A NOTE ON SPECTRUM-PRESERVING MAPS

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Abstract. Let A and B be unital semisimple Banach algebras. If $\phi : M_2(A) \to B$ is a bijective spectrum-preserving linear map, then T is a Jordan homomorphism.

1. Introduction

Is a bijective spectrum-preserving linear map between unital semisimple Banach algebras necessarily a Jordan homomorphism? This intriguing open question has its roots in Kaplansky's discussion [10], and was formulated in this form by Aupetit. Positive answers are known only in some special classes of semisimple Banach algebras, and for all algebras not belonging to one of these classes the problem is quite challenging. See [5] for more details about the history and state-of-the-art. The goal of this note is to obtain a positive answer in the case where one of these two algebras is arbitrary, while the other one must contain a set of 2×2 matrix units; equivalently, it is isomorphic to the algebra $M_2(A)$ of 2×2 matrices over another unital semisimple Banach algebra A. The class of these algebras is quite large; in particular, it is closed under homomorphic images. However, for some nice algebras from this class our result is not new as it can be derived from Aupetit's important result [2] which we will also use in our proof. On the other hand, there is an important example that is not covered by [2]: this is the algebra $B(X)$ of all bounded operators on a Banach space X which is a square, i.e., it is isomorphic to $Y \oplus Y$ for some Banach space Y (and hence $B(X) \cong M_2(B(Y))$). Also, the problem that we consider is more general that the one on describing linear maps $T : A \to B$ such that $T_2 : M_2(A) \to M_2(B)$, $T_2(a_{ij}) = (Ta_{ij})$, is spectrum-preserving (cf. [6, 7]).

In order to make the paper short and readable, we have decided to work in the simplest context and avoid various possible generalizations (like treating $n \times n$ matrix units for any $n \geq 2$) that one might think of.

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2. Tools

The spectrum of an element a from a Banach algebra A will be denoted by $\sigma(a)$. Recall that a map ϕ from one Banach algebra into another one is said to be *spectrum-preserving* if $\sigma(\phi(a)) = \sigma(a)$ for every a in the first algebra. The next theorem gathers together several known results on spectrum-preserving maps, which will be used in our proof.

Theorem 2.1. *Let B and C be unital semisimple Banach algebras, and let $\phi : C \to B$ be a bijective spectrum-preserving linear map. Then:*

(a) *If $e \in C$ is an idempotent, then $\phi(e)$ is an idempotent.*
(b) *If $n \in C$ is such that $n^2 = 0$, then $\phi(n)^2 = 0$.*
(c) *ϕ maps the center $Z(C)$ of C onto the center $Z(B)$ of B.*
(d) *If e is a central idempotent in C, then $\phi(e)$ is a central idempotent in B; moreover, the restriction of ϕ to eC is a bijective unital spectrum-preserving linear map between unital semisimple Banach algebras eC and $\phi(e)B$.*
(e) *ϕ is continuous.*

Proof. (a) Use [2, Theorem 1.2].
(b) Use [12, Corollary 3.2].
(c) Use [11, Corollary 4.4].
(d) Use [13, Theorem 3] and the fact that eu is invertible in eC if and only if $eu + (1 - e)$ is invertible in C.
(e) Use [1, Theorem 5.5.2]. □

From (d) it follows that φ maps a unit element of C into a unit element of B (cf. [2, Proposition 2.1 (ii)]). Let us also mention that in Theorem 2.1 it suffices to assume the surjectivity of ϕ as the injectivity is a consequence of the remaining hypotheses (see, e.g., [2, Proposition 2.1 (i)]). This is a simple consequence of Zemanek's characterization of the radical, just as the next theorem.

Theorem 2.2. *Let A be a semisimple Banach algebra. If $v, w \in A$ do not commute, then there exists $a \in A$ such that $\sigma(a - vw) \neq \sigma(a - wv)$.*

Proof. Suppose $\sigma(a - vw) = \sigma(a - wv)$ for every $a \in A$. Writing $a + vw$ instead of a it follows that $\sigma(a) = \sigma(a + vw - wv)$ for every $a \in A$. But then $vw - wv = 0$ by [1, Theorem 5.3.1]. □

Recall that a linear map $\phi : C \to B$ is a *Jordan homomorphism* if

$$\phi(b^2) = \phi(b)^2$$

for all $b \in C$; this readily implies

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$$

for all $a, b \in C$. We will need two purely algebraic result on such maps. They both actually hold for algebras over any field of characteristic not 2, but we state them only for complex algebras.

Theorem 2.3. *If a linear map ϕ from $M_2(\mathbb{C})$ into another algebra B maps idempotents into idempotents, then ϕ is a Jordan homomorphism; moreover, it is a sum of a homomorphism and an anti-homomorphism.*

Proof. The first assertion follows from [4, Theorem 2.1], and the second assertion follows from [8, Theorem 7]. □

Recall that an ideal I of a semiprime algebra B is said to be *essential* if $bI \neq 0$ and $Ib \neq 0$ for every nonzero $b \in B$. The next result follows immediately from [3, Theorem 2.3].

Theorem 2.4. *Let A, B_0 be complex algebras. If α is a Jordan homomorphism from A onto B_0, then there exist ideals U and V of A such that*

$$\alpha(au) = \alpha(a)\alpha(u), \;\; \alpha(ua) = \alpha(u)\alpha(a),$$
$$\alpha(av) = \alpha(v)\alpha(a), \;\; \alpha(va) = \alpha(a)\alpha(v)$$

for all $a \in A$, $u \in U$, $v \in V$, and $\alpha(U+V)$ is an essential ideal of B_0.

All results mentioned so far are of different complexity, but all of them have nontrivial proofs. The next lemma is elementary.

Lemma 2.5. *Let A be a unital algebra, and let $a, b, c \in A$. Then $\begin{bmatrix} a & b \\ 1 & c \end{bmatrix}$ is invertible in $M_2(A)$ if and only if $b - ac$ is invertible in A.*

Proof. Note that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix}$ is invertible; indeed, its inverse is $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$. Therefore, $\begin{bmatrix} a & b \\ 1 & c \end{bmatrix}$ is invertible if and only if

$$\begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix} \begin{bmatrix} a & b \\ 1 & c \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & b - ac \end{bmatrix}$$

is invertible, which is further equivalent to the invertibility of $b-ac$ in A. □

Let us also mention two folklore results. The first one is that the algebra A is semisimple if and only if $M_2(A)$ is semisimple. The second one is that a unital algebra C is isomorphic to a matrix algebra $M_2(A)$, where A is another unital algebra, if and only if C contains a set of 2×2 matrix units; that is, C contains elements E_{11}, E_{12}, E_{21}, E_{22} such that $E_{ij}E_{kl} = \delta_{jk}E_{il}$ for all $i, j, k, l \in \{1, 2\}$ and $E_{11} + E_{22} = I$, the unit element of C. In what follows we shall denote both unit elements of A and B by 1, while the unit element of $M_2(A)$ will be denoted by I.

3. Main result

Theorem 3.1. *Let A and B be unital semisimple Banach algebras. If $\phi : M_2(A) \to B$ is a bijective spectrum-preserving linear map, then ϕ is a Jordan homomorphism.*

Proof. Obviously, ϕ^{-1} is a spectrum-preserving map between semisimple Banach algebras B and $M_2(A)$. Thus, by (a) and (b) of Theorem 2.1 both ϕ and ϕ^{-1} preserve idempotents and square-zero elements. Moreover, by Theorem 2.3 the restriction of ϕ to the linear span of E_{ij}, $1 \le i,j \le 2$, is the sum of a homomorphism and an anti-homomorphism, $\phi = \phi_1 + \phi_2$. Of course, one of the maps ϕ_1 or ϕ_2 may be zero. Since $1 = \phi(I) = \phi_1(I)+\phi_2(I)$, we see that $\phi_1(I)$ and $\phi_2(I)$ are orthogonal idempotents in B. Set

$$p_1 = \phi_1(E_{11}), \quad n_1 = \phi_1(E_{12}), \quad n_2 = \phi_1(E_{21}), \quad p_2 = \phi_1(E_{22}),$$

and

$$p_3 = \phi_2(E_{11}), \quad m_2 = \phi_2(E_{12}), \quad m_1 = \phi_2(E_{21}), \quad p_4 = \phi_2(E_{22}).$$

It is then clear that p_i are orthogonal idempotents in B with $p_1+\ldots+p_4 = 1$. As ϕ_1 is a homomorphism we have

$$p_1n_1 = n_1, \quad n_1p_1 = 0, \quad p_1n_2 = 0, \quad n_2p_1 = n_2,$$

$$p_2n_1 = 0, \quad n_1p_2 = n_1, \quad p_2n_2 = n_2, \quad n_2p_2 = 0,$$

and

$$n_1n_2 = p_1, \quad \text{and} \quad n_2n_1 = p_2.$$

We use the fact that ϕ_2 is an anti-homomorphism to conclude that

$$p_3m_1 = m_1, \quad m_1p_3 = 0, \quad p_3m_2 = 0, \quad m_2p_3 = m_2,$$

$$p_4m_1 = 0, \quad m_1p_4 = m_1, \quad p_4m_2 = m_2, \quad m_2p_4 = 0,$$

and

$$m_1m_2 = p_3, \quad \text{and} \quad m_2m_1 = p_4.$$

And finally, the product of two elements in B with one factor being any of the elements p_1, p_2, n_1, n_2, and the other factor being any of the elements p_3, p_4, m_1, m_2, has to be zero.

For any $b \in B$ we have $b = (p_1 + \ldots + p_4)b(p_1 + \ldots + p_4)$, and thus

$$b = \sum_{i,j=1}^{4} b_{ij}$$

where $b_{ij} = p_ibp_j$.

Set

$$s(a) = \phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right), \quad a \in A.$$

Choose and fix an invertible element $a \in A$. The map

$$\begin{bmatrix} \lambda & \mu \\ \delta & \tau \end{bmatrix} \mapsto \begin{bmatrix} \lambda & \mu a \\ \delta a^{-1} & \tau \end{bmatrix}$$

from $M_2(\mathbb{C})$ into $M_2(A)$ is a homomorphism. Hence, the map

$$\begin{bmatrix} \lambda & \mu \\ \delta & \tau \end{bmatrix} \mapsto \phi\left(\begin{bmatrix} \lambda & \mu a \\ \delta a^{-1} & \tau \end{bmatrix}\right)$$

from $M_2(\mathbb{C})$ into B preserves idempotents and square-zero elements. From Theorem 2.3 it follows that it is a Jordan homomorphism. In particular,

$$\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right)+\phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right)\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)=\phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right).$$

Thus, $s(a) = (p_1+p_3)s(a)+s(a)(p_1+p_3)$ for every invertible $a \in A$. Multiplying this identity by p_j on the left-hand side and by p_k on the right-hand side, $1 \leq j, k \leq 4$, we see that

$$p_j s(a) p_k = 0$$

whenever a is invertible and both j, k are even or both j, k are odd.

Let us next prove that the same is true for an arbitrary (not necessarily invertible) $a \in A$. Indeed, for a complex number λ with $|\lambda| > \|a\|$ the element $a - \lambda$ is invertible in A, and therefore

$$p_j s(a-\lambda) p_k = 0$$

whenever both j, k are even or both j, k are odd. As $p_j s(\lambda) p_k = 0$ whenever both j, k are even or both j, k are odd, we have

$$s(a) = \sum_{i=1,3;j=2,4} s(a)_{ij} + \sum_{i=2,4;j=1,3} s(a)_{ij}, \quad a \in A. \tag{1}$$

For any complex number λ and any $a \in A$ the matrix

$$\begin{bmatrix} 0 & \lambda + a \\ 0 & 0 \end{bmatrix}$$

is square-zero. It follows that

$$0 = s(\lambda + a)^2 = \left(\lambda(n_1+m_2) + \sum_{i=1,3;j=2,4} s(a)_{ij} + \sum_{i=2,4;j=1,3} s(a)_{ij}\right)^2.$$

Calculating the square on the right-hand side we get a polynomial in λ whose coefficients must be zero. In particular, this is true for the coefficient at λ. Applying the fact that $s(a)_{ij} = p_i s(a) p_j$ and the above formulas for products of p_i, n_j, m_k we see that the coefficient at λ is equal to

$$\begin{aligned} &n_1 s(a)_{21} + n_1 s(a)_{23} + m_2 s(a)_{32} + m_2 s(a)_{34} \\ &+ s(a)_{21} n_1 + s(a)_{41} n_1 + s(a)_{14} m_2 + s(a)_{34} m_2 = 0. \end{aligned}$$

Multiplying this equation by p_1 on the left-hand side and by p_3 on the right-hand side we arrive at

$$0 = n_1 s(a)_{23} + s(a)_{14} m_2, \quad a \in A. \tag{2}$$

Similarly we see that

$$0 = n_2 t(a)_{14} + t(a)_{23} m_1, \quad a \in A, \tag{3}$$

where

$$t(a) = \phi\left(\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}\right), \quad a \in A.$$

Our next goal is to show for every $b \in B$ we have

$$b_{13} = b_{14} = b_{23} = b_{24} = 0.$$

Assume on the contrary that there exists $b \in B$ such that at least one of the elements b_{13}, b_{14}, b_{23}, b_{24} is nonzero. We will first show that then we can find $c \in B$ such that $c = c_{14} \neq 0$. Assume first that there exists $b \in B$ such that $b_{13} \neq 0$. Then

$$0 \neq p_1 b p_3 = p_1 b p_3^2 = (p_1 b m_1 m_2 m_1) m_2$$

and therefore,

$$0 \neq p_1 b m_1 (m_2 m_1) = p_1 b m_1 p_4 = c.$$

Hence, we have found $c \in B$ with $c = c_{14} \neq 0$ in the first case. In the next case that $b_{14} \neq 0$ we simply take $c = p_1 b p_4$. In the case where $b_{23} \neq 0$ we have

$$0 \neq p_2 b p_3 = n_2 (n_1 b p_3),$$

and hence

$$b' = n_1 b p_3 = p_1 n_1 b p_3$$

is a nonzero element with $b'_{13} \neq 0$. So, by the first case we can find $c \in B$ with the desired property. And we treat the last case in the same way.

Hence, we may assume that there exists $c \in B$ with $c = p_1 c p_4 \neq 0$. As ϕ is bijective we have

$$\phi\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = c$$

for some $a_i \in A$. We will show that $a_1 = a_4 = 0$. Assume for a moment that we have already proved this. Then

$$c = c_{14} = s(a_2) + t(a_3).$$

It follows that

$$0 = p_2 c_{14} p_3 = s(a_2)_{23} + t(a_3)_{23},$$

and consequently,

$$0 = n_1 s(a_2)_{23} m_1 + n_1 t(a_3)_{23} m_1.$$

Hence, by (2) and (3),

$$0 = -s(a_2)_{14} m_2 m_1 - n_1 n_2 t(a_3)_{14} = -s(a_2)_{14} - t(a_3)_{14} = -c_{14} = -c,$$

a contradiction.

We have to show that $a_1 = a_4 = 0$. Set $q_j = \phi^{-1}(p_j)$. Then $q_1 + q_3 = E_{11}$. As ϕ^{-1} preserves idempotents, q_1 and q_3 are orthogonal idempotents. Moreover,

$$q_1 = \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix}$$

for some idempotent $r_1 \in A$. Similarly,

$$q_4 = \begin{bmatrix} 0 & 0 \\ 0 & r_4 \end{bmatrix}$$

for some idempotent $r_4 \in A$. Now, for every $\lambda \in \mathbb{C}$ the elements $p_1 + \lambda c = p_1 + \lambda p_1 c p_4$ and $p_4 + \lambda c$ are idempotents. It follows that $q_1 + \lambda\phi^{-1}(c)$ and $q_4 + \lambda\phi^{-1}(c)$ are idempotents for every $\lambda \in \mathbb{C}$. From

$$(q_1 + \lambda\phi^{-1}(c))^2 = q_1 + \lambda\phi^{-1}(c)$$

we get

$$q_1\phi^{-1}(c) + \phi^{-1}(c)q_1 = \phi^{-1}(c)$$

or equivalently,

$$\begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}.$$

Thus, $a_4 = 0$, and similarly, $a_1 = 0$, as desired.

We have proved that for every $b \in B$ we have $b_{13} = b_{14} = b_{23} = b_{24} = 0$. In the same way we see that for every $b \in B$ we have $b_{31} = b_{32} = b_{41} = b_{42} = 0$. It follows that $p_1 + p_2, p_3 + p_4 \in Z(B)$. By Theorem 2.1 (c) we have

$$q_1 + q_2 = \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & r_2 \end{bmatrix} \in Z(M_2(A)).$$

In particular,

$$\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \tag{4}$$

for every $a \in A$. Setting $a = 1$ we get that $r_1 = r_2$. It then follows from (4) that $r = r_1 = r_2$ is a central idempotent in A.

Hence, $A = A_1 \oplus A_2$ where $A_1 = rA$ and $A_2 = (1-r)A$. Both A_1 and A_2 are semisimple unital Banach algebras. Using Theorem 2.1 (d) we see that the restriction of ϕ to $M_2(A_1)$ is a bijective spectrum-preserving linear map from $M_2(A_1)$ onto $(p_1 + p_2)B$ $(= (p_1 + p_2)B(p_1 + p_2))$, and the restriction of ϕ to $M_2(A_2)$ is a bijective spectrum-preserving linear map from $M_2(A_2)$ onto $(p_3 + p_4)B$.

To get a complete description of ϕ we have to study both restrictions above. The goal is to show that the first map is a homomorphism, and the second map is an antihomomorphism. This will tell us even a bit more than that ϕ is a Jordan homomorphism (but this additional information is not new in view of [8, Theorem 7]).

We will consider just the second map. Studying the first map is of course analogous. Thus, we may assume from now on that $A = A_2$ and $B = (p_3 + p_4)B$, and that

$$\phi(E_{11}) = p_3, \quad \phi(E_{12}) = m_2, \quad \phi(E_{21}) = m_1, \quad \phi(E_{22}) = p_4$$

and $p_3 + p_4 = 1$. In fact, p_3, p_4, m_1, m_2 are matrix units of B and hence we may regard B as $M_2(B_0)$ where B_0 is another Banach algebra (isomorphic to p_3Bp_3). In this setting we have

$$p_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ p_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ m_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ m_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We postpone the matrix notation in B for a while, and as before we write every $b \in B$ as $b = b_{33} + b_{34} + b_{43} + b_{44}$, where $b_{jk} = p_j b p_k$. Set

$$w(a) = \phi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right), \quad a \in A.$$

Using the fact that

$$\begin{bmatrix} 1 & \lambda a \\ 0 & 0 \end{bmatrix}$$

is an idempotent for every complex number λ, and $\phi(E_{11}) = p_3$, we conclude that $p_3 w(a) + w(a) p_3 = w(a)$. It follows that $w(a) = w(a)_{34} + w(a)_{43}$, $a \in A$. We also know that $w(\lambda + a)^2 = 0$ for every complex number λ. Thus,

$$(\lambda m_2 + w(a)_{34} + w(a)_{43})^2 = 0.$$

This is a polynomial in λ whose all coefficients are zero. In particular, the coefficient at λ must be zero. Thus,

$$m_2 w(a)_{34} + w(a)_{34} m_2 = 0.$$

Multiplying by m_1 on the left-hand side we arrive at $p_3 w(a)_{34} = 0$, and thus $w(a)_{34} = 0$. Hence, $w(a) = p_4 w(a) p_3$, $a \in A$. Similarly, for

$$u(a) = \phi\left(\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}\right), \quad a \in A,$$

we have $u(a) = u(a)_{34}$. Applying the fact that

$$\frac{1}{2}\begin{bmatrix} 1 & a \\ a^{-1} & 1 \end{bmatrix}$$

is an idempotent for every invertible $a \in A$ we get that

$$u(a^{-1}) w(a) = p_3$$

whenever a is invertible. Since ϕ is continuous (Theorem 2.1 (e)), so are u and w. Now, if $a \in A$ is an arbitrary element and $|\lambda| > \|a\|$, then

$$p_3 = u\left((1-\lambda^{-1}a)^{-1}\right) w(1-\lambda^{-1}a) = u\left(1 + \frac{1}{\lambda}a + \frac{1}{\lambda^2}a^2 + \dots\right) w\left(1 - \frac{1}{\lambda}a\right)$$

$$= \left(m_1 + \frac{1}{\lambda}u(a) + \frac{1}{\lambda^2}u(a^2) + \dots\right)\left(m_2 - \frac{1}{\lambda}w(a)\right).$$

Hence,

$$u(a) m_2 = m_1 w(a), \quad a \in A.$$

Next we apply the fact that

$$\begin{bmatrix} a & -\frac{1}{\lambda}a^2 \\ \lambda & -a \end{bmatrix}$$

is a square-zero matrix for every $a \in A$ and every nonzero complex number λ. Set

$$z(a) = \phi\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\right), \quad a \in A.$$

Then

$$\left(z(a) + \lambda m_1 - \frac{1}{\lambda} w(a^2)\right)^2 = 0.$$

In particular,

$$z(a)m_1 + m_1 z(a) = 0.$$

Hence,

$$0 = p_4 z(a) m_1 m_2 + p_4 m_1 z(a) m_2 = p_4 z(a) p_3$$

and

$$0 = m_1 m_2 z(a) m_1 m_2 + m_1 m_2 m_1 z(a) m_2 = p_3 z(a) p_3 + m_1 z(a) m_2,$$

i.e., $z(a)_{33} = -m_1 z(a) m_2$. Similarly, $z(a)_{44} = -m_2 z(a) m_1$. We have also proved that $z(a)_{43} = 0$, and similarly one can show that $z(a)_{34} = 0$.

We now proceed by using the matrix notation in B. We can summarize all information about the action of ϕ obtained so far in the following conclusion: there exist linear maps $\alpha, \beta : A \to B_0$ such that

$$\phi\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & \beta(c) \\ \beta(b) & -\alpha(a) \end{bmatrix}$$

and $\alpha(1) = \beta(1) = 1$. Using the fact that

$$\begin{bmatrix} a & a^2 \\ -1 & -a \end{bmatrix}$$

is a square-zero element and that ϕ preserves such elements, it follows that $\alpha(a)^2 = \beta(a^2)$. Substituting $\lambda + a$ for a it follows that $\alpha(a) = \beta(a)$, and hence α is a Jordan homomorphism. Thus we have

$$\phi\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & \alpha(c) \\ \alpha(b) & -\alpha(a) \end{bmatrix}$$

for all $a, b, c \in A$.

Suppose that

$$\phi\left(\begin{bmatrix} u & v \\ 0 & w \end{bmatrix}\right) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

for some $u, v, w \in A$ and $x \in B_0$. Then

$$\phi\left(\begin{bmatrix} 1 + \lambda u & \lambda v \\ 0 & \lambda w \end{bmatrix}\right) = \begin{bmatrix} 1 & \lambda x \\ 0 & 0 \end{bmatrix}$$

for every $\lambda \in \mathbb{C}$. The right-hand side element is an idempotent, and hence $\begin{bmatrix} 1 + \lambda u & \lambda v \\ 0 & \lambda w \end{bmatrix}$ must be idempotent. Consequently, $u = v = w = 0$. Therefore only elements from the "lower corner" can be mapped into elements in the "upper corner". This implies the surjectivity of α.

Pick $a \in A$ and set

$$\phi\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}.$$

It is our aim to show that $g_2 = 0$. For every $b \in A$ the spectrum of $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is $\sigma(a) \cup \{0\}$. Together with the surjectivity of α this implies that for each $\lambda \notin \sigma(a) \cup \{0\}$ and for each $x \in B_0$ the matrix

$$\begin{bmatrix} h_1 & g_2 \\ x & h_4 \end{bmatrix},$$

where $h_1 = g_1 - \lambda$ and $h_4 = g_4 - \lambda$, is invertible. We can choose λ in such a way that h_1 is invertible. It follows from

$$\begin{bmatrix} h_1 & g_2 \\ x & h_4 \end{bmatrix} = \begin{bmatrix} 0 & h_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & h_4 \\ 1 & h_1^{-1} g_2 \end{bmatrix}$$

and Lemma 2.5 that $h_4 - x h_1^{-1} g_2$ is invertible in B_0 for every $x \in B_0$. By semisimplicity, $h_1^{-1} g_2 = 0$, and consequently, $g_2 = 0$.

Similarly, $g_3 = 0$. Thus, diagonal matrices are mapped into diagonal matrices. We may identify the subalgebra of diagonal matrices of $M_2(A)$ by $A \times A$. As a diagonal matrix in $M_2(A)$ is invertible in $M_2(A)$ if and only if it is invertible in $A \times A$, it follows that the restriction of ϕ to diagonal matrices is a bijective spectrum-preserving linear map from $A \times A$ onto $B_0 \times B_0$. Since E_{11} is a central idempotent in $A \times A$, Theorem 2.1 (d) tells us that

$$\phi\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \alpha'(a) & 0 \\ 0 & 0 \end{bmatrix}$$

for some map $\alpha' : A \to B_0$. Similarly, there exists $\alpha'' : A \to B_0$ such that

$$\phi\left(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha''(a) \end{bmatrix}.$$

However, in view of the action of ϕ on elements of the form $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ it immediately follows that $\alpha = \alpha' = \alpha''$. Thus we finally have

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \alpha(a) & \alpha(c) \\ \alpha(b) & \alpha(d) \end{bmatrix}$$

for all $a, b, c, d \in A$. Since ϕ is spectrum-preserving, so is α.

Since α is a Jordan isomorphism between semisimple (and thus semiprime) algebras A and B_0, we are now in a position to use Theorem 2.4. Let U and V be ideals from this theorem.

We claim that U is contained in the center of A. Suppose this was not true. Then there would exist $u \in U$ and $a \in A$ such that $au \neq ua$. Theorem 2.2 implies the existence of $b \in A$ such that, say, $b - au$ is invertible while $b - ua$ is not. By Lemma 2.5 the matrix $\begin{bmatrix} a & b \\ 1 & u \end{bmatrix}$ is invertible in $M_2(A)$. Consequently, $\begin{bmatrix} \alpha(a) & 1 \\ \alpha(b) & \alpha(u) \end{bmatrix}$ is invertible in B. Multiplying this matrix from both the left-hand side and the right-hand side by the (invertible) matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we infer that $\begin{bmatrix} \alpha(u) & \alpha(b) \\ 1 & \alpha(a) \end{bmatrix}$ is also invertible. Using Lemma 2.5 again it follows that $\alpha(b) - \alpha(u)\alpha(a)$ is invertible in B_0. However, $\alpha(b) - \alpha(u)\alpha(a) = \alpha(b - ua)$ since $u \in U$. As α is spectrum-preserving, this leads to the contradiction that $b - ua$ is invertible. Our claim is thus proved.

We now have $\alpha(ax) = \alpha(x)\alpha(a)$ for all $a \in A$ and $x \in U+V$. Accordingly, for all $a, b \in A$ and $x \in U + V$ we have

$$\alpha(x)\alpha(b)\alpha(a) = \alpha(bx)\alpha(a) = \alpha(a(bx)) = \alpha((ab)x) = \alpha(x)\alpha(ab).$$

Thus, $\alpha(U+V)((\alpha(ab) - \alpha(b)\alpha(a)) = 0$ for all $a, b \in A$. However, $\alpha(U+V)$ is an essential ideal of B. Therefore $\alpha(ab) = \alpha(b)\alpha(a)$. $\square$

Concluding remark. One of the most influential results in the theory of linear preservers, due to Jafarian and Sourour [9], states that a bijective spectrum-preserving linear map from $B(X)$ onto $B(Y)$ is a Jordan homomorphism; here, X and Y are any Banach spaces. As already indicated in the introduction, our theorem implies that the same is true if we replace $B(Y)$ by an arbitrary semisimple algebra, but, on the other hand, we have to assume that X is a square (as most classical Banach spaces are). The method of the proof, however, is entirely different. While the arguments in [9] are based on operators of finite rank, our approach uses the techniques of the general Banach algebra theory.

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