Lie ideals. If $\mathcal{A} = M_n(\mathcal{B})$, the algebra of all $n \times n$ matrices over an arbitrary unital algebra $\mathcal{B}$, then every Lie ideal of $\mathcal{A}$ is related to an ideal. This was proved by Murphy [Mu] for $n = 2$, and later extended to an arbitrary n by Marcoux

[Ma]. The condition (1.3) will be of some importance also in the present paper, in particular as an intermediate step towards (1.2). Another condition considered in this paper is the following one. We shall say that a subspace L of $\mathcal{A}$ is *commutator equal* to an ideal J of $\mathcal{A}$ if

$$[L, \mathcal{A}] = [J, \mathcal{A}]. \tag{1.4}$$ tri

This condition does not guarantee that L is a Lie ideal, and so it is not entirely appropriate for characterizing Lie ideals. However, if we do know that L is a Lie ideal, then (1.4) implies (1.2). Summarizing the notions introduced above, we have

D1 **Definition 1.5** *Let L be a Lie ideal and J be an ideal of $\mathcal{A}$. We say that*
(i) *L and J are related if* $[J, \mathcal{A}] \subseteq L \subseteq N(J)$;
(ii) *J embraces L if* $[J, \mathcal{A}] \subseteq L \subseteq N([J, \mathcal{A}])$;
(iii) *L and J are commutator equal if* $[L, \mathcal{A}] = [J, \mathcal{A}]$.

From the discussion above it follows that (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i).

In Section 2 we give a more detailed insight into the notions introduced in Definition 1.5. In particular, we show that there exist (a) Lie ideals that are not related to ideals, (b) Lie ideals that are related to ideals but not embraced by ideals, and (c) linear subspaces that are related to and commutator equal to ideals but are not Lie ideals. We also obtain some preparatory results and review the foundations of Herstein's theory of Lie ideals.

In Section 3 we consider a unital algebra $\mathcal{A}$ that contains an idempotent p_1. The main idea is to describe the properties of Lie ideals of $\mathcal{A}$ via the properties of Lie ideals of the algebra $p_1\mathcal{A}p_1 + p_2\mathcal{A}p_2$, where $p_2 = \mathbf{1} - p_1$. We show that this is possible if p_1 satisfies a rather mild condition. The background behind this result is that it makes it possible for us to effectively handle W*-algebras in Section 5. At the same time, the result is of some interest in its own right.

Section 4 is devoted to the study of Lie ideals in the tensor product $\mathcal{B}\otimes\mathcal{P}$ of algebras $\mathcal{B}$ and $\mathcal{P}$, where $\mathcal{B}$ is a (locally) unital algebra. The main restrictions are imposed on the algebra $\mathcal{P}$. Firstly, $\mathcal{P}$ is assumed to be a prime algebra satisfying some further technical conditions. A complete description of Lie ideals in this generality seems to be out of reach. We do, however, associate with every Lie ideal L of $\mathcal{B}\otimes\mathcal{P}$ a pair of ideals of $\mathcal{B}$ and $\mathcal{P}$ that are in some sense connected to L. For a simple $\mathcal{P}$, we construct a large variety of Lie ideals of $\mathcal{B}\otimes\mathcal{P}$ from Lie ideals of $\mathcal{B}$ and $\mathcal{P}$, and thoroughly analyse the case when $\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) \leq 1$. In this situation we are able to precisely describe Lie ideals of $\mathcal{B}\otimes\mathcal{P}$. In the special case where $\mathcal{P} = M_n(\mathbb{F})$ this leads to a characterization of Lie ideals of $M_n(\mathcal{B}) \cong \mathcal{B}\otimes M_n(\mathbb{F})$, which essentially strengthens the results of Murphy and Marcoux mentioned above. Since Lie ideals of $M_n(\mathcal{B})$ are not necessarily embraced by ideals (they are only related to ideals), this characterization is necessarily a bit complicated. With Section 4 a purely algebraic part of the paper ends.

In Section 5 we study Lie ideals (closed and non-closed) of Banach algebras. To analyse relations between Lie ideals and associative ideals of Banach algebras $\mathcal{A}$, we consider topological analogues of the algebraic conditions. We say that a Lie ideal L is *topologically embraced* by an ideal J if

$$\overline{[J, \mathcal{A}]} \subseteq \overline{L} \subseteq N(\overline{[J, \mathcal{A}]}); \tag{1.5}$$ 0.1

and that L is *topologically commutator equal* to J if

$$\overline{[L, \mathcal{A}]} = \overline{[J, \mathcal{A}]}. \tag{1.6}$$ 0.2

Although these conditions are weaker than the corresponding algebraic conditions, they are sufficiently strong to characterize closed Lie ideals of many Banach algebras. Indeed, if all closed Lie ideals of $\mathcal{A}$ are topologically embraced by closed ideals, then the set of all closed Lie ideals of $\mathcal{A}$ consists of all closed subspaces that lie between $\overline{[J,\mathcal{A}]}$ and $N(\overline{[J,\mathcal{A}]})$, where J is an arbitrary closed ideal of $\mathcal{A}$.

For many classes of Banach algebras the link between closed Lie ideals and closed associative ideals is very strong. Miers [Mi] showed that if $\mathcal{A}$ is a W*–algebra then, for each closed Lie ideal L, there is a closed ideal J such that $J \subseteq L + \mathfrak{Z}_{\mathcal{A}} \subseteq J + \mathfrak{Z}_{\mathcal{A}}$. For uniformly hyperfinite triangular operator algebras $\mathcal{A}$, Hudson, Marcoux and Sourour [HMS] proved that each L satisfies the inclusion $J \subseteq L \subseteq J + D$, for some closed ideal J of $\mathcal{A}$ and some closed subalgebra D of the diagonal of $\mathcal{A}$. Marcoux [Ma] established that each simple UHF-algebra $\mathcal{A}$ has only four closed Lie ideals: $\{0\}$, $\mathbb{C}\mathbf{1}$, $\overline{[\mathcal{A},\mathcal{A}]}$ and $\mathcal{A}$. In [MaMu] Marcoux and Murphy showed that it is also true for all unital simple C*-algebras with only one tracial state (for example, for all UHF-algebras, for the irrational rotation algebras $\mathcal{A}_\theta$, for Bunce-Deddens algebras and for the reduced group C*-algebras $C_r(\mathbf{F}_n)$, where $\mathbf{F}_n$ is the free group on $n > 1$ generators).

We describe closed Lie ideals of some Banach *-subalgebras of the C*-algebra $C(H)$ of all compact operators on a Hilbert space H. In particular, we consider symmetrically normed ideals of $C(H)$ and the "differential" subalgebras of $C(H)$, defined by symmetric operators (the term "differential" arises from the fact that these subalgebras are the domains of closed *-derivations of $C(H)$).

We also obtain some Banach algebraic counterpart of the results of Section 4 on tensor products. We prove that, for any unital Banach algebra $\mathcal{B}$, the closed Lie ideals of the projective tensor product $\mathcal{B}\widehat{\otimes}C(H)$ and, more generally, of $\mathcal{B}\widehat{\otimes}\mathcal{J}$, where $\mathcal{J}$ is a separable symmetrically normed ideal of $B(H)$ different from C_1, are ideals.

Going over from the general Banach case to a more special case of C*-algebras, we first consider arbitrary (not necessarily closed) Lie ideals of W*-algebras. Relying heavily on algebraic properties of W*-algebras, we derive from the results of Sections 3 and 4 the desirable characterization of Lie ideals in general W*-algebras: *every Lie ideal of a* W*-*algebra is commutator equal to* (and, hence, embraced by) *an ideal.*

The relation between Lie ideals and associative ideals in general C*-algebras $\mathcal{A}$ is much more complicated and varied than in W*-algebras. We show that, for a closed Lie ideal L and a closed associative ideal J of $\mathcal{A}$, the "topological" conditions (1.5) and (1.6) are equivalent and that they are equivalent to the algebraic condition that L and J are related. Using the characterization of Lie ideals in W*-algebras, we establish the result that completely describes closed Lie ideals of C*-algebras in terms of ideals: *each Lie ideal of a* C*-*algebra is topologically commutator equal to an associative ideal.*

In special cases more transparent descriptions of closed Lie ideals can be possible. Thus if $\mathcal{A}$ has no tracial states then each closed Lie ideal lies between I and $N(I)$, for some closed ideal I of $\mathcal{A}$. Moreover, using the technique developed in the analysis of projective tensor products, we show that any closed Lie ideal of a stable C*-algebra $\mathcal{A}$ ($\mathcal{A} \cong \mathcal{A} \otimes C(H)$) is an ideal.

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2 Lie ideals in general algebras.

As above, $\mathcal{A}$ will be an algebra over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, and by $\mathfrak{Z}_{\mathcal{A}}$ we denote its center. For any subset $\mathcal{S}$ of $\mathcal{A}$, denote by $\mathrm{Id}(\mathcal{S})$ the ideal of $\mathcal{A}$ generated by $\mathcal{S}$ that contains $\mathcal{S}$, that is, $\mathrm{Id}(\mathcal{S})$ is the linear span of all elements s, as, sa, asb with $s \in \mathcal{S}$ and $a, b \in \mathcal{A}$. Let $x, y, z \in \mathcal{A}$. Then

$$[xy, z] = [x, yz] + [y, zx], \tag{2.1}$$ 1

$$[xy, z] = [x, z]y + x[y, z]. \tag{2.2}$$ 2

Let L be a Lie ideal of $\mathcal{A}$. Then $L\mathcal{A} \subseteq L + \mathcal{A}L$ and $\mathcal{A}L \subseteq L + L\mathcal{A}$. Hence

$$\mathrm{Id}(L) = L + \mathcal{A}L = L + L\mathcal{A}, \tag{2.3}$$ e2.1

and

$$\mathcal{A}L\mathcal{A} \subseteq \mathcal{A}L \cap L\mathcal{A}, \text{ so } \mathcal{A}L \text{ and } L\mathcal{A} \text{ are ideals of } \mathcal{A}. \tag{2.4}$$ e2.2

Let us point out that if L, L' are Lie ideals of $\mathcal{A}$, then $[L, L']$ is again a Lie ideal (just use the Jacobi identity to check this). This will be often used in the sequel.

L0 **Lemma 2.1** (i) *For a Lie ideal L and an ideal J of $\mathcal{A}$ conditions in Definition* 1.5 *satisfy the implication* (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(ii) *If $\mathcal{A} = [\mathcal{A}, \mathcal{A}] + \mathfrak{Z}_{\mathcal{A}}$ then conditions* (i), (ii), (iii) *in Definition* 1.5 *are equivalent.*

Proof. Part (i) is evident. Assume that $\mathcal{A} = [\mathcal{A}, \mathcal{A}] + \mathfrak{Z}_{\mathcal{A}}$ and show that condition (i) of Definition 1.5 implies condition (iii). If L is related to J then $[J, \mathcal{A}] \subseteq L$ and $[L, \mathcal{A}] \subseteq J$. By Jacobi identity,

$$[L, \mathcal{A}] = [L, [\mathcal{A}, \mathcal{A}] + \mathfrak{Z}_{\mathcal{A}}] = [L, [\mathcal{A}, \mathcal{A}]] \subseteq [[L, \mathcal{A}], \mathcal{A}] \subseteq [J, \mathcal{A}].$$

Similarly $[J, \mathcal{A}] \subseteq [L, \mathcal{A}]$. Hence $[J, \mathcal{A}] = [L, \mathcal{A}]$. ■

It should be noted that the condition $\mathcal{A} = [\mathcal{A}, \mathcal{A}] + \mathfrak{Z}_{\mathcal{A}}$ holds for many important classes of pure and Banach algebras (for example, all W*-algebras satisfy this condition (see (5.18)).

The next four results are basically due to Herstein, at least they can be extracted from his arguments [H2, pp. 4-5].

P1 **Proposition 2.2** *If L is a Lie ideal of $\mathcal{A}$ then $N(L)$ is a subalgebra and a Lie ideal of $\mathcal{A}$ and*

$$\mathrm{Id}([L, L]) \subseteq \mathrm{Id}([N(L), N(L)]) \subseteq N(L). \tag{2.5}$$ 5

Proof. For $x, y \in N(L)$ and $a \in \mathcal{A}$, we have from (2.1)

$$[xy, a] = [x, ya] + [y, ax] \in L.$$

Hence $xy \in N(L)$, so $N(L)$ is a subalgebra of $\mathcal{A}$. It follows from the definition of $N(L)$ that

$$[N(L), \mathcal{A}] \subseteq L. \tag{2.6}$$ 3

As $L \subseteq N(L)$, (2.6) in particular shows that $N(L)$ is a Lie ideal of $\mathcal{A}$.

As $L \subseteq N(L)$, we only have to prove that $\mathrm{Id}([N(L), N(L)]) \subseteq N(L)$. Further, since $N(L)$ is a Lie ideal of $\mathcal{A}$, so is $[N(L), N(L)]$, and hence in view of (2.3) it suffices to show that $\mathcal{A}[N(L), N(L)] \subseteq$

$N(L)$. Given $a \in \mathcal{A}$ and $x, y \in N(L)$, it follows from (2.2) that $a[x,y] = [ax,y] - [a,y]x$. As $N(L)$ is a Lie ideal and a subalgebra of $\mathcal{A}$, both terms in the right-hand side lie in $N(L)$. Whence $a[x,y] \in N(L)$. ■

Proposition 2.2 yields

contains **Corollary 2.3** *Each non-commutative Lie ideal L of $\mathcal{A}$ contains a Lie ideal of the form $[J,\mathcal{A}]$ where J is a non-zero ideal of $\mathcal{A}$. Moreover, one can take $J = \mathrm{Id}([L,L])$.*

The condition that L is non-commutative can be further weakened in *semiprime* algebras, that is, the algebras in which $a\mathcal{A}a = \{0\}$ implies $a = 0$.

P2 **Proposition 2.4** *If L is a commutative Lie ideal of a semiprime algebra $\mathcal{A}$, then $L \subseteq \mathfrak{Z}_{\mathcal{A}}$.*

Proof. For all $x \in L$ and $a \in \mathcal{A}$, $[x,[x,a]] = 0$. Fix x and write $d_x(a) = [x,a]$. Then $d_x^2(a) = 0$ for all $a \in \mathcal{A}$. Taking into account that, by (2.2), $d_x(ab) = ad_x(b) + d_x(a)b$ for $a, b \in \mathcal{A}$, we have

$$0 = d_x^2(ab) = ad_x^2(b) + 2d_x(a)d_x(b) + d_x^2(a)b = 2d_x(a)d_x(b).$$

As $\mathrm{char}(\mathbb{F}) \neq 2$, it follows that $d_x(a)d_x(b) = 0$ for all $a, b \in \mathcal{A}$. Hence, for all $c \in \mathcal{A}$,

$$d_x(a)cd_x(a) = d_x(a)d_x(ca) - d_x(a)d_x(c)a = 0.$$

Since $\mathcal{A}$ is semiprime, $d_x(a) = [x,a] = 0$ for all $a \in \mathcal{A}$. Thus $x \in \mathfrak{Z}_A$, so $L \subseteq \mathfrak{Z}_A$. ■

A subspace of $\mathcal{A}$ is called *central* if it is contained in $\mathfrak{Z}_{\mathcal{A}}$. Proposition 2.4 thus tells us that in semiprime algebras each non-central Lie ideal is non-commutative. Hence Corollary 2.3 may be extended from non-commutative Lie ideals to non-central ones as follows.

Herstein **Theorem 2.5** (Herstein) *Each non-central Lie ideal L of a semiprime algebra $\mathcal{A}$ contains a Lie ideal of the form $[J,\mathcal{A}]$ where J is a non-zero ideal of $\mathcal{A}$. Moreover, one can take $J = \mathrm{Id}([L,L])$.*

Note that Theorem 1.3 follows immediately from Proposition 2.4 and Theorem 2.5.

So we see that it is not too difficult to find Lie ideals $[J,\mathcal{A}]$, with a non-zero ideal J, inside a non-central Lie ideal L. However, this does not give us any information about how big is $[J,\mathcal{A}]$; in principle it could be a very small portion of L. Our goal is to find J so that $[J,\mathcal{A}]$ is as "close" to L as possible. The condition (1.2) that L is embraced by J can be equivalently written as

$$[L,\mathcal{A}] \subseteq [J,\mathcal{A}] \subseteq L. \tag{2.7}$$ impii

So the problem is to locate $[J,\mathcal{A}]$ between $[L,\mathcal{A}]$ and L. The nice structure of some of the algebras we shall consider will allow us to find J so that actually L is commutator equal to J: $[L,\mathcal{A}] = [J,\mathcal{A}]$.

The weakest condition (1.3) in which we are interested that L is related to J can be rewritten as

$$[J,\mathcal{A}] \subseteq L \text{ and } [L,\mathcal{A}] \subseteq J, \tag{2.8}$$ relii

so L and J appear symmetrically. Note that the set of ideals J satisfying this condition, for a fixed Lie ideal L, is closed under intersections and sums. Therefore, if L is related to some ideal J, then there exist the smallest ideal and the largest ideal to which L is related.

L1.0' **Lemma 2.6** *A Lie ideal L of $\mathcal{A}$ is related to an ideal of $\mathcal{A}$ if and only if* $\mathrm{Id}([L,\mathcal{A}]) \subseteq N(L)$. *Moreover, in this case* $\mathrm{Id}([L,\mathcal{A}])$ *is the smallest ideal to which L is related.*

Proof. If L is related to an ideal J, then we see from the second relation in (2.8) that $\mathrm{Id}([L,\mathcal{A}]) \subseteq J$. So the first relation implies that $\mathrm{Id}([L,\mathcal{A}]) \subseteq N(L)$. The converse is obvious. Also, it is clear that $\mathrm{Id}([L,\mathcal{A}])$ is the smallest ideal related to L. ■

Thus, in order to show that a Lie ideal L is related to some ideal, one just has to check that

$$[\mathrm{Id}([L,\mathcal{A}]),\mathcal{A}] \subseteq L. \tag{2.9}$$ reliii

As $[[L,\mathcal{A}],\mathcal{A}] \subseteq L$ and since, by (2.3), $\mathrm{Id}([L,\mathcal{A}]) = [L,\mathcal{A}] + [L,\mathcal{A}]\mathcal{A}$, (2.9) is equivalent to

$$[[L,\mathcal{A}]\mathcal{A},\mathcal{A}] \subseteq L. \tag{2.10}$$ reliv

T1 **Proposition 2.7** *For a Lie ideal L of $\mathcal{A}$, set* $I_L = \mathrm{Id}([L,L])$.
(i) *If $[L,\mathcal{A}] \subseteq I_L$, then L is related to an ideal.*
(ii) *If the quotient algebra $\mathcal{A}/I_L$ is semiprime or commutative, then $[L,\mathcal{A}] \subseteq I_L$.*

Proof. By Proposition 2.2, $I_L \subseteq N(L)$, so part (i) follows from Lemma 2.6.
Let θ be the canonical map from $\mathcal{A}$ onto $\mathcal{A}/I_L$. Then $\theta(L)$ is a Lie ideal of $\mathcal{A}/I_L$ and $[\theta(L),\theta(L)] = \theta([L,L]) = \{0\}$. If $\mathcal{A}/I_L$ is semiprime then, by Proposition 2.4, $\theta(L)$ lies in the centre of $\mathcal{A}/I_L$. Hence $[L,\mathcal{A}] \subseteq I_L$. If $\mathcal{A}/I_L$ is commutative, $[L,\mathcal{A}] \subseteq [\mathcal{A},\mathcal{A}] \subseteq I_L$. ■

It should be noted that, even for ideals I, the inclusion $[I,\mathcal{A}] \subseteq \mathrm{Id}([I,I])$ does not always hold. For example, in the algebra $\mathcal{A}$ of all upper triangular 2×2 matrices over $\mathbb{F}$, consider the ideal $I = [\mathcal{A},\mathcal{A}]$. Then $[I,I] = \{0\}$ but $[I,\mathcal{A}] \neq \{0\}$.

The condition (2.10) is rather concrete and so the problem of showing that a Lie ideal is related to an ideal is quite accessible. However, the problem of showing that a Lie ideal L is embraced by some ideal is, in general, much harder, since there does not seem to be a way of expressing this condition by a simple formula involving only L and $\mathcal{A}$; that is, the ideal J in (2.7) is "unknown", while in (2.8) we know that $\mathrm{Id}([L,\mathcal{A}])$ is the natural candidate for J.

If a Lie ideal L is related to some ideal, one can then ask whether this ideal is *unique*. In other words, is $\mathrm{Id}([L,\mathcal{A}])$ the *only* ideal to which L is related? The answer is clearly negative in commutative algebras, and, in view of Herstein's theorem, clearly positive in non-commutative simple algebras. A more interesting example where the answer is positive is $B(H)$. Namely, Fong and Murphy showed in [FM] the uniqueness of the ideal J in the Fong-Miers-Sourour theorem (see (1.1)). This uniqueness problem will also be one of the issues in this paper.

The next lemma shows that if one can describe Lie ideals of $\mathcal{A}$ through ideals of $\mathcal{A}$, then the same description carries over to homomorphic images of $\mathcal{A}$.

L5' **Lemma 2.8** *Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$. If each Lie ideal of $\mathcal{A}$ is embraced by* (*related to, commutator equal to*) *an ideal of $\mathcal{A}$, then all Lie ideals of $\mathcal{B}$ have the same property.*

Proof. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a surjective algebra homomorphism. If M is a Lie ideal of $\mathcal{B}$, then $L = \pi^{-1}(M)$ is a Lie ideal of $\mathcal{A}$. Suppose that L is embraced by an ideal J of $\mathcal{A}$, so that (2.7) holds. Since $\pi(L) = M$ and $K = \pi(J)$ is an ideal of $\mathcal{B}$, applying π to (2.7) we get that $[M,\mathcal{B}] \subseteq [K,\mathcal{B}] \subseteq M$. Thus M is embraced by K.

The other two conditions can be handled similarly. ■

We end this section with examples showing the independence and the non-triviality of the concepts that were introduced. In view of Lemma 2.8 it is natural to search for such examples in free algebras (since every algebra is a homomorphic image of a free algebra).

E1.1 **Example 2.9** *Let $\mathcal{A} = \mathbb{F}\langle x, y, z, w\rangle$ be the free algebra in x, y, z, w. Then the Lie ideal*

$$L = \mathbb{F}x + [x, \mathcal{A}] + [[x, \mathcal{A}], \mathcal{A}] + [[[x, \mathcal{A}], \mathcal{A}], \mathcal{A}] + \dots$$

of $\mathcal{A}$ generated by x is not related to any ideal.

Proof. Suppose that L is related to some ideal. Then $\mathrm{Id}([L, \mathcal{A}]) \subseteq N(L)$ by Lemma 2.6, so (2.10) holds. In particular, $[[x, y]z, w] \in L$. Note that this is possible only if there are $\lambda_i \in \mathbb{F}$ such that

$$\begin{aligned}
[[x, y]z, w] \quad = \quad & \lambda_1[x, yzw] + \lambda_2[x, ywz] + \lambda_3[x, zyw] + \lambda_4[x, zwy] \\
& +\lambda_5[x, wyz] + \lambda_6[x, wzy] + \lambda_7[[x, y], zw] + \lambda_8[[x, y], wz] \\
& +\lambda_9[[x, z], yw] + \lambda_{10}[[x, z], wy] + \lambda_{11}[[x, w], yz] + \lambda_{12}[[x, w], zy] \\
& +\lambda_{13}[[x, yz], w] + \lambda_{14}[[x, zy], w] + \lambda_{15}[[x, yw], z] + \lambda_{16}[[x, wy], z] \\
& +\lambda_{17}[[x, zw], y] + \lambda_{18}[[x, wz], y] + \lambda_{19}[[[x, y], z], w] + \lambda_{20}[[[x, y], w], z] \\
& +\lambda_{21}[[[x, z], y], w] + \lambda_{22}[[[x, z], w], y] + \lambda_{23}[[[x, w], y], z] + \lambda_{24}[[[x, w], z], y].
\end{aligned}$$

Comparing the coefficients at the monomials $wyxz$, $zxwy$, $ywxz$ and $zxyw$, respectively, we get

$$\begin{aligned}
-\lambda_{10} - \lambda_{16} + \lambda_{19} + \lambda_{20} + \lambda_{21} &= 1, \\
-\lambda_{10} - \lambda_{16} - \lambda_{22} - \lambda_{23} - \lambda_{24} &= 0, \\
-\lambda_9 - \lambda_{15} + \lambda_{22} + \lambda_{23} + \lambda_{24} &= 0, \\
-\lambda_9 - \lambda_{15} - \lambda_{19} - \lambda_{20} - \lambda_{21} &= 0.
\end{aligned}$$

¿From the first two identities we infer that

$$\lambda_{19} + \lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1,$$

while the last two identities imply that

$$\lambda_{19} + \lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 0.$$

This contradiction shows that L is not related to any ideal of $\mathcal{A}$. ■

E1.2 **Example 2.10** *Let $\mathcal{A} = \mathbb{F}\langle x, y, z\rangle$ be the free algebra in x, y, z. Denote by $\mathcal{A}_n$ the linear span of all monomials of degree greater or equal to n. Then*

$$L = \mathbb{F}x + [x, \mathcal{A}] + [[x, \mathcal{A}], \mathcal{A}] + \mathcal{A}_4$$

is a Lie ideal of $\mathcal{A}$ related to an ideal, but not embraced by any ideal.

Proof. It is clear that L is a Lie ideal of $\mathcal{A}$. As $\mathcal{A} = \mathbb{F}\mathbf{1} + \mathcal{A}_1$ and $[L, \mathcal{A}] \subseteq \mathcal{A}_2$,

$$[[L,\mathcal{A}]\mathcal{A},\mathcal{A}] \subseteq [[L,\mathcal{A}],\mathcal{A}] + [[L,\mathcal{A}]\mathcal{A}_1,\mathcal{A}] \subseteq L + [[L,\mathcal{A}]\mathcal{A}_1,\mathcal{A}_1] \subseteq L + \mathcal{A}_4 = L,$$

so L satisfies (2.10) which means that it is related to an ideal (see Lemma 2.6). Suppose that L is embraced by an ideal J. Pick an arbitrary element $u \in J$ and write $u = \alpha + \beta x + \gamma y + \delta z + u_2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{F}$ and $u_2 \in \mathcal{A}_2$. We have

$$\beta[x,y] + \delta[z,y] + [u_2,y] = [u,y] \in [J,\mathcal{A}] \subseteq L,$$

from which we easily infer that $\delta = 0$. Similarly we see that $\gamma = 0$. Hence

$$\alpha[y,z] + \beta[xy,z] + [u_2y,z] = [uy,z] \in [J,\mathcal{A}] \subseteq L.$$

As $[u_2y,z] \in \mathcal{A}_4 \subseteq L$, it follows that $\alpha[y,z] + \beta[xy,z] \in L$, whence $\alpha = 0$. Suppose $\beta \neq 0$. Then $[xy,z] \in L$. It is easy to see that then we must have

$$[xy,z] = \lambda_1[x,yz] + \lambda_2[x,zy] + \lambda_3[[x,y],z] + \lambda_4[[x,z],y]$$

for some $\lambda_i \in \mathbb{F}$. However, comparing the coefficients at the monomials zxy and yxz we obtain $-1 = -\lambda_3 - \lambda_4$ and $0 = -\lambda_3 - \lambda_4$, which is clearly impossible. Thus $\beta = 0$ as well, and so $u = u_2$. That is to say, $J \subseteq \mathcal{A}_2$. However, this yields $[x,y] \in [L,\mathcal{A}] \subseteq [J,\mathcal{A}] \subseteq \mathcal{A}_3$ – a contradiction. Thus L is not embraced by any ideal. ■

E1.3 **Example 2.11** *Let $\mathcal{A}$ be the algebra of all strictly upper triangular 3×3 matrices over $\mathbb{F}$, that is, $\mathcal{A} = \mathbb{F}e_{12} + \mathbb{F}e_{13} + \mathbb{F}e_{23}$, where e_{ij} denote matrix units. Then $L = \mathbb{F}e_{12}$ is a linear space related to an ideal and commutator equal to an ideal, but is not a Lie ideal.*

Proof. Clearly, L is not a Lie ideal, as $[L,\mathcal{A}] = \mathbb{F}e_{13}$ does not lie in L. As $[\mathcal{A},\mathcal{A}] = \mathbb{F}e_{13}$, L is commutator equal to $\mathcal{A}$. Finally, L is related to the ideal $J = \mathbb{F}e_{13}$. ■

3 Lie ideals in algebras with idempotents.

The presence of an idempotent p_1 in a unital algebra $\mathcal{A}$ makes it possible to reduce the study of the relation between Lie ideals and ideals in $\mathcal{A}$ to the study of this relation in the "block-diagonal subalgebra" $\mathcal{A}_d = p_1\mathcal{A}p_1 + p_2\mathcal{A}p_2$ of $\mathcal{A}$ with $p_2 = \mathbf{1} - p_1$. We will show that, under some mild conditions on p_1, a Lie ideal L of $\mathcal{A}$ is related to (embraced by) an ideal of $\mathcal{A}$ if and only if the same is true for the Lie ideal $L_d = L \cap \mathcal{A}_d$ of $\mathcal{A}_d$. This link, interesting by itself, becomes crucial for the study of Lie ideals of W*-algebras (see Section 5).

Firstly we will consider the simplest case when $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2$ is the direct sum of subalgebras. (In this case we do not require that $\mathcal{A}$ contains idempotents p_i such that $\mathcal{A}_i = p_i\mathcal{A}$. However, if $\mathcal{A}$ has such idempotents then they are central.) If all Lie ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$ are embraced by ideals, then Lie ideals of $\mathcal{A}$ are not necessarily embraced by ideals of $\mathcal{A}$. If, however, all Lie ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutator equal to ideals, then all Lie ideals of $\mathcal{A}$ are also commutator equal to ideals.

P1.4 **Proposition 3.1** *Let $\mathcal{A} = \mathcal{A}_1 \dotplus \mathcal{A}_2$ be the direct sum of algebras $\mathcal{A}_1$ and $\mathcal{A}_2$.*

(i) *Each Lie ideal of $\mathcal{A}$ is commutator equal to an ideal of $\mathcal{A}$ if and only if all Lie ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutator equal to ideals.*

(ii) *Let $\mathcal{A}_1$ or $\mathcal{A}_2$ be unital. Let all Lie ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$ be commutator equal to ideals. Then each Lie ideal of $\mathcal{A}$ is only related to one ideal of $\mathcal{A}$ if and only if each Lie ideal of $\mathcal{A}_1$ and $\mathcal{A}_2$ is only related to one ideal.*

Proof. Denote by P_i the projections on the subalgebras $\mathcal{A}_i$ in $\mathcal{A}$: $P_i(x_1 + x_2) = x_i$, for all $x_i \in \mathcal{A}_i$. (If $\mathcal{A}$ contains idempotents p_i such that $\mathcal{A}_i = p_i\mathcal{A}$, then $P_i(x_1+x_2) = p_ix_1+p_ix_2$). Assume that each Lie ideal of $\mathcal{A}$ is commutator equal to an ideal of $\mathcal{A}$ and show that $\mathcal{A}_1, \mathcal{A}_2$ have the same property. Every Lie ideal L_1 of $\mathcal{A}_1$ is a Lie ideal of $\mathcal{A}$. So there is an ideal J of $\mathcal{A}$ such that $[L_1, \mathcal{A}] = [J, \mathcal{A}]$. As $[L_1, \mathcal{A}] = P_1[L_1, \mathcal{A}] = [L_1, \mathcal{A}_1]$, we have $[L_1, \mathcal{A}_1] = P_1[L_1, \mathcal{A}] = P_1[J, \mathcal{A}] = [J_1, \mathcal{A}_1]$, where $J_1 = P_1J$ is an ideal of $\mathcal{A}_1$. The "only if" part is proved.

Let L be a Lie ideal of $\mathcal{A}$. Then $L_i = P_iL$ is a Lie ideal of $\mathcal{A}_i$,

$$L \subseteq L_1 \dotplus L_2 \text{ and } [L, \mathcal{A}] = [L_1, \mathcal{A}_1] \dotplus [L_2, \mathcal{A}_2]. \tag{3.1}$$ 1.10

If all Lie ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutator equal to ideals, then

$$[J_1, \mathcal{A}_1] = [L_1, \mathcal{A}_1] \text{ and } [J_2, \mathcal{A}_2] = [L_2, \mathcal{A}_2],$$

for some ideals J_i of $\mathcal{A}_i$. Then $J = J_1 \dotplus J_2$ is an ideal of $\mathcal{A}$ and, by (3.1),

$$[J, \mathcal{A}] = [J_1, \mathcal{A}_1] \dotplus [J_2, \mathcal{A}_2] = [L_1, \mathcal{A}_1] \dotplus [L_2, \mathcal{A}_2] = [L, \mathcal{A}].$$

Part (i) is proved.

Suppose now that $\mathcal{A}_1$ is unital and that each Lie ideal of the algebra $\mathcal{A}_1$ and of $\mathcal{A}_2$ is commutator equal to an ideal and is only related to one ideal. Let L be a Lie ideal of $\mathcal{A}$ and let $J = J_1 \dotplus J_2$ be the ideal constructed above that commutator equal to L. Then J is related to L. Assume that there is another ideal I related to L, that is, $[L, \mathcal{A}] \subseteq I$ and $[I, \mathcal{A}] \subseteq L$. As $\mathcal{A}_1$ is unital, the ideals $I_1 = P_1I = \mathbf{1}_{\mathcal{A}_1}I$ of $\mathcal{A}_1$ and $I_2 = P_2I$ of $\mathcal{A}_2$ lie in I and $I = I_1 \dotplus I_2$. By (3.1), we have

$$\begin{aligned} [L_1, \mathcal{A}_1] \dotplus [L_2, \mathcal{A}_2] &= [L, \mathcal{A}] \subseteq I = I_1 \dotplus I_2, \\ [I_1, \mathcal{A}_1] \dotplus [I_2, \mathcal{A}_2] &= [I, \mathcal{A}] \subseteq L \subseteq L_1 \dotplus L_2. \end{aligned}$$

Hence $[L_i, \mathcal{A}_i] \subseteq I_i$ and $[I_i, \mathcal{A}_i] \subseteq L_i$ for $i = 1, 2$. Thus L_1 is related to I_1. The Lie ideal L_1 is also related to the ideal J_1. As L_1 is only related to one ideal, $I_1 = J_1$. Similarly, $I_2 = J_2$. Hence $I = J$, so L is only related to one ideal.

On the other hand, if a Lie ideal M of $\mathcal{A}_1$ is related to ideals I_1 and I_1' of $\mathcal{A}_1$, then M considered as a Lie ideal of $\mathcal{A}$ is related to the ideals I_1 and I_1' considered as ideals of $\mathcal{A}$. ∎

We will consider now the general case of a unital algebra $\mathcal{A}$ with an idempotent p_1. We do not assume that p_1 and $p_2 = \mathbf{1} - p_1$ are equivalent, so $\mathcal{A}$ is not, generally speaking, isomorphic to $M_2(\mathcal{B})$. For a linear subspace L of $\mathcal{A}$, set $L_{ij} = p_iLp_j$. The subspaces L_{ij} of $\mathcal{A}$ do not necessarily lie in L. We have

$$\mathcal{A}_{ii}\mathcal{A}_{ij} = \mathcal{A}_{ij}, \ \mathcal{A}_{ij}\mathcal{A}_{jk} \subseteq \mathcal{A}_{ik} \text{ and } \mathcal{A}_{ij}\mathcal{A}_{km} = \{0\} \text{ if } j \neq k, \tag{3.2}$$

$$[\mathcal{A}_{ij}, L_{jk}] = \mathcal{A}_{ij}L_{jk} \subseteq \mathcal{A}_{ik} \text{ if } i \neq k. \tag{3.3}$$

A linear subspace J of $\mathcal{A}$ is an ideal if and only if

$$J = J_{11} + J_{12} + J_{21} + J_{22}, \text{ with } J_{ij}\mathcal{A}_{jk} \subseteq J_{ik} \text{ and } \mathcal{A}_{ij}J_{jm} \subseteq J_{im}. \tag{3.4}$$ 1.4

Indeed, if J is an ideal, then $J_{ij} = p_iJp_j \subseteq J$ and $J_{ij}\mathcal{A}_{jk} \subseteq J_{ik}$, $\mathcal{A}_{ij}J_{jm} \subseteq J_{im}$. For each $x \in J$, $x = p_1xp_1 + p_1xp_2 + p_2xp_1 + p_2xp_2$, so (3.4) holds. The converse is evident.

L1.5 **Lemma 3.2** (i) *A linear subspace L is a Lie ideal of $\mathcal{A}$ if and only if*

$$L = L_{12} + L_{21} + L_d \textit{ where } L_d = L \cap (\mathcal{A}_{11} + \mathcal{A}_{22}) \subseteq L_{11} + L_{22}, \tag{3.5}$$ 1.5

$$L_{ij}\mathcal{A}_{jj} = [L_{ij}, \mathcal{A}_{jj}] = L_{ij} = \mathcal{A}_{ii}L_{ij} = [\mathcal{A}_{ii}, L_{ij}] \textit{ for } i \neq j, \tag{3.6}$$ 1.6

$$[L_d, \mathcal{A}_{11} + \mathcal{A}_{22}] = [L_{11}, \mathcal{A}_{11}] + [L_{22}, \mathcal{A}_{22}] \subseteq L_d, \tag{3.7}$$ 1.7

$$[L_{ij}, \mathcal{A}_{ji}] \subseteq L_d, \ [L_d, \mathcal{A}_{ij}] \subseteq L_{ij} \textit{ for } i \neq j. \tag{3.8}$$ 1.8

(ii) *If L is a Lie ideal of $\mathcal{A}$ then*

$$[L, \mathcal{A}] = L_{12} + L_{21} + [L_{11}, \mathcal{A}_{11}] + [L_{22}, \mathcal{A}_{22}] + [L_{12}, \mathcal{A}_{21}] + [L_{21}, \mathcal{A}_{12}].$$

Proof. Let L be a Lie ideal of $\mathcal{A}$. For each $x \in L$,

$$p_1xp_2 = \frac{1}{2}([p_1, [p_1, x]] + [p_1, x]) \in L \text{ and } p_2xp_1 = \frac{1}{2}([p_1, [p_1, x]] - [p_1, x]) \in L.$$

Hence $L_{12} + L_{21} \subseteq [L, \mathcal{A}] \subseteq L$. Therefore (3.5) holds.

As $[L_{ij}, \mathcal{A}_{jj}] \subseteq L_{ij}$ and $\mathcal{A}_{jj}$ are unital, $L_{ij}\mathcal{A}_{jj} = [L_{ij}, \mathcal{A}_{jj}] = L_{ij}$ and $\mathcal{A}_{ii}L_{ij} = [\mathcal{A}_{ii}, L_{ij}] = L_{ij}$, for $i \neq j$, so (3.6) holds.

Conditions (3.7) follow from the fact that $L_d \subseteq L_{11} + L_{22}$ and $[L_d, \mathcal{A}_{ii}] = [L_{ii}, \mathcal{A}_{ii}]$.

As $L_{ij}\mathcal{A}_{ji} \subseteq \mathcal{A}_{ii}$ and $[L_{ij}, \mathcal{A}_{ji}] \subseteq L$, for $i \neq j$, we have $[L_{ij}, \mathcal{A}_{ji}] \subseteq L \cap (\mathcal{A}_{11} + \mathcal{A}_{22}) = L_d$. We also have $[L_d, \mathcal{A}_{ij}] \subseteq L \cap \mathcal{A}_{ij} = L_{ij}$, so (3.8) holds.

Conversely, let (3.5)-(3.8) hold. Then

$$[L, \mathcal{A}_{11}] = [L_{12}, \mathcal{A}_{11}] + [L_{21}, \mathcal{A}_{11}] + [L_d, \mathcal{A}_{11}] = L_{12} + L_{21} + [L_{11}, \mathcal{A}_{11}] \subseteq L.$$

Similarly, $[L, \mathcal{A}_{22}] \subseteq L$, $[L, \mathcal{A}_{12}] \subseteq L$, $[L, \mathcal{A}_{21}] \subseteq L$, so L is a Lie ideal of $\mathcal{A}$. Part (i) is proved.

Making use of (3.5)-(3.8), we prove part (ii). ■

We continue to assume that L is a Lie ideal of $\mathcal{A}$. Set

$$\mathcal{A}_d = \mathcal{A}_{11} + \mathcal{A}_{22}, \quad \mathcal{A}_c = \mathcal{A}_{12} + \mathcal{A}_{21}, \ L_c = L_{12} + L_{21}.$$

As $\mathcal{A}$ is unital, it follows from (3.2)-(3.8) that

$$\mathcal{A}_d\mathcal{A}_d = \mathcal{A}_d, \quad \mathcal{A}_d\mathcal{A}_c = \mathcal{A}_c\mathcal{A}_d = \mathcal{A}_c, \quad \mathcal{A}_c\mathcal{A}_c \subseteq \mathcal{A}_d, \tag{3.9}$$

$$L_c\mathcal{A}_d = \mathcal{A}_dL_c = [L_c, \mathcal{A}_d] = L_c, \quad [L_d, \mathcal{A}_c] \subseteq L_c. \tag{3.10}$$

$$[L_d, \mathcal{A}_d] = [L_{11}, \mathcal{A}_{11}] + [L_{22}, \mathcal{A}_{22}] \subseteq L_d, \tag{3.11}$$

$$[L_c, \mathcal{A}_c] = [L_{12}, \mathcal{A}_{21}] + [L_{21}, \mathcal{A}_{12}] \subseteq L_d. \tag{3.12}$$

We have that

$$\begin{aligned}
\mathcal{A}_c[L_d,\mathcal{A}_d] &= \mathcal{A}_{12}[L_{22},\mathcal{A}_{22}]+\mathcal{A}_{21}[L_{11},\mathcal{A}_{11}]=[\mathcal{A}_{12},[L_{22},\mathcal{A}_{22}]]+[\mathcal{A}_{21},[L_{11},\mathcal{A}_{11}]]\\
&\subseteq [\mathcal{A}_{12},L_d]+[\mathcal{A}_{21},L_d]=[\mathcal{A}_c,L_d]\subseteq L_c,\\
[L_d,\mathcal{A}_d]\mathcal{A}_c &= [L_{22},\mathcal{A}_{22}]\mathcal{A}_{21}+[L_{11},\mathcal{A}_{11}]\mathcal{A}_{12}\\
&= [[L_{22},\mathcal{A}_{22}],\mathcal{A}_{21}]+[[L_{11},\mathcal{A}_{11}],\mathcal{A}_{12}]\subseteq L_c.
\end{aligned}\tag{3.13}$$

We also have

$$\begin{aligned}
[L_c\mathcal{A}_c,\mathcal{A}_d] &= [L_{12}\mathcal{A}_{21}+L_{21}\mathcal{A}_{12},\mathcal{A}_{11}+\mathcal{A}_{22}]=[L_{12}\mathcal{A}_{21},\mathcal{A}_{11}]+[L_{21}\mathcal{A}_{12},\mathcal{A}_{22}]\\
&= [[L_{12},\mathcal{A}_{21}],\mathcal{A}_{11}]+[[L_{21},\mathcal{A}_{12}],\mathcal{A}_{22}]\subseteq [L_d,\mathcal{A}_d].
\end{aligned}$$

Similarly, $[\mathcal{A}_cL_c,\mathcal{A}_d]\subseteq[L_d,\mathcal{A}_d]$. Hence

$$[L_c\mathcal{A}_c+\mathcal{A}_cL_c,\mathcal{A}_d]\subseteq[L_d,\mathcal{A}_d]. \tag{3.14}$$ e1.9

Set $I=L_c+L_c\mathcal{A}_c+\mathcal{A}_cL_c$, $\mathcal{M}=\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d$ and $J=I+\mathcal{M}$.

P1.7 **Proposition 3.3** *If $\mathcal{A}_cL_c\mathcal{A}_c\subseteq L_c$, then $J=\mathrm{Id}([L,\mathcal{A}])$ and*

$$[J,\mathcal{A}]=L_c+[L_c,\mathcal{A}_c]+[L_c\mathcal{A}_c+\mathcal{A}_cL_c,\mathcal{A}_d]+[\mathcal{M},\mathcal{A}_d]. \tag{3.15}$$ f2

Proof. As $\mathcal{A}_cL_c\mathcal{A}_c\subseteq L_c$, it follows from (3.9), (3.10), (3.13) that

$$\begin{aligned}
\mathcal{A}_dI &= \mathcal{A}_dL_c+\mathcal{A}_dL_c\mathcal{A}_c+\mathcal{A}_d\mathcal{A}_cL_c=L_c+L_c\mathcal{A}_c+\mathcal{A}_cL_c=I,\\
\mathcal{A}_cI &= \mathcal{A}_cL_c+\mathcal{A}_cL_c\mathcal{A}_c+\mathcal{A}_c\mathcal{A}_cL_c\subseteq\mathcal{A}_cL_c+L_c+\mathcal{A}_dL_c=\mathcal{A}_cL_c+L_c\subseteq I,\\
\mathcal{A}_d\mathcal{M} &= \mathcal{A}_d\mathcal{A}_d[\mathcal{A}_d,L_d]\mathcal{A}_d=\mathcal{A}_d[\mathcal{A}_d,L_d]\mathcal{A}_d=\mathcal{M},\\
\mathcal{A}_c\mathcal{M} &= \mathcal{A}_c\mathcal{A}_d[\mathcal{A}_d,L_d]\mathcal{A}_d=\mathcal{A}_c[\mathcal{A}_d,L_d]\mathcal{A}_d\subseteq L_c\mathcal{A}_d=L_c\subseteq I.
\end{aligned}$$

Thus $\mathcal{A}J\subseteq J$. Similarly, $J\mathcal{A}\subseteq J$, so J is an ideal of $\mathcal{A}$.

It follows from Lemma 3.2(ii) that $\mathrm{Id}([L,\mathcal{A}])\subseteq J$, since

$$[L,\mathcal{A}]=L_c+[L_c,\mathcal{A}_c]+[L_d,\mathcal{A}_d]\subseteq L_c+(L_c\mathcal{A}_c+\mathcal{A}_cL_c)+\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d=J. \tag{3.16}$$ f1

On the other hand, $I\subseteq\mathrm{Id}(L_c)$ and $L_c\subseteq[L,\mathcal{A}]$, so that $I\subseteq\mathrm{Id}([L,\mathcal{A}])$. We also have $\mathcal{M}=\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d\subseteq\mathrm{Id}([L,\mathcal{A}])$. Hence $J\subseteq\mathrm{Id}([L,\mathcal{A}])$. Thus $J=\mathrm{Id}([L,\mathcal{A}])$.

Let us now find $[J,\mathcal{A}]$. As $\mathcal{A}_cL_c\mathcal{A}_c\subseteq L_c$, we have from (3.9) and (3.10) that

$$[L_c\mathcal{A}_c,\mathcal{A}_c]\subseteq L_c\mathcal{A}_c\mathcal{A}_c+\mathcal{A}_cL_c\mathcal{A}_c\subseteq L_c\mathcal{A}_d+L_c=L_c.$$

Similarly, $[\mathcal{A}_cL_c,\mathcal{A}_c]\subseteq L_c$. Hence we have from (3.10) that

$$\begin{aligned}
[I,\mathcal{A}] &= [I,\mathcal{A}_d]+[I,\mathcal{A}_c]=[L_c,\mathcal{A}_d]+[L_c\mathcal{A}_c+\mathcal{A}_cL_c,\mathcal{A}_d]+[L_c,\mathcal{A}_c]+[L_c\mathcal{A}_c+\mathcal{A}_cL_c,\mathcal{A}_c]\\
&= L_c+[L_c\mathcal{A}_c+\mathcal{A}_cL_c,\mathcal{A}_d]+[L_c,\mathcal{A}_c].
\end{aligned}$$

Next, by (3.9), (3.10) and (3.13),

$$\begin{aligned}
[\mathcal{M},\mathcal{A}_c] &= [\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d,\mathcal{A}_c]\subseteq\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d\mathcal{A}_c+\mathcal{A}_c\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d\\
&= \mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_c+\mathcal{A}_c[L_d,\mathcal{A}_d]\mathcal{A}_d\subseteq\mathcal{A}_dL_c+L_c\mathcal{A}_d=L_c.
\end{aligned}$$

Therefore

$$[J,\mathcal{A}] = [I,\mathcal{A}] + [\mathcal{M},\mathcal{A}_c] + [\mathcal{M},\mathcal{A}_d] = L_c + [L_c,\mathcal{A}_c] + [L_c\mathcal{A}_c + \mathcal{A}_cL_c,\mathcal{A}_d] + [\mathcal{M},\mathcal{A}_d]$$

which completes the proof. ■

It should be noted that $\mathcal{M} = \mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d = \mathrm{Id}_{\mathcal{A}_d}([L_d,\mathcal{A}_d])$, so $\mathcal{M}$ is an ideal of $\mathcal{A}_d$.

T1.8 **Theorem 3.4** *Let L be a Lie ideal of $\mathcal{A}$. Set $J = \mathrm{Id}([L,\mathcal{A}])$.*

(i) *L is related to an ideal of $\mathcal{A}$ if and only if the Lie ideal L_d of $\mathcal{A}_d$ is related to an ideal of $\mathcal{A}_d$ and*

$$\mathcal{A}_cL_c\mathcal{A}_c \subseteq L_c. \tag{3.17}$$ f3

(ii) *Let $\mathcal{A}_cL_c\mathcal{A}_c \subseteq L_c$.*

1) *If $\mathrm{Id}_{\mathcal{A}_d}([L_d,\mathcal{A}_d])$ embraces L_d, then J embraces L.*

2) *If $\mathrm{Id}_{\mathcal{A}_d}([L_d,\mathcal{A}_d])$ and L_d are commutator equal, then J and L are commutator equal.*

Proof. Suppose that L is related to an ideal of $\mathcal{A}$. By Lemma 2.6, L is related to J. Hence $[J,\mathcal{A}] \subseteq L$. As $\mathcal{A}$ is unital, $J = \mathrm{Id}([L,\mathcal{A}]) = \mathcal{A}[L,\mathcal{A}]\mathcal{A}$, so $[\mathcal{A}[L,\mathcal{A}]\mathcal{A},\mathcal{A}] \subseteq L$. Hence, by (3.10),

$$[\mathcal{A}_{12}[L_c,\mathcal{A}_d]\mathcal{A}_{12},\mathcal{A}_{11}] = [\mathcal{A}_{12}L_c\mathcal{A}_{12},\mathcal{A}_{11}] = \mathcal{A}_{11}\mathcal{A}_{12}L_c\mathcal{A}_{12} = \mathcal{A}_{12}L_c\mathcal{A}_{12} \subseteq L.$$

Thus $\mathcal{A}_{12}L_c\mathcal{A}_{12} \subseteq L_c$. Similarly, $\mathcal{A}_{21}L_c\mathcal{A}_{21} \subseteq L_c$. Hence $\mathcal{A}_cL_c\mathcal{A}_c \subseteq L_c$.

Set $\mathcal{M} = \mathrm{Id}_{\mathcal{A}_d}([L_d,\mathcal{A}_d])$. We have

$$[\mathcal{A}_d[L_d,\mathcal{A}_d]\mathcal{A}_d,\mathcal{A}_d] = [\mathcal{M},\mathcal{A}_d] \subseteq L_d.$$

As $[L_d,\mathcal{A}_d] \subseteq \mathcal{M}$, the Lie ideal L_d is related to the ideal $\mathcal{M}$.

Conversely, let (3.17) hold and L_d be related to an ideal of $\mathcal{A}_d$. Then it follows from Lemma 2.6 that L_d is related to $\mathcal{M}$, so $[\mathcal{M},\mathcal{A}_d] \subseteq L_d$. It follows from (3.12) that $[L_c,\mathcal{A}_c] \subseteq L_d$ and from (3.14) that

$$[L_c\mathcal{A}_c + \mathcal{A}_cL_c,\mathcal{A}_d] \subseteq [L_d,\mathcal{A}_d] \subseteq L_d.$$

As $\mathcal{A}_cL_c\mathcal{A}_c \subseteq L_c$, we obtain from (3.15) that $[J,\mathcal{A}] \subseteq L$. As $[L,\mathcal{A}] \subseteq J$, we have that J and L are related. Part (i) is proved.

Let $\mathcal{M}$ implement L_d, that is, $[L_d,\mathcal{A}_d] \subseteq [\mathcal{M},\mathcal{A}_d] \subseteq L_d$. By (3.14), (3.15) and (3.16), $[L,\mathcal{A}] \subseteq [J,\mathcal{A}] \subseteq L$, so J embraces L. Part (ii) 1) is proved. Part 2) is proved similarly. ■

We will now study condition (3.17).

D4.4 **Definition 3.5** *An idempotent p_1 in a unital algebra $\mathcal{A}$ is called locally cyclic, if for $(i,j) = (1,2)$ and $(i,j) = (2,1)$ and for each pair $x,y \in \mathcal{A}_{ij}$, there is $z \in \mathcal{A}_{ij}$ such that $x,y \in \mathcal{A}_{ii}z\mathcal{A}_{jj}$.*

L1.9 **Lemma 3.6** *Let p_1 be an idempotent in a unital algebra $\mathcal{A}$ and let L be a Lie ideal of $\mathcal{A}$.*

(i) *If $\mathcal{A}_{12} = \{0\}$ then $\mathcal{A}_cL_c\mathcal{A}_c = \{0\} \subseteq L_c$.*

(ii) *If p_1 is locally cyclic then $\mathcal{A}_cL_c\mathcal{A}_c \subseteq L_c$.*

Proof. If $\mathcal{A}_{12} = \{0\}$ then $L_{12} = \{0\}$. Hence we obtain part (i):

$$\mathcal{A}_c L_c \mathcal{A}_c = \mathcal{A}_{12} L_{21} \mathcal{A}_{12} + \mathcal{A}_{21} L_{12} \mathcal{A}_{21} = \{0\} \subseteq L_c.$$

Let $z \in \mathcal{A}_{12}$ and $b \in L_{21}$. By (3.2), $z^2 = 0$. It follows from (3.8) that $[z, b] \in L_d$, so

$$zbz = \frac{1}{2}(-z^2 b + 2zbz - bz^2) = \frac{1}{2}[[z, b], z] \in L_{12}. \qquad (3.18) \quad \boxed{\texttt{f4}}$$

Let $x, y \in \mathcal{A}_{12}$ and $v \in L_{21}$. If p_1 is locally cyclic then there is $z \in \mathcal{A}_{12}$ such that $x, y \in \mathcal{A}_{11} z \mathcal{A}_{22}$. Hence $x = \sum a_i z b_i$ and $y = \sum c_j z d_j$, for some $a_i, c_j \in \mathcal{A}_{11}$ and $b_i, d_j \in \mathcal{A}_{22}$. Therefore

$$xvy = \left(\sum_i a_i z b_i\right) v \left(\sum_j c_j z d_j\right) = \sum_{i,j} a_i z (b_i v c_j) z d_j.$$

¿From (3.6) it follows that all $b_i v c_j \in L_{21}$. Hence, by (3.18), all $z(b_i v c_j)z \in L_{12}$. From (3.6) we have that all $a_i z(b_i v c_j) z d_j \in L_{12}$. Hence $xvy \in L_{12}$, so $\mathcal{A}_{12} L_{21} \mathcal{A}_{12} \subseteq L_{12}$. Similarly, $\mathcal{A}_{21} L_{12} \mathcal{A}_{21} \subseteq L_{21}$. Thus $\mathcal{A}_c L_c \mathcal{A}_c \subseteq L_c$. ■

Theorem 3.4 and Lemma 3.6 yield

$\boxed{\texttt{T1.10}}$ **Theorem 3.7** *Let p_1 be an idempotent in a unital algebra $\mathcal{A}$. Set $p_2 = \mathbf{1} - p_1$, $\mathcal{A}_{ij} = p_i \mathcal{A} p_j$ and $\mathcal{A}_d = \mathcal{A}_{11} + \mathcal{A}_{22}$. Suppose that p_1 is locally cyclic, or that $\mathcal{A}_{12} = \{0\}$. Let L be a Lie ideal of $\mathcal{A}$ and $L_d = L \cap \mathcal{A}_d$.*

(i) *L is related to an ideal of $\mathcal{A}$ if and only if the Lie ideal L_d of $\mathcal{A}_d$ is related to an ideal of $\mathcal{A}_d$.*

(ii) *If $\mathrm{Id}_{\mathcal{A}_d}([L_d, \mathcal{A}_d])$ embraces L_d, then $\mathrm{Id}([L, \mathcal{A}])$ embraces L.*

(iii) *If $\mathrm{Id}_{\mathcal{A}_d}([L_d, \mathcal{A}_d])$ and L_d are commutator equal, then $\mathrm{Id}([L, \mathcal{A}])$ and L are commutator equal.*

Theorem 3.7 describes the relation between Lie ideals and ideals of an algebra $\mathcal{A}$ in terms of this relation in the "block-diagonal subalgebra" $\mathcal{A}_d = \mathcal{A}_{11} + \mathcal{A}_{22}$ of $\mathcal{A}$. Part (iii) of Theorem 3.7 can be further refined to describe the relation between Lie ideals and ideals of $\mathcal{A}$ in terms of this relation in the subalgebras $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$.

$\boxed{\texttt{C1.10}}$ **Corollary 3.8** *Suppose that an idempotent p_1 in $\mathcal{A}$ is locally cyclic, or that $\mathcal{A}_{12} = \{0\}$. If all Lie ideals M of the algebras $\mathcal{A}_{ii}$, $i = 1, 2$, are commutator equal to the ideals $\mathrm{Id}([M, \mathcal{A}_{ii}])$ of $\mathcal{A}_{ii}$, then each Lie ideal L of $\mathcal{A}$ is commutator equal to the ideal $\mathrm{Id}([L, \mathcal{A}])$ of $\mathcal{A}$.*

Proof. By Lemma 3.6, $\mathcal{A}_c L_c \mathcal{A}_c \subseteq L_c$. If all Lie ideals M of $\mathcal{A}_{ii}$, $i = 1, 2$, are commutator equal to the ideals $\mathrm{Id}([M, \mathcal{A}_{ii}])$ of $\mathcal{A}_{ii}$, then it follows from the proof of Proposition 3.1 that the Lie ideal L_d of the algebra $\mathcal{A}_d$ is commutator equal to the ideal $\mathrm{Id}([L_{11}, \mathcal{A}_{11}])+ \mathrm{Id}([L_{22}, \mathcal{A}_{22}]) = \mathrm{Id}_{\mathcal{A}_d}([L_d, \mathcal{A}_d])$ of $\mathcal{A}_d$. By Theorem 3.7(iii), L is commutator equal to the ideal $\mathrm{Id}([L, \mathcal{A}])$. ■

4 Lie ideals of tensor products of algebras.

We saw that all non-commutative Lie ideals of an algebra $\mathcal{A}$ contain Lie ideals of the form $[J,\mathcal{A}]$ and that this result can be extended to all non-central Lie ideals if $\mathcal{A}$ is semiprime. A similar result was obtained even earlier by Jacobson and Rickart for matrix algebras over an algebra: every non-central Lie ideal of $\mathcal{A} = \mathcal{B} \otimes M_n(\mathbb{F})$, where $\mathcal{B}$ is any unital algebra, contains a Lie ideal of the form $[J,\mathcal{A}]$, where J is a non-zero ideal of $\mathcal{A}$ [JR, Theorem 19]. Murphy [Mu] (for $n = 2$) and Marcoux [Ma] (for all n) proved that any Lie ideal L of $\mathcal{A}$ is related to an ideal J: $[J,\mathcal{A}] \subseteq L \subseteq N(J)$.

In this section we will firstly extend (in a much stronger form) the theorem of Jacobson and Rickart to the tensor products $\mathcal{A} = \mathcal{B}\otimes\mathcal{P}$ with $\mathcal{P}$ prime. Then we will obtain a precise description of Lie ideals of $\mathcal{B}\otimes\mathcal{P}$ for the case that $\mathcal{P}$ is simple and satisfies some additional conditions. Even in case $\mathcal{P} = M_n(\mathbb{F})$ it substantially strengthens the results of Murphy and Marcoux.

4.1 The case of prime algebras $\mathcal{P}$.

In this subsection $\mathcal{P}$ will be a *prime* non-commutative algebra over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. Recall that $\mathcal{P}$ is *prime* if $IJ \neq \{0\}$, for non-zero ideals I, J. Equivalently,

$$a\mathcal{P}b = \{0\}, \text{ for } a, b \in \mathcal{P}, \text{ implies } a = 0 \text{ or } b = 0. \tag{4.1}$$ 4.0

We shall need two results about derivations of prime algebras. The first one is well-known (see Posner [P]).

Lposner **Lemma 4.1** *If a derivation d of $\mathcal{P}$ is such that $[d(x), x] \in \mathfrak{Z}_{\mathcal{P}}$ for all $x \in \mathcal{P}$, then either $d = 0$, or $\mathcal{P}$ is commutative.*

It follows from Lemma 4.1 that, for each $p \notin \mathfrak{Z}_{\mathcal{P}}$, there is $x \in \mathcal{P}$ such that

$$[[p, x], x] \notin \mathfrak{Z}_{\mathcal{P}}, \text{ so } [p, x] \notin \mathfrak{Z}_{\mathcal{P}}. \tag{4.2}$$ d1

The second one is a special case of a result of Lanski [L, Theorem 4].

Llanski **Lemma 4.2** *If derivations d, h of $\mathcal{P}$ are such that $[d(x), h(x)] \in \mathfrak{Z}_{\mathcal{P}}$ for all $x \in \mathcal{P}$, then d and h are linearly dependent over the extended centroid.*

The reader is referred to the book [BMM] for a full account of the theory of the extended centroid and related notions. Let us just mention here that the extended centroid $C(\mathcal{P})$ of a prime $\mathbb{F}$-algebra $\mathcal{P}$ is a field containing $\mathbb{F}$ and the center $\mathfrak{Z}_{\mathcal{P}}$. A prime algebra $\mathcal{P}$ is called *centrally closed over* $\mathbb{F}$ if $C(\mathcal{P}) = \mathbb{F}$. If $\mathcal{P}$ is unital, this means that $C(\mathcal{P}) = \mathfrak{Z}_{\mathcal{P}} = \mathbb{F}\mathbf{1}$; if $\mathcal{P}$ is non-unital, then we have $\mathfrak{Z}_{\mathcal{P}} = \{0\}$. For example, a simple unital ring $\mathcal{P}$ is always centrally closed over its center $\mathfrak{Z}_{\mathcal{P}}$ (here it should be noted that $\mathfrak{Z}_{\mathcal{P}}$ is a field because of the simplicity of $\mathcal{P}$, and so we may consider $\mathcal{P}$ as an algebra over $\mathfrak{Z}_{\mathcal{P}}$). Further, primitive Banach algebras and prime C^*-algebras are centrally closed over $\mathbb{C}$. The only commutative centrally closed algebra is $\mathcal{P} = \mathbb{F}\mathbf{1}$.

Definition 4.3 *We call an algebra $\mathcal{B}$ locally unital if, for every finite subset $b = \{b_i\}$ of $\mathcal{B}$, there is a local identity $e_b \in \mathcal{B}$ such that $b_i e_b = e_b b_i = b_i$ for all b_i.*

There are many examples of non-unital, locally unital algebras: the algebra of functions with compact support on a topological space, the algebra of sequences with finite number of non-zero entries, the algebra of all infinite matrices with finite number of non-zero entries, the algebra of finite rank operators on a linear space etc. If $\mathcal{B}$ is locally unital then, for any subset X of $\mathcal{B}$,

$$X \subseteq X\mathcal{B} \text{ and } X \subseteq \mathcal{B}X. \tag{4.3}$$ e3.1

tensor **Proposition 4.4** *Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra and let $\mathcal{P}$ be a centrally closed non-commutative prime $\mathbb{F}$-algebra. Let L be a Lie ideal of $\mathcal{B} \otimes \mathcal{P}$ and let $a = \sum_{i=1}^{n} b_i \otimes p_i \in L$. If $p_1 \notin \mathfrak{Z}_{\mathcal{P}}$ then there exist $q \in [\mathcal{P}, \mathcal{P}]$, with $q \notin \mathfrak{Z}_{\mathcal{P}}$, and $\lambda_i \in \mathbb{F}$, $i = 2, ..., n$, such that*

$$b \otimes q \in L \text{ with } b = b_1 + \sum_{i=2}^{n} \lambda_i b_i.$$

Proof. Recall that $\mathfrak{Z}_{\mathcal{P}} = \mathbb{F}\mathbf{1}$, if $\mathcal{P}$ is unital, and $\mathfrak{Z}_{\mathcal{P}} = \{0\}$, if $\mathcal{P}$ is non-unital. We will use induction on n. If $n = 1$ then $a = b \otimes p$. For all $x \in \mathcal{P}$, we have $[b \otimes p, e_b \otimes x] = b \otimes [p, x] \in L$. As $p \notin \mathfrak{Z}_{\mathcal{P}}$, it follows from (4.2) that there is x such that $q = [p, x] \notin \mathfrak{Z}_{\mathcal{P}}$.

Let $1 < n$. If all $p_i - \lambda_i p_1 \in \mathfrak{Z}_{\mathcal{P}}$, for some $\lambda_i \in \mathbb{F}$, then

either 1) $a = b \otimes p_1$, if $\mathcal{P}$ is non-unital, or 2) $a = b \otimes p_1 + c \otimes \mathbf{1}$, if $\mathcal{P}$ is unital,

where $b = b_1 + \sum_{i=2}^{n} \lambda_i b_i$ and $b \neq 0$, if $\{b_i\}_{i=1}^{n}$ are linearly independent. Case 1) was considered above. In case 2), let e be the local identity of the set $\{b, c\}$ and let $x \in \mathcal{P}$ be such that $q = [p_1, x] \notin \mathfrak{Z}_{\mathcal{P}}$ (see (4.2)). Then $[a, e \otimes x] = b \otimes [p_1, x] = b \otimes q \in L$. Thus in both cases the result of the proposition holds.

If now, say, $p_n \notin \mathbb{F}p_1 + \mathfrak{Z}_{\mathcal{P}}$, then $x \mapsto [p_1, x]$ and $x \mapsto [p_n, x]$ are linearly independent derivations. Hence, by Lemma 4.2, $[[p_1, x], [p_n, x]] \notin \mathfrak{Z}_{\mathcal{P}}$ for some $x \in \mathcal{P}$. Set $q_i = [[p_i, x], [p_n, x]]$ and let e_B be a local identity for $B = \{b_i\}_{i=1}^{n}$. Then

$$\sum_{i=1}^{n-1} b_i \otimes q_i = [[\sum_{i=1}^{n} b_i \otimes p_i, e_B \otimes x], e_B \otimes [p_n, x]] = [\sum_{i=1}^{n} b_i \otimes [p_i, x], e_B \otimes [p_n, x]] \in L,$$

As $q_1 \notin \mathfrak{Z}_{\mathcal{P}}$ and in view of our assumption, the proof is complete. ■

centr **Corollary 4.5** *Let $\mathcal{B}$ and $\mathcal{P}$ be the same as in Proposition* 4.4 *and let $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$. If $\mathcal{P}$ is unital, then $\mathfrak{Z}_{\mathcal{A}} = \mathfrak{Z}_{\mathcal{B}} \otimes \mathbf{1}$. If $\mathcal{P}$ is non-unital, then $\mathfrak{Z}_{\mathcal{A}} = \{0\}$, so $\mathcal{A}$ has no central Lie ideals.*

Proof. Let $\mathcal{P}$ be unital. Suppose that $\mathfrak{Z}_{\mathcal{A}}$ is not contained in $\mathfrak{Z}_{\mathcal{B}} \otimes \mathbf{1}$. Since it is a Lie ideal, it follows from Proposition 4.4 that $\mathfrak{Z}_{\mathcal{A}}$ contains a non-zero element $b \otimes q$ for some $q \notin \mathbb{F}\mathbf{1}$. Hence there is $p \in \mathcal{P}$ with $[q, p] \neq 0$, so $[b \otimes q, e_b \otimes p] = b \otimes [q, p] \neq 0$, a contradiction. Thus $\mathfrak{Z}_{\mathcal{A}} = \mathfrak{Z}_{\mathcal{B}} \otimes \mathbf{1}$. The proof of the case when $\mathcal{P}$ is non-unital is similar. ■

We need now the following auxiliary results.

Lel **Lemma 4.6** *Let X and Y be Lie ideals of an algebra $\mathcal{A}$. Then the linear span K of all elements of the form $[[y, x], x]$, $x \in X$ and $y \in Y$, is also a Lie ideal of $\mathcal{A}$.*

Proof. Substituting $x + x'$ for x in $[[y, x], x]$, we see that K contains all elements $[[y, x], x'] + [[y, x'], x]$ with $y \in Y$, $x, x' \in X$. Consequently, for each $a \in \mathcal{A}$,

$$[[[y, x], x], a] = ([[y, x], [x, a]] + [[y, [x, a]], x]) + [[[y, a], x], x] \in K.$$

Thus K is a Lie ideal. ■

Le2 **Lemma 4.7** *Any non-zero ideal of a prime non-commutative algebra $\mathcal{P}$ is non-central.*

Proof. Let $J \neq \{0\}$ be an ideal of $\mathcal{P}$. If $J \subseteq \mathfrak{Z}_{\mathcal{P}}$ then $j[x, y] = [jx, y] = 0$ for $j \in J$, $x, y \in \mathcal{P}$. Hence $J[\mathcal{P}, \mathcal{P}] = \{0\}$ and so $J\mathcal{P}[\mathcal{P}, \mathcal{P}] = \{0\}$. As $[\mathcal{P}, \mathcal{P}] \neq \{0\}$, this contradicts (4.1). ■

t3 **Theorem 4.8** *Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra and let $\mathcal{P}$ be a centrally closed, non-commutative, locally unital prime $\mathbb{F}$-algebra. If L is a non-central Lie ideal of the algebra $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$, then there exist non-zero ideals U of $\mathcal{B}$ and V of $\mathcal{P}$ such that*

$$[V, \mathcal{P}] \neq \{0\} \text{ and } U \otimes [V, \mathcal{P}] + [U, \mathcal{B}] \otimes V \subseteq L.$$

Proof. First let us show that L contains an element $b \otimes p \neq 0$ with $p \notin \mathfrak{Z}_{\mathcal{P}}$. If $\mathcal{P}$ is non-unital, then $\mathfrak{Z}_{\mathcal{P}} = \{0\}$ and the result follows from Proposition 4.4. If $\mathcal{P}$ is unital then $\mathfrak{Z}_{\mathcal{P}} = \mathbb{F}\mathbf{1}$. Assume that $L \subseteq \mathcal{B} \otimes \mathbf{1}$. By Corollary 4.5, $L \nsubseteq \mathfrak{Z}_{\mathcal{B}} \otimes \mathbf{1}$, so $\mathcal{B}$ is non-commutative. Hence there is $b \in \mathcal{B} \backslash \mathfrak{Z}_{\mathcal{B}}$ such that $b \otimes \mathbf{1} \in L$. Choosing $p \in \mathcal{P}$ and $b' \in \mathcal{B}$ such that $p \notin \mathbb{F}\mathbf{1}$ and $[b, b'] \neq 0$, we arrive at a contradiction $[b, b'] \otimes p = [b \otimes \mathbf{1}, b' \otimes p] \in L$. Thus $L \nsubseteq \mathcal{B} \otimes \mathbf{1}$ and it follows from Proposition 4.4 that L contains $b \otimes p \neq 0$ with $p \notin \mathfrak{Z}_{\mathcal{P}}$.

Set $T = \{t \in \mathcal{P}\colon\ b \otimes t \in L\}$. Clearly T is a linear subspace of $\mathcal{P}$. Moreover, from $[b \otimes t, e_b \otimes x] = b \otimes [t, x]$ we see that T is a Lie ideal of $\mathcal{P}$. As $p \in T$, T is non-central. Theorem 2.5 therefore yields the existence of a non-zero ideal Y of $\mathcal{P}$ such that $[Y, \mathcal{P}] \subseteq T$. Further, let S be the set of all elements $s \in \mathcal{B}$ such that $s \otimes T \subseteq L$. Then $S \neq \{0\}$, since $b \in S$, and S is a Lie ideal of $\mathcal{B}$. Indeed, as $\mathcal{P}$ is locally unital, $[s, a] \otimes t = [s \otimes t, a \otimes e_t] \in L$, for $s \in S$, $a \in \mathcal{B}$, $t \in T$. Hence $[s, a] \in S$.

Pick $s \in S$, $z \in \mathcal{B}$, $y \in Y$, $x \in \mathcal{P}$. Then $[s, z] \in S$ and $[y, x], [yx, x] \in T$, as Y is an ideal of $\mathcal{P}$ and $[Y, \mathcal{P}] \subseteq T$. So we infer from the identity

$$zs \otimes [[y, x], x] = [s \otimes [y, x], z \otimes x] - [s, z] \otimes [yx, x]$$

that $zs \otimes [[y, x], x] \in L$. Set $U = \mathcal{B}S$ and let K be the linear span of all $[[y, x], x]$, $y \in Y$, $x \in \mathcal{P}$. Then

$$U \otimes K \subseteq L. \tag{4.4}$$ 31

By Lemma 4.6, K is a Lie ideal of $\mathcal{P}$. We have from Lemma 4.7 that Y is non-central. If $y \in Y$ is a non-central element, it follows from (4.2) that there is $x \in \mathcal{P}$ such that $[[y, x], x] \notin \mathfrak{Z}_{\mathcal{P}}$. Hence K is a non-central Lie ideal.

As S is a Lie ideal of $\mathcal{B}$ and $S \neq \{0\}$, we have from (2.4) and (4.3) that U is a non-zero ideal of $\mathcal{B}$. Pick $u \in U$, $x \in \mathcal{B}$, $k \in K$ and $y \in \mathcal{P}$; then $[k, y] \in K$ and so

$$[u, x] \otimes yk = [u \otimes k, x \otimes y] - ux \otimes [k, y] \in [U \otimes K, \mathcal{A}] + U \otimes K \subseteq L.$$

Thus

$$[U, \mathcal{B}] \otimes \mathcal{P}K \subseteq L. \tag{4.5}$$ 32

By Theorem 2.5, there is a non-zero ideal W of $\mathcal{P}$ such that $[W,\mathcal{P}] \subseteq K$. By Lemma 4.7, $W \nsubseteq \mathfrak{Z}_{\mathcal{P}}$, so $[W,\mathcal{P}] \neq \{0\}$. Hence, by (2.4) and (4.3), $V = \mathcal{P}[W,\mathcal{P}]$ is a non-zero ideal of $\mathcal{P}$ contained in $W \cap \mathcal{P}K$. By Lemma 4.7, $V \nsubseteq \mathfrak{Z}_{\mathcal{P}}$, so $\{0\} \neq [V,\mathcal{P}] \subseteq [W,\mathcal{P}] \subseteq K$. By (4.4), it follows that $U \otimes [V,\mathcal{P}] \subseteq L$. By (4.5), it follows that $[U,\mathcal{B}] \otimes V \subseteq L$. ■

Theorem 4.8 yields the following extension of Herstein's result (see Theorem 2.5).

C3.2t **Corollary 4.9** *If $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$ is as in Theorem 4.8, then every non-central Lie ideal L of $\mathcal{A}$ contains a non-zero Lie ideal $[J,\mathcal{A}]$, where J is an ideal of $\mathcal{A}$.*

Proof. Let U and V be as in Theorem 4.8. Then $J = U \otimes V$ is an ideal of $\mathcal{A}$. To prove that $[J,\mathcal{A}] \subseteq L$, consider $u \in U$, $b \in \mathcal{B}$, $v \in V$, and $p \in \mathcal{P}$. We have

$$[u \otimes v, b \otimes p] = ub \otimes [v,p] + [u,b] \otimes pv \in U \otimes [V,\mathcal{P}] + [U,\mathcal{B}] \otimes V \subseteq L.$$

As $[u \otimes V, e_u \otimes \mathcal{P}] = u \otimes [V,\mathcal{P}] \subseteq [J,\mathcal{A}]$ and $[V,\mathcal{P}] \neq \{0\}$, we have $[J,\mathcal{A}] \neq \{0\}$. ■

In general, Lie ideals L of $\mathcal{B} \otimes \mathcal{P}$ are not related to associative ideals: we cannot always choose an ideal J in such a way that L would be contained in $N(J)$. For example, the free algebra $\mathcal{P} = \mathbb{F}\langle x,y,z,w\rangle$ is prime and centrally closed, and the algebra $\mathbb{F} \otimes \mathcal{P} \cong \mathcal{P}$ contains Lie ideals which are not related to any ideal (see Example 2.9). However, if $\mathcal{P}$ is simple, we are able to construct a large variety of Lie ideals of $\mathcal{B} \otimes \mathcal{P}$ each of which is related to an ideal of $\mathcal{B} \otimes \mathcal{P}$ and, moreover, to an exactly one ideal. We will consider this in the next section.

4.2 The case of simple algebras $\mathcal{P}$.

If $\mathcal{P}$ is a simple $\mathbb{F}$-algebra then its extended centroid $C(\mathcal{P})$ coincides with the centroid of $\mathcal{P}$ and can be defined as the algebra of all linear operators T: $\mathcal{P} \to \mathcal{P}$ satisfying the identity $T(x)y = T(xy) = xT(y)$ for all $x,y \in \mathcal{P}$. Moreover, $\mathcal{P}$ can be viewed as a centrally closed simple algebra over $C(\mathcal{P})$. If $\mathfrak{Z}_{\mathcal{P}} \neq \{0\}$ then $C(\mathcal{P}) = \mathfrak{Z}_{\mathcal{P}}$ is a field, so $\mathcal{P}$ is an algebra over its center (such algebras are commonly known as *central simple algebras*). It is possible, of course, that $\mathfrak{Z}_{\mathcal{P}} = \{0\}$. A typical example is the algebra of all finite rank operators on an infinite-dimensional linear space over a field $\mathbb{F}$. Then $\mathfrak{Z}_{\mathcal{P}} = \{0\}$, while $C(\mathcal{P}) \cong \mathbb{F}$.

In this section $\mathcal{P}$ will be a centrally closed simple $\mathbb{F}$-algebra, so $C(\mathcal{P}) \cong \mathbb{F}$. Occasionally we will have to distinguish two cases: 1) when $\mathfrak{Z}_{\mathcal{P}} \neq \{0\}$, so $\mathcal{P}$ is unital and $C(\mathcal{P}) = \mathfrak{Z}_{\mathcal{P}} = \mathbb{F}\mathbf{1}$; and 2) when $\mathfrak{Z}_{\mathcal{P}} = \{0\}$.

The following lemma is well-known, at least for unital $\mathcal{P}$. For the reader's convenience, we will provide a short proof, based on the previous results, that also covers the non-unital case.

L3.5 **Lemma 4.10** *Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra and $\mathcal{P}$ be a centrally closed non-commutative simple $\mathbb{F}$-algebra. If J is an ideal of $\mathcal{B} \otimes \mathcal{P}$, then $J = I \otimes \mathcal{P}$ where I is an ideal of $\mathcal{B}$.*

Proof. If $b \otimes q \in J$, then $b \otimes \sum r_j q s_j = \sum (e_b \otimes r_j)(b \otimes q)(e_b \otimes r_j) \in J$, for $r_j, s_j \in \mathcal{P}$. As $\mathcal{P}$ is simple, $b \otimes \mathcal{P} \subseteq J$. Set $I = \{b \in \mathcal{B}\colon\ b \otimes \mathcal{P} \subseteq J\}$. Then I is a linear space. If $b \in I$ then, for all $x \in \mathcal{B}$ and $p,q \in \mathcal{P}$, $xb \otimes pq = (x \otimes p)(b \otimes q) \in J$. As $\mathcal{P}^2 \neq \{0\}$, we have that $xb \in I$. Similarly, $bx \in I$, so I is an ideal of $\mathcal{B}$ and $I \otimes \mathcal{P} \subseteq J$.

Let us prove by induction on n that, if $a = \sum_{i=1}^n b_i \otimes p_i \in J$ with linearly independent $\{b_i\}$ and $\{p_i\}$, then all $b_i \otimes p_i \in J$. This will imply that $J \subseteq I \otimes \mathcal{P}$ which, in turn, gives us that $I \otimes \mathcal{P} = J$. For

$n = 1$, the result is obviously true. Assume this is true for $n = k - 1$. Let now $n = k$. Since either $\mathfrak{Z}_\mathcal{P} = \mathbb{F}\mathbf{1}$ or $\mathfrak{Z}_\mathcal{P} = \{0\}$ and since all $\{p_i\}$ are linearly independent, we may assume that $p_1 \notin \mathfrak{Z}_\mathcal{P}$. By Proposition 4.4, there exist $q \in [\mathcal{P}, \mathcal{P}]$, with $q \notin \mathfrak{Z}_\mathcal{P}$, and $\lambda_i \in \mathbb{F}$, $i = 2, ..., n$, such that $b \otimes q \in J$ with $b = b_1 + \sum_{i=2}^{n} \lambda_i b_i \neq 0$. Then $b \otimes \mathcal{P} \subseteq J$, so $b \otimes p_1 \in J$. Hence

$$\sum_{i=2}^{n} b_i \otimes (p_i - \lambda_i p_1) = a - b \otimes p_1 \in J.$$

Then $(p_i - \lambda_i p_1) \neq 0$, $i = 2, ..., n$, and they are linearly independent. Hence, by our assumption, all $b_i \otimes (p_i - \lambda_i p_1) \in J$. Therefore $b_i \otimes \mathcal{P} \subseteq J$, for $i = 2, ..., n$, so $b_1 \otimes p_1 \in J$. $\blacksquare$

Denote $Q = [\mathcal{P}, \mathcal{P}]$. A non-zero subspace L of $\mathcal{P}$ is a Lie ideal (see Theorem 1.1) if and only if either L contains Q or $L = \mathbb{F}\mathbf{1}$ (only if $\mathcal{P}$ is unital). One can construct a Lie ideal of $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$ in the following way. Choose proper subspaces $\{Q_\alpha\}_{\alpha \in \mathrm{A}}$ of $\mathcal{P}$, containing Q, and ideals $\{I_\alpha\}_{\alpha \in \mathrm{A}}$ of $\mathcal{B}$. Then $I_\mathrm{A} = \sum_{\alpha \in \mathrm{A}} I_\alpha$ is an ideal of $\mathcal{B}$. Let K, M be Lie ideals of $\mathcal{B}$ satisfying the conditions

$$[M, \mathcal{B}] \subseteq K \subseteq M \text{ and } [I_\alpha, \mathcal{B}] \subseteq K \subseteq I_\mathrm{A}.$$

Set

$$L(K, M, \{I_\alpha\}_{\alpha \in \mathrm{A}}, \{Q_\alpha\}_{\alpha \in \mathrm{A}}) = K \otimes \mathcal{P} + \sum_{\alpha \in \mathrm{A}} I_\alpha \otimes Q_\alpha + M \otimes \mathbf{1}.$$

For a non-unital $\mathcal{P}$, the summand $M \otimes \mathbf{1}$ is absent and we write $L(K, \{I_\alpha\}_{\alpha \in \mathrm{A}}, \{Q_\alpha\}_{\alpha \in \mathrm{A}})$.

genperf **Theorem 4.11** *Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra and let $\mathcal{P}$ be a centrally closed, non-commutative simple $\mathbb{F}$-algebra. If $\mathcal{P}$ is unital then $L = L(K, M, (I_\alpha)_{\alpha \in \mathrm{A}}, (Q_\alpha)_{\alpha \in \mathrm{A}})$ is a Lie ideal of $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$ and $J = I_\mathrm{A} \otimes \mathcal{P}$ is the only ideal of $\mathcal{A}$ related to L. Moreover,* $J = \mathrm{Id}([L, \mathcal{A}])$. *The same holds if $\mathcal{P}$ is non-unital and $L = L(K, (I_\alpha)_{\alpha \in \mathrm{A}}, (Q_\alpha)_{\alpha \in \mathrm{A}})$.*

Proof. Let $\mathcal{P}$ be unital (the non-unital case is similar). For all $b, x \in \mathcal{B}$ and $p, s \in \mathcal{P}$,

$$[b \otimes p, x \otimes s] = [b, x] \otimes ps + xb \otimes [p, s]. \tag{4.6}$$ simfor

If $b \in K$, then $[b, x] \in K$ and $xb \in \sum_{i=1}^{n} I_i$, for some ideals I_i. As $[p, s] \in Q \subseteq \cap_\alpha Q_\alpha$, we have $[b \otimes p, x \otimes s] \in L$. Thus $[K \otimes \mathcal{P}, \mathcal{A}] \subseteq K \otimes \mathcal{P} + \sum_{\alpha \in \mathrm{A}} I_\alpha \otimes Q_\alpha \subseteq L$.

If $b \in I_\alpha$ and $p \in Q_\alpha$, then $[b, x] \in K$ and $xb \in I_\alpha$, whence by (4.6), $[b \otimes p, x \otimes s] \in L$. Thus $[I_\alpha \otimes Q_\alpha, \mathcal{A}] \subseteq K \otimes \mathcal{P} + \sum_{\alpha \in \mathrm{A}} I_\alpha \otimes Q_\alpha \subseteq L$.

Finally, let $b \in M$ and $p = \mathbf{1}$. Then $[b, x] \in K$ and $[p, s] = 0$, so $[b \otimes p, x \otimes s] \in L$. Hence $[M \otimes \mathbf{1}, \mathcal{A}] \subseteq L$, so $[L, \mathcal{A}] \subseteq L$. Thus L is a Lie ideal of $\mathcal{A}$.

Let us prove that $[L, \mathcal{A}] \subseteq J$. As $K \otimes \mathcal{P} + \sum_\alpha I_\alpha \otimes Q_\alpha \subseteq J$, it follows from the above that we need only to show that $[M \otimes \mathbf{1}, \mathcal{A}] \subset J$. But $[m \otimes \mathbf{1}, x \otimes s] = [m, x] \otimes s \in K \otimes \mathcal{P} \subset J$ for all $m \in M$, $x \in \mathcal{B}$, $s \in \mathcal{P}$. Let us establish now that $[J, \mathcal{A}] \subseteq L$. Let $i \in I_\alpha$, $p \in \mathcal{P}$. Then, by (4.6), $[i \otimes p, x \otimes s] = [i, x] \otimes ps + xi \otimes [p, s] \in L$, as $[i, x] \in K$, $xi \in I_\alpha$, $[p, s] \in Q$. Thus J is related to L.

Suppose that another ideal W of $\mathcal{A}$ is related to L. By Lemma 4.10, $W = V \otimes \mathcal{P}$ where V is an ideal of $\mathcal{B}$. We will prove that $V = I_\mathrm{A}$. Let $v \in V$. Then, for each $p, s \in \mathcal{P}$, $v \otimes [p, s] = [v \otimes p, e_v \otimes s] \in [W, \mathcal{A}] \subseteq L$. By (4.2), we can choose p, s such that $[p, s] \notin \mathbb{F}\mathbf{1}$. Let f be a linear functional on $\mathcal{P}$

with $f(\mathbf{1}) = 0$ and $f([p,s]) = 1$. Consider the map $F\colon \mathcal{A} \to \mathcal{B}$ defined by: $F(b \otimes r) = f(r)b$. Then $v = F(v \otimes [p,s]) \in F(L)$ and

$$F(L) = F(K \otimes \mathcal{P} + \sum_{\alpha} I_\alpha \otimes Q_\alpha + M \otimes \mathbf{1}) = F(K \otimes \mathcal{P} + \sum_{\alpha} I_\alpha \otimes Q_\alpha) \subseteq K + \sum_{\alpha} I_\alpha = I_{\mathrm{A}}.$$

Thus $V \subseteq I_{\mathrm{A}}$.

Conversely, for each $i \in I_{\mathrm{A}}$, $q \in Q$, $s \in \mathcal{P}$, we have $i \otimes [q,s] = [i \otimes q, e_i \otimes s] \in [L, \mathcal{A}] \subseteq W$. It follows from (4.2) that there are q and s such that $[q,s] \neq 0$. Hence $i \in V$. Thus $I_{\mathrm{A}} \subseteq V$, so $I_{\mathrm{A}} = V$ and $J = W$. Hence J is the only ideal of $\mathcal{A}$ related to L. Using Lemma 2.6, we have $J = \mathrm{Id}([L, \mathcal{A}])$. ■

4.3 The case of simple algebras $\mathcal{P}$ with $\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) \leq 1$.

The above construction becomes more definite if the codimension of Q in $\mathcal{P}$ does not exceed 1:

$$\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) \leq 1. \qquad (4.7)$$ codim

In this case there is only one possible choice for Q_α: $Q_\alpha = Q$, so we take only one ideal I and use the notation $L(K, M, I)$ ($L(K, I)$ if $\mathcal{P}$ is non-unital). Our aim is to prove that in this case our construction of Lie ideals of algebras is almost universal and gives us all Lie ideals of $\mathcal{B} \otimes \mathcal{P}$.

Note that there are many important examples of simple centrally closed algebras satisfying (4.7). For example, the Weyl algebra (the algebra of all polynomials over $\mathbb{F}$ with generators x, y satisfying the relation $xy - yx = 1$) and simple purely infinite unital C*-algebras (in particular, von Neumann type III factors) satisfy $\mathcal{P} = [\mathcal{P},\mathcal{P}]$. The full matrix algebras $M_n(\mathbb{F})$, the algebra of quaternions, UHF C*-algebras, von Neumann type II_1 factors, the algebra $M_\infty(\mathbb{F})$ of all infinite matrices with finite number of non-zero entries and the algebra of all finite rank operators on an infinite-dimensional linear space satisfy $\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) = 1$.

We will need the following result.

notalg **Lemma 4.12** *If $\mathcal{P}$ is a simple algebra and $[\mathcal{P},\mathcal{P}] \neq \mathcal{P}$ then there is $c \in [\mathcal{P},\mathcal{P}]$ with $c^2 \notin [\mathcal{P},\mathcal{P}]$.*

Proof. By [H2, page 6], the subalgebra generated by $[\mathcal{P},\mathcal{P}]$ coincides with $\mathcal{P}$. Hence, as $[\mathcal{P},\mathcal{P}] \neq \mathcal{P}$, $[\mathcal{P},\mathcal{P}]$ is not a subalgebra of $\mathcal{P}$. Since it is a Lie subalgebra, it can not be closed with respect to Jordan multiplication $a \circ b = (ab + ba)/2$. Since $ab + ba = (a+b)^2 - a^2 - b^2$, there is an element c in $[\mathcal{P},\mathcal{P}]$ with $c^2 \notin [\mathcal{P},\mathcal{P}]$. ■

In what follows $\mathcal{B}$ stands for a locally unital $\mathbb{F}$-algebra and $\mathcal{P}$ for a simple, centrally closed, locally unital $\mathbb{F}$-algebra satisfying condition (4.7). By Theorem 1.1, the only Lie ideals of $\mathcal{P}$ are

$$\{0\},\ Q = [\mathcal{P},\mathcal{P}],\ \mathcal{P} \text{ and } \mathbb{F}\mathbf{1}, \text{ if } \mathcal{P} \text{ is unital.} \qquad (4.8)$$ 5.0

Given a Lie ideal L of $\mathcal{B} \otimes \mathcal{P}$, set

$$\begin{aligned} K(L) &= \{b \in \mathcal{B}\colon b \otimes P \subseteq L\}, \quad I(L) = \{b \in \mathcal{B}\colon b \otimes Q \subseteq L\} \text{ and} \\ M(L) &= \{b \in \mathcal{B}\colon b \otimes \mathbf{1} \in L\} \text{ if } \mathcal{P} \text{ is unital.} \end{aligned}$$

Then $K(L) \subseteq I(L) \cap M(L)$.

mainideals **Lemma 4.13** (i) *If $b\otimes p \in L$, for some $p \notin \mathfrak{Z}_{\mathcal{P}}$, then $b \in I(L)$.*

(ii) *Let $p_1 \notin \mathfrak{Z}_{\mathcal{P}}$ and $p_2 \notin Q$. If $b\otimes p_1 \in L$ and $b\otimes p_2 \in L$ then $b \in K(L)$.*

(iii) *$[M(L),\mathcal{B}] \subseteq K(L)$ and $[I(L),\mathcal{B}] \subseteq K(L)$, so $K(L)$ and $M(L)$ are Lie ideals of $\mathcal{B}$.*

(iv) *$I(L)$ is an ideal of $\mathcal{B}$.*

Proof. Set $K = K(L)$, $I = I(L)$, $M = M(L)$. For $b \in \mathcal{B}$, set $\mathcal{P}(b) = \{p \in \mathcal{P} : b\otimes p \in L\}$. Then $\mathcal{P}(b)$ is a Lie ideal of $\mathcal{P}$. Indeed, let $p \in \mathcal{P}(b)$, $s \in \mathcal{P}$. Then $b\otimes[p,s] = [b\otimes p, e_b\otimes s] \in L$, so $[p,s] \in \mathcal{P}(b)$. The assumption of (i) means that $\mathcal{P}(b) \nsubseteq \mathfrak{Z}_{\mathcal{P}}$. Hence, by (4.8), $\mathcal{P}(b)$ contains Q, so $b \in I$. Part (i) is proved.

Similarly, the assumption of (ii) implies that $\mathcal{P}(b) \nsubseteq \mathfrak{Z}_{\mathcal{P}}$ and $\mathcal{P}(b) \nsubseteq Q$. By (4.8), $\mathcal{P}(b) = \mathcal{P}$. This means that $b \in K$. Part (ii) is proved.

Assume that $\mathcal{P}$ is unital and let $b \in M$, $x \in \mathcal{B}$. Then for each $p \in \mathcal{P}$, $[b,x]\otimes p = [b\otimes\mathbf{1}, x\otimes p] \in L$. Hence $[b,x] \in K$, so $[M,\mathcal{B}] \subseteq K$.

Let us show now that I is a Lie ideal of $\mathcal{B}$. Let $b \in I$, $x \in \mathcal{B}$. For $q \in Q \setminus \mathfrak{Z}_{\mathcal{P}}$, $[b,x]\otimes q = [b\otimes q, x\otimes e_q] \in L$. By (i), $[b,x] \in I$, so that I is a Lie ideal.

If $\mathcal{P} = Q$ then $K = I$ and (iii) is proved. Let $\mathcal{P} \neq Q$. By Lemma 4.12, there is $c \in Q$ such that $c^2 \notin Q$, so $\mathcal{P} = Q + \mathbb{F}c^2$. Then $[b,x]\otimes c^2 = [b\otimes c, x\otimes c] \in L$. Since also $[b,x]\otimes q \in L$, for each $q \in Q$, we have $[b,x] \otimes \mathcal{P} \subseteq L$. Hence $[b,x] \in K$. Thus $[I,\mathcal{B}] \subseteq K$. Part (iii) is proved.

Let $b \in I$. Choose $q \in Q \setminus \mathfrak{Z}_{\mathcal{P}}$. For $x \in \mathcal{B}$ and $p \in \mathcal{P}$, we have from (4.6)

$$bx\otimes[q,p] = [b\otimes q, x\otimes p] - [b,x]\otimes pq.$$

Since $b\otimes q \in L$, we have $[b\otimes q, x\otimes p] \in L$. Since $[b,x] \in K$, the same is true for $[b,x]\otimes pq$. Hence $bx\otimes[q,p] \in L$. Choosing p so that $[q,p] \notin \mathfrak{Z}_{\mathcal{P}}$ (see (4.2)), we conclude from (i) that $bx \in I$. Similarly, $xb \in I$, so that I is an ideal. ■

For each Lie ideal L of $\mathcal{A} = \mathcal{B}\otimes\mathcal{P}$, set

$$L_0 = K(L)\otimes\mathcal{P} + I(L)\otimes Q + M(L)\otimes\mathbf{1}, \text{ or } L_0 = K(L)\otimes\mathcal{P} + I(L)\otimes Q, \text{ if } \mathcal{P} \text{ is non-unital.}$$

It follows from the definitions that $L_0 \subseteq L$. We will see below that $L_0 = L$ when either 1) $\mathcal{P} = Q$, or 2) $\mathcal{P} = Q \dotplus \mathbb{F}\mathbf{1}$, that is, in all cases satisfying (4.7) apart from the case

$$\mathbf{1} \in [\mathcal{P},\mathcal{P}] \neq \mathcal{P}. \tag{4.9}$$ exep

This will give us a full description of Lie ideals of algebras $\mathcal{B}\otimes\mathcal{P}$, satisfying (4.7) and not satisfying (4.9), in terms of Lie ideals of $\mathcal{B}$.

C3.3t **Theorem 4.14** *Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra. Let $\mathcal{P}$ be a centrally closed, locally unital simple $\mathbb{F}$-algebra and let $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$. Let $\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) \leq 1$ and suppose that $\mathcal{P}$ does not satisfy (4.9).*

(i) *If $\mathcal{P}$ is unital and $\mathcal{P} = [\mathcal{P},\mathcal{P}]$, then all Lie ideals L of $\mathcal{A}$ are of the form*

$$L = I \otimes \mathcal{P} + M \otimes \mathbf{1}, \tag{4.10}$$ dop

where I is an ideal of $\mathcal{B}$ and M is a Lie ideal of $\mathcal{B}$ satisfying $[M,\mathcal{B}] \subseteq I \subseteq M$.

(ii) *If $\mathcal{P} = [\mathcal{P},\mathcal{P}] \dotplus \mathbb{F}\mathbf{1}$ then all Lie ideals L of $\mathcal{A}$ are of the form*

$$L = I \otimes [\mathcal{P},\mathcal{P}] + M \otimes \mathbf{1}, \tag{4.11}$$ e6.2

where M is a Lie ideal of $\mathcal{B}$ related to an ideal I of $\mathcal{B}$: $[I,\mathcal{B}] \subseteq M \subseteq N(I)$.

(iii) *If $\mathcal{P}$ is non-unital and $\mathcal{P} = [\mathcal{P},\mathcal{P}]$, then all Lie ideals L of $\mathcal{A}$ are of the form $L = I \otimes \mathcal{P}$ where I is an ideal of $\mathcal{B}$, that is, all Lie ideals of $\mathcal{A}$ are ideals.*

(iv) *If $\mathcal{P}$ is non-unital and $\dim(\mathcal{P}/[\mathcal{P},\mathcal{P}]) = 1$ then all Lie ideals of $\mathcal{A}$ are of the form*

$$L = K\otimes\mathcal{P} + I\otimes[\mathcal{P},\mathcal{P}], \tag{4.12}$$

nonunitideal

where K is a Lie ideal of $\mathcal{B}$ and I is an ideal of $\mathcal{B}$ satisfying $[I,\mathcal{B}] \subseteq K \subseteq I$.

In all cases each Lie ideal L of $\mathcal{A}$ is only related to the ideal $\mathrm{Id}([L,\mathcal{A}]) = I\otimes\mathcal{P}$.

Proof. Using induction on n, we will show that each $a = \sum_{i=1}^{n} b_i\otimes p_i \in L$ belongs to L_0. In (ii) and (iv) $\mathcal{P} = Q \dotplus \mathbb{F}r$ and, if not all p_i lie in Q, then we will assume, without loss of generality, that $p_n = r$ and all $p_i \in Q$ for $i < n$.

Let $n = 1$ and $a = b\otimes p \in L$. If $p \in \mathfrak{Z}_{\mathcal{P}}$, then $\mathcal{P}$ is unital and $p \in \mathbb{F}\mathbf{1}$, so $b \in M(L)$ and $a \in L_0$. Let $p \notin \mathfrak{Z}_{\mathcal{P}}$. Then, by Lemma 4.13(i), $b \in I(L)$. If $p \in Q$ then $a \in L_0$. If $p \notin Q$ then, by Lemma 4.13(ii), $b \in K(L)$. So again $a \in L_0$.

Let $n > 1$ and assume that our hypothesis is true for all $k < n$. Let all $\{b_i\}$ and $\{p_i\}$ be linearly independent. Without loss of generality, one of p_i, say p_1, can be chosen to belong to Q and do not belong to $\mathbb{F}\mathbf{1}$. This is evident in (iii). In (i) all $p_i \in Q$ and, as they linearly independent, at least one of them does not belong to $\mathbb{F}\mathbf{1}$. In (ii) and (iv) all p_i, $i < n$, lie in Q and do not belong to $\mathbb{F}\mathbf{1}$.

By Proposition 4.4, there is $b = b_1 + \sum_{i=2}^{n} \lambda_i b_i$ and $q \in Q \setminus \mathbb{F}\mathbf{1}$ such that $b\otimes q \in L$. By Lemma 4.13(i), $b \in I(L)$, so $b\otimes p_1 \in L_0$ and $a - b\otimes p_1 = \sum_{i=2}^{n} b_i \otimes (p_i - \lambda_i p_1) \in L$. By the induction hypothesis, $a - b\otimes p_1 \in L_0$, whence $a \in L_0$. Thus, if $\mathcal{P}$ is unital and does not satisfy (4.9) then

$$L = L_0 = K(L)\otimes\mathcal{P} + I(L)\otimes[\mathcal{P},\mathcal{P}] + M(L) \otimes \mathbf{1},$$

for each Lie ideal L of $\mathcal{A}$. By Lemma 4.13,

$$[M(L),\mathcal{B}] \subseteq K(L) \subseteq M(L) \text{ and } [I(L),\mathcal{B}] \subseteq K(L) \subseteq I(L).$$

(i) If $\mathcal{P} = [\mathcal{P},\mathcal{P}]$, then $K(L) = I(L)$, so $L = I(L)\otimes\mathcal{P} + M(L) \otimes \mathbf{1}$. On the other hand, by Theorem 4.11, for each ideal I and a Lie ideal M of $\mathcal{B}$ satisfying $[M,\mathcal{B}] \subseteq I \subseteq M$, the space $\widetilde{L} = I\otimes\mathcal{P} + M \otimes \mathbf{1}$ is a Lie ideal of $\mathcal{A}$. It is easy to check that $I(\widetilde{L}) = I$ and $M(\widetilde{L}) = M$. Part (i) is proved.

(ii) If $\mathcal{P} = [\mathcal{P},\mathcal{P}] \dotplus \mathbb{F}\mathbf{1}$ then

$$K(L)\otimes\mathcal{P} = K(L)\otimes Q + K(L)\otimes\mathbf{1} \subseteq I(L)\otimes Q + M(L)\otimes\mathbf{1}.$$

Hence $L = I(L)\otimes[\mathcal{P},\mathcal{P}] + M(L) \otimes \mathbf{1}$ and the rest of the proof is the same as in (i).

If $\mathcal{P}$ is non-unital then, for each Lie ideal L of $\mathcal{A}$,

$$L = L_0 = K(L)\otimes\mathcal{P} + I(L)\otimes[\mathcal{P},\mathcal{P}], \text{ where } [I(L),\mathcal{B}] \subseteq K(L) \subseteq I(L).$$

If $\mathcal{P} \neq [\mathcal{P},\mathcal{P}]$ then, by Theorem 4.11, for each ideal I of $\mathcal{B}$ and Lie ideal K of $\mathcal{B}$ satisfying $[I,\mathcal{B}] \subseteq K \subseteq I$, $\widetilde{L} = K\otimes\mathcal{P} + I\otimes[\mathcal{P},\mathcal{P}]$ is a Lie ideal of $\mathcal{A}$. It is easy to check that $K(\widetilde{L}) = K$ and $I(\widetilde{L}) = I$ which proves part (iv).

If $\mathcal{P} = [\mathcal{P},\mathcal{P}]$, then $K(L) = I(L)$, so $L = I(L)\otimes\mathcal{P}$. By Theorem 4.11, for each ideal I of $\mathcal{B}$, $\widetilde{L} = I\otimes\mathcal{P}$ is a Lie ideal of $\mathcal{A}$. As $I(\widetilde{L}) = I$, part (iii) is proved.

The fact that each Lie ideal of $\mathcal{A}$ is only related to one ideal follows from Theorem 4.11. ■

If $\mathcal{P}$ satisfies (4.9), then the approach of Theorem 4.14 does not work for elements of the form $c\otimes\mathbf{1} + d\otimes p$ with $p \notin Q$. Moreover, we can construct examples of Lie ideals L with $L \neq L_0$ in the following way. Choose an ideal I of $\mathcal{B}$ and Lie ideals K, M with $K \subsetneqq I \subseteq M \subsetneqq N(K)$. Then $L_0 = K\otimes\mathcal{P}+I\otimes Q+M\otimes\mathbf{1}$ is a Lie ideal of $\mathcal{A} = \mathcal{B}\otimes\mathcal{P}$. Let $m = \min(\dim(I/K), \dim(N(K)/M))$. For $n \leq m$, choose elements $\{d_i\}_{i=1}^n$ in I linearly independent modulo K and elements $\{c_i\}_{i=1}^n$ in $N(K)$ linearly independent modulo M. Set $a_i = c_i\otimes\mathbf{1} + d_i\otimes p$ for some $p \notin Q$. Then $L = L_0 + \sum_{i=1}^n \mathbb{F}a_i$ is a Lie ideal of $\mathcal{B} \otimes \mathcal{P}$, as $[a_i, \mathcal{A}] \subseteq L_0$, and

$$K(L) = K,\ I(L) = I,\ M(L) = M \text{ and } L_0 \neq L.$$

Simple algebras satisfying (4.9) do exist. The authors are indebted to Misha Chebotar for the following example.

Example. Let $p = \mathrm{char}(\mathbb{F})$ and $\mathcal{P} = M_p(\mathbb{F})$. It is easy to see that all matrices in $[\mathcal{P}, \mathcal{P}]$ have zero trace, and, moreover, $[\mathcal{P}, \mathcal{P}] = \{a \in \mathcal{P}\colon\ \mathrm{Tr}(a) = 0\}$. (In fact, every element in $[\mathcal{P}, \mathcal{P}]$ can be written as a commutator [AM]). Hence $\mathbf{1} \in [\mathcal{P}, \mathcal{P}] \neq \mathcal{P}$.

All Lie ideals L of the algebras $\mathcal{A} = \mathcal{B}\otimes\mathcal{P}$ considered in Theorem 4.14 are related to the ideals $J = \mathrm{Id}([L, \mathcal{A}])$. We will show that in many cases L is commutator equal to J: $[L, \mathcal{A}] = [J, \mathcal{A}]$.

We begin with the following technical result.

commutators **Lemma 4.15** *If $\mathcal{P}$ is simple with $\dim(\mathcal{P}/Q) \leq 1$, then $[I\otimes\mathcal{P}, \mathcal{A}] = [I\otimes Q, \mathcal{A}]$, for each ideal I of $\mathcal{B}$.*

Proof. We only need to consider the case $\dim(\mathcal{P}/Q) = 1$ and to show that $[I\otimes\mathcal{P}, \mathcal{A}] \subseteq [I\otimes Q, \mathcal{A}]$. By (4.6),

$$[I \otimes \mathcal{P}, \mathcal{B} \otimes \mathcal{P}] \subseteq [I, \mathcal{B}] \otimes \mathcal{P} + I \otimes Q.$$

As $[Q, \mathcal{P}]$ is a Lie ideal of $\mathcal{P}$, it follows from Theorem 1.1 that either $[Q, \mathcal{P}] = Q$ or $[Q, \mathcal{P}] \subseteq \mathbb{F}\mathbf{1}$. As the latter is impossible by (4.2), $Q = [Q, \mathcal{P}]$. For $i \in I$, $q \in Q$, $p \in \mathcal{P}$, we have $i \otimes [q, p] = [i \otimes q, e_i \otimes p] \in [I \otimes Q, \mathcal{A}]$. Thus $I \otimes Q \subseteq [I \otimes Q, \mathcal{A}]$.

For $i \in I$, $x \in \mathcal{B}$ and $q \in Q$, we have $[i, x] \otimes q = [i \otimes q, x \otimes e_q] \in [I \otimes Q, \mathcal{A}]$. Furthermore, if $c \in Q$ with $c^2 \notin Q$ (see Lemma 4.12) then $[i, x] \otimes c^2 = [i \otimes c, x \otimes c] \in [I \otimes Q, \mathcal{A}]$. Since each $p \in \mathcal{P}$ can be written in the form $p = q + \lambda c^2$ with $q \in Q$, $\lambda \in \mathbb{F}$, we conclude that $[I, \mathcal{B}] \otimes \mathcal{P} \subseteq [I \otimes Q, \mathcal{A}]$. Thus $[I \otimes \mathcal{P}, \mathcal{A}] \subseteq [I \otimes Q, \mathcal{A}]$. ■

restricted **Theorem 4.16** *Let $\mathcal{B}$ and $\mathcal{P}$ be as in Theorem 4.14, let L be a Lie ideal of $\mathcal{A} = \mathcal{B}\otimes\mathcal{P}$ and $J = \mathrm{Id}([L, \mathcal{A}])$.*

(i) *If $\mathcal{P} = [\mathcal{P}, \mathcal{P}]$ then $J \subseteq L$ ($J = L$ if $\mathcal{P}$ is non-unital) and $[L, \mathcal{A}] = [J, \mathcal{A}]$.*

(ii) *If $\mathcal{P}$ is non-unital and $\dim \mathcal{P}/[\mathcal{P}, \mathcal{P}] = 1$, then $L \subseteq J$ and $[L, \mathcal{A}] = [J, \mathcal{A}]$.*

(iii) *If $\mathcal{P} = [\mathcal{P}, \mathcal{P}] \dot{+} \mathbb{F}\mathbf{1}$ then $[J, \mathcal{A}] \subseteq [L, \mathcal{A}]$. Moreover, $[J, \mathcal{A}] = [L, \mathcal{A}]$ if and only if $[M, \mathcal{B}] \subseteq [I, \mathcal{B}]$ (in particular, if $\mathcal{B} = [\mathcal{B}, \mathcal{B}] + \mathfrak{z}_{\mathcal{B}}$).*

Proof. Let $\mathcal{P} = [\mathcal{P}, \mathcal{P}]$. If $\mathcal{P}$ is non-unital then (see Theorem 4.14(iii)) $L = J$ is an ideal. If $\mathcal{P}$ is unital then, by Theorem 4.14(i), $L = I \otimes \mathcal{P} + M \otimes \mathbf{1}$, where I is an ideal of $\mathcal{B}$, $[M, \mathcal{B}] \subseteq I \subseteq M$, and $J = I \otimes \mathcal{P}$. Hence $J \subseteq L$ and

$$[J, \mathcal{A}] \subseteq [L, \mathcal{A}] = [J, \mathcal{A}] + [M \otimes \mathbf{1}, \mathcal{A}] = [J, \mathcal{A}] + [M, \mathcal{B}] \otimes \mathcal{P} \subseteq [J, \mathcal{A}] + I \otimes \mathcal{P}.$$

As each $p \in \mathcal{P}$ can be written in the form $p = \sum_k [s_k, t_k]$, we have $b \otimes p = \sum_k [b \otimes s_k, e_b \otimes t_k]$ for each $b \in \mathcal{B}$. This shows that $J = I \otimes \mathcal{P} = [J, \mathcal{A}]$. Hence $[L, \mathcal{A}] = [J, \mathcal{A}]$. Part (i) is proved.

In (ii), by Theorem 4.14(iv), $L = K \otimes \mathcal{P} + I \otimes Q$, where $[I, \mathcal{B}] \subseteq K \subseteq I$, and $J = I \otimes \mathcal{P}$. Hence $L \subseteq J$ and, by Lemma 4.15, $[L, \mathcal{A}] \subseteq [J, \mathcal{A}] = [I \otimes \mathcal{P}, \mathcal{A}] = [I \otimes Q, \mathcal{A}] \subseteq [L, \mathcal{A}]$ which proves (ii).

In (iii), by Theorem 4.14(ii) and its proof, $L = I \otimes Q + M \otimes \mathbf{1}$, where $[M, \mathcal{B}] \subseteq I \subseteq M$, and $J = I \otimes \mathcal{P}$. Hence, by Lemma 4.15, $[J, \mathcal{A}] = [I \otimes Q, \mathcal{A}] \subseteq [L, \mathcal{A}]$.

As $I \otimes Q \subseteq J$, the equality $[L, \mathcal{A}] = [J, \mathcal{A}]$ is equivalent to the inclusion $[M \otimes \mathbf{1}, \mathcal{A}] \subseteq [J, \mathcal{A}]$. As $[M \otimes \mathbf{1}, \mathcal{A}] = [M, \mathcal{B}] \otimes \mathcal{P}$, if $[M, \mathcal{B}] \subseteq [I, \mathcal{B}]$ then

$$[M \otimes \mathbf{1}, \mathcal{A}] \subseteq [I, \mathcal{B}] \otimes \mathcal{P} = [I \otimes \mathcal{P}, \mathcal{B} \otimes \mathbf{1}] \subseteq [J, \mathcal{A}].$$

Conversely, let $[M \otimes \mathbf{1}, \mathcal{A}] \subseteq [J, \mathcal{A}]$. By (4.6), $[J, \mathcal{A}] \subseteq [I, \mathcal{B}] \otimes \mathcal{P} + I \otimes Q$, so

$$[M, \mathcal{B}] \otimes \mathcal{P} = [M \otimes \mathbf{1}, I \otimes \mathcal{P}] \subseteq [J, \mathcal{A}] \subseteq [I, \mathcal{B}] \otimes \mathcal{P} + I \otimes Q.$$

As $\mathcal{P} = Q \dotplus \mathbb{F}\mathbf{1}$,

$$[M, \mathcal{B}] \otimes Q \dotplus [M, \mathcal{B}] \otimes \mathbf{1} \subseteq ([I, \mathcal{B}] \otimes Q + I \otimes Q) \dotplus [I, \mathcal{B}] \otimes \mathbf{1}.$$

Therefore $[M, \mathcal{B}] \subseteq [I, \mathcal{B}]$.

If $\mathcal{B} = [\mathcal{B}, \mathcal{B}] + \mathfrak{Z}_{\mathcal{B}}$ then $[M, \mathcal{B}] = [M, [\mathcal{B}, \mathcal{B}]] \subseteq [[M, \mathcal{B}], \mathcal{B}] \subseteq [I, \mathcal{B}]$. Hence $[L, \mathcal{A}] = [J, \mathcal{A}]$. ■

It is only in case (iii) of Theorem 4.16 that a Lie ideal of the algebra $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$ may be related to, but not commutator equal to, an ideal of $\mathcal{A}$. We will now show that $\mathcal{A}$ may have Lie ideals that are not even embraced by ideals of $\mathcal{A}$.

Corollary 4.17 *There is a unital $\mathbb{F}$-algebra $\mathcal{B}$ such that, for each central simple $\mathbb{F}$-algebra $\mathcal{P}$ satisfying $\mathcal{P} = [\mathcal{P}, \mathcal{P}] \dotplus \mathbb{F}\mathbf{1}$, the algebra $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$ has Lie ideals that are not embraced by ideals.*

Proof. In Example 2.10 we constructed a Lie ideal M of an algebra $\mathcal{B}$ related to an ideal of $\mathcal{B}$ but not embraced by any ideal of $\mathcal{B}$. By Lemma 2.6, M is related to the ideal $I = \mathrm{Id}([M, \mathcal{B}])$, so $[I, \mathcal{B}] \subseteq M$. Therefore, by Theorem 4.14, $L = I \otimes Q + M \otimes \mathbf{1}$ is a Lie ideal of $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$, related to a unique ideal $J = I \otimes \mathcal{P}$. By Lemma 4.15, $[J, \mathcal{A}] = [I \otimes Q, \mathcal{A}] \subseteq [L, \mathcal{A}]$. If L is embraced by J, then $[L, \mathcal{A}] \subseteq [J, \mathcal{A}]$ which implies $[L, \mathcal{A}] = [J, \mathcal{A}]$. Hence, by Theorem 4.16(iii), $[M, \mathcal{B}] \subseteq [I, \mathcal{B}]$, so $[I, \mathcal{B}] \subseteq M \subseteq N([I, \mathcal{B}])$. But this would mean that M is embraced by I, a contradiction. ■

Let us consider now what our results give for matrix algebras over algebras. Let $\mathcal{A} = M_n(\mathcal{B})$, $n \geq 2$, be the full matrix algebra with entries from a locally unital $\mathbb{F}$-algebra $\mathcal{B}$. Then $\mathcal{A} = \mathcal{B} \otimes \mathcal{P}$, where $\mathcal{P} = M_n(\mathbb{F})$ is a central simple algebra, and $\mathcal{P} = [\mathcal{P}, \mathcal{P}] \dotplus \mathbb{F}\mathbf{1}$, where $[\mathcal{P}, \mathcal{P}]$ consists of matrices with zero trace.

By Lemma 4.10, J is an ideal of $\mathcal{A}$ if and only if $J = I \otimes \mathcal{P}$, for some ideal I of $\mathcal{B}$, so that J is isomorphic to $M_n(I)$ and $I \otimes [\mathcal{P}, \mathcal{P}]$ is isomorphic to

$$M_n^0(I) = \{a = (a_{ij}) \in M_n(I) : \sum a_{ii} = 0\}.$$

For a Lie ideal M of $\mathcal{B}$, the space $M \otimes \mathbf{1}$ is isomorphic to

$$\mathcal{D}_M = \{a \in M_n(\mathcal{B}) : a_{ij} = 0, \text{ if } i \neq j, \text{ and } a_{11} = ... = a_{nn} \in M\}$$

and $M_n(I) = M_n^0(I) \dotplus \mathcal{D}_I$.Theorems 4.14(ii) and 4.16(iii) yield

C3 **Corollary 4.18** *Let $\mathcal{A} = M_n(\mathcal{B})$, $n \geq 2$, where $\mathcal{B}$ is a locally unital $\mathbb{F}$-algebra.*

(i) *A linear subspace L of $\mathcal{A}$ is a Lie ideal if and only if there is an ideal I of $\mathcal{B}$ and a Lie ideal M of $\mathcal{B}$ such that*

$$[I, \mathcal{B}] \subseteq M \subseteq N(I) \quad \text{and } L = L(M, I) = M_n^0(I) \dot{+} \mathcal{D}_M.$$

(ii) *Let $L = L(M, I)$ be a Lie ideal of $\mathcal{A}$. Set $J = M_n(I)$. Then J is an ideal of $\mathcal{A}$, $J = \mathrm{Id}([L, \mathcal{A}])$ and it is the only ideal of $\mathcal{A}$ related to L. Moreover,*

$$\begin{aligned} [J, \mathcal{A}] &= [M_n(I), \mathcal{A}] = [M_n^0(I), \mathcal{A}] + [\mathcal{D}_I, \mathcal{A}] = M_n^0(I) \dot{+} \mathcal{D}_{[I,\mathcal{B}]}, &(4.13)\\ [L, \mathcal{A}] &= [M_n^0(I), \mathcal{A}] + [\mathcal{D}_M, \mathcal{A}] = M_n^0(I) \dot{+} \mathcal{D}_{[I+M,\mathcal{B}]}. &(4.14) \end{aligned}$$

If $[M, \mathcal{B}] \subseteq [I, \mathcal{B}]$ (in particular, if $\mathcal{B} = [\mathcal{B}, \mathcal{B}] + \mathfrak{Z}_{\mathcal{B}}$), then J and L are commutator equal: $[J, \mathcal{A}] = [L, \mathcal{A}]$.

Proof. Part (i) follows from Theorem 4.14(ii).

The fact that L is only related to the ideal $J = \mathrm{Id}([L, \mathcal{A}]) = M_n(I)$ follows from Theorem 4.14.

As $[M_n(I), \mathcal{A}]$ is a Lie ideal and $[M_n(I), \mathcal{A}] \subseteq M_n(I)$, it follows from (i) that there are an ideal $I_1 \subseteq I$, and a Lie ideal K of $\mathcal{B}$ such that $[I_1, \mathcal{B}] \subseteq K \subseteq N(I_1)$ and

$$[M_n(I), \mathcal{A}] = M_n^0(I_1) \dot{+} \mathcal{D}_K.$$

Let $\{r_{ij}\}$ be the matrix identity in $\mathcal{P}$. Each $a = (a_{ij}) \in \mathcal{A}$ can written in the form $a = \sum a_{ij} r_{ij}$, where $a_{ij} r_{ij}$ is a matrix such that only (i, j) entry is non-zero and equals $a_{ij} \in \mathcal{B}$. Let i, j, k be all different. For each $b \in I$, $br_{ij} \in M_n^0(I)$ and $[br_{ij}, e_b r_{jk}] = br_{ik} \in M_n^0(I_1)$. Hence $I_1 = I$.

For each $b \in I$ and $c \in \mathcal{B}$ and for $i \neq j$,

$$[br_{ij}, cr_{ji}] = bcr_{ii} - cbr_{jj} = \frac{1}{n} \sum [b, c] r_{kk} - \left(\frac{1}{n} \sum [b, c] r_{kk} - (bcr_{ii} - cbr_{jj}) \right),$$

where $\frac{1}{n} \sum [b, c] r_{kk} - (bcr_{ii} - cbr_{jj}) \in M_n^0(I)$ and $[b, c] \in K$. Hence $[I, \mathcal{B}] \subseteq K$.

For diagonal elements $\sum b_i r_{ii}$ and $\sum c_i r_{ii}$, with $b_i \in I$ and $c_i \in \mathcal{B}$,

$$a = [\sum b_i r_{ii}, \sum c_i r_{ii}] = \sum [b_i, c_i] r_{ii} = \sum g r_{ii} + \sum d_i r_{ii},$$

where $g = \frac{1}{n} \mathrm{Tr}(a) \in \mathcal{D}_{[I,\mathcal{B}]}$ and $d_i = [b_i, c_i] - g$. Hence $\sum d_i r_{ii} \in M_n^0(I)$. This implies that $[I, \mathcal{B}] = K$, so (4.13) is proved.

¿From the above discussion we also have that $[M_n^0(I), \mathcal{A}] = M_n^0(I) \dot{+} \mathcal{D}_{[I,\mathcal{B}]}$. Hence

$$M_n^0(I) \dot{+} \mathcal{D}_{[I,\mathcal{B}]} = [M_n^0(I), \mathcal{A}] \subseteq [L, \mathcal{A}] = [M_n^0(I), \mathcal{A}] + [\mathcal{D}_M, \mathcal{A}].$$

For all $b \in M$ and $a = (a_{ij}) \in \mathcal{A}$,

$$\begin{aligned} [\sum_i br_{ii}, \sum_{k,j} a_{kj} r_{kj}] &= \sum_i [b, a_{ii}] r_{ii} + \sum_{k \neq j} [b, a_{kj}] r_{kj},\\ \sum [b, a_{ii}] r_{ii} &= \sum [b, \frac{1}{n} \mathrm{Tr}(a)] r_{ii} + \sum [b, a_{ii} - \frac{1}{n} \mathrm{Tr}(a)] r_{ii}, \end{aligned}$$

where $\sum_{k\neq j}[b, a_{kj}]r_{kj}$ and $\sum_i [b, a_{ii} - \frac{1}{n}\mathrm{Tr}(a)]r_{ii}$ belong to $M_n^0(I)$, and $\sum_i [b, \frac{1}{n}\mathrm{Tr}(a)]r_{ii} \in [M, \mathcal{B}]$. From this (4.14) follows immediately. ■

Let $B(H)$ be the algebra of all bounded operators on a separable Hilbert space H. Then H is isomorphic to $K \oplus K$, so $B(H)$ is isomorphic to $B(K \oplus K)$ which, in turn, is isomorphic to $M_2(B(K))$. Calkin proved in [C] that, for each ideal I of $B(K)$, $N(I) = I + \mathbb{C}\mathbf{1}$. Hence if a Lie ideal M of $B(K)$ is related to I, then $[I, B(K)] \subseteq M \subseteq N(I) = I + \mathbb{C}\mathbf{1}$. Combining this with Corollary 4.18 yields a refinement of the results of [FMS] and [FM].

C1.3 **Corollary 4.19** *Let $B(H)$ be the algebra of all bounded operators on a separable Hilbert space H, let L be a Lie ideal of $B(H)$ and let $J = \mathrm{Id}([L, B(H)])$. Then*

(i) *[FMS, FM] $[J, B(H)] \subseteq L \subseteq J + \mathbb{C}\mathbf{1}$ and J is the only ideal of $B(H)$ related to L.*

(ii) *L and J are commutator equal: $[L, B(H)] = [J, B(H)]$.*

Let $C(H)$ be the algebra of all compact operators on H and let C_p, $1 \leq p < \infty$, be Schatten ideals of compact operators on H. Then, as above, $C(H)$ is isomorphic to $M_2(C(H))$ and all C_p are isomorphic to $M_2(C_p)$. As the algebra $C(H)$ and the algebras C_p are not locally unital, Corollary 4.18 can not be applied to them.

Problem 4.20 *Are Lie ideals of the algebra $C(H)$ and of the algebras C_p related to (embraced by) ideals of these algebras?*

Remark. De la Harpe in [Ha] (cf. [Mu]) showed that all Lie ideals L of $C(H)$ of finite codimension coincide with $C(H)$.

Denote by $M_\infty(\mathbb{F})$ the algebra of all infinite matrices with only finite number of non-zero entries. Let $\mathcal{B}$ be a locally unital $\mathbb{F}$-algebra and $M_\infty(\mathcal{B})$ be the algebra of all infinite matrices with entries from $\mathcal{B}$, only finitely many of which are non-zero. Then $M_\infty(\mathcal{B}) = \mathcal{B} \otimes \mathcal{P}$ where $\mathcal{P} = M_\infty(\mathbb{F})$. The algebra $\mathcal{P}$ is non-unital, locally unital, simple, centrally closed and $\dim(\mathcal{P}/[\mathcal{P}, \mathcal{P}]) = 1$, where $[\mathcal{P}, \mathcal{P}]$ is the Lie algebra of all matrices with zero trace. Applying the above results and arguments we get

infinmatr **Corollary 4.21** *Each Lie ideal L of $M_\infty(\mathcal{B})$ has form*

$$L = K \otimes \mathcal{P} + I \otimes [\mathcal{P}, \mathcal{P}], \text{ with } [I, \mathcal{B}] \subseteq K \subseteq I,$$

where K is a Lie ideal of $\mathcal{B}$ and I is an ideal of $\mathcal{B}$. It consists of all matrices $A = (a_{ij})$ from $M_\infty(\mathcal{B})$ with entries in I such that $\sum_i a_{ii} \in K$. It is only related to the ideal $J = I \otimes \mathcal{P}$ which consists of all matrices from $M_\infty(\mathcal{B})$ with entries in I. Moreover, L and J are commutator equal.

5 Lie ideals of Banach algebras.

In this section we will consider topological versions of the previous results. Although we will still investigate here arbitrary Lie ideals of Banach algebras, our main objects will be closed Lie ideals. For this we need some modifications of our main definitions and notations.

5.1 Generalities

Throughout this section we denote a Banach algebra by $\mathcal{A}$. The norm closure of a subset S of $\mathcal{A}$ is denoted by $\overline{S}$. We say that a Lie ideal L is *topologically embraced* by an ideal J if

$$\overline{[J,\mathcal{A}]} \subseteq \overline{L} \subseteq N(\overline{[J,\mathcal{A}]}). \tag{5.1}$$ toprestr

Condition (5.1) is weaker than the condition that L is embraced by J in the algebraic sense. However, it is sufficiently strong to characterize closed Lie ideals of $\mathcal{A}$ in terms of closed ideals. Namely if each closed Lie ideal of $\mathcal{A}$ is topologically embraced by a closed ideal, then the set of all closed Lie ideals of $\mathcal{A}$ consists of all closed subspaces that lie between $\overline{[J,\mathcal{A}]}$ and $N(\overline{[J,\mathcal{A}]})$, where J spans the set of all closed ideals of $\mathcal{A}$. If L and J are closed then (5.1) implies that L is related to J.

A stronger condition than (5.1) is the equality

$$\overline{[L,\mathcal{A}]} = \overline{[J,\mathcal{A}]}, \tag{5.2}$$ topcomeq

which again is weaker than the corresponding algebraic condition $[L,\mathcal{A}] = [J,\mathcal{A}]$. We say in this case that L and J are *topologically commutator equal.* For C*-algebras, we will show that a closed Lie ideal L is topologically embraced by an ideal J if and only if it is topologically commutator equal to J and that the both conditions are equivalent to the condition that L and J are related.

L4.1 **Lemma 5.1** *A closed Lie ideal L of $\mathcal{A}$ is related to $\overline{\mathrm{Id}([L,L])}$ if $[L,\mathcal{A}] \subseteq \overline{\mathrm{Id}([L,L])}$.*

Proof. Set $I_L = \mathrm{Id}([L,L])$. By the condition of the lemma, $L \subseteq N(\overline{I_L})$. By (2.5), $I_L \subseteq N(L)$. Since L is closed, $N(L)$ is closed. Hence $\overline{I_L} \subseteq N(L)$, so L is related to $\overline{I_L}$. ■

Clearly, a Banach algebra is semiprime if and only if, for each *closed* ideal I, $I^2 = \{0\}$ implies $I = \{0\}$. Repeating the argument of Proposition 2.7, we have

T4.1 **Proposition 5.2** *Let L be a closed Lie ideal of a Banach algebra $\mathcal{A}$ and let $I_L = \mathrm{Id}([L,L])$. If the Banach algebra $\mathcal{A}/\overline{I_L}$ is semiprime or commutative, then $[L,\mathcal{A}] \subseteq \overline{I_L}$, so L is related to $\overline{I_L}$.*

A Banach algebra $\mathcal{A}$ is *topologically simple* if $\mathcal{A}^2 \neq \{0\}$ and it has no *closed* ideals apart from $\{0\}$ and itself. For unital algebras this is equivalent to algebraic simplicity, but non-unital topologically simple Banach algebras are usually not simple. For example, the algebra $C(H)$ of all compact operators on a Hilbert space H and the Schatten ideals C_p, $1 \leq p < \infty$, are topologically simple Banach algebras but not simple algebras. Any topologically simple algebra is prime.

In the next subsection we will describe closed Lie ideals of some "differential" *-subalgebras of $C(H)$ and, in particular, of all symmetrically normed ideals of $C(H)$. As Banach algebras, they are all either topologically simple or have topologically simple ideals with commutative quotients. So we will establish now some results about Lie ideals of Banach algebras of this type. They can be considered as a topological version of the results on Lie ideals of simple algebras.

A tracial functional on $\mathcal{A}$ is a bounded functional satisfying $f(ab) = f(ba)$ for all $a, b \in \mathcal{A}$. Let $\mathrm{TF}(\mathcal{A})$ be the set of all non-zero tracial functionals on $\mathcal{A}$. As $\overline{[\mathcal{A},\mathcal{A}]} \subseteq \mathrm{Ker}(f)$, for each $f \in \mathrm{TF}(\mathcal{A})$, it follows from Hahn-Banach theorem that

$$\overline{[\mathcal{A},\mathcal{A}]} = \mathcal{A} \text{ if } \mathrm{TF}(\mathcal{A}) = \emptyset, \text{ and } \overline{[\mathcal{A},\mathcal{A}]} = \cap\ \{\mathrm{Ker}(f)\colon\ f \in\ \mathrm{TF}(\mathcal{A})\}. \tag{5.3}$$ 6.01

C4.2 **Theorem 5.3** *Let $\mathcal{A}$ be a topologically simple Banach algebra.*

(i) *A closed subspace L of $\mathcal{A}$ is a Lie ideal if and only if either $L \subseteq \mathfrak{Z}_{\mathcal{A}}$ or $\overline{[\mathcal{A},\mathcal{A}]} \subseteq L$.*

(ii) *If L is commutative then $L \subseteq \mathfrak{Z}_{\mathcal{A}}$; otherwise $\overline{[L,\mathcal{A}]} = \overline{[\mathcal{A},\mathcal{A}]}$.*

(iii) *$\mathcal{A}$ has only one closed non-central Lie ideal - $\mathcal{A}$, if and only if it has no non-zero tracial functionals.*

(iv) *Let $\mathcal{B}$ be a dense subalgebra of $\mathcal{A}$ and* $\mathrm{tr}(\cdot)$ *be a trace functional on $\mathcal{B}$ such that* $\mathrm{Ker}(\mathrm{tr}) \subseteq \overline{[\mathcal{B},\mathcal{B}]}$.

a) *If* tr *is unbounded on $\mathcal{B}$, then $\mathcal{A}$ has only one closed non-central Lie ideal - $\mathcal{A}$.*

b) *If* tr *is bounded, then $\mathcal{A}$ has two closed non-central Lie ideals: $\mathcal{A}$ and $\overline{[\mathcal{A},\mathcal{A}]}$; moreover, the Lie ideal $\overline{[\mathcal{A},\mathcal{A}]}$ has codimension* 1.

Proof. Part "if" in (i) is evident. Set $I_L = \mathrm{Id}([L,L])$. As $\mathcal{A}$ is topologically simple, either $\overline{I_L} = \mathcal{A}$ or $I_L = \{0\}$. Let $\overline{I_L} = \mathcal{A}$. By Proposition 2.2, $I_L \subseteq N(L)$. As L is closed, $N(L)$ is closed. Hence $N(L) = \mathcal{A}$, so $[\mathcal{A},\mathcal{A}] \subseteq L$. Part (i) is proved.

If $I_L = \{0\}$ then $[L,L] = \{0\}$, so L is commutative. As $\mathcal{A}$ is prime, by Proposition 2.4, $L \subseteq \mathfrak{Z}_{\mathcal{A}}$.

Let L be non-commutative, so there is $l \in L$ such that $l \notin \mathfrak{Z}_{\mathcal{A}}$. Consider the closed Lie ideal $\overline{[L,\mathcal{A}]}$. By the above argument, either $\overline{[L,\mathcal{A}]} \subseteq \mathfrak{Z}_{\mathcal{A}}$ or $[\mathcal{A},\mathcal{A}] \subseteq \overline{[L,\mathcal{A}]}$. As $\mathcal{A}$ is prime, it follows from (4.2) that there is $x \in \mathcal{A}$ such that $[l,x] \notin \mathfrak{Z}_{\mathcal{A}}$. Hence the first case does not hold. In the second case $\overline{[L,\mathcal{A}]} = \overline{[\mathcal{A},\mathcal{A}]}$. Part (ii) is proved.

Part (iii) follows from (i) and (5.3).

For any subspace E of $\mathcal{A}$, $\overline{[E,E]} = \overline{[\overline{E},\overline{E}]}$. Hence $\overline{\mathrm{Ker}(\mathrm{tr})} = \overline{[\mathcal{B},\mathcal{B}]} = \overline{[\mathcal{A},\mathcal{A}]}$. Any linear functional $f \neq 0$ on a dense linear subspace of a Banach space X is bounded if and only if $\overline{\mathrm{Ker}(f)} \neq X$, in which case $\overline{\mathrm{Ker}(f)}$ has codimension 1 in X. This completes the proof. ■

C5.1 **Corollary 5.4** *Let $\mathcal{A}$ be a semiprime Banach algebra and let it have a closed topologically simple ideal I such that $\mathcal{A}/I$ is commutative.*

(i) *If L is a closed non-central Lie ideal of $\mathcal{A}$ then $I = \overline{\mathrm{Id}([L,L])}$ and I and L are related.*

(ii) *If $\overline{[I,I]} = I$ then the set of all closed non-central Lie ideals of $\mathcal{A}$ consists of all closed subspaces L satisfying $I \subseteq L \subseteq \mathcal{A}$.*

Proof. If L is a closed Lie ideal of $\mathcal{A}$ then $[L,L] \subseteq [L,\mathcal{A}] \subseteq [\mathcal{A},\mathcal{A}] \subseteq I$. Hence $\overline{\mathrm{Id}[L,L]} \subseteq \overline{\mathrm{Id}[L,\mathcal{A}]} \subseteq \overline{\mathrm{Id}[\mathcal{A},\mathcal{A}]} \subseteq I$. As I is topologically simple, either $[L,L] = \{0\}$ or $\overline{\mathrm{Id}[L,L]} = I$. If $[L,L] = \{0\}$ then, by Proposition 2.4, $L \subseteq \mathfrak{Z}_{\mathcal{A}}$. If $\overline{\mathrm{Id}[L,L]} = I$ then, by Proposition 5.2, I and L are related.

Clearly, all L satisfying $I \subseteq L \subseteq \mathcal{A}$ are Lie ideals. Let $\overline{[I,I]} = I$. By (i), each closed non-central Lie ideal L is related to I, so $[I,\mathcal{A}] \subseteq L$. Hence $I = \overline{[I,I]} \subseteq \overline{[I,\mathcal{A}]} \subseteq L$ ■

5.2 Closed Lie ideals of Banach *-algebras of compact operators.

Let H be a separable Hilbert space. For $x, y \in H$, denote by $y \otimes x$ the rank one operator:

$$(y \otimes x)z = (z, y)x \text{ for } z \in H.$$

Then, for $x, y, u, v \in H$ and $A \in B(H)$,

$$(y \otimes x)(u \otimes v) = (v, y)(u \otimes x), \; A(y \otimes x) = y \otimes Ax \text{ and } (y \otimes x)A = A^* y \otimes x.$$

If R is a subspace of H, we denote by $\mathcal{F}(R)$ the linear subspace of $B(H)$ generated by all rank one operators $y \otimes x$, $x, y \in R$. In particular, $\mathcal{F}(H)$ is the algebra of all finite rank operators on H.

L6.4 **Lemma 5.5** *Let $\mathcal{B}$ be a *-subalgebra of $\mathcal{F}(H)$ dense in $C(H)$. Then there exists a dense subspace R of H such that $\mathcal{B} = \mathcal{F}(R)$.*

Proof. Let $a = a^* \in \mathcal{B}$. Then $a = \sum_{i=1}^n \lambda_i p_i$, where $n < \infty$, all λ_i are distinct and p_i are finite-dimensional mutually orthogonal projections. As $\lambda_k p_k \prod_{i \neq k}(\lambda_k - \lambda_i) = a \prod_{i \neq k}(a - \lambda_i \mathbf{1}) \in \mathcal{B}$, all $p_k \in \mathcal{B}$. Let a projection p belong to $\mathcal{B}$ and $e \in \mathrm{Range}(p)$. As $\mathcal{B}$ is dense in $C(H)$, $\|e \otimes e - a_n\| \to 0$ for some $a_n \in \mathcal{B}$. Thus $\|e \otimes e - p a_n p\| = \|p(e \otimes e - a_n)p\| \to 0$. As $p\mathcal{B}p$ is a subalgebra of the finite-dimensional algebra $p\mathcal{F}(H)p$, we have $e \otimes e \in \mathcal{B}$. Hence $\mathcal{B}$ is the linear space generated by all rank one operators $e \otimes e \in \mathcal{B}$.

Let $e \otimes e, f \otimes f \in \mathcal{B}$. For $a \in \mathcal{B}$,

$$(f, a^* e)(f \otimes e) = (a^* e \otimes e)(f \otimes f) = (e \otimes e)a(f \otimes f) \in \mathcal{B}.$$

As $\mathcal{B}$ is dense in $C(H)$, $(f, a^* e) \neq 0$ for some a, so $f \otimes e \in \mathcal{B}$. Similarly, $e \otimes f \in \mathcal{B}$, so $(e+f) \otimes (e+f) \in \mathcal{B}$. Thus there is a dense linear subspace R of H such that $\mathcal{B} = \mathcal{F}(R)$. $\blacksquare$

We denote by $\mathrm{Tr}(\cdot)$ the standard trace functional on $\mathcal{F}(H)$ and set $\mathcal{F}_0(R) := [\mathcal{F}(R), \mathcal{F}(R)]$ for any subspace R of H. It is well known that

$$\mathcal{F}_0(R) = \{a \in \mathcal{F}(R) \colon \mathrm{Tr}(a) = 0\} \text{ and } \mathcal{F}(R) = \mathcal{F}_0(R) \dot{+} \mathbb{C}(e \otimes e), \tag{5.4}$$ e5.2

for each $e \in R$. We will use now Theorem 5.3 to obtain the following result.

P6.4 **Proposition 5.6** *Let $\mathcal{A}$ be a dense *-subalgebra of $C(H)$ and a Banach *-algebra in norm $\|\cdot\|_{\mathcal{A}}$. Assume that $\mathcal{A} \cap \mathcal{F}(H)$ is dense in $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. Then $\mathcal{A}$ is topologically simple and*

(i) *if $\mathrm{Tr}(\cdot)$ is unbounded on $\mathcal{A} \cap \mathcal{F}(H)$, then $\mathcal{A}$ has only two closed Lie ideals: $\{0\}$ and $\mathcal{A}$;*

(ii) *if $\mathrm{Tr}(\cdot)$ is bounded on $\mathcal{A} \cap \mathcal{F}(H)$, then $\mathcal{A}$ has three closed Lie ideals: $\{0\}$, $\mathcal{A}$ and $\overline{[\mathcal{A}, \mathcal{A}]}$ that has codimension 1 in $\mathcal{A}$.*

Proof. As $\|\cdot\|_{\mathcal{A}}$ majorizes the operator norm in $C(H)$, $\mathcal{A} \cap \mathcal{F}(H)$ is dense in $C(H)$. By Lemma 5.5, $\mathcal{A} \cap \mathcal{F}(H) = \mathcal{F}(R)$ for some dense subspace R of H. Let I be an ideal of $\mathcal{A}$ and $0 \neq B \in I$. Then $Bx \neq 0$ for some $x \in R$. For all $e \in R$,

$$\|Bx\|^2 (e \otimes e) = (Bx \otimes e)(e \otimes Bx) = (x \otimes e)B^* B(e \otimes x) \in I.$$

Hence $e \otimes e \in I$, so $\mathcal{A} \cap \mathcal{F}(H) \subseteq I$. Therefore $\overline{I} = \mathcal{A}$ and $\mathcal{A}$ is topologically simple.

The proof of (i) and (ii) follows from (5.4) and Theorem 5.3(iv) a) and b), respectively. ■

Now we will consider various examples of Banach *-algebras of compact operators. A two-sided ideal $\mathcal{J}$ of a C*-algebra $(\mathfrak{A}, \|\cdot\|)$ is *symmetrically normed* (s. n.) if it is a Banach space with respect to a norm $\|\cdot\|_{\mathcal{J}}$ and

$$\|axb\|_{\mathcal{J}} \le \|a\|\|x\|_{\mathcal{J}}\|b\|, \text{ for all } a, b \in \mathfrak{A} \text{ and } x \in \mathcal{J}. \tag{5.5}$$ 7.8

Denote by Φ the set of all symmetric norming functions on the space of all sequences $\xi = \{\xi_i\}$ of real numbers converging to 0. Each $\phi \in \Phi$ defines a symmetrically normed ideal $(J^\phi, \|\cdot\|_\phi)$ of $B(H)$ (for the detailed discussion, see [GK]). For example, the functions

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{\frac{1}{p}}, \text{ for } 1 \le p < \infty$$

define the Schatten ideals C_p: $J^{\phi_p} = C_p$. The ideal $\mathcal{F}(H)$ lies in each J^ϕ, its closure $\overline{\mathcal{F}(H)}^\phi$ in $\|\cdot\|_\phi$ is a separable s. n. ideal J_0^ϕ and $J_0^\phi \subseteq J^\phi$. Each separable s. n. ideal of $B(H)$ is isomorphic to J_0^ϕ for some $\phi \in \Phi$. In many cases (for example, for all ϕ_p above) the ideals J_0^ϕ and J^ϕ coincide. It was proved in [BKS, Theorem 3.6] that

$$(J^\phi)^2 \subseteq J_0^\phi. \tag{5.6}$$ 6.1

It is well known that the functional $\mathrm{Tr}(\cdot)$ on $\mathcal{F}(H)$ is only bounded in $\|\cdot\|_{\phi_1}$. For the convenience of the reader we will prove it.

L6.1n **Lemma 5.7** *The linear functional* $\mathrm{Tr}(\cdot)$ *is not bounded in* $\|\cdot\|_\phi$, *if* $\phi \neq \phi_1$.

Proof. Let $\{e_n\}$ be an orthonormal basis in H. Consider the finite rank operators $A_n = \frac{1}{n}(e_1 \otimes e_1 + ... + e_n \otimes e_n)$. Then $\mathrm{Tr}(A_n) = 1$. If $\phi \neq \phi_1$ then (see [GK, III.4])

$$\|A_n\|_\phi = \frac{1}{n}\|e_1 \otimes e_1 + ... + e_n \otimes e_n\|_\phi = \frac{1}{n}\phi(\underbrace{1, ..., 1}_{n}, 0, ...) \to 0, \tag{5.7}$$ 7.10

as it follows from (3.18) of [GK] that $\sup_n \frac{n}{\phi(\underbrace{1, ..., 1}_{n}, 0, ...)} = \infty$. Hence $\mathrm{Tr}(\cdot)$ is not bounded in $\|\cdot\|_\phi$. ■

For each s. n. ideal $\mathcal{J}$ of $B(H)$ (see [GK]), there exists a function $\phi \in \Phi$ such that

$$J_0^\phi \subseteq \mathcal{J} \subseteq J^\phi; \tag{5.8}$$ 71

the first inclusion is isometric and the second one is continuous. It is known (see, for example, Corollary 4.10 in [KS]) that each s. n. ideal $\mathcal{J}$ of $C(H)$ also satisfies (5.8). (Note that ideals of $C(H)$ are not necessarily ideals of $B(H)$ (see [FM, p. 451]).)

T4.2 **Theorem 5.8** (i) *Let* $\mathcal{J} = J_0^\phi \neq C_1$ *be a separable s. n. ideal of* $B(H)$ *or* $\mathcal{J} = C(H)$. *Then* $\mathcal{J}$ *has only two closed Lie ideals*: $\{0\}$ *and* $\mathcal{J}$.

(ii) C_1 *has only three closed non-zero Lie ideals*: $\{0\}$, $C_1^0 = \overline{\mathcal{F}_0(H)}$ *and* C_1; *furthermore*

$$\overline{[C_1^0, C_1^0]} = \overline{[C_1^0, C_1]} = \overline{[C_1, C_1]} = C_1^0 \subsetneqq C_1.$$

(iii) *Let* $\mathcal{J}$ *be a non-separable s. n. ideal of* $C(H)$ *and* $J_0^\phi \subsetneqq \mathcal{J} \subseteq J^\phi$. *A closed subspace* $L \neq \{0\}$ *of* $\mathcal{J}$ *is a Lie ideal if and only if it contains* J_0^ϕ. *Moreover, in this case* L *is an ideal of* $\mathcal{J}$ *and*

$$\overline{[L, L]} = \overline{[L, \mathcal{J}]} = \overline{[\mathcal{J}, \mathcal{J}]} = J_0^\phi.$$

Proof. By Lemma 5.7, $\mathrm{Tr}(\cdot)$ is not bounded in $\|\cdot\|_\phi$, for $\phi \neq \phi_1$. It is well known that $\mathrm{Tr}(\cdot)$ is bounded on C_1. Hence (i) and (ii) follow from Proposition 5.6.

The "if" part in (iii) follows from (5.6). Each non-zero ideal I of $\mathcal{J}$ contains $\mathcal{F}(H)$ and $\mathcal{F}(H)^2 = \mathcal{F}(H)$. Hence $\mathcal{J}$ is a prime algebra and $\mathfrak{Z}_{\mathcal{J}} = \{0\}$. As $\mathrm{Tr}(\cdot)$ is not bounded in $\|\cdot\|_\phi$, for $\phi \neq \phi_1$, we have $\overline{\mathcal{F}_0(H)}^\phi = \overline{\mathcal{F}(H)}^\phi = J_0^\phi$. Therefore

$$J_0^\phi = \overline{\mathcal{F}_0(H)}^\phi = \overline{[\mathcal{F}(H), \mathcal{F}(H)]}^\phi \subseteq \overline{[J_0^\phi, J_0^\phi]}^\phi \subseteq J_0^\phi,$$

so

$$\overline{[J_0^\phi, J_0^\phi]}^\phi = J_0^\phi. \tag{5.9}$$ e5.5

Hence the "only if" part in (iii) follows from (5.6) and Corollary 5.4. ■

Let S be a densely defined, symmetric, closed operator on H with domain $D(S)$. Set

$$\begin{aligned} \mathcal{K}_S \;=\; & \{a \in C(H)\colon\ aD(S) \subseteq D(S),\ a^*D(S) \subseteq D(S) \text{ and } [S,a]|_{D(S)} \\ & \text{extends to a bounded operator } a_S \text{ on } H\}; \end{aligned}$$

$$\text{and } \mathcal{J}_S = \{a \in \mathcal{K}_S\colon\ a_S \in C(H)\}.$$

The subalgebra $\mathcal{F}^S$ of $\mathcal{F}(H)$ generated by all rank one operators $x \otimes y$ with $x, y \in D(S)$ is dense in $C(H)$ and $\mathcal{F}^S = \mathcal{F}(D(S))$. Let Φ_S be the closure of $\mathcal{F}^S$ in the norm

$$\|a\|_S = \|a\| + \|[S, a]\|.$$

Then $\Phi_S \subseteq \mathcal{J}_S \subseteq \mathcal{K}_S$. It was proved in [KS1] that $\mathcal{K}_S$, $\mathcal{J}_S$ and Φ_S are Banach *-algebras, that $\mathcal{F}^S$ is contained in every ideal of Φ_S and $(\mathcal{F}^S)^2 = \mathcal{F}^S$. Moreover,

$$\overline{(\mathcal{K}_S)^2} = \overline{(\mathcal{J}_S)^2} = \Phi_S.$$

Hence the algebras Φ_S, $\mathcal{J}_S$ and $\mathcal{K}_S$ are prime. If S is selfadjoint then $\Phi_S = \mathcal{J}_S \neq \mathcal{K}_S$. Thus Φ_S is topologically simple and it follows from Proposition 5.6 that

(i) Φ_S has only two closed Lie ideals: $\{0\}$ and Φ_S, if $\mathrm{Tr}(\cdot)$ is unbounded on $\mathcal{F}^S$,

(ii) Φ_S has three closed Lie ideals: $\{0\}$, $\overline{\mathcal{F}_0^S} = \overline{[\mathcal{F}^S, \mathcal{F}^S]}$ and Φ_S, if $\mathrm{Tr}(\cdot)$ is bounded on $\mathcal{F}^S$.

P5.4e **Proposition 5.9** *Let* S *be a symmetric, closed operator. Suppose that there are* $c > 0$ *and unit vectors* $\{e_n\}_{n=1}^\infty$ *in* $D(S)$ *such that all subspaces* $\{e_n, Se_n\}$ *are mutually orthogonal and* $\|[S, e_n \otimes e_n]\| \leq c$. *Then* $\mathrm{Tr}(\cdot)$ *is unbounded on* $\mathcal{F}^S$, *so*

(i) Φ_S *has only two closed Lie ideals*: $\{0\}$ *and* Φ_S;

(ii) *a closed subspace* L *of* $\mathcal{K}_S$ *is a Lie ideal if and only if* $\Phi_S \subseteq L \subseteq \mathcal{K}_S$. *Moreover,* L *is an ideal of* $\mathcal{K}_S$ *and*

$$\overline{[L,L]} = \overline{[L,\mathcal{K}_S]} = \overline{[\mathcal{K}_S,\mathcal{K}_S]} = \Phi_S.$$

The same is true for the algebra $\mathcal{J}_S$.

Proof. Consider $a_n = \frac{1}{n}(e_1 \otimes e_1 + ... + e_n \otimes e_n)$. Then $\mathrm{Tr}(a_n) = 1$ and

$$\begin{aligned} \|a_n\|_S &= \|a_n\| + \|[S,a_n]\| = \frac{1}{n}\|e_1 \otimes e_1 + ... + e_n \otimes e_n\| \\ &\quad + \frac{1}{n}\|[S,e_1 \otimes e_1] + ... + [S,e_n \otimes e_n]\| \\ &= \frac{1}{n}\sup\|e_i \otimes e_i\| + \frac{1}{n}\sup\|[S,e_i \otimes e_i]\| \to 0, \end{aligned}$$

as $n \to \infty$. Hence Tr is unbounded on $\mathcal{F}^S$. This implies (i). Part (ii) can be proved in the same way as part (iii) in Theorem 5.8. ■

In particular, all selfadjoint and all maximal symmetric operators satisfy conditions of Proposition 5.9.

Problem 5.10 *Do there exist symmetric operators* S *for which* $\mathrm{Tr}(\cdot)$ *is bounded on* $\mathcal{F}^S$?

5.3 Closed Lie ideals of projective tensor products $\mathcal{B}\widehat{\otimes}\mathcal{J}$.

In the theory of C*-algebras an important role is played by stable C*-algebras, that is, by the C*-tensor products $\mathcal{B} \otimes_{\min} C(H)$, where $\mathcal{B}$ is a C*-algebra. In the category of Banach algebras a natural tensor product is the projective one. We will show that in the projective tensor products $\mathcal{B}\widehat{\otimes}C(H)$, where $\mathcal{B}$ are unital Banach algebras, and, more generally, in $\mathcal{B}\widehat{\otimes}\mathcal{J}$, where $\mathcal{J} \neq C_1$ are separable s. n. ideals of $B(H)$, all closed Lie ideals are ideals. This can be viewed as a Banach algebraic counterpart of some results in Section 4. A similar result for stable C*-algebras will be obtained in Section 5.5.

For each algebra $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{op}}$ the opposite algebra, that is, the same linear space with multiplication $a \circ b = ba$. The following auxiliary result is taken from [SS]; we will give its proof for the convenience of the reader.

tanya **Lemma 5.11** *Let* $M = M_n(\mathbb{C})$ *and let* V *be the subalgebra of the algebra* $M \otimes M^{\mathrm{op}}$ *generated by all elements* $a \otimes \mathbf{1} - \mathbf{1} \otimes a$ *with* $a \in M$. *Then*

$$V = \left\{ \sum_i a_i \otimes b_i : \sum_i a_i b_i = \sum_i b_i a_i = 0 \right\}. \qquad (5.10)$$ descr

Proof. Denote the right hand side of (5.10) by U. One can easily check that U is a subalgebra of $M \otimes M^{\mathrm{op}}$ and $V \subseteq U$. Define the representation π of $M \otimes M^{\mathrm{op}}$ on the space M by the equality: $\pi(a \otimes b)(x) = axb$. Then π is irreducible and faithful, as $M \otimes M^{\mathrm{op}}$ is simple (see Lemma 4.10). So it suffices to show that $\pi(U) = \pi(V)$.

Set $M^0 = \{x \in M\colon \operatorname{Tr}(x) = 0\}$. Then $M^0 = [M, M]$ and $M = M^0 \dotplus \mathbb{C}\mathbf{1}$. For $T = \sum_i a_i \otimes b_i \in U$,

$$\operatorname{Tr}\,(\pi(T)x) = \operatorname{Tr}\left(\sum_i a_i x b_i\right) = \operatorname{Tr}\left(x \sum_i b_i a_i\right) = 0, \text{ for each } x \in M.$$

Therefore $\pi(U)M \subseteq M^0$. Clearly, $\mathbb{C}\mathbf{1} \subseteq \operatorname{Ker}\,\pi(U) \subseteq \operatorname{Ker}\,\pi(V)$. As

$$\pi(a \otimes \mathbf{1} - \mathbf{1} \otimes a)x = ax - xa, \text{ for } a, x \in M, \tag{5.11}$$ 7.4

$x \in \operatorname{Ker}\,\pi(V)$ only if $xa = ax$ for all $a \in M$. Hence $\mathbb{C}\mathbf{1} = \operatorname{Ker}\,\pi(U) = \operatorname{Ker}\,\pi(V)$.

We have from (5.11) that if L is an invariant subspace for $\pi(V)$, then L is a Lie ideal of M. Thus if L is an invariant subspace of $\pi(V)$ in M^0, it follows from Theorem 1.1 that either $L = \{0\}$ or $L = M^0$. Hence $\pi(V)$ has no non-trivial invariant subspaces in M^0. By Burnside's Theorem, $\pi(V)|M^0$ coincides with the algebra $B(M^0)$ of all operators on M^0. Therefore $\pi(V) = B(M^0) \dotplus \{0\}$ on $M = M^0 \dotplus \mathbb{C}\mathbf{1}$. As $\pi(V) \subseteq \pi(U)$ and $\operatorname{Ker}\,\pi(U) = \mathbb{C}\mathbf{1}$, we have $\pi(V) = \pi(U)$. ■

We will assume below that H is an infinite dimensional Hilbert space.

L7.1 **Lemma 5.12** *Let K be a subspace of H, $\dim(K) = n < \infty$ with an orthonormal basis $\{f_i\}_{i=1}^n$. For each N, there is a subspace K_N of H with an orthonormal basis $\{e_k\}_{k=1}^{nN}$ such that $K \subset K_N$ and $|(f_i, e_k)| \le \frac{1}{\sqrt{N}}$ for all i and k.*

Proof. Let $\{R_i\}_{i=1}^n$ be N-dimensional mutually orthogonal subspaces of H such that $f_i \in R_i$. Let $\{g_i^m\}_{m=1}^N$ be an orthonormal basis in R_i. Set $g_i = \frac{1}{\sqrt{N}} \sum_{m=1}^N g_i^m$. Then $\|g_i\| = 1$. Let U_i, $i = 1, .., n$, be unitary operators on R_i such that $U_i g_i = f_i$. Then $\{e_{im} = U_i g_i^m\colon 1 \le i \le n, 1 \le m \le N\}$ is the required basis in $K_N = R_1 \oplus ... \oplus R_n$. ■

For each unit vector $e \in H$, by $p_e = e \otimes e$ is the orthogonal projection onto $\mathbb{C}e$. If $\mathcal{E} = \{e_i\}_{i=1}^n$ is a finite family of pairwise orthogonal unit vectors in H, we set $p_{\mathcal{E}} = \sum_i p_{e_i}$. Let $\mathcal{J} = J_0^\phi$ be a separable s. n. ideal of $B(H)$. We define operators $s_{\mathcal{E}}$ and $t_{\mathcal{E}}$ on $\mathcal{J}$ by

$$s_{\mathcal{E}}(x) = \sum_i p_{e_i} x p_{e_i} \text{ and } t_{\mathcal{E}}(x) = p_{\mathcal{E}} x p_{\mathcal{E}}, \text{ for } x \in \mathcal{J}. \tag{5.12}$$ 5.55

As all projections p_{e_i}, for $e_i \in \mathcal{E}$, are mutually orthogonal, it can be deduced from Theorem III.4.2 of [GK] that $\|s_{\mathcal{E}}(x)\|_\phi \le \|x\|_\phi$ for $x \in \mathcal{J}$. It also follows from (5.5) that $\|t_{\mathcal{E}}(x)\|_\phi = \|p_{\mathcal{E}} x p_{\mathcal{E}}\|_\phi \le \|x\|_\phi$. Hence

$$\|s_{\mathcal{E}}\| = \|t_{\mathcal{E}}\| = 1. \tag{5.13}$$ ogr

L7.2 **Lemma 5.13** *Let $\mathcal{J} = J_0^\phi \neq C_1$ be a separable s. n. ideal or $\mathcal{J} = C(H)$. Then for each finite family $(x_1, ..., x_m)$ in $\mathcal{F}(H)$ and each $\varepsilon > 0$, there is $\mathcal{E}$ with*

$$\|s_{\mathcal{E}}(x_i)\|_J < \varepsilon \text{ and } x_i = t_{\mathcal{E}}(x_i), \text{ for } 1 \le i \le m. \tag{5.14}$$ 7.12

Proof. As all $x_i \in \mathcal{F}(H)$, there is a finite-dimensional subspace K of H such that $x_i H \subset K$ and $x_i^* H \subset K$, $1 \le i \le m$. Let $\{f_j\}_{j=1}^n$ be an orthonormal basis in K. Then all x_i are linear combinations of the rank one operators $f_p \otimes f_j$. Let K_N, $K \subset K_N$, be a subspace of H constructed in Lemma 5.12 and $\mathcal{E} = \{e_k\}_{k=1}^{nN}$ be the basis in K_N such that $|(e_k, f_j)| \leq \frac{1}{\sqrt{N}}$. Then $t_{\mathcal{E}}(x_i) = p_{\mathcal{E}} x_i p_{\mathcal{E}} = x_i$. As $\phi \neq \phi_1$ (in particular, when $\phi = \phi_\infty$, we have $\mathcal{J} = C(H)$), we obtain as in (5.7) (see [GK, III.4]) that

$$\begin{aligned} \|s_{\mathcal{E}}(f_p \otimes f_j)\|_{\mathcal{J}} &= \left\| \sum_{k=1}^{nN} p_{e_k}(f_p \otimes f_j) p_{e_k} \right\|_{\mathcal{J}} = \left\| \sum_{k=1}^{nN} (e_k, f_p)(f_j, e_k) p_{e_k} \right\|_{\mathcal{J}} \\ &\leq \left\| \sum_{k=1}^{nN} \frac{1}{N} p_{e_k} \right\|_{\mathcal{J}} = \frac{1}{N} \phi(\underbrace{1, ..., 1}_{nN}, 0, ...) = n \left(\frac{1}{nN} \phi(\underbrace{1, ..., 1}_{nN}, 0, ...) \right) \to 0. \end{aligned}$$

as $N \to \infty$. Hence $\|s_{\mathcal{E}}(f_p \otimes f_j)\|_\phi \to 0$, as $N \to \infty$, for all p and j. As $s_{\mathcal{E}}$ is linear and n is fixed, for each $\varepsilon > 0$, we can find N such that $\|s_{\mathcal{E}}(x_i)\|_\phi < \varepsilon$, for $1 \leq i \leq m$. Thus (5.14) is proved. ■

Let $\mathcal{B}$ be a unital Banach algebra and $\mathcal{J}$ be a separable s. n. ideal of $B(H)$. Let τ be an algebraic cross-norm on the algebraic tensor product $\mathcal{B} \otimes \mathcal{J}$. Then its completion $\mathcal{A} = \mathcal{B} \otimes_\tau \mathcal{J}$ with respect to τ is a Banach algebra. We consider the algebraic tensor product $\mathcal{A}_0 = \mathcal{B} \otimes \mathcal{F}(H)$ as a subalgebra of $\mathcal{A}$. As $\mathcal{J}$ is separable, $\mathcal{F}(H)$ is dense in $\mathcal{J}$, so $\mathcal{A} = \overline{\mathcal{A}_0}^\tau$. Let id be the identity operator on $\mathcal{B}$. For $\mathcal{E} = \{e_i\}_{i=1}^k$, consider the operators

$$S_{\mathcal{E}} = \text{ id} \otimes s_{\mathcal{E}} \text{ and } T_{\mathcal{E}} = \text{ id} \otimes t_{\mathcal{E}}$$

on $\mathcal{A}_0$. We will impose on τ the condition that the families of operators $S_{\mathcal{E}}$ and $T_{\mathcal{E}}$ are uniformly bounded: there is $k > 0$ such that, with respect to the norm τ,

$$\|S_{\mathcal{E}}\| \leq k \text{ and } \|T_{\mathcal{E}}\| \leq k \text{ for each } \mathcal{E}. \tag{5.15}$$ 5.u

Our aim is to show that under this condition all closed Lie ideals of $\mathcal{A}$ are ideals.

density **Proposition 5.14** *Let $\mathcal{J} \neq C_1$ be a separable s. n. ideal of $B(H)$ or $\mathcal{J} = C(H)$. Suppose that* (5.15) *holds for a norm τ. Then $L \cap \mathcal{A}_0$ is dense in L for each Lie ideal L of $\mathcal{A}$.*

Proof. Let $\mathcal{E} = \{e_i\}_{i=1}^k$. As the operators $S_{\mathcal{E}}$ and $T_{\mathcal{E}}$ are bounded on $\mathcal{A}_0$, they extend to $\mathcal{A}$; we denote their extensions by $S_{\mathcal{E}}$ and $T_{\mathcal{E}}$. They map $\mathcal{A}$ into $\mathcal{A}_0$. Indeed, let R be the linear span of $\mathcal{E}$ and $\mathcal{F}(R)$ be the subspace of all operators $y \in \mathcal{F}(H)$ such that $yH \subset R$, $y^*H \subset R$. For each $x \in J$, we have $t_{\mathcal{E}}(x), s_{\mathcal{E}}(x) \in \mathcal{F}(R)$. If $a = \sum_{i=1}^m b_i \otimes x_i \in \mathcal{A}_0$ then $T_{\mathcal{E}}(a) = \sum_{i=1}^m b_i \otimes t_{\mathcal{E}}(x_i) \in \mathcal{B} \otimes \mathcal{F}(R)$. As $\mathcal{F}(R)$ is finite-dimensional, $\mathcal{B} \otimes \mathcal{F}(R)$ is a closed subspace of $\mathcal{A}$ and $\mathcal{B} \otimes \mathcal{F}(R) \subset \mathcal{A}_0$. As $T_{\mathcal{E}}$ is bounded, it maps $\mathcal{A}$ into $\mathcal{B} \otimes \mathcal{F}(R)$. The same is true for $S_{\mathcal{E}}$. Hence $T_{\mathcal{E}}(a) - S_{\mathcal{E}}(a) \in \mathcal{A}_0$, for each $a \in \mathcal{A}$.

Let us show that $T_{\mathcal{E}}(a) - S_{\mathcal{E}}(a) \in L$ if $a \in L$. For each matrix $u = (u_{ij}) \in M_k(\mathbb{C}) = M_k$, the operator $\widehat{u} = \sum u_{ij}(e_j \otimes e_i)$ lies in $\mathcal{F}(R)$. We define a bilinear map Ψ from $M_k \times M_k$ into the space $B(\mathcal{A})$ of all bounded operators on $\mathcal{A}$ by setting $\Psi(u, v)(b \otimes x) = b \otimes \widehat{u} x \widehat{v}$. It extends to the homomorphism from $M_k \otimes M_k^{\text{op}}$ into $B(\mathcal{A})$ which will be also denoted by Ψ.

As $\Psi(\widehat{u} \otimes \mathbf{1}_k - \mathbf{1}_k \otimes \widehat{u})(b \otimes x) = b \otimes [\widehat{u}, x] = [\mathbf{1}_{\mathcal{B}} \otimes \widehat{u}, b \otimes x]$, we have

$$\Psi(u \otimes \mathbf{1}_k - \mathbf{1}_k \otimes u)a = [\mathbf{1}_{\mathcal{B}} \otimes \widehat{u}, a] \text{ for all } a \in \mathcal{A}.$$

Hence L is invariant under all operators $\Psi(u\otimes \mathbf{1}_k - \mathbf{1}_k\otimes u)$, $u\in M_k$. By Lemma 5.11, L is invariant under all operators $\{\Psi(\sum_i u_i\otimes v_i)\colon\ \sum_i u_iv_i = \sum_i v_iu_i = 0,\ u_i, v_i\in M_k\}$. As $T_{\mathcal{E}} = \Psi(\mathbf{1}_k\otimes\mathbf{1}_k)$ and $S_{\mathcal{E}} = \Psi(\sum_{i=1}^k p_i\otimes p_i)$,

$$T_{\mathcal{E}} - S_{\mathcal{E}} = \Psi(\mathbf{1}_k\otimes\mathbf{1}_k - \sum_{i=1}^k p_i\otimes p_i)$$

and $\mathbf{1} - \sum_{i=1}^k p_ip_i = 0$, so L is invariant under $T_{\mathcal{E}} - S_{\mathcal{E}}$. Thus $T_{\mathcal{E}} - S_{\mathcal{E}}$ maps L into $L\cap\mathcal{A}_0$.

We claim that, for each $a\in\mathcal{A}$ and $\varepsilon > 0$, there is $\mathcal{E}$ such that

$$\|S_{\mathcal{E}}(a)\|_{\mathcal{A}} < \varepsilon \text{ and } \|a - T_{\mathcal{E}}(a)\|_{\mathcal{A}} < \varepsilon. \tag{5.16}$$ verymalo

As $\mathcal{A}_0$ is dense in $\mathcal{A}$ and all $\|S_{\mathcal{E}}\|\le k$ and $\|T_{\mathcal{E}}\|\le k$, it suffices to prove (5.16) for $a\in\mathcal{A}_0$. Let $a = \sum_{i=1}^m b_i\otimes x_i$ with all $x_i\in\mathcal{F}(H)$. Then $S_{\mathcal{E}}(a) = \sum_{i=1}^m b_i\otimes s_{\mathcal{E}}(x_i)$ and

$$\|S_{\mathcal{E}}(a)\|_{\mathcal{A}} \le \sum_{i=1}^m \|b_i\|_{\mathcal{B}}\,\|s_{\mathcal{E}}(x_i)\|_{\mathcal{J}} \le \beta\max_i\|s_{\mathcal{E}}(x_i)\|_{\mathcal{J}},$$

where $\beta = \sum_i\|b_i\|_{\mathcal{B}}$. To obtain (5.16), take the set $\mathcal{E}$ constructed in Lemma 5.13 for $(x_1, ..., x_m)$ and ε/β instead of ε. Then $\|S_{\mathcal{E}}(a)\|_{\mathcal{A}} < \varepsilon$ and $T_{\mathcal{E}}(a) = \sum_i b_i\otimes t_{\mathcal{E}}(x_i) = \sum_i b_i\otimes x_i = a$. Thus this claim is also proved.

As a consequence of (5.16) we have that, for each $a\in\mathcal{A}$ and $\varepsilon > 0$, there is $\mathcal{E}$ such that

$$\|a - (T_{\mathcal{E}}(a) - S_{\mathcal{E}}(a))\|_{\mathcal{A}} < 2\varepsilon.$$

This concludes the proof. ■

C7.3 **Theorem 5.15** *Let $\mathcal{B}$ be a unital Banach algebra, let $\mathcal{J}\neq C_1$ be a separable s. n. ideal of $B(H)$ or $\mathcal{J} = C(H)$ and let $\mathcal{A} = \mathcal{B}\otimes_\tau\mathcal{J}$ be a Banach algebra. Suppose that the families of operators $\{S_{\mathcal{E}}\}$ and $\{T_{\mathcal{E}}\}$ are uniformly bounded in the norm τ (see* (5.15)*). Then a closed subspace L of $\mathcal{A}$ is a Lie ideal if and only if it is an ideal. Moreover, L is of the form $L = \overline{I\otimes\mathcal{J}}^\tau \cong I\otimes_\tau\mathcal{J}$, where I is a closed ideal of $\mathcal{B}$.*

Proof. The "if" part is evident. Let L be a closed Lie ideal of $\mathcal{A}$. Set $\mathcal{F} = \mathcal{F}(H)$, $\mathcal{A}_0 = \mathcal{B}\otimes\mathcal{F}$ and $L_0 = L\cap\mathcal{A}_0$. As $\mathcal{F}$ is a simple non-unital algebra and $\dim(\mathcal{F}/[\mathcal{F},\mathcal{F}]) = 1$, Theorem 4.14(iv) implies that there is a Lie ideal K of $\mathcal{B}$ and an ideal I_0 of $\mathcal{B}$ such that

$$L_0 = K\otimes\mathcal{F} + I_0\otimes[\mathcal{F},\mathcal{F}] \text{ and } [I_0,\mathcal{B}]\subseteq K\subseteq I_0.$$

Let $\mathcal{J} = J_0^\phi$ (in particular, $\phi = \phi_\infty$, so $\mathcal{J} = C(H)$). By Lemma 5.7, $\overline{[\mathcal{F},\mathcal{F}]}^\phi = \mathcal{J}$. Hence

$$L_0\subseteq I_0\otimes\mathcal{J}\subseteq\overline{I_0\otimes[\mathcal{F},\mathcal{F}]}^\tau\subseteq\overline{L_0}^\tau.$$

Set $I = \overline{I_0}$. By Proposition 5.14, $\overline{L_0}^\tau = L$. Hence $L = \overline{I_0\otimes\mathcal{J}}^\tau = \overline{I\otimes\mathcal{J}}^\tau$. ■

Let A and B be bounded operators on Banach spaces X and Y respectively. It is well known that the operator $A\otimes B$ on the projective tensor product $X\widehat{\otimes}Y$ has norm $\|A\|\|B\|$. The same is true for $A\otimes B$ on the tensor product $X\otimes_\lambda Y$ with respect to the injective cross norm λ. Hence, by (5.13), $\|S_{\mathcal{E}}\| = \|T_{\mathcal{E}}\| = 1$ on $\mathcal{B}\widehat{\otimes}\mathcal{J}$ and on $\mathcal{B}\otimes_\lambda\mathcal{J}$. We formulate the result of Theorem 5.15 for these tensor products separately.

C7.4 **Corollary 5.16** *Let $\mathcal{B}$ be a unital Banach algebra and let $\mathcal{J} \neq C_1$ be a separable s. n. ideal of $B(H)$ or $\mathcal{J} = C(H)$.*

(i) *A closed subspace L of the projective tensor product $\mathcal{B}\widehat{\otimes}\mathcal{J}$ is a Lie ideal if and only if it is an ideal. Moreover, $L \cong I\widehat{\otimes}\mathcal{J}$ where I is a closed ideal of $\mathcal{B}$.*

(ii) *If $\mathcal{B} \otimes_\lambda \mathcal{J}$ is a Banach algebra, then its closed subspace L is a Lie ideal if and only if L is an ideal. Moreover, $L \cong I \otimes_\lambda \mathcal{J}$ where I is a closed ideal of $\mathcal{B}$.*

Consider now the projective tensor product $\mathcal{A} = \mathcal{B}\widehat{\otimes}C_1$. By Theorem 5.8, $C_1^0 = \{a \in C_1\colon \mathrm{Tr}(a) = 0\}$ is the only non-trivial closed Lie ideal of C_1 and $C_1 = \mathbb{C}(e_1 \otimes e_1) \dotplus C_1^0$. Let K be a closed Lie ideal of $\mathcal{B}$ and I be a closed ideal of $\mathcal{B}$ such that $[I, \mathcal{B}] \subseteq K \subseteq I$. Then it follows from (4.6) that

$$L(K, I) = K\widehat{\otimes}C_1 + I\widehat{\otimes}C_1^0 = K \otimes (e_1 \otimes e_1) \dotplus I\widehat{\otimes}C_1^0$$

is a closed Lie ideal of $\mathcal{A}$. Repeating the proof of Theorem 5.15, we also obtain that if L is a closed Lie ideal of $\mathcal{A}$ and $L \cap \mathcal{A}_0 \neq \{0\}$, where $\mathcal{A}_0 = \mathcal{B} \otimes \mathcal{F}(H)$, then L contains a closed Lie ideal $L(K, I)$ for some K and I.

Problem 5.17 *Do all closed Lie ideals of $\mathcal{B}\widehat{\otimes}C_1$ have form $L(K, I) = K \otimes (e_1 \otimes e_1) \dotplus I\widehat{\otimes}C_1^0$ for some K and I?*

5.4 Lie ideals of W*-algebras.

In this subsection we study the algebraic relation between Lie ideals and ideals of W*-algebras $\mathcal{A}$. In particular, we show that each Lie ideal L of $\mathcal{A}$ is commutator equal to the ideal $J = \mathrm{Id}([L, \mathcal{A}])$: $[L, \mathcal{A}] = [J, \mathcal{A}]$. To do this we first establish the following result.

5.p **Proposition 5.18** *Each projection in a W*-algebra $\mathcal{A}$ is locally cyclic* (*see Definition* 3.5).

Proof. Let p and q be orthogonal projections in $\mathcal{A}$ and $p + q = \mathbf{1}$. Assume first that $p \prec q$, that is, there is a projection $p_1 \leq q$ and a partial isometry u such that $u^*u = p$, $uu^* = p_1$. Let $x_1, x_2 \in p\mathcal{A}q$. As $p_1u = u$, we have $y_i = ux_i \in p_1\mathcal{A}q$. Let $y_i = |y_i|\, v_i$ be their polar decompositions, where $|y_i| = (y_iy_i^*)^{1/2} \in p_1\mathcal{A}p_1$ and $v_i \in p_1\mathcal{A}q \subseteq q\mathcal{A}q$. As $p_1v_i = v_i$,

$$x_i = px_i = u^*y_i = u^*\,|y_i|\,v_i = u^*\,|y_i|\,p_1v_i = (u^*\,|y_i|\,u)(u^*p_1)v_i = (u^*\,|y_i|\,u)ev_i,$$

where $u^*\,|y_i|\,u \in p\mathcal{A}p$ and $e = u^*p_1 \in p\mathcal{A}p_1 \subseteq p\mathcal{A}q$. Thus $x_i \in (p\mathcal{A}p)e(q\mathcal{A}q)$.

Similarly, if $x_1, x_2 \in q\mathcal{A}p$, then $y_i = x_iu^* \in q\mathcal{A}p_1$, as $u^*p_1 = p_1$. Let $y_i = w_i\,|y_i|$ be their polar decompositions, where $|y_i| = (y_i^*y_i)^{1/2} \in p_1\mathcal{A}p_1$ and $w_i \in q\mathcal{A}p_1 \subseteq q\mathcal{A}q$. Then, as $w_ip_1 = w_i$,

$$x_i = x_ip = y_iu = w_i\,|y_i|\,u = w_ip_1\,|y_i|\,u = w_i(p_1u)(u^*\,|y_i|\,u) = w_ie(u^*\,|y_i|\,u),$$

where $u^*\,|y_i|\,u \in p\mathcal{A}p$ and $e = p_1u \in p_1\mathcal{A}p \subseteq q\mathcal{A}p$. Thus $x_i \in (q\mathcal{A}q)e(p\mathcal{A}p)$. Therefore p and q are locally cyclic.

Let now p and q be not comparable. Then (see [S, Theorem 2.1.3]) there is a central projection z such that $zp \prec zq$ and $(\mathbf{1} - z)q \prec (\mathbf{1} - z)p$. By the above argument, for each pair $x_1, x_2 \in p\mathcal{A}q$, there exist $e_1 \in zp\mathcal{A}q$ and $e_2 \in (\mathbf{1} - z)p\mathcal{A}q$ such that $zx_i \in (zp\mathcal{A}p)e_1(zq\mathcal{A}q)$ and $(\mathbf{1} - z)x_i \in ((\mathbf{1} - z)p\mathcal{A}p)e_2((\mathbf{1} - z)q\mathcal{A}q)$, for $i = 1, 2$. Then $e = e_1 + e_2 \in p\mathcal{A}q$, $e_1 = ze$, $e_2 = (\mathbf{1} - z)e$ and

$$x_i = zx_i + (\mathbf{1} - z)x_i \in (p\mathcal{A}p)ze(q\mathcal{A}q) + (p\mathcal{A}p)(\mathbf{1} - z)e(q\mathcal{A}q) \subseteq (p\mathcal{A}p)e(q\mathcal{A}q).$$

Similarly, we can prove that, for each pair $x_1, x_2 \in q\mathcal{A}p$, there exists $e \in q\mathcal{A}p$ such that $x_i \in (p\mathcal{A}p)e(q\mathcal{A}q)$. Thus p and q are locally cyclic.

As each projection $q \in \mathcal{A}$ is similar to an orthogonal projection $p \in \mathcal{A}$ ($q = apa^{-1}$ for some invertible $a \in \mathcal{A}$), all projections in $\mathcal{A}$ are locally cyclic. ■

Each W*-algebra $\mathcal{A}$ has the decomposition $\mathcal{A} = \mathcal{A}_{\mathrm{I}} \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}}$ where $\mathcal{A}_{\mathrm{I}}, \mathcal{A}_{\mathrm{II}}, \mathcal{A}_{\mathrm{III}}$ are W*-algebras of type I,II and III. The algebra $\mathcal{A}_{\mathrm{I}}$ decomposes uniquely into the direct sum of W*-algebras $\mathcal{A}_n$ of type I_n:

$$\mathcal{A}_{\mathrm{I}} = \oplus_{n\in\mathbb{N}(\mathcal{A})}\mathcal{A}_n, \tag{5.17}$$ 1.11

where n is the number of mutually orthogonal abelian equivalent projections p_i in $\mathcal{A}_n$ such that $\mathbf{1}_{\mathcal{A}_n} = \sum p_i$ and $\mathbb{N}(\mathcal{A})$ is a subset of $(\mathbb{N} - \{0\}) \cup \infty$.

T51 **Theorem 5.19** *Let $\mathcal{A}$ be a W*-algebra.*

(i) *Each Lie ideal L of $\mathcal{A}$ is commutator equal to the ideal $J = \mathrm{Id}([L,\mathcal{A}])$, that is, $[J,\mathcal{A}] = [L,\mathcal{A}]$.*

(ii) *Let there exist a finite number of prime numbers $q_1, ..., q_k$ such that each $n \in \mathbb{N}(\mathcal{A})$ is divisible by one of them. Then each Lie ideal L of $\mathcal{A}$ is related to only one ideal $\mathrm{Id}([L,\mathcal{A}])$.*

Proof. It was proved in [Su] that if $\mathcal{C}$ is a properly infinite W*-algebra then $\mathcal{C} = [\mathcal{C},\mathcal{C}]$. For a finite W*-algebra $\mathcal{C}$ it was proved in [PT, Theorem 1] that $[\mathcal{C},\mathcal{C}] = \{c \in \mathcal{C}\colon\ c^\natural = 0\}$, where $c \to c^\natural$ is the center-valued trace ($c^\natural \in \mathfrak{Z}_{\mathcal{C}}$). Then (see the proof of [Mi, Lemma 3]), for each $c \in \mathcal{C}$, $c - c^\natural \in [\mathcal{C},\mathcal{C}]$, as $(c - c^\natural)^\natural = 0$. As $c^\natural \in \mathfrak{Z}_{\mathcal{C}}$, $\mathcal{C} = [\mathcal{C},\mathcal{C}] + \mathfrak{Z}_{\mathcal{C}}$. Since each W*-algebra $\mathcal{C}$ is the direct sum of a properly infinite and a finite algebras, this implies that

$$\mathcal{C} = [\mathcal{C},\mathcal{C}] + \mathfrak{Z}_{\mathcal{C}}. \tag{5.18}$$ 5.1

The algebra $\mathcal{B} = \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}}$ has no abelian projections. By Proposition 2.2.13 [S], there is a projection p equivalent to $\mathbf{1}_{\mathcal{B}} - p$. Hence $\mathcal{B}$ is isomorphic to $M_2(p\mathcal{B}p)$.

Split $\mathbb{N}(\mathcal{A})$ into odd and even parts: $\mathbb{N}(\mathcal{A}) = \mathbb{N}^e(\mathcal{A}) \cup \mathbb{N}^o(\mathcal{A})$. Then $\mathcal{A}_{\mathrm{I}} = \mathcal{A}_{\mathrm{I}}^o \oplus \mathcal{A}_{\mathrm{I}}^e$ with $\mathcal{A}_{\mathrm{I}}^e = \oplus_{n\in\mathbb{N}^e(\mathcal{A})}\mathcal{A}_n$ and $\mathcal{A}_{\mathrm{I}}^o = \oplus_{n\in\mathbb{N}^o(\mathcal{A})}\mathcal{A}_n$. The W*-algebra $\mathcal{A}_{\mathrm{I}}^e$ is isomorphic to $M_2(\mathcal{U})$ for some W*-algebra $\mathcal{U}$. Hence $\mathfrak{M} = \mathcal{A}_{\mathrm{I}}^e \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}}$ is isomorphic to $M_2(\mathcal{C})$ where $\mathcal{C} = \mathcal{U} \oplus p\mathcal{B}p$. It follows from Corollary 4.18(ii) and (5.18) that each Lie ideal L of the W*-algebra $\mathfrak{M}$ is commutator equal to the ideal $J = \mathrm{Id}([L,\mathcal{A}])$.

For each $n \in \mathbb{N}^o(\mathcal{A})$, there are n mutually orthogonal abelian equivalent projections $p_i^{(n)}$ in $\mathcal{A}_n$ such that $\mathbf{1}_{\mathcal{A}_n} = \sum_{i=1}^n p_i^{(n)}$, so $\mathcal{A}_n$ is isomorphic to $M_n(\mathcal{B}_n)$, for some commutative W*-algebra $\mathcal{B}_n$. Set

$$p = \oplus_{n\in\mathbb{N}^o(\mathcal{A})}p_1^{(n)}$$

and $q = \mathbf{1} - p$. Then $\mathcal{A}_{11} = p\mathcal{A}_{\mathrm{I}}^o p = \oplus_{n\in\mathbb{N}^o(\mathcal{A})}\mathcal{B}_n$ is a commutative W*-algebra, so each Lie ideal L of $\mathcal{A}_{11}$ is commutator equal to $\mathrm{Id}([L,\mathcal{A}_{11}]) = \{0\}$. As all $n-1$ are even, the W*-algebra $\mathcal{A}_{22} = q\mathcal{A}_{\mathrm{I}}^o q = \oplus_{n\in\mathbb{N}^o(\mathcal{A})}M_{n-1}(\mathcal{B}_n)$ is isomorphic to $M_2(\mathcal{D})$ for some W*-algebra $\mathcal{D}$. Hence it follows from Corollary 4.18(ii) and (5.18) that each Lie ideal L of $\mathcal{A}_{22}$ is commutator equal to $\mathrm{Id}([L,\mathcal{A}_{22}])$. By Proposition 5.18, p is locally cyclic. Therefore it follows from Corollary 3.8 that each Lie ideal L of $\mathcal{A}_{\mathrm{I}}^o$ is commutator equal to $\mathrm{Id}([L,\mathcal{A}_{\mathrm{I}}^o])$.

As $\mathcal{A} = \mathcal{A}_{\mathrm{I}}^o \oplus \mathfrak{M}$, it follows from Proposition 3.1(i) that each Lie ideal L of $\mathcal{A}$ is commutator equal to the ideal $\mathrm{Id}([L,\mathcal{A}])$. Part (i) is proved.

If $\mathcal{A}$ satisfies the condition in (ii), then $\mathcal{A}_{\mathrm{I}} = M_{q_1}(\mathcal{B}_1) \oplus ... \oplus M_{q_k}(\mathcal{B}_k)$, where $\mathcal{B}_i$ are some W*-algebras. Hence

$$\mathcal{A} = \mathcal{A}_{\mathrm{I}} \oplus \mathcal{B} = M_{q_1}(\mathcal{B}_1) \oplus ... \oplus M_{q_k}(\mathcal{B}_k) \oplus M_2(p\mathcal{B}p). \tag{5.19}$$ 4.2

We have from Corollary 4.18(ii) that each Lie ideal of the W*-algebra $M_n(\mathcal{C})$, $n \geq 2$, is only related to one ideal, so each summand in (5.19) has this property. Using now Proposition 3.1, we obtain that each Lie ideal L of $\mathcal{A}$ is only related to the ideal $J = \mathrm{Id}([L, \mathcal{A}])$. ■

Theorem 5.19 is one of two main results of this section. The other one is Theorem 5.27 below. Applying (5.18) and Lemma 2.1, we obtain

Corollary 5.20 *For a Lie ideal L and an ideal I of a W*-algebra the following conditions are equivalent.*

(i) *L is related to I;*

(ii) *L is embraced by I;*

(iii) *L is commutator equal to I.*

If a W*-algebra $\mathcal{A}$ has a commutative component $\mathcal{A}_1$, then each Lie ideal of $\mathcal{A}_1$ is related to more than one ideal. Hence, by Proposition 3.1(ii), $\mathcal{A}$ has Lie ideals related to more than one ideal.

Problem 5.21 *Let a W*-algebra $\mathcal{A}$ have no commutative summand $\mathcal{A}_1$. Is every Lie ideal of $\mathcal{A}$ related to only one ideal?*

5.5 Lie ideals of C*-algebras.

We will start this subsection by showing that all closed Lie ideals of stable C*-algebras are associative ideals. Let $\mathcal{B}$ be a unital C*-algebra. Identify it with its faithful representation on a Hilbert space K. The C*-algebra $\mathcal{A} = \mathcal{B} \otimes_{\min} C(H)$ (the completion of $\mathcal{B} \otimes C(H)$ in the minimal and, hence, in any C*-norm, as $C(H)$ is nuclear) can be considered as an operator algebra on $K \otimes H$. For each $\mathcal{E} = \{e_i\}_{i=1}^n$, $e_i \in H$, the map $T_{\mathcal{E}}$ on $\mathcal{B} \otimes C(H)$ acts (see (5.12)) by

$$T_{\mathcal{E}}(b \otimes c) = b \otimes t_{\mathcal{E}}(c) = (\mathbf{1}_K \otimes p_{\mathcal{E}})(b \otimes c)(\mathbf{1}_K \otimes p_{\mathcal{E}}),$$

so $\|T_{\mathcal{E}}\| = \|\mathbf{1}_K \otimes p_{\mathcal{E}}\| = 1$. The map $S_{\mathcal{E}}$ on $\mathcal{B} \otimes C(H)$ acts (see (5.12)) by

$$S_{\mathcal{E}}(b \otimes c) = b \otimes s_{\mathcal{E}}(c) = \sum_i (\mathbf{1}_K \otimes p_{e_i})(b \otimes c)(\mathbf{1}_K \otimes p_{e_i}).$$

The projections $\mathbf{1}_K \otimes p_{e_i}$ on $K \otimes H$ are mutually orthogonal. Hence $\|S_{\mathcal{E}}\| = 1$. Thus the families of operators $T_{\mathcal{E}}$ and $S_{\mathcal{E}}$ are uniformly bounded. Therefore from Theorem 5.15 we obtain

stable **Corollary 5.22** *Let $\mathcal{B}$ be a unital C*-algebra. All closed Lie ideals L of the stable C*-algebra $\mathcal{B} \otimes_{\min} C(H)$ are ideals and have form $L = \overline{I \otimes C(H)}^{\min} \cong I \otimes_{\min} C(H)$ where I is a closed ideal of $\mathcal{B}$.*

We will mention now some applications of Theorem 5.3 to C*-algebras. These results were already published (see [MaMu]), but in somewhat different form. Denote by $\mathrm{T}(\mathcal{A})$ the set of all tracial states on a C*-algebra $\mathcal{A}$. Cuntz and Pedersen proved in [CP] that

$$\overline{[\mathcal{A},\mathcal{A}]} = \bigcap_{f\in \mathrm{T}(\mathcal{A})} \mathrm{Ker}(f) \tag{5.20}$$ 6.0

(cf. (5.3)). For unital C*-algebras, Pop proved in [Po] that

$$\mathrm{T}(\mathcal{A}) = \emptyset \text{ implies } [\mathcal{A},\mathcal{A}] = \mathcal{A}. \tag{5.21}$$ 62

For example, the Cuntz algebras O_n and infinite simple C*-algebras ($aa^* \neq a^*a = \mathbf{1}$, for some $a \in \mathcal{A}$) satisfy (5.21). Earlier Fack [F] showed that $[\mathcal{A},\mathcal{A}] = \mathcal{A}$, if $\mathcal{A} \cong \mathcal{A} \otimes_{\min} C(H)$ or if $\mathcal{A}$ is an infinite simple C*-algebra, and that $[\mathcal{A},\mathcal{A}] = \overline{[\mathcal{A},\mathcal{A}]}$ if $\mathcal{A}$ is a unital simple AF-algebra.

Thomsen [Th] extended the last result and proved that $[\mathcal{A},\mathcal{A}] = \overline{[\mathcal{A},\mathcal{A}]}$ for simple, infinite-dimensional, unital inductive limits of certain C*-algebras. This includes all unital simple AF-algebras, Bunce-Deddens algebras and the irrational rotation algebras $\mathcal{A}_\theta$. Moreover, Bunce-Deddens algebras, the algebras $\mathcal{A}_\theta$ and UHF-algebras have exactly one tracial state.

We will call the Lie ideals $\{0\}$, $\mathcal{A}$ and $\mathbb{C}\mathbf{1}$ (if $\mathcal{A}$ is unital) *trivial*.

6.5 **Proposition 5.23** *Let $\mathcal{A}$ be a unital simple C*-algebra.*

(i) *If $\mathrm{T}(\mathcal{A}) = \emptyset$, then $\mathcal{A}$ has only trivial Lie ideals.*

(ii) *Let $\mathrm{T}(\mathcal{A})$ consist of one functional. Then*

a) *[MaMu] $\mathcal{A}$ has only one non-trivial closed Lie ideal $\overline{[\mathcal{A},\mathcal{A}]}$ and its codimension is* 1;

b) *if $[\mathcal{A},\mathcal{A}]$ is closed, then $\mathcal{A}$ has only one non-trivial Lie ideal $[\mathcal{A},\mathcal{A}]$.*

Proof. For unital simple C*-algebra $\mathcal{A}$, $\mathfrak{Z}_{\mathcal{A}} = \mathbb{C}\mathbf{1}$. If $\mathrm{T}(\mathcal{A}) = \emptyset$ then (see (5.21)) $[\mathcal{A},\mathcal{A}] = \mathcal{A}$. Applying Theorem 1.1, we obtain part (i).

Let $\mathrm{T}(\mathcal{A}) = \{\mathrm{tr}(\cdot)\}$. By (5.20), $\overline{[\mathcal{A},\mathcal{A}]} = \mathrm{Ker}(\mathrm{tr}(\cdot))$ and part (ii) a) follows from Theorem 5.3(iv) b).

If $[\mathcal{A},\mathcal{A}] = \overline{[\mathcal{A},\mathcal{A}]}$ then, by a), $[\mathcal{A},\mathcal{A}]$ has codimension 1 in $\mathcal{A}$. Applying Theorem 1.1, we conclude the proof. ∎

Corollary 5.4 allows us to describe closed Lie ideals of extensions of $C(H)$ by commutative C*-algebras. The simplest non-trivial example of such an extension is the Toeplitz C*-algebra $\mathcal{T}$ generated by the unilateral shift. It is unital, contains the ideal $C(H)$ and $\mathcal{T}/C(H)$ is a commutative algebra isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle.

C7.2 **Corollary 5.24** *Let a C*-subalgebra $\mathcal{A}$ of $B(H)$ contain $C(H)$ and let $\mathcal{A}/C(H)$ be commutative. A closed subspace L of $\mathcal{A}$ is a non-trivial Lie ideal if and only if $C(H) \subseteq L$.*

Proof. The "if" part is evident. By (5.9), $\overline{[C(H),C(H)]} = C(H)$. As $\mathfrak{Z}_{\mathcal{A}} = \{0\}$ or $\mathfrak{Z}_{\mathcal{A}} = \mathbb{C}\mathbf{1}$ (if $\mathcal{A}$ is unital), the result follows from Corollary 5.4(ii). ∎

We will consider now Lie ideals of general C*-algebras. Let $\mathcal{A}^*$ be the dual space of a C*-algebra $\mathcal{A}$ (the space of all bounded functionals on $\mathcal{A}$). The bidual space $\mathcal{A}^{**}$ of $\mathcal{A}$ is a W*-algebra and the natural inclusion of $\mathcal{A}$ into $\mathcal{A}^{**}$ is a *-homomorphism. We will consider $\mathcal{A}$ as a subalgebra of $\mathcal{A}^{**}$.

It follows from the general duality theory of Banach spaces that for any subspace E of $\mathcal{A}$,

$$\overline{E} = \mathcal{A} \cap \overline{E}^{\sigma} \tag{5.22}$$ inter

where $\overline{E}$ is the closure of E in $\mathcal{A}$ and $\overline{E}^{\sigma}$ is the closure of E in $\mathcal{A}^{**}$ with respect to the *-weak topology, that is, $\sigma(A^{**}, A^{*})$-topology. We also have $\mathcal{A}^{**} = \overline{\mathcal{A}}^{\sigma}$.

For each $a \in \mathcal{A}^{**}$, the mappings $x \to xa$ and $x \to ax$ are σ-continuous, so $\overline{E}^{\sigma} F \subseteq \overline{EF}^{\sigma}$ for all subspaces E and F of $\mathcal{A}$. Hence $\overline{\overline{E}^{\sigma}\overline{F}^{\sigma}}^{\sigma} \subseteq \overline{\overline{E}^{\sigma} F}^{\sigma} \subseteq \overline{EF}^{\sigma}$. On the other hand, $\overline{EF}^{\sigma} \subseteq \overline{\overline{E}^{\sigma}\overline{F}^{\sigma}}^{\sigma}$, so

$$\overline{\overline{E}^{\sigma}\overline{F}^{\sigma}}^{\sigma} = \overline{EF}^{\sigma}. \tag{5.23}$$ 7.6

Similarly,

$$\overline{[\overline{E}^{\sigma}, \overline{F}^{\sigma}]}^{\sigma} = \overline{[E, F]}^{\sigma}. \tag{5.24}$$ 7.5

Bunce (see [Mi, Lemma 1]) proved that if I is a closed ideal of a C*-algebra $\mathcal{A}$ then $\overline{[I, \mathcal{A}]} = I \cap \overline{[\mathcal{A}, \mathcal{A}]}$. The following proposition is a refinement of this result.

P7.2 **Proposition 5.25** *Let I be a closed ideal of C*-algebra $\mathcal{A}$. Then*

$$\overline{[I, I]} = \overline{[[I, \mathcal{A}], \mathcal{A}]} = \overline{[I, \mathcal{A}]} = I \cap \overline{[\mathcal{A}, \mathcal{A}]}, \tag{5.25}$$ numer

so $N(\overline{[I, \mathcal{A}]}) = N(I)$.

Moreover, if $\mathcal{A}$ has no tracial states then $\overline{[I, \mathcal{A}]} = I$.

Proof. We only need to prove the first and second equalities. It follows from (5.18) that $[\mathcal{C}, \mathcal{C}] = [[\mathcal{C}, \mathcal{C}], \mathcal{C}]$, for each W*-algebra $\mathcal{C}$. As $\overline{I}^{\sigma}$ is a *-weakly closed ideal of the W*-algebra $\overline{\mathcal{A}}^{\sigma}$, there is a projection P in $\mathfrak{Z}_{\overline{\mathcal{A}}^{\sigma}}$ such that $\overline{I}^{\sigma} = P\overline{\mathcal{A}}^{\sigma}$. Hence, as P commutes with $\overline{\mathcal{A}}^{\sigma}$,

$$\begin{aligned}
[\overline{I}^{\sigma}, \overline{\mathcal{A}}^{\sigma}] &= [P\overline{\mathcal{A}}^{\sigma}, \overline{\mathcal{A}}^{\sigma}] = [P\overline{\mathcal{A}}^{\sigma}, P\overline{\mathcal{A}}^{\sigma}] = [\overline{I}^{\sigma}, \overline{I}^{\sigma}] \\
&= P[\overline{\mathcal{A}}^{\sigma}, \overline{\mathcal{A}}^{\sigma}] = P[[\overline{\mathcal{A}}^{\sigma}, \overline{\mathcal{A}}^{\sigma}], \overline{\mathcal{A}}^{\sigma}] = [[P\overline{\mathcal{A}}^{\sigma}, \overline{\mathcal{A}}^{\sigma}], \overline{\mathcal{A}}^{\sigma}] = [[\overline{I}^{\sigma}, \overline{\mathcal{A}}^{\sigma}], \overline{\mathcal{A}}^{\sigma}].
\end{aligned}$$

Therefore we obtain from (5.22) and (5.24) that

$$\begin{aligned}
\overline{[I, I]} &= \mathcal{A} \cap \overline{[I, I]}^{\sigma} = \mathcal{A} \cap \overline{[\overline{I}^{\sigma}, \overline{I}^{\sigma}]}^{\sigma} = \mathcal{A} \cap \overline{[\overline{I}^{\sigma}, \overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \mathcal{A} \cap \overline{[I, \mathcal{A}]}^{\sigma} = \overline{[I, \mathcal{A}]}, \\
\overline{[I, \mathcal{A}]} &= \mathcal{A} \cap \overline{[\overline{I}^{\sigma}, \overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \mathcal{A} \cap \overline{[[\overline{I}^{\sigma}, \overline{\mathcal{A}}^{\sigma}], \overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \mathcal{A} \cap \overline{[\overline{[I, \mathcal{A}]}^{\sigma}, \overline{\mathcal{A}}^{\sigma}]}^{\sigma} \\
&= \mathcal{A} \cap \overline{[[I, \mathcal{A}], \mathcal{A}]}^{\sigma} = \overline{[[I, \mathcal{A}], \mathcal{A}]}
\end{aligned}$$

which establish (5.25).

If $\mathcal{A}$ has no tracial states then, by (5.20), $\overline{[\mathcal{A}, \mathcal{A}]} = \mathcal{A}$ and it follows from (5.25) that $\overline{[I, \mathcal{A}]} = I$.
■

bezr **Corollary 5.26** *Let $\mathcal{A}$ be a C*-algebra. For a closed ideal I of $\mathcal{A}$ and for a closed Lie ideal L of $\mathcal{A}$, the following conditions are equivalent*:

(i) *L is related to I : $[I, \mathcal{A}] \subseteq L \subseteq N(I)$;*

(ii) *L is topologically embraced by I : $\overline{[I, \mathcal{A}]} \subseteq L \subseteq N(\overline{[I, \mathcal{A}]})$;*

(iii) *L and I are topologically commutator equal*: $\overline{[I, \mathcal{A}]} = \overline{[L, \mathcal{A}]}$.

Proof. (i) $\Rightarrow$ (iii). By Proposition 5.25, $N(I) = N(\overline{[I,\mathcal{A}]})$. Hence if L is related to I then

$$\overline{[I,\mathcal{A}]} \subseteq L \subseteq N(I) = N(\overline{[I,\mathcal{A}]}).$$

Therefore, by Proposition 5.25, $\overline{[I,\mathcal{A}]} = \overline{[\overline{[I,\mathcal{A}]},\mathcal{A}]} \subseteq \overline{[L,\mathcal{A}]} \subseteq \overline{[I,\mathcal{A}]}$, so $\overline{[L,\mathcal{A}]} = \overline{[I,\mathcal{A}]}$. Thus (i) implies (iii). The inclusions (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are evident. ■

We will prove now the second main result of this section.

C*-cl **Theorem 5.27** *Let L be a closed Lie ideal of a C*-algebra $\mathcal{A}$ and $I = \overline{\mathrm{Id}([L,\mathcal{A}])}$. Then*
(i) *L is topologically commutator equal to the ideal I: $\overline{[L,\mathcal{A}]} = \overline{[I,\mathcal{A}]}$.*
(ii) *if $\mathcal{A}$ has no tracial states then*

$$I \subseteq L \subseteq N(I). \tag{5.26}$$ 7.7

Proof. It follows from (5.24) that $\overline{L}^{\sigma}$ is a *-weakly closed Lie ideal of the W*-algebra $\overline{\mathcal{A}}^{\sigma}$. By Theorem 5.19(i), the ideal $J = \mathrm{Id}([\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}])$ of $\overline{\mathcal{A}}^{\sigma}$ satisfies $[\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}] = [J,\overline{\mathcal{A}}^{\sigma}]$. Hence, by (5.24),

$$\overline{[L,\mathcal{A}]}^{\sigma} = \overline{[\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \overline{[J,\overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \overline{[\overline{J}^{\sigma},\overline{\mathcal{A}}^{\sigma}]}^{\sigma}. \tag{5.27}$$ top

By (2.3), $I = \overline{[L,\mathcal{A}] + \mathcal{A}[L,\mathcal{A}]}$ and $J = [\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}] + \overline{\mathcal{A}}^{\sigma}[\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}] = \overline{\mathcal{A}}^{\sigma}[\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}]$, as $\overline{\mathcal{A}}^{\sigma}$ is unital. As $\mathcal{A}[L,\mathcal{A}] \subseteq I \subseteq J$, we have from (5.24) and (5.23)

$$\overline{J}^{\sigma} \supseteq \overline{I}^{\sigma} \supseteq \overline{\mathcal{A}[L,\mathcal{A}]}^{\sigma} = \overline{\overline{\mathcal{A}}^{\sigma}\overline{[L,\mathcal{A}]}^{\sigma}}^{\sigma} = \overline{\overline{\mathcal{A}}^{\sigma}\overline{[\overline{L}^{\sigma},\overline{\mathcal{A}}^{\sigma}]}^{\sigma}}^{\sigma} = \overline{J}^{\sigma}.$$

Hence $\overline{I}^{\sigma} = \overline{J}^{\sigma}$. Thus, by (5.24), $\overline{[\overline{J}^{\sigma},\overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \overline{[\overline{I}^{\sigma},\overline{\mathcal{A}}^{\sigma}]}^{\sigma} = \overline{[I,\mathcal{A}]}^{\sigma}$. Therefore (5.27) implies $\overline{[L,\mathcal{A}]}^{\sigma} = \overline{[I,\mathcal{A}]}^{\sigma}$. Taking the intersection of both parts with $\mathcal{A}$ and using (5.22), we complete the proof of part (i):

$$\overline{[L,\mathcal{A}]} = \mathcal{A} \cap \overline{[L,\mathcal{A}]}^{\sigma} = \mathcal{A} \cap \overline{[I,\mathcal{A}]}^{\sigma} = \overline{[I,\mathcal{A}]}.$$

If $\mathcal{A}$ has no tracial states then, by Proposition 5.25, $\overline{[I,\mathcal{A}]} = I$, so $\overline{[L,\mathcal{A}]} = I$. This implies (5.26). ■

Theorem 5.27 and Corollary 5.26 give a full description of the set all closed Lie ideals of a C*-algebra $\mathcal{A}$ as the set of all closed subspaces that lie between $\overline{[I,\mathcal{A}]}$ and $N(\overline{[I,\mathcal{A}]})$, where I is an arbitrary closed ideal of $\mathcal{A}$. If $\mathcal{A}$ has no tracial states, then the set of all closed Lie ideals of $\mathcal{A}$ coincides with the set of all closed subspaces that lie between I and $N(\overline{[I,\mathcal{A}]})$, for all closed ideals I of $\mathcal{A}$.

If $\mathcal{A}$ has a tracial state then (5.26) does not necessarily hold. Indeed, let $\mathcal{A}$ be an UHF-algebra and $L = \overline{[\mathcal{A},\mathcal{A}]}$ (see Proposition 5.23). As $\mathcal{A}$ is simple, it has only two closed ideals $I_1 = \mathcal{A}$ and $I_2 = \{0\}$. As $L \neq \mathcal{A}$, (5.26) holds neither for I_1 nor for I_2.

Let $\mathcal{A}$ be a dense subalgebra of a C*-algebra $\mathfrak{A}$ and L be a Lie ideal of $\mathcal{A}$. Then

$$\overline{[L,\mathcal{A}]} = \overline{[J,\mathcal{A}]}, \text{ where } J = \mathrm{Id}([L,\mathcal{A}]). \tag{5.28}$$ subal

Indeed, as $\overline{L}$ is a closed Lie ideal of $\mathfrak{A}$, by Theorem 5.27, $\overline{[\overline{L},\mathfrak{A}]} = \overline{[I,\mathfrak{A}]}$ where $I = \mathrm{Id}([\overline{L},\mathfrak{A}])$. Then $\overline{J} \subseteq \overline{I}$. By (2.3), $I = [\overline{L},\mathfrak{A}] + \mathfrak{A}[\overline{L},\mathfrak{A}]$ and $J = [L,\mathcal{A}] + \mathcal{A}[L,\mathcal{A}]$. Hence $I \subseteq \overline{J}$, so $\overline{I} = \overline{J}$. As $\overline{[\overline{E},\mathfrak{A}]} = \overline{[E,\mathcal{A}]}$, for each subspace E of $\mathcal{A}$, we have $\overline{[L,\mathcal{A}]} = \overline{[\overline{L},\mathfrak{A}]} = \overline{[I,\mathfrak{A}]} = \overline{[\overline{I},\mathfrak{A}]} = \overline{[\overline{J},\mathfrak{A}]} = \overline{[J,\mathcal{A}]}$.

Let $\mathcal{A}$ be also a Banach *-algebra in some norm $\|\cdot\|_{\mathcal{A}}$. It would be interesting to know under what conditions (5.28) holds if the closure there is taken in $\|\cdot\|_{\mathcal{A}}$.

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