JORDAN DERIVATIONS REVISITED

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Abstract. Let d be a Jordan derivation from a ring $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. Our main result in particular shows that the restriction of d to the ideal of $\mathcal{A}$ generated by certain higher commutators of $\mathcal{A}$ is a derivation. This general statement is used for proving that under various additional conditions d must be a derivation on $\mathcal{A}$. Furthermore, several examples of proper Jordan derivations are given, C^*-algebras admitting proper additive Jordan derivations are characterized, and the connections with the related problems on Jordan homomorphisms and Jordan $\mathcal{A}$-module homomorphisms are discussed.

1. Introduction

Let $\mathcal{A}$ be a ring (resp. algebra) and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. An additive (resp. linear) map $d : \mathcal{A} \to \mathcal{M}$ is called a *Jordan derivation* if

$$d(x^2) = d(x)x + xd(x) \quad \text{and} \quad d(xyx) = d(x)yx + xd(y)x + xyd(x)$$

for all $x, y \in \mathcal{A}$. The standard problem is to find out whether a Jordan derivation is necessarily a derivation, that is, does

$$d(xy) = d(x)y + xd(y)$$

hold for all $x, y \in \mathcal{A}$. Starting with the results by Jacobson and Rickart [17] and Herstein [14], this problem has been an active area of research for more than 50 years (see [1, 3, 5, 6, 8, 11, 12, 13, 19, 22, 24] and references therein).

There certainly exist *proper Jordan derivations*, i.e. such that they are not derivations. Our main purpose, however, is to show that Jordan derivations are derivations at least on some "piece of $\mathcal{A}$". The largeness of this piece depends on the structure of certain higher commutators of $\mathcal{A}$. Specifically, in particular we show that $d(ux) = d(u)x + ud(x)$ for every $x \in \mathcal{A}$ and every u from the ideal generated by $[[[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{A}]], [[\mathcal{A}, \mathcal{A}], [\mathcal{A}, \mathcal{A}]]]$. If $\mathcal{A}$ is commutative, or "close" to be commutative, then this information is of course useless. If, however, $\mathcal{A}$ is "fairly noncommutative", then one may expect that d acts as a derivation on a considerable piece of $\mathcal{A}$.

We shall illustrate the usefulness of our basic result on Jordan derivations, Theorem 3.1, by deriving generalizations and new proofs of some known results from it. On the other hand, we shall obtain Theorem 3.1 as a corollary

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to a more general result, Theorem 2.1, which considers biadditive maps satisfying certain identities. We remark that there is some analogy with our treatment of Lie derivations [9] where the result was also derived from a more abstract theorem on biadditive maps.

In Section 4 we shall present a variety of examples of proper Jordan derivations, and thereby indicate the limitations of the problem whether a Jordan derivation is (or at least it is close to) a derivation. Our construction of proper Jordan derivations on some commutative rings (Subsection 4.4) seems to be of particular interest for two reasons. On the one hand, it justifies the exclusion of commutative rings in some of our main results, and on the other hand it connects the Jordan derivation problem with the classical problem on the existence of nontrivial derivations on commutative rings and algebras. Moreover, it will also make it possible for us to state some remarks on Jordan derivations on C^*-algebras in Section 5. In particular, we shall characterize C^*-algebras on which there exist proper additive Jordan derivations (Theorem 5.1).

For the most part, $\mathcal{M}$ will play an entirely formal role in this paper, so the majority of our results depend only on the structure of $\mathcal{A}$. One of the advantages of such approach is that thereby we obtain the solution of the analogous problem for Jordan $\mathcal{A}$-module homomorphisms (which are defined in Section 6) as direct consequences of the results on Jordan derivations. The related problem on Jordan homomorphism is, on the contrary, more general (see Theorem 6.1).

To the best of our knowledge, this paper brings a new approach to the study of Jordan derivations. It has been motivated by the recent work on Jordan ideals [10]; as in [10] we wish to show in the present paper that although Jordan structures in associative rings have already been thoroughly studied by a number of authors, one can still obtain new and in our opinion somewhat surprising results by entirely elementary means.

2. A theorem on biadditive maps

Let $\mathcal{A}$ be an associative ring. We set $[x,y]=xy-yx$ and $x\circ y=xy+yx$ for $x,y\in\mathcal{A}$. We shall write $[x,m]$ for $xm-mx$ also in the case where $x\in\mathcal{A}$ and m is from some $\mathcal{A}$-bimodule. We set $\mathcal{A}^{(0)}=\mathcal{A}$, and inductively, $\mathcal{A}^{(n+1)}=[\mathcal{A}^{(n)},\mathcal{A}^{(n)}]$ for every $n\geq 0$. Given a subset $\mathcal{S}$ of $\mathcal{A}$, we denote by $\mathfrak{R}(\mathcal{S})$ (resp. $\mathfrak{I}(\mathcal{S})$) the subring (resp. ideal) of $\mathcal{A}$ generated by $\mathcal{S}$.

If $\mathcal{L}$ is a Lie ideal of $\mathcal{A}$, then $\mathfrak{I}([\mathcal{L},\mathcal{L}])\subseteq\mathfrak{R}(\mathcal{L})$. This observation is essentially due to Herstein (cf. [15, pp. 4-5]) and can be easily proved. Indeed, from $x[u,v]=[xu,v]-[x,v]u$ we see that $\mathcal{A}[\mathcal{L},\mathcal{L}]\subseteq\mathfrak{R}(\mathcal{L})$, similarly we show that $[\mathcal{L},\mathcal{L}]\mathcal{A}\subseteq\mathfrak{R}(\mathcal{L})$, and finally, noting that $\mathfrak{R}(\mathcal{L})$ is also a Lie ideal of $\mathcal{A}$ and using $x[u,v]y=[x[u,v],y]+yx[u,v]$ one concludes that $\mathcal{A}[\mathcal{L},\mathcal{L}]\mathcal{A}\subseteq\mathfrak{R}(\mathcal{L})$. Using the Jacobi identity we see that $[\mathcal{L},\mathcal{L}]$ is also a Lie ideal of $\mathcal{A}$. Consequently, $\mathcal{A}^{(n)}$ is a Lie ideal and so we have

$$\mathfrak{I}(\mathcal{A}^{(n+1)})\subseteq\mathfrak{R}(\mathcal{A}^{(n)})\quad\text{for every } n\geq 0.$$

In particular, $\mathfrak{I}(\mathcal{A}^{(3)}) \subseteq \mathfrak{R}(\mathcal{A}^{(2)})$, which explains the last statement in the following theorem.

Theorem 2.1. *Let $\mathcal{A}$ be a ring and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. If $\{\,.\,,\,.\,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ is a biadditive map such that*

$$\{x, x\} = 0, \tag{1}$$

$$\{xy, x\} + \{x, y\}x = 0, \tag{2}$$

$$\{xy, z\} + \{x, y\}z = \{x, yz\} + x\{y, z\} \tag{3}$$

for all $x, y, z \in \mathcal{A}$, then $\{\mathfrak{R}(\mathcal{A}^{(2)}), \mathcal{A}\} = 0$ (and hence $\{\mathfrak{I}(\mathcal{A}^{(3)}), \mathcal{A}\} = 0$).

Proof. By (1), $\{\,.\,,\,.\,\}$ is skew-symmetric. Consequently, the linearized form of (2) can be written as

$$\{xy, z\} + \{x, y\}z = \{x, zy\} + \{y, z\}x.$$

Comparing this identity with (3) we get

$$\{x, [y, z]\} + [x, \{y, z\}] = 0 \tag{4}$$

for all $x, y, z \in \mathcal{A}$. Using (4) together with the fact that both $\{\,.\,,\,.\,\}$ and $[\,.\,,\,.\,]$ are skew-symmetric one easily infers that

$$\{[x, y], [u, v]\} = -[[x, y], \{u, v\}] = -[\{x, y\}, [u, v]] \tag{5}$$

for all $x, y, u, v \in \mathcal{A}$.

We shall now compute $\{[x, y]z, [u, v]\}$ in two different ways. On the one hand, using (4) we have

$$\begin{aligned} \{[x, y]z, [u, v]\} &= -[[x, y]z, \{u, v\}] \\ &= -[x, y][z, \{u, v\}] - [[x, y], \{u, v\}]z \\ &= [x, y]\{z, [u, v]\} - [[x, y], \{u, v\}]z. \end{aligned}$$

On the other hand, first applying (3) and after that (4) we get

$$\begin{aligned} \{[x, y]z, [u, v]\} &= \{[x, y], z[u, v]\} - \{[x, y], z\}[u, v] + [x, y]\{z, [u, v]\} \\ &= -\{z[u, v], [x, y]\} + \{z, [x, y]\}[u, v] + [x, y]\{z, [u, v]\} \\ &= [z[u, v], \{x, y\}] - [z, \{x, y\}][u, v] + [x, y]\{z, [u, v]\} \\ &= z[[u, v], \{x, y\}] + [x, y]\{z, [u, v]\}. \end{aligned}$$

Comparing both expressions we obtain

$$z[[u, v], \{x, y\}] = -[[x, y], \{u, v\}]z,$$

which can be, in view of (5), rewritten as $[z, \{[x, y], [u, v]\}] = 0$. Hence (4) implies that $\{z, [[x, y], [u, v]]\} = 0$; that is, $\{\mathcal{A}, \mathcal{A}^{(2)}\} = \{\mathcal{A}^{(2)}, \mathcal{A}\} = 0$. From (3) we see that the set of all $u \in \mathcal{A}$ such that $\{u, \mathcal{A}\} = 0$ is a subring of $\mathcal{A}$. Consequently, $\{\mathfrak{R}(\mathcal{A}^{(2)}), \mathcal{A}\} = 0$. □

Let us mention another two useful observations concerning $\{\,.\,,\,.\,\}$. First, using (3) it is straightforward to note

Remark 2.2. *If $\mathcal{I}$ is an ideal of $\mathcal{A}$ such that $\{\mathcal{I},\mathcal{A}\}=0$, then $\mathcal{I}\{\mathcal{A},\mathcal{A}\}=0$. Accordingly, $\mathfrak{I}(\mathcal{A}^{(3)})\{\mathcal{A},\mathcal{A}\}=0$.*

Let $e\in\mathcal{A}$ be an idempotent, and let $x\in\mathcal{A}$. By (2) we have $\{ex,e\}+\{e,x\}e=0$, and by (3) and (1) we have $\{xe,e\}+\{x,e\}e=\{x,e\}$. Now assume that e and x commute. Comparing both relations and using the fact that $\{\,.\,,\,.\,\}$ is skew-symmetric we infer that $2\{e,x\}e=\{e,x\}$. Multiplying this identity from the right by e it follows that $\{e,x\}e=\{e,x\}=0$. We have proved

Remark 2.3. *Let $e\in\mathcal{A}$ be an idempotent and let $x\in\mathcal{A}$. Then $[e,x]=0$ implies $\{e,x\}=0$.*

3. Jordan derivations on noncommutative rings

First we remark that in the literature often only the first condition, $d(x^2)=d(x)x+xd(x)$, is required in the definition of a Jordan derivation, since the second condition, $d(xyx)=d(x)yx+xd(y)x+xyd(x)$, follows from it provided that $\mathcal{M}$ is 2-torsionfree (i.e. $2m=0$ with $m\in\mathcal{M}$ implies $m=0$). This is a simple consequence of the identity $2xyx=x\circ(y\circ x)-x^2\circ y$. Moreover, in the 2-torsionfree case the definition of a Jordan derivation is equivalent to

$$d(x\circ y)=d(x)\circ y+x\circ d(y)\quad\text{for all } x,y\in\mathcal{A}. \tag{6}$$

Given a Jordan derivation $d:\mathcal{A}\to\mathcal{M}$, we set

$$\{x,y\}=d(xy)-d(x)y-xd(y)$$

for all $x,y\in\mathcal{A}$. Of course, d is a derivation if and only if $\{\mathcal{A},\mathcal{A}\}=0$, so $\{\,.\,,\,.\,\}$ measures how far is d from being a derivation. It is straightforward to check that $\{\,.\,,\,.\,\}$ satisfies the conditions (1), (2), and (3). Therefore, Theorem 2.1 yields the following result.

Theorem 3.1. *If d is a Jordan derivation from a ring $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$, then*

$$d(ux)=d(u)x+ud(x)$$

for all $x\in\mathcal{A}$ and all $u\in\mathfrak{R}(\mathcal{A}^{(2)})$ (and so, in particular, for all $u\in\mathfrak{I}(\mathcal{A}^{(3)})$).

Let us mention that the concept behind the proof of Theorem 3.1 has been the well-known formula $[[x,y],z]=x\circ(y\circ z)-y\circ(x\circ z)$ which implies that every Jordan derivation is also Lie triple derivation, i.e. it satisfies

$$d([[x,y],z])=[[d(x),y],z]+[[x,d(y)],z]+[[x,y],d(z)]$$

(this is hidden in (4)). The reader can notice the similarity to the recent study of Jordan ideals in [10].

An immediate but noteworthy corollary to Theorem 3.1 is

Corollary 3.2. *If a ring $\mathcal{A}$ is such that $\mathfrak{R}(\mathcal{A}^{(2)})=\mathcal{A}$ (in particular, if $\mathfrak{I}(\mathcal{A}^{(3)})=\mathcal{A}$), then every Jordan derivation from $\mathcal{A}$ into an arbitrary $\mathcal{A}$-bimodule is a derivation.*

We continue by stating corollaries to Remarks 2.2 and 2.3.

Remark 3.3. *Every Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ satisfies*

$$\mathfrak{I}(\mathcal{A}^{(3)})\big(d(xy) - d(x)y - xd(y)\big) = 0$$

for all $x, y \in \mathcal{A}$. Accordingly, if a bimodule $\mathcal{M}$ is such that $\mathfrak{I}(\mathcal{A}^{(3)})m \neq 0$ for every nonzero $m \in \mathcal{M}$, then every Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$ is a derivation.

Remark 3.4. *Every Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ satisfies $d(ex) = d(e)x + ed(x)$ whenever $e \in \mathcal{A}$ is an idempotent and $x \in \mathcal{A}$ commutes with e.*

Although Remark 3.4 is just an elementary observation, it is important since it makes it possible for one to reduce the study of Jordan derivations from rings to their direct summands. Namely, in particular it implies that $d(e)f + ed(f) = 0$ for every pair of orthogonal idempotents e and f, and using this one can easily infer

Remark 3.5. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be unital rings and let d be a Jordan derivation from $\mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_n$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. If $d|\mathcal{A}_i$ is a derivation for every i, then d is a derivation.*

Remark 3.5 partially generalizes [1, Lemma 6.4] (it seems that the fact that Remark 3.4 holds was overlooked in [1]). The usefulness of both remarks will be illustrated in Subsection 5 where an alternative proof of Johnson's theorem on Jordan derivations on C^*-algebras will be given.

Using the very definition it is easy to see that every Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ satisfies

$$d(x^k) = \sum_{i=1}^{k} x^{i-1} d(x) x^{k-i}$$

for all $k \geq 1$ and all $x \in \mathcal{A}$. This readily implies that

$$d(x^n x^m) = d(x^n)x^m + x^n d(x^m)$$

whenever $n, m \geq 1$. Moreover, if $\mathcal{A}$ is unital, then Remark 3.4 shows that that the same is true if $n = 0$ or $m = 0$. Consequently, we have

Remark 3.6. *Let $\mathcal{A}$ be a unital ring and let $x \in \mathcal{A}$. If $d : \mathcal{A} \to \mathcal{M}$ is a Jordan derivation, then $d|\mathfrak{R}(\{1, x\})$ is a derivation.*

Theorem 3.1 somehow reduces the question on the structure of Jordan derivations to the question on the largeness of the subring $\mathfrak{R}(\mathcal{A}^{(2)})$. The latter is of course intimately connected with the question on the largeness of the ideal $\mathfrak{I}(\mathcal{A}^{(3)})$. It is difficult to expect reasonable answers to these questions in arbitrary noncommutative rings (just take, for example, nilpotent rings), so we shall confine ourselves to some of their special classes.

We recall that an ideal $\mathcal{E}$ of a ring $\mathcal{A}$ is said to be *essential* if $\mathcal{E} \cap \mathcal{I} \neq 0$ for every nonzero ideal $\mathcal{I}$ of $\mathcal{A}$. It is well-known and easy to see that in the case when $\mathcal{A}$ is a semiprime ring, this condition is equivalent to the one that

$\mathcal{E}$ has trivial (right) annihilator, i.e. $\mathcal{E}c = 0$, where $c \in \mathcal{A}$, implies $c = 0$. In this connection we note that the right and the left annihilator of any ideal $\mathcal{I}$ of a semiprime ring $\mathcal{A}$ coincide, that is, $\mathcal{I}c = 0$ of and only $c\mathcal{I} = 0$ (so, in particular, $[c, \mathcal{I}] = 0$ holds in this case). Of course, the annihilator of an ideal is again an ideal.

Is $\mathfrak{I}(\mathcal{A}^{(3)})$ an essential ideal of a semiprime ring $\mathcal{A}$? If the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ contains a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$, then the answer is no. Namely, in this case we have $\mathcal{A}\mathcal{J} \subseteq \mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$ and so $[\mathcal{A}, \mathcal{A}]\mathcal{J} = [\mathcal{A}, \mathcal{A}\mathcal{J}] = 0$, showing that $\mathcal{J}$ is contained in the annihilator of $\mathfrak{I}(\mathcal{A}^{(1)})$ (and hence also in the annihilator of $\mathfrak{I}(\mathcal{A}^{(n)})$ for every $n \geq 1$). But otherwise the answer is affirmative. To prove this we shall need [4, Lemma 3] which states that for every Lie ideal $\mathcal{L}$ of a prime ring $\mathcal{A}$ with $\mathrm{char}(\mathcal{A}) \neq 2$ and every $c \in \mathcal{A}$, $[c, [\mathcal{L}, \mathcal{L}]] = 0$ implies $[c, \mathcal{L}] = 0$.

Lemma 3.7. *Let $\mathcal{A}$ be a 2-torsionfree semiprime ring, and let $n \geq 1$. If $c \in \mathcal{A}$ is such that $[c, \mathcal{A}^{(n)}] = 0$, then $c \in \mathcal{Z}(\mathcal{A})$. In particular, the annihilator of $\mathfrak{I}(\mathcal{A}^{(n)})$ is contained in $\mathcal{Z}(\mathcal{A})$. Accordingly, $\mathfrak{I}(\mathcal{A}^{(n)})$ is an essential ideal of $\mathcal{A}$ if and only if $\mathcal{Z}(\mathcal{A})$ does not contain nonzero ideals of $\mathcal{A}$.*

Proof. In view of the above discussion it suffices to prove that $[c, \mathcal{A}^{(n)}] = 0$ implies $c \in \mathcal{Z}(\mathcal{A})$.

A standard argument shows that there exist prime ideals $\{\mathcal{P}_\lambda \,|\, \lambda \in \Lambda\}$ such that $\bigcap_{\lambda\in\Lambda} \mathcal{P}_\lambda = 0$ and the prime rings $\mathcal{A}/\mathcal{P}_\lambda$ are of characteristic different from 2. Since $[c, \mathcal{A}^{(n)}] = 0$ implies $[c+\mathcal{P}_\lambda, (\mathcal{A}/\mathcal{P}_\lambda)^{(n)}] = 0$ we see that without loss of generality we may assume that $\mathcal{A}$ is a prime ring with $\mathrm{char}(\mathcal{A}) \neq 2$. But then, in view of [4, Lemma 3], $[c, [\mathcal{A}^{(n-1)}, \mathcal{A}^{(n-1)}]] = [c, \mathcal{A}^{(n)}] = 0$ implies $[c, \mathcal{A}^{(n-1)}] = 0$. Therefore, inductively we arrive at $[c, \mathcal{A}] = [c, \mathcal{A}^{(0)}] = 0$, i.e. $c \in \mathcal{Z}(\mathcal{A})$. □

We remark that the assumption in Lemma 3.7 that $\mathcal{A}$ is 2-torsionfree is really necessary: just consider the matrix ring $M_2(\mathbb{F})$ with $\mathrm{char}(\mathbb{F}) = 2$.

Corollary 3.8. *Let $\mathcal{A}$ be a 2-torsionfree semiprime ring and let d be a Jordan derivation from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$.*

(i) If $\mathcal{Z}(\mathcal{A})$ does not contain nonzero ideals of $\mathcal{A}$, then the restriction of d to some essential ideal of $\mathcal{A}$ is a derivation.

(ii) Suppose that $\mathcal{M}$ satisfies the following two conditions: (a) for every essential ideal $\mathcal{E}$ of $\mathcal{A}$ and every $m \in \mathcal{M}$, $\mathcal{E}m = 0$ implies $m = 0$, and (b) $cm = mc$ for all $c \in \mathcal{Z}(\mathcal{A})$ and all $m \in \mathcal{M}$. Then d is a derivation.

Proof. (i) Use Theorem 3.1 and Lemma 3.7.

(ii) Let $\mathcal{I} = \mathfrak{I}(\mathcal{A}^{(3)})$ and let $\mathcal{J} = \{c \in \mathcal{A} \,|\, \mathcal{I}c = 0\}$. Note that $\mathcal{E} = \mathcal{I} \oplus \mathcal{J}$ is an essential ideal of $\mathcal{A}$. By Remark 3.3, $\mathcal{I}\{\mathcal{A}, \mathcal{A}\} = 0$, where of course $\{x, y\}$ stands for $d(xy) - d(x)y - xd(y)$. Therefore, in view of *(a)*, it suffices to show that also $\mathcal{J}\{\mathcal{A}, \mathcal{A}\} = 0$.

By Lemma 3.7 we have $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{A})$. Pick $c \in \mathcal{J}$ and $x \in \mathcal{A}$. Using (6) and *(b)* it follows that

$$2d(cx) = d(c) \circ x + 2cd(x) \quad \text{for all } c \in \mathcal{J},\, x \in \mathcal{A}. \tag{7}$$

For every $c' \in \mathcal{J}$ we have $c'x \in \mathcal{J}$ and so both $c', c'x \in \mathcal{Z}(\mathcal{A})$. By *(b)* we then have $c'd(c)x = d(c)(c'x) = c'xd(c)$. Therefore, multiplying (7) from the left by c' we obtain $2c'\{c, x\} = 0$. Thus $2\mathcal{J}\{c, x\} = 0$ and so, since $\mathcal{I}\{\mathcal{A}, \mathcal{A}\} = 0$, also $2\mathcal{E}\{c, x\} = 0$. As $2\mathcal{E}$ is also an essential ideal of $\mathcal{A}$ it follows that $\{c, x\} = 0$. Thus, $\{J, \mathcal{A}\} = 0$, and so Remark 2.2 tells us that $\mathcal{J}\{\mathcal{A}, \mathcal{A}\} = 0$. □

The conditions *(a)* and *(b)* are trivially fulfilled in the case where $\mathcal{M} = \mathcal{A}$. Thus, the assertion *(ii)* is a generalization of the result of Cusack [12] (which was later also proved in [5]) stating that a Jordan derivation from a 2-torsionfree ring $\mathcal{A}$ into itself is necessarily a derivation (the prime ring case of this result is the classical Herstein's theorem [14]). Let us mention that the arguments in [12] and [5] do not lead to *(ii)* since they both indirectly make some computations with d^2.

There is another, more general instance when *(ii)* is applicable.

Corollary 3.9. *Let $\mathcal{A}$ be a 2-torsionfree semiprime ring and let $\mathcal{Q}$ be the maximal left (or right) ring of quotients of $\mathcal{A}$. Then every Jordan derivation from $\mathcal{A}$ into $\mathcal{Q}$ is a derivation.*

Proof. It is a fact that $\mathcal{M} = \mathcal{Q}$ satisfies *(a)* and *(b)* (see, for example, [2, Proposition 2.1.7, Remark 2.3.1]). □

In the case when $\mathcal{A}$ is a prime ring, the assumption in *(i)* of Corollary 3.8 converts into a very simple one: $\mathcal{A}$ must be noncommutative. Furthermore, if $\mathcal{A}$ is a simple ring, then we get the definitive conclusion:

Corollary 3.10. *Let $\mathcal{A}$ be a noncommutative simple ring with* $\operatorname{char}(\mathcal{A}) \neq 2$. *Then every Jordan derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule is a derivation.*

In the next section we shall see that the conclusion of this corollary does not hold even for rings that are just slightly more general than the noncommutative simple ones.

Corollary 3.10 improves [8, Corollary 1]; in particular, it removes the assumption that $\mathcal{A}$ must be unital (on which the argument in [8] is based). For unital rings, however, we are now in a position to state a stronger result.

Corollary 3.11. *Let $\mathcal{A}$ be a unital ring with* $\operatorname{char}(\mathcal{A}) \neq 2$. *Suppose there exists a noncommutative simple subring $\mathcal{A}_0$ of $\mathcal{A}$ which contains the unity of $\mathcal{A}$. Then every Jordan derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule is a derivation.*

Proof. The ideal of $\mathcal{A}_0$ generated by $\mathcal{A}_0^{(3)}$ is equal to $\mathcal{A}_0$ and so it contains the unity of $\mathcal{A}$. Since this ideal is clearly contained in $\mathfrak{I}(\mathcal{A}^{(3)})$, we have $\mathfrak{I}(\mathcal{A}^{(3)}) = \mathcal{A}$. Now apply Corollary 3.2. □

For the special case where $\mathcal{A}_0 = M_n(\mathbb{C})$, Corollary 3.11 was proved by Johnson [19, Theorem 7.1]. The next corollary is motivated by another result of Johnson who proved its special where $\mathcal{A}$ is the norm closure of the algebra of all finite rank operators on a Banach space [19, Theorem 6.4].

Corollary 3.12. *Let $\mathcal{A}$ be a normed (real or complex) algebra containing a dense simple subalgebra. Then every continuous Jordan derivation from $\mathcal{A}$ into any normed $\mathcal{A}$-bimodule is a derivation.*

Proof. Let $d : \mathcal{A} \to \mathcal{M}$ be a continuous Jordan derivation and let $\mathcal{A}_0$ be a dense simple subalgebra of $\mathcal{A}$. If $\mathcal{A}_0$ is commutative, then $\mathcal{A} = \mathcal{A}_0$ is a field ($\mathbb{R}$ or $\mathbb{C}$) and so one can check directly that d is a derivation (in the case when $\mathcal{A} = \mathbb{C}$ is considered as an $\mathbb{R}$-algebra this perhaps does not appear so evident, but note that Remark 3.4 can be applied to yield the desired conclusion). So let $\mathcal{A}_0$ be noncommutative. Then $\mathcal{A}_0 = \mathfrak{I}(\mathcal{A}_0^{(3)})$ and hence Theorem 3.1 tells us that d satisfies $d(ux) = d(u)x + ud(x)$ for all $u \in \mathcal{A}_0$, $x \in \mathcal{A}$. Since d is continuous and $\mathcal{A}_0$ is dense, d must be a derivation. □

4. Examples of proper Jordan derivations

We begin with with a simple but important general observation which can be extracted from [19, p. 465]. One example of a proper Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ easily generates further examples on other rings (or algebras). Namely, if there exists a proper Jordan derivation d from a ring (resp. algebra) $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$, then there also exists a proper Jordan derivation from every ring (resp. algebra) $\mathcal{B}$ that has $\mathcal{A}$ as a quotient into $\mathcal{M}$. Indeed, if π is a homomorphism from $\mathcal{B}$ onto $\mathcal{A}$, then $\mathcal{M}$ becomes an $\mathcal{B}$-bimodule in the canonical way, and $d\pi$ is a proper Jordan derivation from $\mathcal{B}$ into $\mathcal{M}$.

Some examples of proper Jordan derivations were found already in [3, 12, 19]. In the first two subsections we shall recall (in some modified form) those of them that particularly nicely illustrate the results obtained in the previous section.

In order to make it clear what are in our opinion the main features of the particular example considered, we shall describe them in a short statement.

Throughout this section, $\mathbb{F}$ will denote a field with $\mathrm{char}(\mathbb{F}) \neq 2$. By $T_n(\mathbb{F})$ we denote the subalgebra of $M_n(\mathbb{F})$ consisting of all upper triangular matrices, and by e_{ij} we denote the matrix units.

4.1. **Constructing antiderivations.**

In [19, p. 465] Johnson constructed an example of a proper Jordan derivation from $T_2(\mathbb{F})$ into a certain $T_2(\mathbb{F})$-bimodule. Proceeding from this example, Benkovič [3] recently discovered a general method for constructing proper Jordan derivations, which we shall now briefly survey. In fact, these proper Jordan derivations are the so-called *antiderivations*, i.e. they satisfy the condition $d(xy) = d(y)x + yd(x)$ for all x and y.

Let $\mathcal{A}$ be an algebra. Suppose that there is an $\mathcal{A}$-bimodule $\mathcal{N}$ (we denote the multiplication in this bimodule by $\cdot$) and a derivation $\delta : \mathcal{A} \to \mathcal{N}$ satisfying the following two conditions:

$$[\mathcal{A},\mathcal{A}] \cdot \mathcal{N} = \mathcal{N} \cdot [\mathcal{A},\mathcal{A}] = 0 \quad \text{and} \quad \delta([\mathcal{A},\mathcal{A}]) \neq 0. \tag{8}$$

The first condition guarantees that the vector space of $\mathcal{N}$ becomes an $\mathcal{A}$-bimodule if we define the new multiplication by $xn = n \cdot x$ and $nx = x \cdot n$ for all $x \in \mathcal{A}$, $n \in \mathcal{N}$. Let us denote this bimodule by $\mathcal{M}$. Note that $d : x \mapsto \delta(x)$ defines an antiderivation from $\mathcal{A}$ into $\mathcal{M}$. Given $x, y \in \mathcal{A}$, we see that $d(xy) = d(x)y + xd(y)$ holds if and only if $\delta([x, y]) = 0$. Therefore, in view of the second condition in (8), d is a proper Jordan derivation.

Let us now consider a concrete situation where (8) occurs (cf. [3, Remark 2.4]). Let $\mathcal{A} = T_n(\mathbb{F})$, $n \geq 2$, and let $\mathcal{S}$ be the ideal of $\mathcal{A}$ consisting of all strictly upper triangular matrices. Consider the ideals $\mathcal{S}$ and $\mathcal{S}^2$ as $\mathcal{A}$-bimodules, and let $\mathcal{N} = \mathcal{S}/\mathcal{S}^2$ be the quotient module. Clearly, $\mathcal{N}$ satisfies the first condition in (8). Define a linear map $\delta : \mathcal{A} \to \mathcal{N}$ by $\delta(e_{i\,i+1}) = e_{i\,i+1} + \mathcal{S}^2$, $i = 1, \dots, n$, and $\delta(e_{ij}) = 0$ whenever $j \neq i + 1$. Note that δ is a derivation satisfying the second condition in (8). Moreover, setting, for example, $e = e_{11}$ and $u = e_{12}$ we arrive at the following conclusion concerning the corresponding antiderivation d.

Example 4.1. *Let $\mathcal{A} = T_n(\mathbb{F})$, $n \geq 2$. Then there exists an $\mathcal{A}$-bimodule $\mathcal{M}$ and a proper Jordan derivation (in fact, an antiderivation) $d : \mathcal{A} \to \mathcal{M}$ such that $d(ue) \neq d(u)e + ud(e)$ for some $u \in \mathcal{A}^{(1)}$ and some idempotent $e \in \mathcal{A}$.*

Example 4.1 shows, on the one hand, that the assumption that e commutes with x is really necessary in Remark 3.4, and on the other hand, that, using the above notation, $\{\mathcal{A}^{(1)}, \mathcal{A}\}$ is not zero for every Jordan derivation. Thus, the involvement of higher commutators in Theorem 3.1, and hence also in Theorem 2.1, is really necessary.

The main result in [3] states that every Jordan derivation from $T_n(\mathbb{F})$ into an $T_n(\mathbb{F})$-bimodule is the sum of a derivation and an antiderivation. A thorough analysis of antiderivations on $T_n(\mathbb{F})$ shows that these maps always vanish on $\mathcal{S}^2$. Therefore, in a loose manner we can say that Jordan derivations on $T_n(\mathbb{F})$ act as derivations on a rather large part of the algebra. Thus, the results from [3] nicely illustrate the philosophy of the present paper.

4.2. **Constructing proper Jordan derivations on some noncommutative algebras, I.** In [12, p. 324] Cusack gave two simple examples of proper Jordan derivations from a ring into itself. We shall now present a modified version of his second example. Let $\mathcal{A}$ be a 3-dimensional algebra over $\mathbb{F}$ generated by elements a and b such that $a^2 = b^2 = 0$ and $ab = -ba \neq 0$ (for example, one can take $\mathcal{A} \subset T_4(\mathbb{F})$ with $a = e_{12} - e_{34}$, $b = e_{13} + e_{24}$, and so $ab = e_{14}$). Of course, $\mathcal{A}$ is noncommutative, but $\mathcal{A}^{(2)} = 0$. Since $x \circ y = 0$ for all $x, y \in \mathcal{A}$, every linear map from $\mathcal{A}$ into itself is a Jordan derivation. However, not every map is a derivation. In particular, consider $d : \mathcal{A} \to \mathcal{A}$ defined by $d(\lambda a + \mu b + \nu ab) = \nu a$. In the next statement we state a special property of this map which is of some interest in light of Theorem 3.1.

Example 4.2. *There exists a 3-dimensional noncommutative algebra $\mathcal{A}$ admitting a proper Jordan derivation $d : \mathcal{A} \to \mathcal{A}$ such that for every $u \neq 0$ in $\mathcal{A}$ there is $x \in \mathcal{A}$ satisyfing $d(ux) \neq d(u)x + ud(x)$.*

4.3. **Constructing proper Jordan derivations on some noncommutative algebras, II.** Let $\mathcal{U}$ be an algebra over $\mathbb{F}$ such that $\mathcal{U} \circ \mathcal{U}$, the linear span of all elements of the form $u \circ v$, $u, v \in \mathcal{U}$, is a proper subspace of $\mathcal{U}^2$. Clearly, $\mathcal{U}$ cannot be commutative or unital. Further, let $\mathcal{M}$ be any nonzero vector space over $\mathbb{F}$. We make an $\mathcal{U}$-bimodule of it by defining the trivial multiplication $\mathcal{M}\mathcal{U} = \mathcal{U}\mathcal{M} = 0$. Now let $d : \mathcal{U} \to \mathcal{M}$ be a linear map such that $d(\mathcal{U} \circ \mathcal{U}) = 0$ and $d(\mathcal{U}^2) \neq 0$. In particular, $d(u^2) = d(u)u = ud(u) = 0$ for all $u \in \mathcal{U}$, so that d is a Jordan derivation. On the other hand, since $d(uv) \neq 0$ for some $u, v \in \mathcal{U}$ and $d(u)v = ud(v) = 0$ for all $u, v \in \mathcal{U}$, d is not a derivation.

One might find this example a bit artificial because of the triviality of the module multiplication. However, we may consider the unitization $\mathcal{A}$ of $\mathcal{U}$, extend the module multiplication so that $\mathcal{M}$ becomes a unital $\mathcal{A}$-bimodule, and extend d to $\mathcal{A}$ by defining $d(1) = 0$. Note that we can interpret the resulting construction in the following way.

Example 4.3. *Let $\mathcal{A}$ be a unital algebra over $\mathbb{F}$ containing an ideal $\mathcal{U}$ of codimension 1 such that $\mathcal{U} \circ \mathcal{U} \neq \mathcal{U}^2$. Then there exist a proper Jordan derivation from $\mathcal{A}$ into some unital $\mathcal{A}$-bimodule.*

Algebras containing an ideal of codimension 1 will also appear in a different construction below.

A simple concrete example of an algebra satisfying the conditions of Example 4.3 is the free noncommutative algebra (here, $\mathcal{U}$ consists of elements of constant term zero). An example of a proper Jordan derivation on this algebra was found already in [3], and its purpose was to show that there exist Jordan derivations on some rings that cannot be expressed as the sums of derivations and antiderivations. One can notice that this also applies to both proper Jordan derivations constructed in Examples 4.2 and 4.3.

4.4. **Constructing proper Jordan derivations on some commutative rings.** Let $\mathcal{A}$ be a ring, let $\mathcal{N}$ be an $\mathcal{A}$-bimodule and let $\delta : \mathcal{A} \to \mathcal{N}$ be a derivation. Note that the additive group $\mathcal{N} \times \mathcal{A}$ becomes an $\mathcal{A}$-bimodule if we define

$$x(n, y) = (xn + \delta(x)y, xy), \quad (n, y)x = (nx - y\delta(x), yx).$$

We denote this $\mathcal{A}$-bimodule by $\mathcal{M}$. Further, let $\gamma : \mathcal{A} \to \mathcal{A}$ be a derivation and define $d : \mathcal{A} \to \mathcal{M}$ by

$$d(x) = (0, \gamma(x)).$$

It is straightforward to check that d is a derivation if and only if

$$\delta(x)\gamma(y) = \gamma(x)\delta(y) \quad \text{for all } x, y \in \mathcal{A}, \tag{9}$$

and d is a Jordan derivation if and only if

$$\delta(x)\gamma(x) = \gamma(x)\delta(x) \quad \text{for all } x \in \mathcal{A}. \tag{10}$$

Our point here is that the condition (10) seems to be considerably weaker than the condition (9). Thus, one may expect that this construction can be used to produce examples of proper Jordan derivations.

From now on we assume that $\mathcal{N} = \mathcal{A}$ (the more general situation when $\mathcal{N}$ is an $\mathcal{A}$-bimodule was mentioned only because of possible applications elsewhere). If $\mathcal{A}$ is a noncommutative prime ring with $\operatorname{char}(\mathcal{A}) \neq 2$, then, unfortunately, (9) and (10) are equivalent. This follows, for example, from [21, Theorem 4]. On the other hand, if $\mathcal{A}$ is a commutative ring, then (10) is automatically fulfilled, while (9) holds only expectionally. For example, if $\mathcal{A}$ is an integral domain then (9) is readily equivalent to the condition that δ and γ are linearly dependent over the field of fractions $\mathcal{F}$ of $\mathcal{A}$ (i.e. $\delta = 0$ or $\gamma(x) = \lambda\delta(x)$ for all $x \in \mathcal{A}$ and some $\lambda \in \mathcal{F}$). Thus, we have

Example 4.4. *Let $\mathcal{A}$ be an integral domain. Suppose there exist two derivations from $\mathcal{A}$ into itself that are linearly independent over the field of fractions of $\mathcal{A}$. Then there exist a proper Jordan derivation from $\mathcal{A}$ into some $\mathcal{A}$-bimodule $\mathcal{M}$.*

It is clear from our construction that $\mathcal{M}$ is a unital bimodule if $\mathcal{A}$ is a unital ring. We also remark that it is not enough to assume only the existence of only one nonzero derivation on $\mathcal{A}$. For example, there are certainly nonzero derivations on the polynomial ring $\mathbb{Z}[x]$, but applying Remark 3.6 we see that every Jordan derivation d from $\mathbb{Z}[x]$ into any $\mathbb{Z}[x]$-bimodule is a derivation.

Example 4.4 justifies the necessity of some of the assumptions in certain results in Section 3. In particular, it shows that Corollary 3.10 really does not hold for commutative rings. This follows from the discussion in the next subsection.

4.5. **Constructing proper additive Jordan derivations on some real and complex algebras.** It is well-known that nontrivial derivations can be constructed on many fields (see, for example, sections on derivations in [16] or [27]). Let us briefly outline one such construction which is sufficient for our present purposes. For simplicity we assume that our fields have characteristic 0. Let $\mathbb{E}/\mathbb{F}$ be a field extension such that a transcendence basis B for $\mathbb{E}$ over $\mathbb{F}$ has at least two elements. Set $\mathbb{K} = \mathbb{F}(B)$. Pick different $a, b \in B$ and let $\gamma, \delta : B \to \mathbb{E}$ be any maps such that $\delta(a) = \gamma(b) = 1$ and $\delta(b) = \gamma(a) = 0$. Note that δ and γ can be extended uniquely to $\mathbb{F}$-linear derivations from $\mathbb{K}$ into itself. Moreover, since every element in $\mathbb{E}$ is algebraic over $\mathbb{K}$, every derivation, say ϵ, on $\mathbb{K}$ can be extended uniquely to a derivation on $\mathbb{E}$. Indeed, given $u \in \mathbb{E}$, let $f(x) = \sum_{i=0}^{n-1} a_i x^i + x^n \in \mathbb{K}[x]$ be its minimal polynomial, and define $\epsilon(u) = -\left(\sum_{i=0}^{n-1} \epsilon(a_i)u^i\right)f'(u)^{-1}$ where $f'(x)$ is the formal derivative of $f(x)$. Therefore, γ and δ can be extended to ($\mathbb{F}$-linear) derivations on $\mathbb{E}$, and of course they are linearly independent over $\mathbb{E}$.

In particular this shows that there are (say, $\mathbb{Q}$-linear) linearly independent derivations on $\mathbb{R}$ and $\mathbb{C}$. This fact gives rise examples of proper Jordan derivations on many other important rings. By an *additive (Jordan) derivation* on an algebra $\mathcal{A}$ we mean an additive (and not necessarily linear) map that satisfies the usual (Jordan) derivation law; that is, it is a (Jordan) derivation of the ring $\mathcal{A}$. Let us mention that there has been some interest in additive derivations on Banach algebras, see e.g. [20]. In view of the observation at the beginning of Section 4 we can now state

Example 4.5. *Let $\mathcal{A}$ be a unital real or complex algebra. If $\mathcal{A}$ contains an ideal of codimension* 1*, then there exists a proper additive Jordan derivation from $\mathcal{A}$ into some unital $\mathcal{A}$-bimodule $\mathcal{M}$.*

So, while there are no proper additive Jordan derivations on a simple real or complex algebra $\mathcal{A}$ without unity (Corollary 3.10), this is no longer true for the unitization of $\mathcal{A}$. This indicates the delicate nature of the Jordan derivation problem.

5. On Jordan derivations on C^*-algebras

Is every Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule a derivation? This is an intriguing open question. Unfortunately our methods do not seem to lead to the final solution, but at least we can give some new insight into the problem. In particular we are able to give the definitive answer to a related (but more algebraic) question on additive Jordan derivations.

Johnson proved that a *continuous* Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is a derivation [19, Theorem 6.3]. Since derivations from C^*-algebras into their Banach $\mathcal{A}$-bimodules are automatically continuous [23], the above question is equivalent to the following one: *Is every Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule continuous?* (This is Question 14 in Villena's survey [25]).

In the recent paper [1] the affirmative answer was obtained for various classes of C^*-algebras, including von Neumann algebras and commutative C^*-algebras. On the other hand, in 1992 the present author proved that every additive Jordan derivation from a unital C^*-algebra $\mathcal{A}$ with no multiplicative functionals into any $\mathcal{A}$-bimodule is a derivation [8, Corollary 2] (in fact, the result was not stated for *additive* Jordan derivations, but it is clear from the proof that it holds true for them). This result also follows from Corollary 3.2 and the fact that a unital C^*-algebra $\mathcal{A}$ with no multiplicative functionals satisfies $\mathfrak{I}(\mathcal{A}^{(n)}) = \mathcal{A}$ for every $n \geq 1$. The latter can be proved easily by combining Lemma 3.7 with the arguments from the proof of [8, Lemma 4] (we omit details).

The condition that a unital algebra $\mathcal{A}$ has a multiplicative functional is clearly equivalent to the condition that $\mathcal{A}$ contains an ideal of codimension 1. Therefore, in view of Example 4.5 we have

Theorem 5.1. *Let $\mathcal{A}$ be a unital C^*-algebra. Then there exists a proper additive Jordan derivation from $\mathcal{A}$ into some unital $\mathcal{A}$-bimodule if and only if $\mathcal{A}$ contains an ideal of codimension* 1*.*

The theorem of Johnson [19, Theorem 6.3] can be easily derived from Theorem 5.1 and Remarks 3.4 and 3.5. Indeed, let d be a continuous Jordan derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$. First, from the argument in [19, p. 466] we see that in order to show that d is a derivation we may assume without loss of generality that $\mathcal{A}$ is a von Neumann algebra. Every von Neumann algebra $\mathcal{A}$ can be represented as $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where $\mathcal{A}_1$ is a commutative von Neumann algebra and $\mathcal{A}_2$ is a von Neumann algebra without commutative direct summands (i.e., $\mathcal{A}_2$ has no central portion of type I_1). Since $\mathcal{A}_2$ does not have multiplicative functionals (this follows for example from [7, Lemma 2.6]), $d|\mathcal{A}_2$ is a derivation. Remark 3.4 implies that $d(u_1x_1) = d(u_1)x_1 + u_1d(x_1)$ for all $x_1 \in \mathcal{A}_1$ and every $u_1 \in \mathcal{A}_1$ which can be expressed as a linear combination of projections in $\mathcal{A}_1$. However, since d is continuous and the linear span of projections in $\mathcal{A}_1$ is dense in $\mathcal{A}_1$, it follows that $d|\mathcal{A}_1$ is a derivation (in fact, the use of the continuity assumption can be avoided at this point, but this makes the proof incomparably more complicated, see [1]). Applying Remark 3.5 we now see that d is a derivation.

Of course, our construction of proper Jordan derivations in Subsection 4.4 also works in the context of algebras (i.e., for linear and not only additive maps). The result from [1] stating that there are no proper Jordan derivations from a commutative C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ can now be viewed from a different perspective: it has some relations with the well-known fact that there are no nonzero derivations from commutative C^*-algebras into themselves. Actually, nonzero derivations do not exist even on commutative semisimple Banach algebras [18]. Now it would be interesting to know whether this result from [1] can be generalized to more general algebras. This question, however, requires some carefulness. In the noncommutative context it is clear that Johnson's result cannot be extended from C^*-algebras to semisimple Banach algebras. Namely, it is easy to find a semisimple (even primitive) Banach algebra $\mathcal{A}$ having $T_2(\mathbb{C})$ as a quotient [19, p. 465], and so there are continuous proper Jordan derivations (in fact, antiderivations) on $\mathcal{A}$.

6. Related problems on Jordan homomorphisms and Jordan $\mathcal{A}$-module homomorphisms

Recall that a *Jordan homomorphism* is an additive map h from a ring $\mathcal{A}$ into a ring $\mathcal{B}$ satisfying

$$h(x^2) = h(x)^2 \quad \text{and} \quad h(xyx) = h(x)h(y)h(x) \quad \text{for all } x, y \in \mathcal{A}.$$

The basic examples are homomorphisms, antihomomorphisms, and their sums. It was observed already by Jacobson and Rickart that the problem on Jordan derivations can be often reduced to the one on Jordan homomorphisms [17, Theorem 22]. The idea of this reduction can be easily described. Let $\mathcal{A}$ be a ring and let $\mathcal{M}$ an $\mathcal{A}$-bimodule (we remark that in [17] only the case when $\mathcal{M} = \mathcal{A}$ was considered, but the following facts are true also in the case where $\mathcal{M}$ is an $\mathcal{A}$-bimodule). Note that the set of all matrices matrices of the form

$$\begin{pmatrix} x & m \\ 0 & x \end{pmatrix}, \qquad x \in \mathcal{A},\, m \in \mathcal{M},$$

forms a ring under the usual matrix operations. We denote this ring by $\mathcal{B}$. Given a map $d : \mathcal{A} \to \mathcal{M}$ we define $h : \mathcal{A} \to \mathcal{B}$ by

$$h(x) = \begin{pmatrix} x & d(x) \\ 0 & x \end{pmatrix}.$$

Note that d is a derivation (resp. Jordan derivation) if and only if h is a homomorphism (resp. Jordan homomorphism). So, knowing the structure of Jordan homomorphisms one can also get some conclusion concerning Jordan derivations. All these are well-known facts. What we would like to add here is that h also satisfies

$$\text{(11)} \quad \big(h(xy) - h(x)h(y)\big)\big(h(zw) - h(z)h(w)\big) = 0 \quad \text{for all } x, y, z, w \in \mathcal{A}.$$

So, if we know that h is a Jordan homomorphism, then it is much more likely that h is a homomorphism rather than an antihomomorphism. The condition *(i)* below therefore seems to be a natural one.

Now let $\mathcal{M}_0$ be a right $\mathcal{A}$-module. We shall call an additive map $f : \mathcal{A} \to \mathcal{M}_0$ a *Jordan $\mathcal{A}$-module homomorphism* if it satisfies

$$\text{(12)} \qquad f(x^2) = f(x)x \quad \text{and} \quad f(xyx) = f(x)yx \quad \text{for all } x, y \in \mathcal{A}$$

(similarly as in the Jordan derivation case, in the 2-torsionfree setting the second condition follows from the first one). A natural question is of course whether a Jordan $\mathcal{A}$-module homomorphism is an $\mathcal{A}$-module homomorphism, i.e. does $f(xy) = f(x)y$ hold for all $x, y \in \mathcal{A}$. This question was studied by Zalar [26] (for the case where $\mathcal{M}_0 = \mathcal{A}$) who used a method similar to the one that has been used in the study of Jordan derivations. Appropriate modifications of our above arguments would also give results for Jordan $\mathcal{A}$-module homomorphisms. However, instead of doing this we will show that the Jordan derivation problem is in fact more general. Indeed, let $f : \mathcal{A} \to \mathcal{M}_0$ be a Jordan $\mathcal{A}$-module homomorphism. We can turn $\mathcal{M}_0$ into an $\mathcal{A}$-bimodule by defining the trivial multiplication $\mathcal{A}\mathcal{M}_0 = 0$. We denote this bimodule by $\mathcal{M}$ and note that $d : x \mapsto f(x)$ is a Jordan derivation from $\mathcal{A}$ into $\mathcal{M}$. If d is actually a derivation from $\mathcal{A}$ into $\mathcal{M}$, then clearly f is an $\mathcal{A}$-module homomorphism from $\mathcal{A}$ into $\mathcal{M}_0$.

We summarize the above discussion in

Theorem 6.1. *Let $\mathcal{A}$ be a ring. Consider the following conditions:*

(i) every Jordan homomorphism from $\mathcal{A}$ into an arbitrary ring that satisfies (11) is a homomorphism;

(ii) every Jordan derivation from $\mathcal{A}$ into an arbitrary $\mathcal{A}$-bimodule is a derivation;

(iii) every Jordan $\mathcal{A}$-module homomorphism from $\mathcal{A}$ into an arbitrary right $\mathcal{A}$-bimodule is an $\mathcal{A}$-module homomorphism.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

In order to justify the relevance of Theorem 6.1 we have to show that *(i)* really holds in some rings and that *(iii)* does not hold in every ring.

We claim that the matrix ring $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}$ is any unital ring and $n \geq 2$, satisfies *(i)*. Indeed, if $h : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism then by [17, Theorem 7] there are a homomorphism $h_1 : \mathcal{A} \to \mathcal{B}$ and an antihomomorphism $h_2 : \mathcal{A} \to \mathcal{B}$ such that $h = h_1 + h_2$ and $h_1(\mathcal{A})h_2(\mathcal{A}) = h_2(\mathcal{A})h_1(\mathcal{A}) = 0$. If h also satisfies (11), then we have $[h_2(y), h_2(x)][h_2(w), h_2(z)] = 0$, that is, $[x, y][z, w]$ lies in the kernel $\mathcal{K}$ of h_2 for all $x, y, z, w \in \mathcal{A}$. In particular, $\mathcal{K}$ contains all matrix units e_{ii} since $e_{ii} = [e_{ii}, e_{ij}][e_{jj}, e_{ji}]$ for every $j \neq i$. Accordingly, $\mathcal{K}$ contains the unity of $\mathcal{A}$. However, $\mathcal{K}$ is an ideal of $\mathcal{A}$ and so $\mathcal{K} = \mathcal{A}$; that is, $h_2 = 0$ and so $h = h_1$ is a homomorphism. Thus, $\mathcal{A} = M_n(\mathcal{D})$ satisfies *(i)*. Theorem 6.1 now implies that there are no proper Jordan derivations from $\mathcal{A}$ into $\mathcal{A}$-bimodules. Under some slight additional assumptions, this could also be deduced from the results in Section 3.

If $\mathcal{A}$ is a unital ring, then taking $x = 1$ in the second identity in (12) we see that $f(y) = m_0 y$ for all $y \in \mathcal{A}$ and some $m_0 \in \mathcal{M}_0$. Therefore it makes sense to search for an example of a ring not satisfying *(iii)* only among rings that are not unital. But in fact such an example has already been found. Note that the Jordan derivation $d : \mathcal{U} \to \mathcal{M}$ defined in Subsection 4.3 satisfies $d(u^2) = d(u)u = 0$ for all $u \in \mathcal{U}$, and hence (since $\mathrm{char}(\mathbb{F}) \neq 2$) also $d(uvu) = d(u)vu = 0$ for all $u, v \in \mathcal{U}$. Thus d is a Jordan $\mathcal{A}$-module homomorphism. However, d is not an $\mathcal{A}$-module homomorphism. In the algebra considered in Subsection 4.2 the desired examples exist even for maps from the algebra into itself: just consider $f(\lambda a + \mu b + \nu ab) = \nu ab$.

In view of Theorem 6.1 we now see that some of the results in Section 3 imply that in certain rings Jordan $\mathcal{A}$-module homomorphisms on $\mathcal{A}$ are necessarily $\mathcal{A}$-module homomorphisms, and the examples in Section 4 generate examples of nontrivial Jordan homomorphisms.

References

[1] J. Alaminos, M. Brešar, A. R. Villena, The strong degree and the structure of Lie and Jordan derivations from von Neumann algebras, *Math. Proc. Camb. Phil. Soc.*, to appear.

[2] K. I. Beidar, W. S. Martindale 3rd, A. V. Mikhalev, "Rings with Generalized Identities", *Marcel Dekker, Inc.*, New York-Basel-Hong Kong, 1996.

[3] D. Benkovič, Jordan derivations and antiderivations of triangular matrices, *Linear Algebra Appl.*, to appear.
[4] J. Bergen, I. N. Herstein, J. W. Kerr, Lie ideals and derivations of prime rings, *J. Algebra* **71** (1981), 259-267.
[5] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **104** (1988), 1003-1006.
[6] M. Brešar, Jordan mappings of semiprime rings, *J. Algebra* **127** (1989), 218-228.
[7] M. Brešar, Centralizing mappings on von Neumann algebras, *Proc. Amer. Math. Soc.* **111** (1991), 501-510.
[8] M. Brešar, Characterizations of derivations on some normed algebras with involution, *J. Algebra* **152** (1992), 454-462.
[9] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.* **335** (1993), 525-546.
[10] M. Brešar, A. Fošner, M. Fošner, Jordan ideals revisited, preprint.
[11] M. Brešar, J. Vukman, Jordan derivations on prime rings, *Bull. Austral. Math. Soc.* **37** (1988), 321-322.
[12] J. M. Cusack, Jordan derivations on rings, *Proc. Amer. Math. Soc.* **53** (1975), 321-324.
[13] M. Fošner, Jordan superderivations, *Comm Algebra* **31** (2003), 4533-4545.
[14] I. N. Herstein, Jordan derivations of prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1104-1110.
[15] I. N. Herstein, *Topics in ring theory*, The University of Chicago Press, Chicago, 1969.
[16] N. Jacobson, "Lectures in abstract algebra, Vol. III", *D. Van Nostrand Company, Inc.*, Princeton-Toronto-London-New York, 1964.
[17] N. Jacobson, C. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.* **69** (1950), 479-502.
[18] B. E. Johnson, Continuity of derivations on commutative Banach algebras, *Amer. J. Math.* **91** (1969), 1-10.
[19] B. E. Johnson, Symmetric amenability and the nonexistence of Lie and Jordan derivations, *Math. Proc. Camb. Phil. Soc.* **120** (1996), 455-473.
[20] B. E. Johnson, A. M. Sinclair, Continuity of derivations and a problem of Kaplansky, *Amer. J. Math.* **90** (1968), 1067-1073.
[21] C. Lanski, Differential identities of prime rings, Kharchenko's theorem, and applications, *Contemporary Math.* **124** (1992), 111-128.
[22] K. McCrimmon, The Zelmanov approach to Jordan homomorphisms of associative rings, *J. Algebra* **123** (1989), 457-477.
[23] J. R. Ringrose, Automatic continuity of derivations on operator algebras, *J. London Math. Soc.* **5** (1972), 432-438.
[24] A. M. Sinclair, Jordan homomorphisms and derivations on semisimple Banach algebras, *Proc. Amer. Math. Soc.* **24** (1970), 209-214.
[25] A. R. Villena, Automatic continuity in associative and nonassociative context, *Irish Math. Soc. Bull.* **46** (2001), 43-76.
[26] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae* **32** (1991), 609-614.
[27] O. Zariski, P. Samuel, "Commutative algebra", *van Nostrand Comp. Inc.*, New York, 1958.

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