A UNIFIED APPROACH TO THE STRUCTURE THEORY OF PI-RINGS AND GPI-RINGS

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Dedicated to Yuri Bahturin on his 65th birthday

Abstract. We give short proofs, based only on basic properties of the extended centroid of a prime ring, of Martindale's theorem on prime GPI-rings and (a strengthened version of) Posner's theorem on prime PI-rings.

1. Introduction

In the recent paper [7] the author has exposed a rather simple and direct approach to the structure theory of prime PI-rings. Unlike the standard approach which combines various tools, this one basically rests upon only one concept: the extended centroid of a prime ring. It was remarked in the paper that the method of the proof is also applicable to generalized polynomial identities.

The present paper is an expanded version of [7], which, in particular, also covers generalized polynomial identities. It is organized as follows. Section 2 gives a short survey of the symmetric Martindale ring of quotients and the extended centroid of a prime ring. Section 3 reveals the essence of our approach. Its sole goal is an elementary lemma treating a special functional identity. In Section 4 we give a new proof of Martindale's theorem on prime GPI-rings [15]. Finally, in Section 5 we derive the structure theorem on prime PI-rings from Martindale's theorem. The idea to apply generalized polynomial identities to polynomial identities is not new. Already in [15] Martindale noticed that Posner's theorem on prime PI-rings can be derived from his result. However, we will be able to recover a significant strengthening of Posner's theorem, established by Rowen [19] and others. It is worthwhile mentioning that our arguments also yield a nonconstructive proof of the existence of central polynomials for matrices over infinite fields.

2. The symmetric Martindale ring of quotients and the extended centroid

By a ring we mean an associative ring, not necessarily with 1. Let R be a *prime ring*. Then one can construct the *symmetric Martindale ring of quotients* $Q = Q_s(R)$ of R, which is, up to isomorphism, characterized by the following four properties:

(a) R is a subring of Q;
(b) for every $q \in Q$ there exists a nonzero ideal I of R such that $qI \cup Iq \subseteq R$;
(c) if I is a nonzero ideal of R and $0 \neq q \in Q$, then $qI \neq 0$ and $Iq \neq 0$;
(d) if I is a nonzero ideal of R, $f : I \to R$ is a right R-module homomorphism, and $g : I \to R$ is a left R-module homomorphism such that $xf(y) = g(x)y$ for all $x, y \in I$, then there exists $q \in Q$ such that $f(y) = qy$ and $g(x) = xq$ for all $x, y \in I$.

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Remark 2.1. Note that (b) can be extended as follows: If $q_1, \dots, q_n \in Q$, then there exist a nonzero ideal I of R such that $q_i I \bigcup I q_i \subseteq R$ for every $i = 1, \dots, n$. Indeed, if I_i is a nonzero ideal of R such that $q_i I_i \bigcup I_i q_i \subseteq R$, then $I = I_1 \bigcap \dots \bigcap I_n$ is nonzero since R is prime and obviously satisfies the desired condition.

For details and some illustrative examples we refer the reader to [5] and [14]. We will be primarily interested in the center C of Q, called the *extended centroid* of R. It is a field containing the center Z of R. We remark that Z has no zero divisors, and therefore, provided it is nonzero, one can form its field of fractions. This is a subfield of C; examples where it is a proper subfield can be easily constructed. For example, if R is the ring of all countably infinite complex matrices of the form $A + \lambda$, where A is a matrix with only finitely many nonzero entries and λ is a real scalar matrix, then $Z \cong \mathbb{R}$ and $C \cong \mathbb{C}$.

We may consider Q as an algebra over C. The subalgebra of Q generated by R is called the *central closure* of R. We will denote it by A. Both Q and A are prime rings. The extended centroid of A, as well as of any nonzero ideal of A, is nothing but C. If $C \subseteq A$, i.e., if A is unital, then C is the center of A.

The main property of C that we need is given in the following theorem. It is one of the cornerstones of the theory of generalized polynomial identities as well as of the theory of functional identities. Its original version was proved by Martindale in [15]. The version that we state is, as one can see from [8, Theorem A.4], a special case of [8, Theorem A.7].

Theorem 2.2. *Let R be a prime ring with extended centroid C, and let I be a nonzero ideal of R. Assume that $a_i, b_i, c_j, d_j \in Q_s(R)$ satisfy $\sum_{i=1}^n a_i x b_i = \sum_{j=1}^m c_j x d_j$ for all $x \in I$. If $a_1, \dots, a_n$ are linearly independent over C, then each b_i is a linear combination of $d_1, \dots, d_m$. (In particular, $b_i = 0$ if the d_j's are 0.)*

We have thereby gathered together all prerequsities that we need. The proofs of the aforementioned results are self-contained and quite simple. See [5, Chapter 2] for a detailed, and [8, Appendix A] for an informal survey on this subject.

3. A lemma on functional identities

The theory of functional identities deals with identities on rings that involve arbitrary functions. A functional identity is formally more general than a polynomial identity, but in practice the theory of functional identities is most often complementary to the theory of polynomial identities, rather than being its generalization. For a full account on functional identities and their applications we refer the reader to the book [8]. See also [3, 4] for some of the most recent applications.

The next lemma treats a special functional identity by elementary means. It is independent of the general theory from [8] (at least technically, if not philosophically).

Lemma 3.1. *Let S be a set, Q be a ring with 1, and $\mathcal{F}$ be a set of functions from S into Q. For every π in the symmetric group S_n, $n \geq 2$, we write*

$$\{x_1, \dots, x_n\}_\pi = \sum_i F^i_{\pi 1}(x_{\pi(1)}) F^i_{\pi 2}(x_{\pi(2)}) \dots F^i_{\pi n}(x_{\pi(n)}),$$

where $F^i_{\pi k} \in \mathcal{F}$. If

$$\sum_{\pi \in S_n} \{x_1, \dots, x_n\}_\pi = 0 \quad \textit{for all } x_1, \dots, x_n \in S,$$

then one of the following assertions holds:

(a) *For every $\pi \in S_n$ we have $\{x_1, \dots, x_n\}_\pi = 0$ for all $x_1, \dots, x_n \in S$, or*

(b) *There exist* $a_k, b_k, c_k, d_l, e_l, f_l \in Q$ *and* $F_k, G_k, H_l, K_l \in \mathcal{F}$ *such that*
$$\varphi(x,y) := \sum_k a_k F_k(x) b_k G_k(y) c_k = \sum_l d_l H_l(y) e_l K_l(x) f_l \quad \text{for all } x, y \in S,$$
and $\varphi(s_1, s_2) \neq 0$ *for some* $s_1, s_2 \in S$.

Proof. Let us set
$$\Phi(x_1,\dots,x_n) := \sum_{\substack{\pi\in S_n,\\ \pi^{-1}(1)<\pi^{-1}(2)}} \{x_1,\dots,x_n\}_\pi.$$
Suppose there exist $s_1,\dots,s_n \in S$ such that $\Phi(s_1,\dots,s_n) \neq 0$. Let us define $\varphi(x,y) = \Phi(x,y,s_3,\dots,s_n)$, and note that $\varphi(x,y)$ consists of summands of the form $aF^i_{\pi k}(x)bF^i_{\pi l}(y)c$ where a,b,c are either equal to 1 or are products of elements from $\mathcal{F}(s_i)$, $i \geq 3$. On the other hand, since, by our assumption,
$$\Phi(x_1,\dots,x_n) = -\sum_{\substack{\pi\in S_n,\\ \pi^{-1}(2)<\pi^{-1}(1)}} \{x_1,\dots,x_n\}_\pi,$$
we see that $\varphi(x,y)$ can be also represented as a sum of summands of the form $dF^i_{\pi k}(y)eF^i_{\pi l}(x)f$. As $\varphi(s_1,s_2) \neq 0$, (b) holds.

We may therefore assume that $\Phi(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in S$, and also that $n \geq 3$. Now we set
$$\Psi(x_1,\dots,x_n) := \sum_{\substack{\pi\in S_n,\\ \pi^{-1}(1)<\pi^{-1}(2),\\ \pi^{-1}(2)<\pi^{-1}(3)}} \{x_1,\dots,x_n\}_\pi = -\sum_{\substack{\pi\in S_n,\\ \pi^{-1}(1)<\pi^{-1}(2),\\ \pi^{-1}(3)<\pi^{-1}(2)}} \{x_1,\dots,x_n\}_\pi.$$
If $\Psi(t_1,\dots,t_n) \neq 0$ for some $t_1,\dots,t_n \in S$, then one shows, just as in the preceding paragraph, that $\varphi(x,y) = \Psi(t_1,x,y,t_4,\dots,t_n)$ gives rise to (b). Thus we may assume that $\Psi(x_1,\dots,x_n) = 0$ for all $x_1,\dots,x_n \in S$. We can now continue this procedure by considering the summation over permutations π that additionally satisfy $\pi^{-1}(3) < \pi^{-1}(4)$. Assuming that it is nonzero we arrive at (b), otherwise we make another step. If (b) does not hold we finally arrive at $\pi^{-1}(1) < \pi^{-1}(2) < \dots < \pi^{-1}(n)$, which, of course, holds only for $\pi = 1$. Thus, if (b) is not true then $\{x_1,\dots,x_n\}_1 = 0$ for all $x_1,\dots,x_n \in S$. Analogously we see that $\{x_1,\dots,x_n\}_\pi = 0$ for every $\pi \in S_n$ if (b) does not hold. □

A special case of this lemma, where $\mathcal{F}$ consists of scalar multiples of the identity map (so that the identity treated can be interpreted as a multilinear polynomial identity), indirectly appeared in [7]. In what follows we will need another special case where $\mathcal{F}$ consists of two-sided multiplications (which corresponds to a multilinear generalized polynomial identity). Perhaps the lemma shall turn out to be useful in some other instances.

4. Prime GPI-rings

Let R be a prime ring with extended centroid C and symmetric Martindale ring of quotients $Q = Q_s(R)$. By $Q_C\langle X_1, X_2, \dots\rangle$ we denote the coproduct of the C-algebra Q and the free algebra $C\langle X_1, X_2, \dots\rangle$. Informally we can consider elements in $Q_C\langle X_1, X_2, \dots\rangle$ as sums of "monomials" of the form $a_0 X_{i_1} a_1 X_{i_2} a_2 \dots a_{n-1} X_{i_n} a_n$ with $a_i \in Q$. We say that $f = f(X_1,\dots,X_n) \in Q_C\langle X_1, X_2, \dots\rangle$ is a *generalized polynomial identity* (GPI) on R if $f(r_1,\dots,r_n) = 0$ for all $r_1,\dots,r_n \in R$. If there exists a nonzero GPI on R, then R is said to be a GPI-*ring*. We refer to [5] for a full treatise of GPI's.

Let A be any algebra. For $a, b \in A$ we define $L_a, R_b : A \to A$ by $L_a(x) = ax$ and $R_b(x) = xb$. Clearly, $L_{aa'} = L_a L_{a'}$, $R_{bb'} = R_{b'} R_b$, and $L_a R_b = R_b L_a$. By $M(A)$ we denote the algebra of all operators of the form $\sum_i L_{a_i} R_{b_i}$, $a_i, b_i \in A$.

The fundamental result in the theory of GPI-rings is the following theorem by Martindale from 1969 [15]. (The condition (ii) is usually not stated in the theorem, but in our opinion it does deserve a special attention.)

Theorem 4.1. (Martindale) *Let R be a prime ring with extended centroid C and central closure A. The following statements are equivalent:*

(i) *R is a GPI-ring.*
(ii) *$M(A)$ contains a nonzero finite rank operator.*
(iii) *A contains an idempotent e such that Ae is a minimal left ideal of A and eAe is a finite dimensional division algebra over C.*

Proof. (i)$\Longrightarrow$(ii). By a standard linearization process we see that R satisfies a multilinear generalized polynomial identity $f = f(X_1, \ldots, X_n) \in Q_C\langle X_1, X_2, \ldots\rangle$, $f \neq 0$. We can write $f = \sum_{\pi \in S_n} f_\pi$ where f_π consists of summands of the form

$$a_0 X_{\pi(1)} a_1 X_{\pi(2)} a_2 \ldots a_{n-1} X_{\pi(n)} a_n, \ a_i \in Q.$$

Since $f \neq 0$, at least one of the f_π's is not 0. We may assume that

$$f_1 = \sum_i a_{0i} X_1 a_{1i} X_2 a_{2i} \ldots a_{n-1i} X_n a_{ni} \neq 0.$$

We claim that f_1 cannot be a generalized polynomial identity of R. We proceed by induction on n. The case where $n = 0$, i.e., $f = a_0$ with $a_0 \in Q$, is trivial. We may therefore assume that our claim is true for all nonnegative integers smaller than n. Let us write f_1 as $f_1 = \sum_i a_{0i} X_1 h_i$ where $h_i = h_i(X_2, \ldots, X_n)$. There is no loss of generality in assuming that the set of the elements a_{0i} is linearly independent, since otherwise we can choose its maximal linearly independent subset, write each a_{0i} as a linear combination of elements from this subset, and accordingly rewrite f_1 as $f_1 = \sum_i a'_{0i} X_1 h'_i$ where the set of the elements a'_{0i} now is linearly independent. If f_1 was a generalized polynomial identity of R, we would have $\sum_i a_{i0} x_1 h_i(x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$, hence $h_i(x_2, \ldots, x_n) = 0$ by Theorem 2.2, and so, by induction assumption, $h_i = 0$ as an element of $Q_C\langle X_1, X_2, \ldots\rangle$. This contradicts $f_1 \neq 0$.

The identity $\sum_{\pi \in S_n} f_\pi(x_1, \ldots, x_n) = 0$ makes it possible for us to apply Lemma 3.1 to the case where $\mathcal{F}$ consists of maps from R into Q of the form $x \mapsto axb$, $a, b \in Q$. As the possibility (a) has been ruled out in the preceding paragraph, (b) must hold. Note that this can be interpreted as follows: There exist $p_i, q_j \in Q$ and $F_i, G_j \in M(Q)$ such that

$$\varphi(x, y) := \sum_i p_i x F_i(y) = \sum_j G_i(y) x q_j \quad \text{for all } x, y \in R$$

and $\varphi(s_1, s_2) \neq 0$ for some $s_1, s_2 \in R$. A similar argument as in the preceding paragraph shows that without loss of generality we may assume that the elements p_i are linearly independent. We may also assume that $F_1 \neq 0$. Theorem 2.2 tells us that $F_1(y) \in \sum_j C q_j$ for every $y \in R$, implying that $F_1(A)$ is a finite dimensional space, as desired. The only problem is that F_1 lies in $M(Q)$ rather than in $M(A)$. But we can easily remedy this. We have $F_1(y) = \sum_l s_l y t_l$ with $s_l, t_l \in Q$. By Remark 2.1 there exists a nonzero ideal I of R such that $I s_l \bigcup t_l I \subseteq R$ for every l. Of course, $I F_1(R) I \neq 0$, so that there are $u, v \in I$ and $y_0 \in R$ such that $u F_1(y_0) v \neq 0$. Define $F(y) := \sum_l u s_l y t_l v$. Since $u s_l, t_l v \in R \subseteq A$, we can consider F as an element of $M(A)$. Clearly, $F(A) = u F_1(A) v$ is a finite dimensional space.

(ii)$\Longrightarrow$(iii). Let $W = \sum_{i=1}^n L_{a_i} R_{b_i}$ be a nonzero finite rank operator in $M(A)$. Without loss of generality we may assume that the set $\{a_1, \ldots, a_n\}$ is linearly independent and that $b_1 \neq 0$.

Suppose first that $n = 1$. Let us write a for a_1 and b for b_1. Thus, $1 \leq \dim_C aAb < \infty$. If L_0 is nonzero left ideal of A with $L_0 \subseteq Ab$, and R_0 is a nonzero right ideal of A with $R_0 \subseteq aA$, then $0 \neq R_0 L_0 \subseteq aAb$ and hence $1 \leq \dim_C R_0 L_0 < \infty$. Choose L_0

and R_0 so that R_0L_0 is of minimal dimension. We claim that AR_0L_0 is a minimal left ideal of A. Let L_1 be a left ideal such that $0 \neq L_1 \subseteq AR_0L_0$. Then $L_1 \subseteq L_0$, hence $R_0L_1 \subseteq R_0L_0$, and so $R_0L_1 = R_0L_0$ in view of the dimension assumption. Consequently, $L_1 \supseteq AR_0L_1 = AR_0L_0$ which proves that AR_0L_0 is indeed minimal. As it is well-known (see, e.g., [5, Proposition 4.3.3]), this implies the existence of an idempotent $e \in A$ such that $Ae = AR_0L_0$ and eAe is a division algebra. Moreover, since $e \in AR_0L_0 \subseteq AaAb$ it follows that $\dim_C eAe < \infty$.

Now let $n > 1$. If each b_i, $i \geq 2$, is a scalar multiple of b_1, then we are back to the $n = 1$ case. We may therefore assume that b_2 and b_1 are linearly independent. By Theorem 2.2 there exists $c \in A$ such that $b_1cb_2 \neq b_2cb_1$. Define $W' \in M(A)$ by $W' = WR_{b_1c} - R_{cb_1}W$. Obviously, W' has finite rank, and we have $W' = \sum_{i=2}^n L_{a_i}R_{c_i}$ where $c_i = b_1cb_i - b_icb_1$. Since $a_2, \dots, a_n$ are linearly independent and $c_2 \neq 0$, Theorem 2.2 shows that $W' \neq 0$. By induction, the proof is complete.

(iii)$\Longrightarrow$(i). Let $d = \dim_C eAe$. Then the elements $ex_1e, \dots, ex_{d+1}e$ are linearly dependent for each $x \in R$, so that $St_{d+1}(eX_1e, \dots, eX_{d+1}e)$, where St_{d+1} is the standard polynomial in $d+1$ variables, is a GPI on R. □

The essence of the theorem is that the central closure A of a prime GPI-ring has minimal left ideals Ae, so A is a primitive algebra having a particularly nice structure; moreover, the corresponding division algebra eAe is finite dimensional.

The main novelty is the proof of (i)$\Longrightarrow$(ii), although, of course, it is based on ideas from [7]. The proof of (ii)$\Longrightarrow$(iii) is similar to those from [15] and [5], yet some modifications taken from [10] were used.

5. Prime PI-rings

It is convenient to define that a *prime ring* R is a PI-*ring* if a nonzero polynomial in $C\langle X_1, X_2, \dots\rangle$, where C is the extended centroid of R, is a polynomial identity of R. The structure of prime PI-rings was first described in 1960 by Posner [17]. Later, after the discovery of central polynomials in the 1970's, Posner's theorem was sharpened by Rowen and others (cf. [19]) as follows.

Theorem 5.1. (Posner) *Let R be a prime PI-ring with extended centroid C and central closure A. Then:*

(a) *A is a finite-dimensional central simple algebra over C.*
(b) *Every nonzero ideal of R intersects the center Z of R nontrivially.*
(c) *C is the field of fractions of Z.*

Accordingly, every element in A is of the form $z^{-1}r$ with $0 \neq z \in Z$ and $r \in R$ (thus, $A = S^{-1}R$ where $S = Z \setminus \{0\}$).

Proof. (a) Let U be a nonzero ideal of A. Since A is clearly a prime PI-ring (as it satisfies the same multilinear identities as R), so is U. Let $f = f(X_1, \dots, X_n)$ be a multilinear polynomial identity of U of minimal degree n. Write

$$f = gX_n + \sum_i g_iX_nh_i$$

where each h_i is a monomial of degree ≥ 1 and with leading coefficient 1, and g and g_i are multilinear polynomials. We may assume that $g \neq 0$. As the degree of g is $n-1$, g is not an identity of U. Pick $u_1, \dots, u_{n-1} \in U$ so that $u = g(u_1, \dots, u_{n-1}) \neq 0$. The identity $f(u_1, \dots, u_n) = 0$ shows that $ux1 = ux = \sum v_ixw_i$ for all $x \in U$, where $v_i \in A + C \subseteq Q_s(A)$ and $w_i \in U$. Hence Theorem 2.2 tells us that 1 lies in the C-linear span of the w_i's. This in particular shows that $1 \in A$, hence $C \subseteq A$, and so $1 \in \sum Cw_i \subseteq U$. Thus A is a simple algebra over its center C.

By Theorem 4.1 there exist $a, b \in A$ such that $V = aAb$ is a finite dimensional space (we may take $a = b = e = e^2$, but we do not need this). Since A is simple, we have $\sum_j a_j a b_j = \sum_k c_k b d_k = 1$ for some $a_j, b_j, c_k, d_k \in A$. Consequently,

$$x = 1x1 = \Big(\sum_j a_j a b_j\Big) x \Big(\sum_k c_k b d_k\Big) \in \sum_{j,k} a_j V d_k$$

for every $x \in A$. Therefore $\dim_C A < \infty$.

(b) Let $\{a_1, \dots, a_d\}$ be a basis of A. Suppose $\sum_{i,j=1}^d \lambda_{ij} L_{a_i} R_{a_j} = 0$ for some $\lambda_{ij} \in C$. Rewriting this as $\sum_{i=1}^d L_{a_i}\big(\sum_{j=1}^d \lambda_{ij} R_{a_j}\big) = 0$ we see, by using Theorem 2.2, that $\sum_{j=1}^d \lambda_{ij} R_{a_j} = 0$, which in turn yields $\lambda_{ij} = 0$ for all i, j. Therefore $\dim_C M(A) = d^2 = \dim_C \mathrm{End}_C(A)$, and so $M(A) = \mathrm{End}_C(A)$. Consequently, given a nonzero C-linear functional ζ on A there exists $T \in M(A)$ such that $T(x) = \zeta(x)1$ for all $x \in A$. Let $p_i, q_i \in A$ be such that $T = \sum_{i=1}^m L_{p_i} R_{q_i}$ with $\{p_1, \dots, p_m\}$ linearly independent and $q_1 \neq 0$. Let J be a nonzero ideal of R such that $p_i J \bigcup J q_i \subseteq R$, $i = 1, \dots, m$ (Remark 2.1). Now take an arbitrary nonzero ideal I of R. Then $I' = JIJ$ is again a nonzero ideal of R, and note that $T(I') \subseteq I \cap C$. Theorem 2.2 shows that $T(I') \neq 0$, and so $I \cap C \neq 0$. Since $I \subseteq R$ we actually have $I \cap C = I \cap Z$.

(c) Let $\lambda \in C$. Choose a nonzero ideal I of R such that $\lambda I \subseteq R$. Picking $0 \neq z \in I \cap Z$, we thus have $\lambda z \in R \cap C = Z$. Therefore $\lambda = z^{-1} z'$ with $z, z' \in Z$.

Finally, we now know that every element in $a \in A$ can be written as $a = \sum_i z_i^{-1} r_i$ for some $z_i \in Z \setminus \{0\}$ and $r_i \in R$. Hence $a = (\Pi_i z_i)^{-1} r$ with $r \in R$. $\square$

The usual proof of Theorem 5.1, as given in several graduate algebra textbooks (e.g., in [2, 16, 20]), is a beautiful illustration of the power and applicability of the classical structure theory of rings. Its main appeal lies in a surprising combination of different tools and concepts. On the other hand, the proof we gave is more streamlined. In particular, it completely avoids representing elements in our rings as matrices or linear operators. One of its main advantages is that it does not depend on two classical results that the usual proof uses, Kaplansky's theorem on primitive PI-rings [12] and the existence of central polynomials for matrices [11, 18]. As we will indicate in the next two paragraphs, these two results can be easily derived from Theorem 5.1. Therefore our approach leads to a shortcut to the basic structure theory of PI-rings.

Kaplansky's theorem says that a primitive PI-ring R is a finite dimensional central simple algebra over its center. Proving the simplicity is an easy application of the Jacobson Density Theorem; see the first paragraph of the proof of [20, Theorem 23.31]. Now, if R is simple, then its center is a field, and so the desired conclusion follows immediately from Theorem 5.1.

It is easy to see that the algebra of generic $n \times n$ matrices is a prime ring; see, e.g., [20, Corollary 23.52] (i.e., this is easier than showing that it is actually a domain). Therefore its center is nonzero by Theorem 5.1, which immediately implies the existence of central polynomials for $M_n(K)$ with K an infinite field. The author is thankful to L. Rowen and A. Braun for pointing out this simple fact to him. He is also thankful to L. Rowen and (resp.) V. Drensky for drawing his attention to the papers by Braun [6] and (resp.) Kharchenko [13], which also contain nonconstructive proofs of the existence of central polynomials. These proofs are in fact similar to our proof of the assertion (b) in Theorem 5.1. However, they use a version of Posner's theorem for domains, proved by Amitsur already in 1955 [1], i.e., before the discovery of central polynomials by Formanek and Razmyslov in the early 1970's. At any rate, it seems interesting in its own right that a consideration of abstract rings leads to a nontrivial result on matrices.

A downside of nonconstructive proofs of the existence of central polynomials is the limitation to infinite fields. But this can be remedied. In the most recent short note [9] it is shown, by elementary combinatorial methods, that, given an infinite field K of positive characteristic p and a central polynomial c for $M_n(K)$, there exists a multihomogeneous polynomial c_0 with coefficients in the prime field $\mathbb{F}_p$ such that, for an arbitrary (possibly finite) field F of characteristic p, c_0 is central for $M_n(F)$.

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