

# COMMUTATORS AND SQUARE-ZERO ELEMENTS IN BANACH ALGEBRAS

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ABSTRACT. Our initial result states that, in a certain class of Banach algebras which includes  $C^*$ -algebras and group algebras of locally compact groups, every commutator lies in the closed linear span of square-zero elements. The proof relies on the theory of Banach algebras with property  $\mathbb{B}$ . We then study several variations and extensions of this result. For instance, we show that in a von Neumann algebra every commutator is actually a finite sum of square zero elements. We also consider the commutator ideal and the existence of some special square-zero elements.

## 1. INTRODUCTION

A complex Banach algebra  $A$  is said to have *property  $\mathbb{B}$*  if, for every continuous bilinear map  $f: A \times A \rightarrow X$  where  $X$  is an arbitrary Banach space, the condition that for all  $x, y \in A$ ,

$$(1) \quad xy = 0 \implies f(x, y) = 0,$$

implies that

$$(2) \quad f(xy, z) = f(x, yz) \quad \text{for all } x, y, z \in A.$$

This concept was introduced in [1] and has since turned out to be applicable to a variety of problems; see for example [1, 2, 3, 27] and references therein. The main message of [1] is that the class of Banach algebras with property  $\mathbb{B}$  is quite large, in particular it includes  $C^*$ -algebras and group algebras of arbitrary locally compact groups.

The starting point of this paper is the observation that every commutator in a Banach algebra  $A$  with property  $\mathbb{B}$  which also satisfies  $\overline{A^2} = A$  (this is fulfilled in  $C^*$ -algebras and group algebras of locally compact groups) is contained in the closed linear span of square-zero elements (Theorem 2.1). The proof is fairly easy, but the result itself is perhaps a bit surprising. In particular it generalizes and unifies two classical results: the one by Kaplansky stating that a  $C^*$ -algebra  $A$  has a nonzero nilpotent element if and only if  $A$  is noncommutative (see, e.g., [19, p. 292]), and the one by Behncke stating that the group algebra  $L^1(G)$  of a locally compact group  $G$  has a nonzero nilpotent element if and only if  $G$  is not abelian [5]. On the other

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hand, there are several results in the literature stating that, in certain algebras, every commutator lies in the (non-closed) linear span of square-zero elements, i.e., it can be written as a finite sum of such elements. For example, this is true in the algebra  $B(H)$  of all bounded linear operators on an infinite-dimensional Hilbert space [14, 25]; in fact, an element in  $B(H)$  is a commutator if and only if it can be written as the sum of 4 square-zero elements [29]. Percy and Topping showed that commutators are sums of (a certain number of) square-zero elements in some classes of von Neumann algebras [25, 26], Kataoka established this for stable and properly infinite  $C^*$ -algebras [21], and Marcoux for  $C^*$ -algebras containing some special projections [23, 24]. There has also been some interest in this topic in pure algebra. Recently, Chebotar, Lee, and Puczylowski [10] showed that every commutator in a simple ring with a nontrivial idempotent is the sum of square-zero elements, but, on the other hand, there exists a simple ring with zero-divisors in which this is not the case; the latter gave the answer to a long-standing problem by Herstein [16].

The question that now presents itself is whether every commutator in an arbitrary  $C^*$ -algebra  $A$  is a finite sum of square-zero elements in  $A$ . In Section 3 we show that (1) does not always imply (2) if  $f$  is a discontinuous bilinear map on a  $C^*$ -algebra, and thereby rule out the possibility to get a positive answer to this question by using the same approach as in the proof of Theorem 2.1. However, by using an algebraic approach based on idempotents, we prove, in Section 4, that the answer is positive at least for von Neumann algebras. For general  $C^*$ -algebras the question remains open. In Section 5 we prove that, in a certain class of Banach algebras with property  $\mathbb{B}$ , in particular in  $C^*$ -algebras and some Banach algebras associated with a compact group, the commutator ideal coincides with the closed subalgebra generated by square-zero elements. Finally, in Section 6 we study the existence of some special square-zero elements in Banach algebras with property  $\mathbb{B}$ . It follows almost immediately from the definition that a noncommutative unital Banach algebra with property  $\mathbb{B}$  contains a pair of elements  $a, b$  such that  $ab = 0$  and  $ba \neq 0$  (and so  $ba$  is a nonzero square-zero element). We will establish some somewhat deeper results of a similar type.

## 2. COMMUTATORS AND SQUARE-ZERO ELEMENTS IN BANACH ALGEBRAS WITH PROPERTY $\mathbb{B}$

Let  $A$  be an algebra. As usual, the *commutator*  $xy - yx$  of elements  $x, y \in A$  will be denoted by  $[x, y]$ . We write  $[A, A]$  for the linear span of all commutators in  $A$ . By a *square-zero element* we mean any element  $x \in A$  such that  $x^2 = 0$ . The *centre* of  $A$  will be denoted by  $Z(A)$ .

A Banach algebra  $A$  is said to be *essential* if  $\overline{A^2} = A$  (i.e., every element in  $A$  is the limit of sums of elements of the form  $xy$  with  $x, y \in A$ ). For example, every Banach algebra with a (left, right) approximate identity is essential.  $C^*$ -algebras and group algebras of locally compact groups therefore have property  $\mathbb{B}$  and are essential; for further examples of such Banach algebras we refer the reader to [1].

**Theorem 2.1.** *Let  $A$  be an essential Banach algebra with property  $\mathbb{B}$ . Then every commutator in  $A$  lies in the closed linear span of square-zero elements.*

*Proof.* Denote the closed linear span of all square-zero elements in  $A$  by  $V$ , and consider the map  $f: A \times A \rightarrow A/V$  given by  $f(x, y) = yx + V$ . Note that  $f$  is bilinear, continuous, and satisfies the condition (1). Consequently,  $f$  satisfies (2), meaning that  $[zx, y] \in V$ . Since  $A$  is essential, this implies that  $V$  contains all commutators in  $A$ .  $\square$

We now show that, for any  $C^*$ -algebra and some Banach algebras associated with a compact group, the square-zero elements lie in the closure of the linear span of the commutators.

**Proposition 2.2.** *Let  $A$  be a  $C^*$ -algebra. Then the closed linear span of all square-zero elements in  $A$  coincides with the closure of  $[A, A]$ .*

*Proof.* As observed by Kataoka, every square-zero element  $x$  in a  $C^*$ -algebra  $A$  is a commutator [21, Proposition 6]. The proof is as follows. We consider the polar decomposition  $x = u|x|$  of  $x$  in the enveloping von Neumann algebra of  $A$ . Since  $|x|^{1/2}$  lies in the  $C^*$ -subalgebra of  $A$  generated by  $|x|$ , it follows that there exists a sequence  $(f_n)$  of polynomials such that  $|x|^{1/2}$  is the limit in norm of the sequence  $(|x|f_n(|x|))$ . Consequently,

$$u|x|^{1/2} = \lim u|x|f_n(|x|) = \lim x f_n(|x|) \in A.$$

Since  $u^*x = |x|$ , we see that  $|x|u|x| = u^*x^2 = 0$  and therefore we have

$$|x|^{1/2}u|x|^{1/2} = \lim f_n(|x|)|x|u|x|f_n(|x|) = 0.$$

Consequently, we have  $x = [u|x|^{1/2}, |x|^{1/2}]$ , which is a commutator in  $A$ . This of course implies that the closed linear span of square zero elements is contained in the closed linear span of commutators, while the converse inclusion follows from Theorem 2.1.  $\square$

In the remainder of this section we are concerned with a compact group  $G$ . The Haar measure  $\lambda$  on  $G$  is normalized, so that  $\lambda(G) = 1$ , and  $\lambda$  is both left and right invariant. We write  $\int_G f(t) dt$  for the integral of  $f \in L^1(G)$  with respect to  $\lambda$ . The Banach spaces  $L^p(G)$ , with  $1 \leq p \leq \infty$ , and  $C(G)$  are Banach algebras under convolution.

Let  $\pi$  be an irreducible continuous unitary representation of  $G$  on a Hilbert space  $H_\pi$ . Then  $d_\pi = \dim(H_\pi) < \infty$  and the character  $\chi_\pi$  of  $\pi$  is the continuous function on  $G$  defined by

$$\chi_\pi(t) = \text{trace}(\pi(t)) \quad (t \in G).$$

By [18, Theorem 27.24(i)],  $\chi_\pi \in Z(L^1(G))$ . We write  $\mathfrak{T}_\pi(G)$  for the linear span of the set of continuous functions on  $G$  of the form  $t \mapsto \langle \pi(t)u|v \rangle$  as  $u, v$  vary over  $H_\pi$ . On account of [18, Theorems 27.21 and 27.24], it follows that  $\mathfrak{T}_\pi(G)$  is a minimal ideal of  $L^1(G)$  isomorphic to the full matrix algebra  $M_{d_\pi}(\mathbb{C})$ ,  $d_\pi\chi_\pi$  is the unit of  $\mathfrak{T}_\pi(G)$ , and

$$\text{trace}(f * d_\pi\chi_\pi) = \int_G f(t)\overline{\chi_\pi(t)} dt$$

for each  $f \in L^1(G)$ . We write  $\mathfrak{T}(G)$  for the linear span of the functions in  $\mathfrak{T}_\pi(G)$  as  $\pi$  varies over the irreducible continuous unitary representations of

$G$ . Functions in  $\mathfrak{T}(G)$  are called *trigonometric polynomials* on  $G$ . By [15, Theorem 5.11],  $\mathfrak{T}(G)$  is dense in  $L^p(G)$  for  $1 \leq p < \infty$  and  $C(G)$ . Further  $\mathfrak{T}(G) * \mathfrak{T}(G) = \mathfrak{T}(G)$ . These facts entail that any of the Banach algebras  $L^p(G)$  for  $1 \leq p < \infty$  and  $C(G)$  is essential and has property  $\mathbb{B}$  (see [1, Example 1.3.3(4)]). We will denote by  $\mathcal{X}(G)$  the set of the characters of the irreducible continuous unitary representations of  $G$ .

**Proposition 2.3.** *Let  $G$  be a compact group, and let  $A$  be any of the Banach algebras  $L^p(G)$ , with  $1 \leq p < \infty$ , or  $C(G)$ . Then the following conditions on  $f \in A$  are equivalent:*

- (i)  $f$  lies in the closure of  $[A, A]$ ;
- (ii)  $f$  lies in the closed linear span of all square-zero elements of  $A$ ;
- (iii)  $\int_G f(t) \overline{\chi(t)} dt = 0$  for each  $\chi \in \mathcal{X}(G)$ .

*Proof.* Theorem 2.1 shows that (i) implies (ii).

We now assume that (ii) holds. Let  $\pi$  be an irreducible continuous unitary representation of  $G$ . Then  $f * d_\pi \chi_\pi \in \mathfrak{T}_\pi(G)$  can be thought of as a square-zero  $d_\pi \times d_\pi$  matrix and hence

$$0 = \text{trace}(f * d_\pi \chi_\pi) = \int_G f(t) \overline{\chi_\pi(t)} dt,$$

which shows that  $f$  satisfies (iii).

Finally, we assume that  $f$  satisfies (iii). Let  $\varepsilon > 0$ . Let  $\mathcal{U}$  denote the family of all compact, symmetric neighbourhoods of the identity  $e$  of  $G$  that are invariant under the inner automorphisms of  $G$ . On account of [17, Theorem 4.9],  $\mathcal{U}$  is a neighbourhoods base at  $e$ . For every  $U \in \mathcal{U}$ , let  $\varrho_U$  be  $\lambda(U)^{-1}$  times the characteristic function of  $U$ . By [15, Proposition 2.42], the net  $(f * \varrho_U)_{U \in \mathcal{U}}$  (where  $\mathcal{U}$  is ordered by reverse inclusion) converges to  $f$  in  $A$ . Let  $U \in \mathcal{U}$  be such that  $\|f * \varrho_U - f\| < 2^{-1}\varepsilon$ . Since  $U$  is invariant, [18, Theorem 28.49] shows that  $\varrho_U \in Z(L^1(G))$  and [15, Proposition 5.25] now shows that there exist finitely many irreducible continuous unitary representations  $\pi_1, \dots, \pi_n$  of  $G$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  such that

$$\left\| \varrho_U - \sum_{k=1}^n \alpha_k \chi_{\pi_k} \right\|_1 < 2^{-1} (1 + \|f\|)^{-1} \varepsilon.$$

Then we have

$$\begin{aligned} \left\| f - \sum_{k=1}^n \alpha_k f * \chi_{\pi_k} \right\| &\leq \|f - f * \varrho_U\| + \left\| f * \left( \varrho_U - \sum_{k=1}^n \alpha_k \chi_{\pi_k} \right) \right\| \\ &< 2^{-1}\varepsilon + \|f\| \left\| \varrho_U - \sum_{k=1}^n \alpha_k \chi_{\pi_k} \right\|_1 < \varepsilon. \end{aligned}$$

Since  $f * \chi_{\pi_k} \in \mathfrak{T}_{\pi_k}(G)$  and

$$\text{trace}(f * \chi_{\pi_k}) = d_{\pi_k}^{-1} \int_G f(t) \overline{\chi_{\pi_k}(t)} dt = 0,$$

it follows that  $f * \chi_{\pi_k} = [g_k, h_k]$  for some  $g_k, h_k \in \mathfrak{T}_{\pi_k}(G)$  for  $k \in \{1, \dots, n\}$ . Therefore  $\sum_{k=1}^n \alpha_k f * \chi_{\pi_k} \in [A, A]$ .  $\square$

3. THE ALGEBRAIC PROPERTY  $\mathbb{B}$ 

Is the closure unnecessary in both Propositions 2.2 and 2.3, i.e., is every commutator in  $A$  a finite sum of square-zero elements? A rather obvious way to attack this question is to consider the algebraic version of property  $\mathbb{B}$ . Unfortunately, as we will see in Theorem 3.3 below, this cannot lead to a definitive conclusion.

Let us precise what we mean by the “algebraic property  $\mathbb{B}$ ”. We will say that an algebra  $A$  over a commutative unital ring  $C$  has property  $\mathbb{B}_{\text{alg}}$  if, for every bilinear map  $f$  from  $A \times A$  into an arbitrary  $C$ -module  $X$ , the condition (1) implies the condition (2). This notion is in fact just a small modification of the notion of a *zero product determined algebra*, introduced in [9] and later discussed in a series of papers — yet mostly in the context of nonassociative algebras. The difference in the definition is that the condition (2) is replaced by the condition that there exists a  $C$ -linear map  $T: A^2 \rightarrow X$  such that  $f(x, y) = T(xy)$  for all  $x, y \in A$ . It is clear that every (associative) zero product determined algebra has property  $\mathbb{B}_{\text{alg}}$ , and, conversely, a unital algebra with property  $\mathbb{B}_{\text{alg}}$  is zero product determined (just set  $z = 1$  in (2)). An example of such an algebra is  $M_n(B)$ , the algebra of  $n \times n$  matrices (with  $n \geq 2$ ) over a unital algebra  $B$  [9, Theorem 2.1]. Further, every unital algebra which is generated by its idempotents has property  $\mathbb{B}_{\text{alg}}$  [6, Theorem 4.1]. The algebra  $M_n(B)$  is in fact generated by its idempotents (see Proposition 4.2 below), so this second result is more general.

The following result is an algebraic analogue of Theorem 2.1.

**Theorem 3.1.** *Let  $A$  be an algebra with property  $\mathbb{B}_{\text{alg}}$  and such that  $A^2 = A$ . Then every commutator in  $A$  is the sum of square-zero elements.*

*Proof.* Define  $V$  as the  $C$ -module generated by all square-zero elements in  $A$ , and proceed as in the proof of Theorem 2.1.  $\square$

**Corollary 3.2.** *Let  $A$  be a unital algebra generated by its idempotents. Then every commutator in  $A$  is the sum of square-zero elements.*

Theorem 3.1 covers various examples of  $C^*$ -algebras. However, we will now show that this approach has limitations.

**Theorem 3.3.** *Let  $A$  be an infinite-dimensional, semisimple, commutative Banach algebra. Then  $A$  does not have property  $\mathbb{B}_{\text{alg}}$ .*

*Proof.* Let  $\Omega$  denote the character space of  $A$ . If  $x \in A$ , then we write  $\hat{x}$  for the Gelfand transform of  $x$  and  $\mathfrak{h}(x) = \{s \in \Omega : \hat{x}(s) = 0\}$ . We think of the algebraic tensor product  $A \otimes A$  as a subalgebra of  $C_0(\Omega \times \Omega)$  through the usual identification given by  $(x \otimes y)(s, t) = \hat{x}(s)\hat{y}(t)$  for all  $s, t \in \Omega$  and  $x, y \in A$ .

Since  $\dim(A) = \infty$ , it follows that  $\Omega$  is infinite. Let  $\Lambda$  be a countable infinite subset of  $\Omega$ . By [11, Corollary 2.2.26], there exists  $u \in A$  such that  $\hat{u}(s) \neq 0$  ( $s \in \Lambda$ ) and  $\hat{u}(s) \neq \hat{u}(t)$  ( $s, t \in \Lambda$ ,  $s \neq t$ ).

We claim that

$$u^2 \otimes u - u \otimes u^2 \notin \text{span}(\{x \otimes y : x, y \in A, xy = 0\}).$$

On the contrary, suppose that  $u^2 \otimes u - u \otimes u^2 = \sum_{k=1}^n x_k \otimes y_k$  for some  $x_1, \dots, x_n, y_1, \dots, y_n \in A$  with  $x_k y_k = 0$  for each  $k \in \{1, \dots, n\}$ . We thus

have

$$\widehat{u}(s)\widehat{u}(t)(\widehat{u}(s) - \widehat{u}(t)) = \sum_{k=1}^n \widehat{x}_k(s)\widehat{y}_k(t) \quad (s, t \in \Omega).$$

Since  $x_1y_1 = 0$ , it follows that  $\Lambda \subset \mathfrak{h}(x_1) \cup \mathfrak{h}(y_1)$  and therefore either  $\mathfrak{h}(x_1) \cap \Lambda$  or  $\mathfrak{h}(y_1) \cap \Lambda$  is infinite. We define  $\Lambda_1$  to be any of the infinite sets of the pair  $\{\mathfrak{h}(x_1) \cap \Lambda, \mathfrak{h}(y_1) \cap \Lambda\}$ . Observe that  $\widehat{x}_1(s)\widehat{y}_1(t) = 0$  for all  $s, t \in \Lambda_1$ . Since  $x_2y_2 = 0$ , it follows that  $\Lambda_1 \subset \mathfrak{h}(x_2) \cup \mathfrak{h}(y_2)$  and therefore either  $\mathfrak{h}(x_2) \cap \Lambda_1$  or  $\mathfrak{h}(y_2) \cap \Lambda_1$  is infinite. We define  $\Lambda_2$  to be any of the infinite sets of the pair  $\{\mathfrak{h}(x_2) \cap \Lambda_1, \mathfrak{h}(y_2) \cap \Lambda_1\}$ . We now observe that  $\widehat{x}_1(s)\widehat{y}_1(t) = \widehat{x}_2(s)\widehat{y}_2(t) = 0$  for all  $s, t \in \Lambda_2$ . By repeating the process we get infinite sets  $\Lambda \supset \Lambda_1 \supset \dots \supset \Lambda_n$  such that

$$\widehat{x}_1(s)\widehat{y}_1(t) = \dots = \widehat{x}_k(s)\widehat{y}_k(t) = 0 \quad (s, t \in \Lambda_k, k = 1, \dots, n).$$

Accordingly,  $\widehat{u}(s)\widehat{u}(t)(\widehat{u}(s) - \widehat{u}(t)) = 0$  and so  $\widehat{u}(s) = \widehat{u}(t)$  for all  $s, t \in \Lambda_n$ , a contradiction.

Since  $u^2 \otimes u - u \otimes u^2 \notin \text{span}(\{x \otimes y : x, y \in A, xy = 0\})$ , it may be concluded that there exists a linear functional  $\varphi: A \otimes A \rightarrow \mathbb{C}$  such that  $\varphi(u^2 \otimes u - u \otimes u^2) \neq 0$  and  $\varphi(x \otimes y) = 0$  whenever  $x, y \in A$  are such that  $xy = 0$ . Then the bilinear functional  $f: A \times A \rightarrow \mathbb{C}$  defined by  $f(x, y) = \varphi(x \otimes y)$  for all  $x, y \in A$  obviously satisfies (1). Since  $f(u^2, u) \neq f(u, u^2)$ , it follows that  $f$  does not satisfy (2).  $\square$

We remark that the idea of the proof of this theorem was used in the paper by the second author [7], written at about the same time as the present paper. The main result of [7] states that a finite dimensional unital algebra over an arbitrary field satisfies  $\mathbb{B}_{\text{alg}}$  (if and) only if it is generated by idempotents.

#### 4. IDEMPOTENTS, COMMUTATORS AND SQUARE-ZERO ELEMENTS IN MATRIX AND VON NEUMANN ALGEBRAS

The ultimate goal of this section is to prove that commutators in von Neumann algebras can be written as sums of square-zero elements. We start, however, with a purely algebraic consideration of matrix algebras  $M_n(B)$ . Theorem 3.1 implies that commutators in these algebras are sums of square-zero elements. We will establish this in a more explicit fashion, which will enable the passing to von Neumann algebras.

**Lemma 4.1.** *Let  $B$  be a unital algebra. Then every element in  $M_2(B)$  can be written as  $e_1e_2 + e_3e_4 - e_5 - e_6$  for some idempotents  $e_i \in M_2(B)$  ( $i = 1, \dots, 6$ ).*

*Proof.* We have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & a_{11} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{22} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -a_{12} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -a_{21} & 1 \end{bmatrix}$$

and all matrices appearing on the right-hand side are idempotent.  $\square$

**Proposition 4.2.** *Let  $B$  be a unital algebra, and  $n \geq 2$ . Then every element in  $M_n(B)$  can be written as  $e_1e_2 + e_3e_4 + e_5e_6 + e_7 - e_8 - e_9 - e_{10} - e_{11}$  for some idempotents  $e_i \in M_n(B)$  ( $i = 1, \dots, 11$ ).*

*Proof.* If  $n = 2k$  then  $M_n(B) \cong M_2(M_k(B))$  and we may use Lemma 4.1. Therefore we may assume that  $n = 2k + 1$  for some  $k \geq 1$ .

Take  $a = (a_{ij}) \in M_n(B)$  and set

$$a' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad a'' = \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \dots & 0 \end{bmatrix}.$$

Note that

$$a' = \begin{bmatrix} 1 & a_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} - \begin{bmatrix} 1 & -a_{12} & \dots & -a_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

$$a'' = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \dots & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

and all matrices on the right-hand sides of these two identities are idempotent. Since  $a - a' - a''$  lies in a subalgebra of  $M_n(B)$  isomorphic to  $M_2(M_k(B))$ , it satisfies the conclusion of Lemma 4.1. Thus  $a = (a - a' - a'') + a' + a''$  can be written in the desired form.  $\square$

*Remark 4.3.* Suppose  $B$  is a unital  $C^*$ -algebra and  $M_n(B)$  is endowed with the standard  $C^*$ -norm. An inspection of the proof of Lemma 4.2 shows that each of the idempotents  $e_i$  has norm less or equal to  $\max\{2, 1 + \|a\|\}$ .

Following the approach from [10] we will now pass from idempotents to square-zero elements.

**Theorem 4.4.** *Let  $B$  be a unital algebra, and  $n \geq 2$ . Then every commutator in  $M_n(B)$  can be written as the sum of 22 square-zero elements.*

*Proof.* If  $e$  is an idempotent and  $x$  is an arbitrary element, then the commutator  $[e, x]$  is the sum of two square-zero elements, namely

$$[e, x] = ex(1 - e) + (e - 1)xe.$$

If  $e'$  is another idempotent, then  $[ee', x]$  can be written as the sum of four square-zero elements:

$$[ee', x] = ee'x(1 - e) + (e - 1)e'xe + e'xe(1 - e') + (e' - 1)xee'.$$

The desired conclusion can be now derived from Proposition 4.2.  $\square$

The number 22 could probably be lowered. In fact, from the proof of [25, Theorem 2] is evident that every element in  $M_2(B)$  is the sum of 5 square-zero elements. However, in this paper we will not address the question about the minimal number of summands.

Now we proceed to von Neumann algebras.

**Proposition 4.5.** *Let  $A$  be a von Neumann algebra without abelian central summands. Then every element in  $A$  can be written as  $e_1e_2 + e_3e_4 + e_5e_6 + e_7 - e_8 - e_9 - e_{10} - e_{11}$  for some idempotents  $e_i \in A$  ( $i = 1, \dots, 11$ ).*

*Proof.* We claim that  $A$  can be decomposed into a direct sum  $\bigoplus_{i \in I} M_i(B_i)$ , where  $I \subset \mathbb{N} \setminus \{1\}$  and  $B_i$  is a von Neumann algebra for each  $i \in I$ . Indeed, by [28, Theorems V.1.19 and V.1.27],  $A$  can be decomposed into a direct sum  $A_1 \oplus A_2 \oplus A_3$  of a properly infinite von Neumann algebra  $A_1$ , a type  $\text{II}_1$  von Neumann algebra  $A_2$ , and a finite type I von Neumann algebra  $A_3$ . Further,  $A_3 = \bigoplus_{j \in J} M_j(C_j)$ , where  $J \subset \mathbb{N}$  and  $C_j$  is an abelian von Neumann algebra for each  $j \in J$ . Since  $A$  has no abelian central summands, it follows that  $1 \notin J$ . On the other hand, the unity of  $A_1$ , as well as the unity of  $A_2$ , is the sum of two equivalent orthogonal projections ([28, Proposition V.1.36] and [28, Proposition V.1.35], respectively). From [28, Proposition V.1.22] it follows that  $A_1 = M_2(D_1)$  and  $A_2 = M_2(D_2)$  for some von Neumann algebras  $D_1$  and  $D_2$ . If  $2 \in J$ , then we take  $I = J$ ,  $B_2 = D_1 \oplus D_2 \oplus C_2$ , and  $B_i = C_i$  for each  $i \in I \setminus \{2\}$ , and otherwise we take  $I = J \cup \{2\}$ ,  $B_2 = D_1 \oplus D_2$  and  $B_i = C_i$  for each  $i \in J$ .

Let  $x \in A$ . Then  $x = (x_i)_{i \in I} \in \bigoplus_{i \in I} M_i(B_i)$  and Proposition 4.2 together with Remark 4.3 show that for each  $i \in I$  there are idempotents  $e_{1,i}, \dots, e_{11,i} \in M_i(B_i)$  such that

$$x_i = e_{1,i}e_{2,i} + e_{3,i}e_{4,i} + e_{5,i}e_{6,i} + e_{7,i} - e_{8,i} - e_{9,i} - e_{10,i} - e_{11,i}$$

and

$$\|e_{1,i}\|, \dots, \|e_{11,i}\| \leq \max\{2, 1 + \|x_i\|\} \leq \max\{2, 1 + \|x\|\}.$$

The elements  $e_1, \dots, e_{11} \in \bigoplus_{i \in I} M_i(B_i)$  defined by  $e_j = (e_{j,i})_{i \in I}$  ( $j = 1, \dots, 11$ ) satisfy our requirements.  $\square$

**Theorem 4.6.** *Let  $A$  be a von Neumann algebra. Then every commutator in  $A$  can be written as the sum of 22 square-zero elements.*

*Proof.* On account of [28, Theorems V.1.19 and V.1.27],  $A$  can be decomposed into a direct sum  $A_1 \oplus A_2$ , where  $A_1$  is the type  $I_1$  part of  $A$  (and therefore it is an abelian von Neumann algebra) and  $A_2$  is a von Neumann algebra without abelian central summands. Let  $x, y \in A$  and write  $x = x_1 + x_2$  and  $y = y_1 + y_2$  with  $x_1, y_1 \in A_1$  and  $x_2, y_2 \in A_2$ . Then  $[x, y] = [x_2, y_2] \in A_2$  and Proposition 4.5 now shows that  $[x, y]$  can be written as the sum of 22 square-zero elements, as claimed.  $\square$

*Remark 4.7.* Let  $A$  be a von Neumann algebra and  $x \in A$ . On account of the canonical decomposition of  $A$  and known results from the literature (see the introduction of [12]), it follows that there exists  $z \in Z(A)$  such that  $x - z$  is the sum of finitely many commutators. Theorem 4.6 then shows that  $x - z$  is the sum of finitely many square-zero elements.

## 5. SQUARE-ZERO ELEMENTS AND THE COMMUTATOR IDEAL

The purpose of this section is to apply our seminal result, Theorem 2.1, for characterizing the *commutator ideal* of certain Banach algebras  $A$ . By the commutator ideal we mean the closed ideal generated by all commutators in  $A$ .

**Lemma 5.1.** *Let  $A$  be a Banach algebra. Suppose that the quotient algebra  $A/I$  is semisimple for each proper closed ideal  $I$  of  $A$ . Then the commutator ideal of  $A$  coincides with the closed subalgebra of  $A$  generated by all commutators in  $A$ .*



*Proof.* The proof is based on the following fact: the ideal generated by the commutators  $[[A, A], A]$  is contained in the subalgebra  $B$  generated by all commutators  $[A, A]$ . Let us repeat the proof given in [8, p. 2]. Given  $x, y, z, w \in A$  we have

$$x[[y, z], w] = [x[y, z], w] - [x, w][y, z],$$

which shows that the left ideal generated by  $[[A, A], A]$  is contained in  $B$ . Similarly we see that the same is true for the right ideal generated by  $[[A, A], A]$ . Finally, from  $xuy = [xu, y] + (yx)u$ , with  $x, y \in A$  and  $u \in [[A, A], A]$ , we get the desired conclusion.

Hence we see that it suffices to show that the closed ideal of  $A$  generated by  $[[A, A], A]$ , which we denote by  $C'$ , is equal to the commutator ideal  $C$ . That is, we have to show that  $C'$  contains all commutators in  $A$ . We may assume that  $C' \neq A$ . Then the quotient Banach algebra  $B = A/C'$  is semisimple and satisfies the property  $[[B, B], B] = 0$ . This implies that  $\delta_b(B) \subset Z(B)$  for each  $b \in B$ , where  $\delta_b$  stands for the inner derivation implemented by  $b$ . The Singer-Wermer theorem [11, Corollary 2.7.20] now shows that  $\delta_b(B) = 0$  for each  $b \in B$ , which clearly implies that  $B$  is commutative and therefore that  $[A, A] \subset C'$ , as required.  $\square$

Let us show that Lemma 5.1 does not hold for all Banach algebras.

*Example 5.2.* Let  $A$  be the (complex) Grassmann algebra in three generators  $x_1, x_2, x_3$ . That is,  $A$  is the 8-dimensional algebra with basis

$$\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\}$$

whose multiplication is determined by  $x_i^2 = x_ix_j + x_jx_i = 0$  for all  $i, j = 1, 2, 3$ . Note that the commutator ideal of  $A$  is equal to

$$C = \mathbb{C}x_1x_2 + \mathbb{C}x_1x_3 + \mathbb{C}x_2x_3 + \mathbb{C}x_1x_2x_3,$$

the subalgebra generated by all commutators is equal to

$$B = \mathbb{C}x_1x_2 + \mathbb{C}x_1x_3 + \mathbb{C}x_2x_3,$$

and the ideal generated by the commutators  $[[A, A], A]$  is  $C' = \{0\}$ .

**Theorem 5.3.** *Let  $A$  be an essential Banach algebra with property  $\mathbb{B}$ . Suppose that the quotient algebra  $A/I$  is semisimple for each proper closed ideal  $I$  of  $A$ . Then the commutator ideal of  $A$  is equal to the closed subalgebra generated by all square-zero elements in  $A$ .*

*Proof.* Let  $C$  denote the commutator ideal of  $A$ , and let  $D$  denote the closed subalgebra generated by all square-zero elements. Theorem 2.1 and Lemma 5.1 imply that  $C \subseteq D$ . The proof will be completed by showing that every square-zero element  $a \in A$  lies in  $C$ . We may assume that  $C \neq A$ . Since  $a + C$  is a square-zero element of the semisimple commutative Banach algebra  $A/C$ , it follows that  $a \in C$ .  $\square$

*Examples 5.4.* Let us point out some basic special cases of Banach algebras that satisfy the requirements in Theorem 5.3.

- (i) Arbitrary  $C^*$ -algebras.

- (ii) For a compact group  $G$ , any of the Banach algebras  $L^p(G)$ , with  $1 \leq p < \infty$ , or  $C(G)$  (under convolution) has property  $\mathbb{B}$  and further [20, Theorem 15] shows that also satisfy the semisimplicity condition of its quotient algebras. In [13, Example 3.1] it is given an example of a non-compact, non-abelian group  $G$  for which the group algebra  $L^1(G)$  has spectral synthesis and this property, in particular, yields the semisimplicity of its quotient algebras.

## 6. SQUARE-ZERO ELEMENTS AND CENTRAL ELEMENTS

Once again, we return to Theorem 2.1. In particular it tells us that a noncommutative essential Banach algebra with property  $\mathbb{B}$  has nonzero square-zero elements. In fact, there is no need to involve the linear span of square-zero elements to establish this. All one has to do is to consider the map  $(x, y) \mapsto yx$  in light of the definition of property  $\mathbb{B}$ . Let us record the result which we obtain in this way.

**Proposition 6.1.** *Let  $A$  be an essential Banach algebra with property  $\mathbb{B}$ . If  $A$  is not commutative, then there exists  $a, b \in A$  such that  $ab = 0$  and  $ba \neq 0$  (and hence  $A$  contains nonzero square-zero elements).*

We will now develop further the simple idea upon which this proposition is based. Our goal is to give two characterizations of elements from the centre of a Banach algebra  $A$  with property  $\mathbb{B}$  which satisfies some mild additional assumptions. In our first result we will assume that  $A$  is both *left* and *right faithful*, i.e., for every  $x \in A$ , each of the conditions  $xA = \{0\}$  and  $Ax = \{0\}$  implies  $x = 0$ .

**Proposition 6.2.** *Let  $A$  be a Banach algebra with property  $\mathbb{B}$ . Suppose that  $A$  is both left and right faithful. Then the following conditions on  $c \in A$  are equivalent:*

- (i)  $c \in Z(A)$ ;
- (ii) For all  $x, y \in A$ ,  $xy = 0$  implies  $xcy = 0$ .

*Proof.* It is enough to show that (ii) implies (i). Consider the continuous bilinear map  $f: A \times A \rightarrow A$  defined by  $f(x, y) = xcy$  for all  $x, y \in A$ . Then (ii) implies that  $f$  satisfies condition (1) in the introduction. Since  $A$  has property  $\mathbb{B}$ , it follows that  $f(xy, z) = f(x, yz)$  and so that  $(xy)cz = xc(yz)$  for all  $x, y, z \in A$ . This gives  $x[y, c]z = 0$ , and since  $A$  is both left and right faithful, it follows that  $[y, c] = 0$  for every  $y \in A$ . That is,  $c \in Z(A)$ .  $\square$

The second result is somewhat less straightforward to prove. We will sharpen (ii), but at the price of assuming that the algebra  $A$  is *semiprime*, i.e., it does not contain nonzero nilpotent ideals (equivalently, for every  $x \in A$ ,  $xAx = \{0\}$  implies  $x = 0$ ). Our main examples of Banach algebras with property  $\mathbb{B}$ ,  $C^*$ -algebras and group algebras over locally compact groups, are semiprime.

We need two lemmas. The first one was implicitly proved in [2] and was later used in [4] to study the orthogonality preserving linear maps on group algebras.

**Lemma 6.3.** *Let  $A$  be a Banach algebra with the property  $\mathbb{B}$ , let  $X$  be a Banach space, and let  $f: A \times A \rightarrow X$  be a continuous bilinear map such that for all  $x, y \in A$ ,*

$$xy = yx = 0 \Rightarrow f(x, y) = 0.$$

*Then*

$$\begin{aligned} & f(z_1x_2y_2, z_2x_1y_1) - f(z_1x_2, y_2z_2x_1y_1) \\ & + f(y_1z_1x_2, y_2z_2x_1) - f(y_1z_1x_2y_2, z_2x_1) = 0 \end{aligned}$$

*for all  $x_1, x_2, y_1, y_2, z_1, z_2 \in A$ .*

*Proof.* This is exactly what is proved in the first part of the proof of [2, Theorem 2.2] (indeed the result is stated for  $C^*$ -algebras only, but this part of the proof obviously works for any Banach algebra with property  $\mathbb{B}$ ).  $\square$

**Lemma 6.4.** *Let  $A$  be a semiprime algebra. Then each of the following conditions on  $c \in A$ :*

- (i)  $cx \in Z(A)$  for every  $x \in A$ ;
- (ii)  $xc \in Z(A)$  for every  $x \in A$ ;
- (iii)  $[c, x] \in Z(A)$  for every  $x \in A$ ;

*implies that  $c \in Z(A)$ .*

*Proof.* Assuming (i) we have  $cxy = ycx$  for all  $x, y \in A$ . Replacing  $x$  by  $xz$  it follows that  $cxzy = ycxz$ . However, on the other hand  $(ycx)z = (cxy)z$ , and so comparing the last two relations we get  $cx[z, y] = 0$  for all  $x, y, z \in A$ . This clearly implies that  $[c, y]x[c, y] = c(yx)[c, y] - y(cx[c, y]) = 0$  for all  $x, y \in A$ . Since  $A$  is semiprime this yields  $[c, y] = 0$  for every  $y \in A$ , i.e.,  $c \in Z(A)$ .

Of course, (ii) can be handled in a similar fashion. So let us assume (iii). Then we have  $[c, x]c = [c, xc] \in Z(A)$  for all  $x \in A$ . In particular,  $[[c, x]c, x] = 0$ . Since  $[c, x]$ , as a central element, commutes with  $x$ , it follows that  $[c, x]^2 = 0$ . However, the centre of a semiprime algebra cannot contain nonzero square-zero elements, so we obtain  $[c, x] = 0$ .  $\square$

**Theorem 6.5.** *Let  $A$  be a semiprime Banach algebra with property  $\mathbb{B}$ . Then the following conditions on  $c \in A$  are equivalent:*

- (i)  $c \in Z(A)$ ;
- (ii) For all  $x, y \in A$ ,  $xy = yx = 0$  implies  $xcy = 0$ .

*Proof.* Of course, it suffices to show that (ii) implies (i). As in the proof of Proposition 6.2, we consider the continuous bilinear map  $f: A \times A \rightarrow A$  defined by  $f(x, y) = xcy$  for all  $x, y \in A$ . Assuming that (ii) holds, it follows from Lemma 6.3 that

$$z_1x_2y_2cz_2x_1y_1 - z_1x_2cy_2z_2x_1y_1 + y_1z_1x_2cy_2z_2x_1 - y_1z_1x_2y_2cz_2x_1 = 0$$

for all  $x_1, x_2, y_1, y_2, z_1, z_2 \in A$ . We can rewrite this identity as

$$[z_1x_2[y_2, c]z_2x_1, y_1] = 0,$$

which means that  $z_1x_2[y_2, c]z_2x_1 \in Z(A)$  for all  $x_1, x_2, y_2, z_1, z_2 \in A$ . Using Lemma 6.4 (i) twice it follows that  $z_1x_2[y_2, c] \in Z(A)$ . Similarly, using Lemma 6.4 (ii) twice we now get that  $[y_2, c] \in Z(A)$  for all  $y_2 \in A$ . But then Lemma 6.4 (iii) implies  $c \in Z(A)$ .  $\square$

Another way of stating Theorem 6.5 is that every non-central element  $c \in A$  gives rise to a nonzero square-zero element in  $A$  of the form  $xcy$  for some  $x, y \in A$  such that  $xy = yx = 0$ . We remark that this is a generalization of the result by Magajna [22, Corollary 2.8] stating that for every non-central element  $c$  in a  $C^*$ -algebra  $A$  there exist  $x, y \in A$  such that  $xcy \neq 0$  and  $(xcy)^2 = 0$ .

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