ZERO LIE PRODUCT DETERMINED BANACH ALGEBRAS

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Abstract. A Banach algebra A is said to be zero Lie product determined if every continuous bilinear functional $\varphi\colon A\times A\to\mathbb{C}$ with the property that $\varphi(a,b)=0$ whenever a and b commute is of the form $\varphi(a,b)=\tau(ab-ba)$ for some $\tau\in A^*$. In the first part of the paper we give some general remarks on this class of algebras. In the second part we consider amenable Banach algebras and show that all group algebras $L^1(G)$ with G an amenable locally compact group are zero Lie product determined.

1. Introduction

Let A be a Banach algebra and let $\varphi\colon A\times A\to\mathbb{C}$ be a continuous bilinear functional satisfying

$$a,b\in A,\ [a,b]=0 \quad\Longrightarrow\quad \varphi(a,b)=0 \tag{1.1}$$

(here and subsequently, $[a,b]$ stands for the commutator $ab-ba$). This is certainly fulfilled if φ is of the form

$$\varphi(a,b)=\tau([a,b])\quad (a,b\in A) \tag{1.2}$$

for some τ in A^*, the dual of A. We will say that A is a *zero Lie product determined Banach algebra* if, for every continuous bilinear functional $\varphi\colon A\times A\to\mathbb{C}$ satisfying (1.1), there exists $\tau\in A^*$ such that (1.2) holds. This is an analytic analogue of the purely algebraic notion of a zero Lie product determined algebra, first indirectly considered in [8] and, slightly later, more systematically in [7] (see also subsequent papers [10, 15]). Further, the concept of a zero Lie product determined Banach algebra can be seen as the Lie version of the notion of a Banach algebra having property $\mathbb{B}$ (see [1]), which will also play an important role in this paper. Another motivation for us for studying this concept is the similarity with the group-theoretic notion of triviality of Bogomolov multiplier (see, e.g., [13]), which made us particularly interested in considering it in the context of group algebras.

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The paper is organized as follows. In Section 2 we provide motivating examples. Firstly, by applying a result by Goldstein [9] we show that C^*-algebras are zero Lie product determined Banach algebras. Secondly, we find a Banach algebra, even a finite dimensional one, that is not zero Lie product determined. In Section 3 we prove that in the definition of a zero Lie product determined Banach algebra one can replace the role of $\mathbb{C}$ by any Banach space. The bulk of the paper is Section 4 in which we show that the group algebra $L^1(G)$ of any amenable locally compact group G is a zero Lie product determined Banach algebra. We actually obtain this as a byproduct of the result concerning the condition

$$a, b \in A,\ ab = ba = 0 \implies \varphi(a, b) = 0, \tag{1.3}$$

where A is an amenable Banach algebras with property $\mathbb{B}$. We remark that (1.3) has also been already studied in the literature, but definitive results were so far obtained only for finite dimensional algebras [3, 12].

2. Examples

The goal of this section is to provide examples indicating the nontriviality of the concept of a zero Lie product determined Banach algebra.

Proposition 2.1. *Every C^*-algebra is a zero Lie product determined Banach algebra.*

Proof. Let A be a C^*-algebra, and let $\varphi\colon A \times A \to \mathbb{C}$ be a continuous bilinear functional satisfying (1.1). Then the map $\psi\colon A \times A \to \mathbb{C}$ defined by $\psi(a, b) = \varphi(a, b^*)$ for all a, $b \in A$ is a continuous sesquilinear functional. Further, if $a, b \in A$ are self-adjoint and $ab = 0$, then $ba = 0$, which in turn implies that $[a, b] = 0$ and therefore $\psi(a, b) = \varphi(a, b) = 0$. This shows that ψ is orthogonal in the sense of [9] (see [9, Definition 1.1]). By [9, Theorem 1.10], A is $\mathbb{C}$-stationary, which means ([9, Definition 1.5]) that there exist $\tau_1, \tau_2 \in A^*$ such that $\psi(a, b) = \tau_1(ab^*) + \tau_2(b^*a)$ for all $a, b \in A$ (see also [11, Section 3]). Consequently, we have

$$\varphi(a, b) = \tau_1(ab) + \tau_2(ba) \quad (a, b \in A). \tag{2.1}$$

On the other hand, if $a \in A$, then $[a, a] = 0$ and therefore $\varphi(a, a) = 0$. Hence φ is skew-symmetric and taking into account (2.1) we get

$$\varphi(a, b) = -\varphi(b, a) = -\tau_1(ba) - \tau_2(ab) \quad (a, b \in A). \tag{2.2}$$

Adding (2.1) and (2.2), we obtain

$$2\varphi(a, b) = \tau_1([a, b]) - \tau_2([a, b]) \quad (a, b \in A),$$

which shows that φ is of the form (1.2), where $\tau \in A^*$ is defined by $\tau = \frac{1}{2}(\tau_1 - \tau_2)$. □

We will now give an example of a finite dimensional Banach algebra that is not zero Lie product determined. This is of interest also from a purely algebraic viewpoint. Namely, so far only examples of infinite dimensional algebras that are not zero Lie product determined were found [7] (since bilinear functionals are automatically continuous in finite dimension, in this framework there is no difference between "zero Lie product determined Banach algebra" and "zero Lie product determined algebra"). The algebra from the next proposition can be thought of as the Grassmann algebra with four generators to which we add another relation.

Proposition 2.2. *The 10-dimensional Banach algebra*

$$A = \mathbb{C}\big\langle x_1, x_2, x_3, x_4 \,|\, x_1x_2 = x_3x_4, x_i^2 = 0, x_ix_j = -x_jx_i,\ i,j = 1,2,3,4\big\rangle$$

is not zero Lie product determined.

Proof. It is easy to check that the elements

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_2x_3,\ x_2x_4$$

form a basis of A (so that $\dim_{\mathbb{C}} A = 10$). Note that 1 and all x_ix_j lie in Z, the center of A.

Define a bilinear functional $\varphi\colon A \times A \to \mathbb{C}$ by

$$\varphi(x_1, x_2) = -\varphi(x_2, x_1) = 1$$

and

$$\varphi(u, v) = 0$$

for all other pairs of elements from our basis. Take a pair of commuting elements a, $b \in A$. We can write

$$a = \sum_{i=1}^{4} \lambda_i x_i + z \quad \text{and} \quad b = \sum_{j=1}^{4} \mu_j x_j + w,$$

where $\lambda_i, \mu_j \in \mathbb{C}$ and $z, w \in Z$. Our goal is to show that $\varphi(a,b) = \lambda_1\mu_2 - \lambda_2\mu_1$ is 0. From $[a,b] = 0$ we obtain

$$\Big[\sum_{i=1}^{4} \lambda_i x_i, \sum_{j=1}^{4} \mu_j x_j\Big] = 0,$$

which yields

$$\begin{aligned}&\big((\lambda_1\mu_2-\lambda_2\mu_1)+(\lambda_3\mu_4-\lambda_4\mu_3)\big)x_1x_2\\&+(\lambda_1\mu_3-\lambda_3\mu_1)x_1x_3+(\lambda_2\mu_3-\lambda_3\mu_2)x_2x_3\\&+(\lambda_1\mu_4-\lambda_4\mu_1)x_1x_4+(\lambda_2\mu_4-\lambda_4\mu_2)x_2x_4=0.\end{aligned}$$

Consequently,

$$(\lambda_1\mu_2-\lambda_2\mu_1)+(\lambda_3\mu_4-\lambda_4\mu_3)=0, \tag{2.3}$$

$$\lambda_1\mu_3=\lambda_3\mu_1,\ \ \lambda_2\mu_3=\lambda_3\mu_2, \tag{2.4}$$

$$\lambda_1\mu_4=\lambda_4\mu_1,\ \ \lambda_2\mu_4=\lambda_4\mu_2. \tag{2.5}$$

Note that (2.4) yields

$$(\lambda_1\mu_2-\lambda_2\mu_1)\mu_3=0.$$

and, similarly, (2.5) yields

$$(\lambda_1\mu_2-\lambda_2\mu_1)\mu_4=0.$$

But then we infer from (2.3) that $\lambda_1\mu_2-\lambda_2\mu_1=0$, as desired. We have thereby proved that φ satisfies (1.1). However, since $\varphi(x_1,x_2)\neq\varphi(x_3,x_4)$, we see from $[x_1,x_2]=[x_3,x_4]$ that φ does not satisfy (1.2). □

3. An alternative definition

From now on, we write $[A,A]$ for the linear span of all commutators of the Banach algebra A.

Proposition 3.1. *Let A be a Banach algebra. Then the following properties are equivalent:*

1. *the algebra A is a zero Lie product determined Banach algebra,*
2. *for each Banach space X, every continuous bilinear map $\varphi\colon A\times A\to X$ with the property that $\varphi(a,b)=0$ whenever $a,b\in A$ are such that $[a,b]=0$ is of the form $\varphi(a,b)=T([a,b])$ $(a,b\in A)$ for some continuous linear map $T\colon[A,A]\to X$.*

Proof. Suppose that (1) holds. Let X be a Banach space and let $\varphi\colon A\times A\to X$ be a continuous bilinear map with the property that $\varphi(a,b)=0$ whenever $a,b\in A$ are such that $[a,b]=0$. For each $\xi\in X^*$, the continuous bilinear functional $\xi\circ\varphi\colon A\times A\to\mathbb{C}$ satisfies (1.1). Therefore there exists a unique $\tau(\xi)\in[A,A]^*$ such that $\xi(\varphi(a,b))=\tau(\xi)([a,b])$ for all $a,b\in A$. It is clear that the map $\tau\colon X^*\to[A,A]^*$ is linear. We next show that τ is continuous.

Let (ξ_n) be a sequence in X^* with $\lim \xi_n = 0$ and $\lim \tau(\xi_n) = \xi$ for some $\xi \in [A,A]^*$. For each $a, b \in A$, we have

$$0 = \lim \xi_n(\varphi(a,b)) = \lim \tau(\xi_n)([a,b]) = \xi([a,b]).$$

We thus have $\xi = 0$, and the closed graph theorem yields the continuity of τ.

For all $a_1, \dots, a_n, b_1, \dots, b_n \in A$ and $\xi \in X^*$ we have

$$\begin{aligned} \xi\Big(\sum_{k=1}^n \varphi(a_k,b_k)\Big) &= \sum_{k=1}^n \xi\big(\varphi(a_k,b_k)\big) = \sum_{k=1}^n \tau(\xi)([a_k,b_k]) \\ &= \tau(\xi)\Big(\sum_{k=1}^n [a_k,b_k]\Big). \end{aligned} \tag{3.1}$$

Consequently, if $a_1, \dots, a_n, b_1, \dots, b_n \in A$ are such that $\sum_{k=1}^n [a_k,b_k] = 0$, then $\xi\big(\sum_{k=1}^n \varphi(a_k,b_k)\big) = 0$ for each $\xi \in X^*$, and hence $\sum_{k=1}^n \varphi(a_k,b_k) = 0$. We thus can define a linear map $T\colon [A,A] \to X$ by

$$T\Big(\sum_{k=1}^n [a_k,b_k]\Big) = \sum_{k=1}^n \varphi(a_k,b_k)$$

for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$. Of course, $\varphi(a,b) = T([a,b])$ for all $a, b \in A$. Our next concern is the continuity of T. Let $a_1, \dots, a_n, b_1, \dots, b_n \in A$. Then there exists $\xi \in X^*$ such that

$$\xi\Big(\sum_{k=1}^n \varphi(a_k,b_k)\Big) = \Big\|\sum_{k=1}^n \varphi(a_k,b_k)\Big\|.$$

On account of (3.1), we have

$$\begin{aligned} \Big\|T\Big(\sum_{k=1}^n [a_k,b_k]\Big)\Big\| &= \Big\|\sum_{k=1}^n \varphi(a_k,b_k)\Big\| = \xi\Big(\sum_{k=1}^n \varphi(a_k,b_k)\Big) \\ &= \Big|\tau(\xi)\Big(\sum_{k=1}^n [a_k,b_k]\Big)\Big| \le \|\tau(\xi)\|\Big\|\sum_{k=1}^n [a_k,b_k]\Big\| \\ &\le \|\tau\|\Big\|\sum_{k=1}^n [a_k,b_k]\Big\|, \end{aligned}$$

which shows the continuity of T, and hence that property (2) holds.

We now assume that (2) holds. Let $\varphi\colon A \times A \to \mathbb{C}$ be a continuous bilinear functional satisfying (1.1). By applying property (2) with $X = \mathbb{C}$, we get $\tau \in [A,A]^*$ such that $\varphi(a,b) = \tau([a,b])$ $(a, b \in A)$. The functional τ can be extended to a continuous linear functional on A so that (1) is obtained. □

4. Amenable Banach algebras with property $\mathbb{B}$

We say that a Banach algebra A has *property* $\mathbb{B}$ if for every continuous bilinear functional $\varphi\colon A \times A \to \mathbb{C}$, the condition

$$a, b \in A,\ ab = 0 \quad \Rightarrow \quad \varphi(a, b) = 0 \tag{4.1}$$

implies the condition

$$\varphi(ab, c) = \varphi(a, bc) \quad (a, b, c \in A). \tag{4.2}$$

According to [5, Remark 2.1], this definition agrees with the one given in the seminal paper [1], i.e., the Banach algebra A has property $\mathbb{B}$ if and only if for each Banach space X and for each continuous bilinear map $\varphi\colon A \times A \to X$ the condition (4.1) implies the condition (4.2). We remark that if A has a bounded approximate identity, (4.2) is equivalent to the condition that $\varphi(a, b) = \tau(ab)$ for some $\tau \in A^*$ (see [1, Lemma 2.3]). In [1] it was shown that many important examples of Banach algebras, including C^*-algebras, group algebras on arbitrary locally compact groups, and the algebra $\mathcal{A}(X)$ of all approximable operators on any Banach space X, have property $\mathbb{B}$, and that this property can be applied to a variety of problems. Since then, a number of papers treating property $\mathbb{B}$ have been published; see the last paper in the series [4] and references therein.

The class of amenable Banach algebras is of great significance. We refer the reader to [14] for the necessary background on amenability. There are different characterizations of amenable Banach algebras. The seminal one comes from B. E. Johnson: vanishing of a certain cohomology group. For our purposes here, the best way to introduce the amenability is the following. Let A be a Banach algebra. The projective tensor product $A\widehat{\otimes}A$ becomes a Banach A-bimodule for the products defined by

$$a \cdot (b \otimes c) = (ab) \otimes c$$

and

$$(b \otimes c) \cdot a = b \otimes (ca)$$

for all $a, b, c \in A$. There is a unique continuous linear map $\pi\colon A\widehat{\otimes}A \to A$ such that

$$\pi(a \otimes b) = ab$$

for all $a, b \in A$. The map π is the projective induced product map, and it is an A-bimodule homomorphism. An *approximate diagonal* for A is a bounded net $(u_\lambda)_{\lambda\in\Lambda}$ in $A\widehat{\otimes}A$ such that, for each $a \in A$, we have

$$\lim_{\lambda\in\Lambda}(a \cdot u_\lambda - u_\lambda \cdot a) = 0 \tag{4.3}$$

and

$$\lim_{\lambda\in\Lambda} \pi(u_\lambda)a = a. \tag{4.4}$$

We point out that (4.3) together with (4.4) implies that also $\lim a\pi(u_\lambda) = a$ for each $a \in A$. Consequently, the net $(\pi(u_\lambda))_{\lambda\in\Lambda}$ is a bounded approximate identity for A. The Banach algebra A is *amenable* if and only if A has an approximate diagonal.

Throughout this section we are notably interested in amenable Banach algebras having property $\mathbb{B}$. According to [14], the following are examples of amenable Banach algebras (which we already know to have property $\mathbb{B}$): nuclear C^*-algebras, the group algebra $L^1(G)$ for each amenable locally compact group G, and the algebra $\mathcal{A}(X)$ for Banach spaces with certain approximation properties (this includes the Banach space $C_0(\Omega)$ for each locally compact Hausdorff space Ω and the Banach space $L^p(\mu)$ for each measure space (Ω, Σ, μ) and each $p \in [1, \infty]$).

We begin with a lemma whose version appears also in [2].

Lemma 4.1. *Let A be a Banach algebra with property $\mathbb{B}$ and having a bounded approximate identity, let X be a Banach space, and let $\varphi\colon A\times A \to X$ be a continuous bilinear map satisfying the condition:*

$$a, b \in A,\ \ ab = ba = 0 \ \Rightarrow\ \varphi(a, b) = 0.$$

Then

$$\varphi(ab, cd) - \varphi(a, bcd) + \varphi(da, bc) - \varphi(dab, c) = 0 \quad (a, b, c, d \in A) \tag{4.5}$$

and there exists a continuous linear operator $S\colon A \to X$ such that

$$\varphi(ab, c) - \varphi(b, ca) + \varphi(bc, a) = S(abc) \quad (a, b, c \in A). \tag{4.6}$$

Proof. Let $\mathcal{B}^2(A;X)$ denote the Banach space of all continuous bilinear maps from $A \times A$ to X, and let $\mathcal{B}^2_0(A;X)$ denote the closed subspace of $\mathcal{B}^2(A;X)$ consisting of those bilinear maps φ which satisfy (4.1). We define

$$\psi\colon A \times A \to \mathcal{B}^2(A;X)$$

by

$$\psi(a, b)(s, t) = \varphi(bs, ta) \quad (a, b, s, t \in A).$$

It is immediate to check that $\psi(a, b) \in \mathcal{B}^2_0(A;X)$ whenever $a, b \in A$ are such that $ab = 0$. Consequently, the continuous bilinear map

$$\widetilde{\psi}\colon A \times A \to \mathcal{B}^2(A;X)/\mathcal{B}^2_0(A;X)$$

defined by

$$\widetilde{\psi}(a, b) = \psi(a, b) + \mathcal{B}^2_0(A;X) \quad (a, b \in A)$$

satisfies (4.1). Property $\mathbb{B}$ then gives

$$\psi(ab,c) - \psi(a,bc) \in \mathcal{B}_0^2(A;X) \quad (a,b,c \in A).$$

For each $a, b, c \in A$, property $\mathbb{B}$ now yields

$$\big(\psi(ab,c) - \psi(a,bc)\big)(rs,t) = \big(\psi(ab,c) - \psi(a,bc)\big)(r,st)$$

for all r, s, $t \in A$. Hence

$$\varphi(crs,tab) - \varphi(bcrs,ta) - \varphi(cr,stab) + \varphi(bcr,sta) = 0 \tag{4.7}$$

for all a, b, c, r, s, $t \in A$.

Let $(\rho_\lambda)_{\lambda\in\Lambda}$ be an approximate identity of A of bound C. For each a, b, c, r, $s \in A$, we apply (4.7) with the element t replaced by ρ_λ ($\lambda \in \Lambda$) and then we take the limit to arrive at

$$\varphi(crs,ab) - \varphi(bcrs,a) - \varphi(cr,sab) + \varphi(bcr,sa) = 0. \tag{4.8}$$

We now replace r by ρ_λ ($\lambda \in \Lambda$) in (4.8) and take the limit to get

$$\varphi(cs,ab) - \varphi(bcs,a) - \varphi(c,sab) + \varphi(bc,sa) = 0,$$

which gives (4.5).

By applying (4.5) with the element c replaced by ρ_λ ($\lambda \in \Lambda$) we see that the net $\varphi(dab,\rho_\lambda)_{\lambda\in\Lambda}$ is convergent and by taking the limit in (4.5) we arrive at

$$\begin{aligned} &\varphi(ab,d) - \varphi(a,bd) + \varphi(da,b) - \lim_{\lambda\in\Lambda}\varphi(dab,\rho_\lambda) \\ &= \lim_{\lambda\in\Lambda}\big(\varphi(ab,\rho_\lambda d) - \varphi(a,b\rho_\lambda d) + \varphi(da,b\rho_\lambda) - \varphi(dab,\rho_\lambda)\big) = 0 \end{aligned} \tag{4.9}$$

for all a, b, $d \in A$. By Cohen's factorization theorem (see [6, Corollary 11 in §11]), each $c \in A$ can be written in the form $c = dab$ with a, b, $d \in A$, and hence the net $(\varphi(c,\rho_\lambda))_{\lambda\in\Lambda}$ is convergent. We can thus define a linear operator $S\colon A \to X$ by

$$S(a) = \lim_{\lambda\in\Lambda}\varphi(a,\rho_\lambda)$$

for each $a \in A$. Since $\|\varphi(a,\rho_\lambda)\| \le C\|\varphi\|\|a\|$ for all $a \in A$ and $\lambda \in \Lambda$, it follows that $\|S(a)\| \le C\|\varphi\|\|a\|$ for each $a \in A$, which implies that S is continuous. Further, (4.9) gives (4.6). □

Lemma 4.2. *Let A be an amenable Banach algebra, let X be a Banach space, and let $\varphi\colon A \times A \to X$ be a continuous bilinear map. Suppose that there exists a continuous linear operator $S\colon A \to X$ such that*

$$\varphi(ab,c) - \varphi(b,ca) + \varphi(bc,a) = S(abc) \quad (a,b,c \in A). \tag{4.10}$$

Then there exist continuous linear operators $\Phi\colon [A,A] \to X$ and $\Psi\colon A \to X$ such that

$$\varphi(a,b) = \Phi([a,b]) + \Psi(a \circ b) \quad (a,b \in A).$$

Here and subsequently, $a \circ b$ stands for $ab + ba$.

Proof. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate diagonal for A of bound C, and let $\mathcal{U}$ be an ultrafilter on Λ refining the order filter. On account of the Banach-Alaoglu theorem, each bounded subset of the bidual X^{**} of X is relatively compact with respect to the weak*-topology. Consequently, each bounded net $(x_\lambda)_{\lambda\in\Lambda}$ in X has a unique limit in X^{**} with respect to the weak*-topology along the ultrafilter $\mathcal{U}$, and we write $\lim_{\mathcal{U}} x_\lambda$ for this limit.

Let $\widehat{\varphi}\colon A\widehat{\otimes}A \to X$ be the unique continuous linear map such that

$$\widehat{\varphi}(a\otimes b) = \varphi(a, b)$$

for all $a, b \in A$. We define $T\colon A \to X^{**}$ by

$$T(a) = \lim_{\mathcal{U}} \widehat{\varphi}(u_\lambda \cdot a)$$

for each $a \in A$. For each $a \in A$, we have

$$\|\widehat{\varphi}(u_\lambda \cdot a)\| \le \|\widehat{\varphi}\|\|u_\lambda\|\|a\| \le C\|\varphi\|\|a\| \quad (\lambda\in\Lambda). \tag{4.11}$$

Hence the net $(\widehat{\varphi}(u_\lambda \cdot a))_{\lambda\in\Lambda}$ is bounded and the map T is well-defined. The linearity of the limit along an ultrafilter on a topological linear space gives the linearity of T. Further, from (4.11) we deduce that $\|T(a)\| \le C\|\varphi\|\|a\|$ for each $a \in A$, which gives the continuity of T.

We now claim that

$$\widehat{\varphi}(u\cdot a) = \widehat{\varphi}(a\cdot u) + \widehat{\varphi}(\pi(u)\otimes a) - S(a\pi(u)) \tag{4.12}$$

for all $a \in A$ and $u \in A\widehat{\otimes}A$. Of course, it suffices to prove (4.12) for the simple tensor products $u = b\otimes c$ with $b, c \in A$. Observe that (4.10) can be written as

$$\widehat{\varphi}(a\cdot(b\otimes c)) - \widehat{\varphi}((b\otimes c)\cdot a) + \widehat{\varphi}(\pi(b\otimes c)\otimes a) = S(a\pi(b\otimes c))$$

and this gives (4.12).

For each $\lambda \in \Lambda$, we apply (4.12) with u replaced by $u_\lambda \cdot a$ and a replaced by b to get the following

$$\begin{aligned}\widehat{\varphi}(u_\lambda\cdot(ab)) &= \widehat{\varphi}((u_\lambda\cdot a)\cdot b)\\ &= \widehat{\varphi}(b\cdot u_\lambda\cdot a) + \widehat{\varphi}(\pi(u_\lambda\cdot a)\otimes b) - S(b\pi(u_\lambda\cdot a))\\ &= \widehat{\varphi}(b\cdot u_\lambda\cdot a) + \widehat{\varphi}((\pi(u_\lambda)a)\otimes b) - S(b\pi(u_\lambda)a).\end{aligned}$$

We thus have

$$\begin{aligned}&\widehat{\varphi}(u_\lambda\cdot(ab)) - \widehat{\varphi}(u_\lambda\cdot(ba))\\ =&\widehat{\varphi}(b\cdot u_\lambda\cdot a) - \widehat{\varphi}(u_\lambda\cdot(ba)) + \widehat{\varphi}((\pi(u_\lambda)a)\otimes b) - S(b\pi(u_\lambda)a)\\ =&\widehat{\varphi}((b\cdot u_\lambda - u_\lambda\cdot b)\cdot a) + \widehat{\varphi}((\pi(u_\lambda)a)\otimes b) - S(b\pi(u_\lambda)a).\end{aligned} \tag{4.13}$$

On account of (4.3), we have $\lim_{\lambda\in\Lambda}(b\cdot u_\lambda - u_\lambda\cdot b) = 0$ and therefore $\lim_{\lambda\in\Lambda}(b\cdot u_\lambda - u_\lambda\cdot b)\cdot a = 0$, which implies that $\lim_{\lambda\in\Lambda}\widehat{\varphi}((b\cdot u_\lambda - u_\lambda\cdot b)\cdot a) = 0$. Since $\mathcal{U}$ refines the order filter on Λ, it follows that $\lim_{\mathcal{U}}\widehat{\varphi}((b\cdot u_\lambda - u_\lambda\cdot b)\cdot a) = 0$.

According to (4.4), we have $\lim_{\lambda\in\Lambda}\pi(u_\lambda)a = a$. Hence

$$\lim_{\lambda\in\Lambda}(\pi(u_\lambda)a)\otimes b = a\otimes b \quad \text{and} \quad \lim_{\lambda\in\Lambda} b\pi(u_\lambda)a = ba.$$

The continuity of both $\widehat{\varphi}$ and S then gives

$$\lim_{\lambda\in\Lambda}\widehat{\varphi}((\pi(u_\lambda)a)\otimes b) = \widehat{\varphi}(a\otimes b) \quad \text{and} \quad \lim_{\lambda\in\Lambda} S(b\pi(u_\lambda)a) = S(ba).$$

Since $\mathcal{U}$ refines the order filter on Λ, we conclude that

$$\lim_{\mathcal{U}}\widehat{\varphi}((\pi(u_\lambda)a)\otimes b) = \widehat{\varphi}(a\otimes b) \quad \text{and} \quad \lim_{\mathcal{U}} S(b\pi(u_\lambda)a) = S(ba).$$

We now prove that

$$\varphi(a,b) = T([a,b]) + S(ba) \quad (a,b\in A). \tag{4.14}$$

Indeed, by taking the limit along $\mathcal{U}$ in (4.13) we arrive at

$$\begin{aligned}
T([a,b]) = T(ab) - T(ba) &= \lim_{\mathcal{U}}\widehat{\varphi}(u_\lambda\cdot(ab)) - \lim_{\mathcal{U}}\widehat{\varphi}(u_\lambda\cdot(ba)) \\
&= \lim_{\mathcal{U}}(\widehat{\varphi}(u_\lambda\cdot(ab)) - \widehat{\varphi}(u_\lambda\cdot(ba))) \\
&= \lim_{\mathcal{U}}\widehat{\varphi}((b\cdot u_\lambda - u_\lambda\cdot b)\cdot a) + \lim_{\mathcal{U}}\widehat{\varphi}((\pi(u_\lambda)a)\otimes b) \\
&\quad - \lim_{\mathcal{U}} S(b\pi(u_\lambda)a) \\
&= \widehat{\varphi}(a\otimes b) - S(ba) = \varphi(a,b) - S(ba).
\end{aligned}$$

Define $\Phi\colon [A,A]\to X^{**}$ and $\Psi\colon A\to X$ by

$$\Phi(a) = (T - \tfrac{1}{2}S)(a) \quad (a\in[A,A])$$

and

$$\Psi = \tfrac{1}{2}S.$$

Note that, on account of (4.14), T maps $[A,A]$ into X and therefore Φ does not map merely into X^{**}, but actually into X. From (4.14) we see that $\varphi(a,b) = \Phi([a,b]) + \Psi(a\circ b)$ for all $a,b\in A$. □

Theorem 4.3. *Let A be an amenable Banach algebra with property $\mathbb{B}$, let X be a Banach space, and let $\varphi\colon A\times A\to X$ be a continuous bilinear map satisfying the condition:*

$$a,b\in A,\ ab = ba = 0 \ \Rightarrow\ \varphi(a,b) = 0.$$

Then there exist continuous linear operators $\Phi\colon [A,A]\to X$ and $\Psi\colon A\to X$ such that

$$\varphi(a,b) = \Phi([a,b]) + \Psi(a\circ b)$$

for all a, $b \in A$.

Proof. A straightforward consequence of Lemmas 4.1 and 4.2. □

Corollary 4.4. *If A is an amenable Banach algebra with property $\mathbb{B}$, then A is a zero Lie product determined Banach algebra.*

Proof. Let $\varphi\colon A\times A \to \mathbb{C}$ be a continuous bilinear functional satisfying (1.1). If $a, b \in A$ are such that $ab = ba = 0$, then $[a, b] = 0$ and therefore $\varphi(a, b) = 0$. Consequently, the functional φ satisfies the condition in Theorem 4.3. Hence there exist continuous linear functionals $\tau_1\colon [A, A] \to \mathbb{C}$ and $\tau_2\colon A \to \mathbb{C}$ such that

$$\varphi(a, b) = \tau_1([a, b]) + \tau_2(a \circ b) \quad (a, b \in A). \tag{4.15}$$

Of course, the functional τ_1 extends to a continuous linear functional on A. On the other hand, if $a \in A$, then $[a, a] = 0$ and therefore $\varphi(a, a) = 0$. Hence φ is skew-symmetric and (4.15) yields

$$\begin{aligned}\varphi(a, b) &= -\varphi(b, a) = -\tau_1([b, a]) - \tau_2(b \circ a)\\ &= \tau_1([a, b]) - \tau_2(a \circ b) \quad (a, b \in A).\end{aligned} \tag{4.16}$$

Adding (4.15) and (4.16), we obtain

$$\varphi(a, b) = \tau_1([a, b]) \quad (a, b \in A),$$

which shows that φ is of the form (1.2). □

Since the group algebra $L^1(G)$ has property $\mathbb{B}$ for each locally compact group G and further it is amenable exactly in the case when G is amenable, the following result follows.

Theorem 4.5. *Let G be an amenable locally compact group. Then the group algebra $L^1(G)$ is a zero Lie product determined Banach algebra.*

It should be pointed out that we do not know whether or not Theorem 4.5 fails to be true without the assumption of amenability.

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References

[1] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, *Maps preserving zero products*, Studia Math. 193 (2009), 131–159.

[2] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, *Characterizing Jordan maps on C^*-algebras through zero products*, Proc. Edinb. Math. Soc. 53 (2010), 543–555.

[3] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, *On bilinear maps determined by rank one idempotents*, Linear Algebra Appl. 432 (2010), 738–743.

[4] J. Alaminos, M. Brešar, J. Extremera, Š. Špenko, A. R. Villena, *Commutators and square-zero elements in Banach algebras*, Quart. J. Math. 67 (2016), 1–13.

[5] J. Alaminos, J. Extremera, A. R. Villena, *Disjointness-preserving linear maps on Banach function algebras associated to a locally compact group*, Ann. Funct. Anal. 7 (2016), 442–451.

[6] F. F. Bonsall, J. Duncan, *Complete normed algebras,* Springer-Verlag, 1973.

[7] M. Brešar, M. Grašič, J. Sanchez, *Zero product determined matrix algebras*, Linear Algebra Appl. 430 (2009), 1486–1498.

[8] M. Brešar, P. Šemrl, *On bilinear maps on matrices with applications to commutativity preservers*, J. Algebra 301 (2006), 803–837.

[9] S. Goldstein, *Stationarity of operator algebras*, J. Funct. Anal. 118 (1993), 275–308.

[10] M. Grašič, *Zero product determined classical Lie algebras*, Linear Multilinear Algebra 58 (2010), 1007–1022.

[11] U. Haagerup, N. J. Laustsen, *Weak amenability of C^*-algebras and a theorem of Goldstein.* Banach algebras '97 (Blaubeuren), 223–243, de Gruyter, Berlin, 1998.

[12] M. T. Koşan, T.-K. Lee, Y. Zhou, *Bilinear forms on matrix algebras vanishing on zero products of xy and yx*, Linear Algebra Appl. 453 (2014), 110–124.

[13] P. Moravec, *Unramified Brauer groups of finite and infinite groups*, Amer. J. Math. 134 (2012), 1679–1704.

[14] V. Runde, *Lectures on amenability,* Springer, 2002.

[15] D. Wang, X. Yu, Z. Chen, *A class of zero product determined Lie algebras*, J. Algebra 331 (2011), 145–151.

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