JORDAN GRADINGS ON ASSOCIATIVE ALGEBRAS

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Abstract. In this paper we apply the method of functional identities to the study of group gradings by an abelian group G on simple Jordan algebras, under very mild restrictions on the grading group or the base field of coefficients.

1. Introduction

In this paper we seek to reduce the determination of the gradings by abelian groups on simple Jordan algebras to the same question about associative algebras. In the case of finite-dimensional algebras with certain restrictions on the base field of coefficients this was done in [3, 4, 5, 8]. Recently a new approach was found, using so called Functional Identities [11], which enables one to get rid of most restrictions on the dimension or on the base field of coefficients. Functional identities help one to reduce Jordan maps on associative algebras to a combination of associative homomorphisms and antihomomorphisms. The classical results about Jordan maps deal with simple or prime rings but the latest achievements reflected in [11] include much wider classes of rings sufficient to settle some questions about graded Jordan algebras.

Specifically, the situation in the theory of graded algebras is the following. Suppose a Jordan algebra J over a field F is graded by a group G. This is well known [16] to be equivalent to J being a (right) H-comodule Jordan algebra over the group algebra $H = FG$, that is, to the existence of a Jordan homomorphism $\rho : J \to J \otimes H$ such that

$$(\rho \otimes \mathrm{id}_H)\rho = (\mathrm{id}_J \otimes \Delta)\rho \tag{1}$$

and

$$(\mathrm{id}_J \otimes \varepsilon)\rho = \mathrm{id}_J. \tag{2}$$

In the case of a graded algebra, ρ is determined by $\rho(a_g) = a_g \otimes g$ where a_g is a homogeneous element of degree g. Here Δ and ε are the coproduct and the counit of H, respectively. If J is a Jordan subalgebra generating an associative algebra A and ρ extends to an associative homomorphism $\rho : A \to A \otimes H$ (with (1), (2) preserved!) then A also becomes G-graded. Since both J_g and A_g are defined as the sets of elements x in J and A satisfying $\rho(x) = x \otimes g$ we have $J_g = J \cap A_g$.

In what follows we will use techniques of [11] to show the existence of such extension under certain natural restrictions on J and G. Notice, however, that the natural extension of a Jordan homomorphism is not an associative homomorphism but rather the direct sum of a homomorphism and an antihomomorphism. The

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grading theory counterpart of this situation is the so called *involution grading* on an associative algebra with involution. In the case of matrix algebras such gradings have been completely described in [8]. In what follows we will show that this approach works in a more general situation considered in this paper.

The paper is organized as follows. In Section 2 we recall some definitions, fix the notation, and indicate the main idea upon which this paper is based. Then, in Sections 3 - 5, we apply the techniques of [11] to the study of Jordan maps of tensor products. In the remaining Sections 6 - 8 we give some applications of the results obtained to the grading theory.

This paper was written at the same time as [2], where we dealt with Lie gradings on associative algebras. Therefore, there are a lot of similarities, and some technical steps are the same. They appear in both papers for the sake of completeness of exposition. The only exception is that in this paper we shall state two technical results from [2] without proofs, since in both papers they are needed in exactly the same form and [2] was submitted earlier.

2. Preliminaries

Let A be a not necessarily associative algebra over a field F, and let G be a group. We say that A is *graded by* G if $A = \bigoplus_{g\in G} A_g$ and $A_g A_h \subset A_{gh}$, for any $g, h \in G$. An element $a \in A_g$ is called *homogeneous of degree* g and we write $\deg a = g$. A subspace M is called *graded* if $M = \bigoplus_{g\in G}(M \cap A_g)$. The set $\operatorname{Supp} A = \{g \in G \,|\, A_g \neq 0\}$ is called the *support* of the grading. Let H be the group algebra $H = FG$. This is a Hopf algebra with coproduct $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$, for any $g \in G$. As mentioned above, A becomes a right H-comodule algebra with a structure homomorphism $\rho : A \to A \otimes H$.

Suppose A is an associative algebra and $A^{(+)}$ is the Jordan algebra attached to A. As mentioned above, a grading by an abelian group G on $A^{(+)}$ is equivalent to a comodule mapping $\rho : A^{(+)} \to A^{(+)} \otimes H$, $H = FG$. We remark that $A^{(+)} \otimes H$ is actually equal to $(A \otimes H)^{(+)}$. Thus, ρ is a Jordan homomorphism between associative algebras A and $A \otimes H$. Its range is a rather "small" subset of $A \otimes H$, which makes the results and the methods from [11] more or less inapplicable. However, ρ can be extended to a Jordan automorphism $\widetilde{\rho}$ of the algebra $A \otimes H$, defined as follows:

$$\widetilde{\rho}(a \otimes h) = \sum_{g\in G} a_g \otimes (gh), \text{ where } a = \sum_{g\in G} a_g \text{ and } a_g \in A_g.$$

Since A is spanned by the elements of A_g, it follows that $A \otimes H$ is spanned by $a_g \otimes h = \widetilde{\rho}(a_g \otimes (g^{-1}h))$, $g \in G$, $h \in H$. Thus $\widetilde{\rho}$ is surjective. Now, this makes the theory exposed in [11] applicable. Actually, the results are not directly applicable, but the methods are. More precisely, the approach based on the concept of the fractional degree works, as we shall see.

The previous paragraph reveals the main idea of our approach. It will be used not only for Jordan algebras $A^{(+)}$, A being an associative algebra, but also for the Jordan algebra of symmetric elements of an associative algebra with involution.

By an algebra we shall always mean an algebra over a fixed field F. We shall mostly deal with prime and simple (associative) algebras, and notions such as the maximal algebra of quotients, the extended centroid etc. will appear frequently in our exposition. We refer the reader to [10] for a full account on these notions.

Let us just recall that a prime algebra is said to be *centrally closed* if its extended centroid coincides with F. In case of simple algebras, the extended centroid is equal to the centroid. A more standard term for a centrally closed simple algebra is a *central simple algebra* (we are not assuming that such an algebra is unital, just that its centroid is F).

Let A be an algebra. By Z_A we denote its center. If X is a subset of A, then by $\langle X\rangle$ we denote the subalgebra of A generated by X. By an *involution* on A we shall mean an F-linear map $*: A \to A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. If A is an algebra with involution $*$, then by S_A (resp. K_A) we denote the set of all symmetric (resp. skew-symmetric) elements in A: $S_A = \{a \in A \mid a^* = a\}$, $K_A = \{a \in A \mid a^* = -a\}$. The grading $A = \bigoplus_{g\in G} A_g$ is called an *involution grading* if $(A_g)^* = A_g$, for any $g \in G$. If A is not unital, then we denote by $A^\sharp$ the algebra obtained by adjoining the unity to A. If, however, A is unital, then we set $A^\sharp = A$.

3. Fractional degree and functional identities

The fractional degree of an element of a ring was introduced in [9] as a technical concept needed for handling functional identities in certain rings; specifically, the existing results on functional identities on prime rings were extended to semiprime rings by making use of the fractional degree. As we shall see, this concept is also suitable for the purposes of this paper. The first aim of this section is to give a fragmentary survey on fractional degree which is sufficient for our needs. For more details see [9] or [11, Section 5.1].

First we record basic definitions. Let A be a subalgebra of an algebra Q. We say that an element $a \in A$ is *fractionable* in Q if the following two conditions hold:

(i) If $\varphi : A \to Q$ is an additive map such that $\varphi(xay) = ax\varphi(y)$ for all $x, y \in A$, then there exists $q \in Q$ such that $\varphi(x) = axq$ for all $x \in A$;

(ii) If $q \in Q$ is such that $qAa = 0$ or $aAq = 0$, then $q = 0$.

By $\mathcal{M}(A)$ we denote the multiplication algebra of A, i.e., the algebra of linear operators on A of the form $x \mapsto \sum_i a_ixb_i$ where $a_i, b_i \in A$. These operators can be extended to $A^\sharp$ in the obvious way. We say that the *fractional degree* of an element $t \in A$ is greater than n (in Q), where $n \geq 0$, if for every $i = 0, 1, \dots, n$ there exists $\mathcal{E}_i \in \mathcal{M}(A)$ such that

$$\mathcal{E}_i(t^j) = 0 \text{ if } j \neq i, \text{ and } \mathcal{E}_i(t^i) \text{ is fractionable in } Q$$

(here, of course, is should be understood that $t^0 = 1 \in A^\sharp$). We write this as $f\text{-}\deg_{A,Q}(t) > n$. Of course, we define that $f\text{-}\deg_{A,Q}(t) = n$ if $f\text{-}\deg_{A,Q}(t) > n-1$ but $f\text{-}\deg_{A,Q}(t) \not> n$. If $f\text{-}\deg_{A,Q}(t) > n$ for every positive integer n, then we write $f\text{-}\deg_{A,Q}(t) = \infty$.

The next two lemmas can be easily extracted from [11].

Lemma 3.1. *Let A be a centrally closed prime algebra, and let Q be its maximal left algebra of quotients. If $\dim_F A \geq d^2$ (possibly ∞), then A contains an element a with $f\text{-}\deg_{A,Q}(a) \geq d$.*

Proof. By [11, Theorems C.1 and C.2] A contains elements such that their degree of algebraicity over F is $\geq d$. Now use [11, Lemma 5.10]. □

Lemma 3.2. *Let $\operatorname{char} F \neq 2$, let A be a centrally closed prime algebra with involution, and let Q be its maximal left algebra of quotients. If $\dim_F A \geq d^2$, then $S_A \cup K_A$ contains an element a with $f\text{-}\deg_{A,Q}(a) \geq d$.*

Proof. We repeat the argument of Lemma 3.1 and also use [11, Lemma C.6] which shows that a can be chosen in $S_A \cup K_A$. $\square$

Proposition 3.3. [2, Proposition 3.6] *Let $A \subseteq Q$ be arbitrary algebras, and H a unital algebra. Then*

$$f\text{-}\deg_{A\otimes H, Q\otimes H}(t\otimes 1) \geq f\text{-}\deg_{A,Q}(t)$$

holds for every $t \in A$, provided that one of the following two conditions holds:

(a) *H is finite dimensional;*
(b) *A is a simple unital algebra and Q has the same identity element as A.*

See also [2, Example 3.4] which justifies the presence of the assumption (a) in case A is not simple unital.

Let us add a few words about the meaning of the fractional degree in the theory of functional identities. Roughly speaking, if A contains elements whose fractional degree is $\geq d$, then certain basic functional identities in sufficiently small number (connected to d) of variables can be handled in A. Using the right technical term, such rings are *d-free*. The notion of d-frenees is fundamental in the theory of functional identities. Let us introduce it in a brief and non-rigorous manner.

Let Q be a unital ring, let us write C for Z_Q, and let X be a subset of Q. Let $x_1, \ldots, x_d \in X$. For $1 \leq i \leq d$ we write

$$\overline{x}_d^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in X^{d-1} = X \times \ldots \times X,$$

and for $1 \leq 1 \leq i < j \leq d$ we write

$$\overline{x}_d^{ij} = \overline{x}_d^{ji} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in X^{d-2}.$$

A functional identity on X is, roughly, an identical relation holding for all elements in X which involves some arbitrary (unknown) functions. An important and illustrative example is a functional identity

$$\sum_{i=1}^{d} E_i(\overline{x}_d^i)x_i + \sum_{j=1}^{d} x_j F_j(\overline{x}_d^j) = 0 \quad \text{for all } x_i \in X,$$

where $E_i, F_j : X^{d-1} \to Q$. One wishes to describe these functions. More precisely, the desirable conclusion is the so called standard solution of this functional identity, meaning that there exist functions $p_{ij} : X^{d-2} \to Q$, $i \neq j$, and $\lambda_i : X^{d-1} \to C$ such that

$$E_i(\overline{x}_d^i) = \sum_{\substack{j=1\\ j\neq i}}^{d} x_j p_{ij}(\overline{x}_d^{ij}) + \lambda_i(\overline{x}_d^i), \quad F_j(\overline{x}_d^j) = -\sum_{\substack{i=1\\ i\neq j}}^{d} p_{ij}(\overline{x}_d^{ij})x_i - \lambda_j(\overline{x}_d^j).$$

If this conclusion holds, and if also some related functional identities have only (similarly defined) standard solutions, then we say that X is a d-free subset of Q.

In order to establish the d-freeness of a set, one usually has to deal with a more general (yet auxiliary) concept of *$(t; d)$-freeness*. Here t is a fixed element from a ring in question. The definition of a $(t; d)$-free set includes functional identities such as

$$\sum_{i=1}^{d}\sum_{u=0}^{a} E_{iu}(\overline{x}_d^i)x_i t^u + \sum_{j=1}^{d}\sum_{v=0}^{b} t^v x_j F_{jv}(\overline{x}_d^j) = 0 \quad \text{for all } x_i \in X,$$

so for $a = b = 0$ this coincides with the functional identity stated above. It is trivial, but essential, that a set is d-free if it is $(t; d)$-free for some t.

The main result connecting functional identities and fractional degree says the following: If A is a subring of Q such that the centralizer of A in Q is equal to C, then the existence of $t \in A$ such that $f\text{-}\deg_{A,Q}(t) \geq d$ implies that A is a $(t;d)$-free (and hence d-free) subset of Q [11, Theorem 5.6].

The results in [11, Section 6.4] show that Jordan homomorphisms in d-free rings (for specific d) can be described through homomorphisms and antihomomorphisms.

We hope that the informal explanation just given makes it possible for a non-specialist to follow the next sections superficially. For understanding all details, however, one has to study the book [11] to which we shall continually refer.

4. Jordan maps on associative algebras

Proposition 4.1. *Let* $\operatorname{char} F \neq 2$*,* R *a unital algebra and* T *its subalgebra such that the centralizer of* T *in* R *is equal to* Z_R*. Let* B *be an algebra and let* $\rho : B \to T$ *be a surjective Jordan homomorphism. If there exists* $t \in T$ *such that* $f\text{-}\deg_{T,R}(t) \geq 4$*, then there exists an idempotent* $\epsilon \in Z_R$ *such that* $x \mapsto \epsilon\rho(x)$ *is a homomorphism and* $x \mapsto (1-\epsilon)\rho(x)$ *is an antihomomorphism.*

Proof. By [11, Theorem 5.6] T is a 4-free subset of R, and so the result follows from [11, Theorem 6.23]. □

Theorem 4.2. *Let* $\operatorname{char} F \neq 2$*,* A *a centrally closed prime algebra and* H *a unital commutative algebra. Assume that either* H *is finite dimensional or* A *is central simple unital. Let* B *be an algebra and* $\rho : B \to A \otimes H$ *a surjective Jordan homomorphism. If* $\dim_F A \geq 16$*, then there exists an idempotent* $e \in H$ *such that* $x \mapsto (1 \otimes e)\rho(x)$ *is a homomorphism and* $x \mapsto (1 \otimes (1-e))\rho(x)$ *is an antihomomorphism.*

Proof. Let Q be the maximal left algebra of quotients of A, and set $R = Q \otimes H$, $T = A \otimes H$. Suppose that $r \in R$ is such that $[r, T] = 0$. Writing r as $\sum_i q_i \otimes h_i$, where $q_i \in Q$ and the h_i's are linearly independent elements in H, it follows from $[r, A \otimes 1] = 0$ that $[q_i, A] = 0$ for every i. Since A is a centrally closed prime F-algebra, it follows that each q_i is a scalar multiple of 1 (see [10, Remark 2.3.1]). Therefore $r \in 1 \otimes H = Z_R$. We have thereby showed that the centralizer of T in R is Z_R. Also, it is clear that every idempotent in Z_R is of the form $1 \otimes e$ with $e = e^2 \in H$.

By Lemma 3.1 there exists $a \in A$ with $f\text{-}\deg_{A,Q}(a) \geq 4$, and so $t = a \otimes 1 \in T$ satisfies $f\text{-}\deg_{T,R}(t) \geq 4$ by Proposition 3.3. Using Proposition 4.1 one easily infers the desired conclusion. □

We are primarily interested in the situation when B is actually equal $A \otimes H$. In this setting we can get rid of the restriction on the dimension of A by using a more classical approach based on an old result by Jacobson and Rickart on Jordan homomorphisms on matrix algebras [13].

Theorem 4.3. *Let* $\operatorname{char} F \neq 2$*,* A *a centrally closed prime algebra and* H *a unital commutative algebra. Assume that either* H *is finite dimensional or* A *is central simple unital. If* $\rho : A \otimes H \to A \otimes H$ *is a surjective Jordan homomorphism, then there exists an idempotent* $e \in H$ *such that* $x \mapsto (1 \otimes e)\rho(x)$ *is a homomorphism and* $x \mapsto (1 \otimes (1-e))\rho(x)$ *is an antihomomorphism.*

Proof. In view of Theorem 4.2, it suffices to consider the case when A is finite dimensional. Actually, it would be enough to assume that $\dim_F A < 16$, but this additional restriction would not simplify the proof that follows. We may also assume that $\dim_F A > 1$ since otherwise $A \otimes H \cong H$ is commutative and ρ is trivially a homomorphism.

Set $T = A \otimes H$ and consider its scalar extension $\overline{T} = T \otimes \overline{F}$ where $\overline{F}$ is the algebraic closure of F. We extend ρ to a Jordan homomorphism $\overline{\rho} : \overline{T} \to \overline{T}$ in the obvious way, $\overline{\rho}(t \otimes \lambda) = \rho(t) \otimes \lambda$. Note that $A \otimes \overline{F} \cong M_n(\overline{F})$ for some $n > 1$ by Wedderburn theorem. Accordingly, $\overline{T} \cong M_n(H \otimes \overline{F})$. We may now apply [13, Theorem 7] to conclude that there is a central idempotent $\overline{e} \in \overline{T}$ such that $x \mapsto (1 \otimes \overline{e})\overline{\rho}(x)$ is a homomorphism and $x \mapsto (1 \otimes (1 - \overline{e}))\overline{\rho}(x)$ is an antihomomorphism. It is easy to see (as in the preceding proof) that $\overline{e}$ is of the form $1 \otimes \widetilde{e}$ where $\widetilde{e}$ is an idempotent in $H \otimes \overline{F}$. Let us write $\widetilde{e}$ as $e \otimes 1 + \sum_{i=1}^n h_i \otimes \lambda_i$ where $e, h_i \in H$ and $1, \lambda_1, \ldots, \lambda_n \in \overline{F}$ are linearly independent over F. Now, for all $s, t \in T$ we have

$$\begin{aligned}
\rho(st) \otimes 1 &= \overline{\rho}(st \otimes 1) = \overline{e}\overline{\rho}((s \otimes 1)(t \otimes 1)) + (1 - \overline{e})\overline{\rho}((s \otimes 1)(t \otimes 1)) \\
&= \overline{e}\overline{\rho}(s \otimes 1)\overline{\rho}(t \otimes 1) + (1 - \overline{e})\overline{\rho}(t \otimes 1)\overline{\rho}(s \otimes 1) \\
&= (1 \otimes \widetilde{e})(\rho(s)\rho(t) \otimes 1) + (1 \otimes (1 - \widetilde{e}))(\rho(t)\rho(s) \otimes 1) \\
&= \Big((1 \otimes e)\rho(s)\rho(t) + (1 \otimes (1 - e))\rho(t)\rho(s)\Big) \otimes 1 \\
&\quad + \sum_{i=1}^n (1 \otimes h_i)[\rho(s), \rho(t)] \otimes \lambda_i.
\end{aligned}$$

This readily implies that

$$\rho(st) = (1 \otimes e)\rho(s)\rho(t) + (1 \otimes (1 - e))\rho(t)\rho(s) \tag{3}$$

and $(1 \otimes h_i)[\rho(s), \rho(t)] = 0$ for every i. Since ρ is surjective, the latter actually means that $(1 \otimes h_i)[T, T] = 0$, and so in particular $(1 \otimes h_i)([A, A] \otimes 1) = 0$. However, since A is noncommutative (in view of $\dim_F A > 1$) this is possible only if every $h_i = 0$. Therefore $\widetilde{e} = e \otimes 1$ and so e is an idempotent. The desired conclusion now follows from (3). □

We remark that in the twin Lie grading paper [2] the use of functional identities also led to certain dimension restrictions, which, however, the authors were unable to remove by some alternative approach. Thus, from this point of view the present Jordan grading paper is more definite.

5. Jordan maps on symmetric elements

Proposition 5.1. *Let* $\operatorname{char} F \neq 2$, *$R$ a unital algebra and T a subalgebra of R. Assume that T has an involution and that the centralizer of T in R is equal to Z_R. Further, let B be an arbitrary algebra with involution, and let $\rho : S_B \to S_T$ be a surjective Jordan homomorphism. If there exists $t \in S_T \cup K_T$ with $f\text{-}\deg_{T,R}(t) \geq 15$, then ρ can be extended to a homomorphism from $\langle S_B \rangle$ onto $\langle S_T \rangle$.*

Proof. By [11, Theorem 5.6] T is a $(t; 15)$-free subset of R. Accordingly, S_T is a $(t; 7)$-free subset of R by [11, Theorem 3.28], and so in particular a 7-free subset of R. It is now easy to see that all conditions of [11, Theorem 6.26] are met, and the result follows. □

Theorem 5.2. *Let* $\operatorname{char} F \neq 2$*,* A *a centrally closed prime algebra with involution, and* B *be an arbitrary algebra with involution. Further, let* H *be a unital commutative algebra. Assume that either* H *is finite dimensional or* A *is central simple unital. If* $\rho : S_B \to S_A \otimes H$ *is a surjective Jordan homomorphism and* $\dim_F A \geq 225$*, then* ρ *can be extended to a homomorphism from* $\langle S_B \rangle$ *onto* $\langle S_A \rangle \otimes H$*.*

Proof. As in the proof of Theorem 4.2, we set $R = Q \otimes H$, where Q is the maximal left algebra of quotients of A, and $T = A \otimes H$. We recall from that proof that the centralizer of T in R is equal to Z_R.

By Lemma 3.2 there exists $a \in S_A \cup K_A$ with $f\text{-}\deg_{A,Q}(a) \geq 15$. Therefore $t = a \otimes 1 \in T$ has $f\text{-}\deg_{T,R}(t) \geq 15$ by Proposition 3.3. Define an involution on T by $(x \otimes h)^* = x^* \otimes h$, note that $S_T = S_A \otimes H$ and $K_T = K_A \otimes H$, and hence $t \in S_T \cup K_T$. Now use Proposition 5.1. □

One cannot entirely remove the restriction on the dimension of A - see e.g. [15, p. 243] where an example on a 16-dimensional algebra $A = M_2(\mathbb{H})$ is given. However, if $B = A \otimes H$ then we can substantially decrease the number 225, as we shall now see. We will rely on a result by Martindale [15] (which we find in this context somewhat more convenient than an earlier result by Jacobson and Rickart [14] on Jordan homomorphisms of symmetric elements of matrix algebras).

Theorem 5.3. *Let* $\operatorname{char} F \neq 2$*,* A *a centrally closed prime algebra with involution, and* H *a unital commutative algebra. Assume that either* H *is finite dimensional or* A *is central simple unital. If* $\rho : S_A \otimes H \to S_A \otimes H$ *is a surjective Jordan homomorphism and* $\dim_F A > 16$*, then* ρ *can be extended to a homomorphism from* $\langle S_A \rangle \otimes H$ *onto* $\langle S_A \rangle \otimes H$*.*

Proof. In view of Theorem 5.2 we may assume that A is finite dimensional. As in the proof of Theorem 4.3 we set $T = A \otimes H$ and $\overline{T} = T \otimes \overline{F}$ where $\overline{F}$ is the algebraic closure of F, and extend ρ to $S_{\overline{T}}$ by $\overline{\rho}(s \otimes \lambda) = \rho(s) \otimes \lambda$; here it should be understood that the involution on the $\overline{F}$-algebra $\overline{T}$ is given by $(a \otimes h \otimes \lambda)^* = a^* \otimes h \otimes \lambda$. Note that $A \otimes \overline{F} \cong M_n(\overline{F})$ for some $n > 4$. In view of [10, Corollary 4.6.13] it is enough to consider just two involutions on $M_n(\overline{F})$: the transpose and the symplectic one. Since $n > 4$, in each case one easily infers that $M_n(\overline{F})$ contains three nonzero orthogonal symmetric idempotents e_1, e_2, e_3 with $e_1 + e_2 + e_3 = 1$. We may identify $\overline{T}$ with $M_n(\overline{F}) \otimes H$. Clearly, $E_i = e_i \otimes 1$ are nonzero orthogonal symmetric idempotents in $\overline{T}$ with $E_1 + E_2 + E_3 = 1$. Further, the (2-sided) ideal of $\overline{T}$ generated by E_i is equal to $\overline{T}$; this follows from the fact that the ideal of $M_n(\overline{F})$ generated by e_i is equal to $M_n(\overline{F})$ (because of the simplicity of $M_n(\overline{F})$). Thus, all conditions of [15, Theorem 1] are met. Hence we can extend $\overline{\rho}$ to a homomorphism on $S_{\overline{T}}$. Its restriction to $\langle S_T \rangle = \langle S_A \rangle \otimes H$ is of course a homomorphism whose range is $\langle S_A \rangle \otimes H$. □

Corollary 5.4. *Let* $\operatorname{char} F \neq 2$*, let* A *be a central simple algebra with involution, and assume that* $\dim_F A > 16$*. Let* H *be a unital commutative algebra which is finite dimensional if* A *is not unital. Then every surjective Jordan homomorphism* $\rho : S_A \otimes H \to S_A \otimes H$ *can be extended to a homomorphism from* $A \otimes H$ *onto* $A \otimes H$*.*

Proof. Use Theorem 5.3 together with the fact that $\langle S_A \rangle = A$ by Herstein's theorem [12, Theorem 1.6]. □

6. Applications to graded algebras: Jordan structure on prime algebras

Let A be an associative algebra, $J = A^{(+)}$ and G an abelian group. In this section we consider the question when is a group grading of J induced from a group grading of A. We know that the grading by an abelian group G on J is completely equivalent to a comodule map $\rho : J \to J \otimes H$, $H = FG$, which can be viewed as a Jordan homomorphism from A to $A \otimes H$. We also know that ρ extends to a Jordan homomorphism from $A \otimes H$ to itself which we denote by the same letter: $\rho(a \otimes h) = \rho(a)(1 \otimes h)$. As mentioned in Section 2, $\rho : A \otimes H \to A \otimes H$ is surjective. Now one can apply the results of Section 4: if A is centrally closed prime use Theorem 4.2 to derive that $\rho : A \otimes H \to A \otimes H$ is the sum of an automorphism and an antiautomorphism of this latter algebra. If G is not finite we have additionally to assume that A is central simple unital.

Now because $A = J$, the properties of the comodule map ρ are the same for A as they were for J which makes A into a G-graded vector space. If in addition ρ is an associative homomorphism, A becomes G-graded and the grading of J is induced from A. In this section we will look at the conditions on G which enable one to conclude that ρ, indeed, is a homomorphism.

We start with an example showing that, in general, ρ does not need to be a homomorphism.

Example 6.1. Let A be an algebra with involution, let $J = A^{(+)}$, and let $G = \mathbb{Z}_2$. Thus, we have $G = \{1, t\}$ and $t^2 = 1$. Suppose that $\operatorname{char} F \neq 2$. Setting $J_1 = S_A$ and $J_t = K_A$ we see that J becomes graded by G. Thus, ρ is given by $\rho(a_1 + a_t) = a_1 \otimes 1 + a_t \otimes t$. Note that $\epsilon = 1 \otimes \frac{1+t}{2}$ is a nontrivial central idempotent in $A \otimes H$, $H = FG$, such that $a \mapsto \epsilon\rho(a)$ is a homomorphism and $a \mapsto (1-\epsilon)\rho(a)$ is an antihomomorphism. In particular, ρ is not a homomorphism.

In this section we are interested in the case where central idempotents are trivial. We need a result from [2].

Proposition 6.2. [2, Proposition 6.7] *Let A be an algebra with $Z_{A^\sharp} = F$, and let $H = FG$ be the group algebra of an abelian group G. We define a subgroup T_1 of G as follows. Let T be the subgroup of elements of G of finite order. In the case* $\operatorname{char} F = 0$ *we set $T_1 = T$. In the case* $\operatorname{char} F = p > 0$ *we define T_1 as the subgroup of T consisting of all elements whose order is coprime to p. Then any central idempotent of $A^\sharp \otimes H$ is of the form $1 \otimes e$ where the idempotent e lies in the group algebra of T_1.*

Some easy but important consequences of this result are as follows.

Theorem 6.3. *Let A be a centrally closed prime algebra,* $\operatorname{char} F \neq 2$, *and G an abelian group which is finite if A is not simple and unital. Suppose $J = A^{(+)} = \bigoplus_{g \in G} J_g$ is graded by G and denote by G' be a subgroup of G generated by the support of J. Suppose G' has no periodic elements, if* $\operatorname{char} F = 0$ *or no periodic elements whose order is coprime to p if* $\operatorname{char} F = p > 0$. *Then setting $A_g = J_g$ turns A into a G-graded associative algebra: $A = \bigoplus_{g \in G} A_g$. Thus under these conditions, every Jordan grading of J is induced from an associative grading of A.*

Proof. We use Proposition 6.2 to obtain that any central idempotent of $A^\sharp \otimes H$ is trivial. Given a Jordan grading of J we consider the comodule map $\rho : J \to J \otimes H$.

As pointed out at the beginning of the section, by Theorem 4.3 ρ is the sum of a homomorphism and an antihomomorphism $\rho : A \to A \otimes H$. The central idempotent ϵ yielding the decomposition of ρ lies in $A^\sharp \otimes H$. However, Proposition 6.2 tells us that $A^\sharp \otimes H$ has only trivial central idempotents, so that $\epsilon = 0$ or $\epsilon = 1$. The case $\epsilon = 0$ can be ruled out since using (2) we see that in this case the identity on $J = A$ is an antihomomorphism, which is possible only when A is commutative (and hence ρ is a simultaneously a homomorphism). Thus $\epsilon = 1$ and so ρ is a homomorphism. This, together with $J = A$, implies that ρ also satisfies the axioms (1) and (2). Now since both L_g and A_g are defined in the same way, using ρ, say, $A_g = \{x \mid \rho(x) = x \otimes g\}$, we obviously obtain $J_g = J \cap A_g$, for all $g \in G$. □

Now let $A = M_n(F)$ be a matrix algebra of order n over a field F, $\overline{A} = M_n(\overline{F})$ where $\overline{F}$ is the algebraic closure of F. Any grading of A by a group G naturally induces a grading of $\overline{A}$ by G if one sets $\overline{A}_g = A_g \otimes \overline{F}$. We say that a grading of the matrix algebra $A = M_n(F)$ is *elementary* if there is an n-tuple of elements $(g_1, \ldots, g_n) \in G^n$ such that for a certain choice of matrix units E_{ij} of $\overline{A}$ one has $\overline{A}_g = \mathrm{Span}\,\{E_{ij} \mid g = g_i^{-1} g_j\}$.

Theorem 6.4. *Let $J = M_n(F)^{(+)}$ be graded by an abelian p-group G. If $\mathrm{char}\, F = p \neq 2$ then any grading of J is induced from an (elementary) grading of $A = M_n(F)$.*

Proof. By [6] all gradings of a matrix algebra over a field of characteristic $p > 0$ by a finite abelian p-group are elementary. Applying Theorem 6.3 we easily derive our result. □

7. Applications to graded algebras: Jordan algebras of symmetric elements

Let A be an algebra with involution $*$, $\mathrm{char}\, F \neq 2$, and $J = S_A$ be the Jordan algebra of symmetric elements under $*$. Again, given a grading of J by an abelian group G, we have a comodule Jordan homomorphism $\rho : J \to J \otimes H$. As before, we can extend ρ to a surjective Jordan homomorphism $\rho : J \otimes H \to J \otimes H$. Applying Theorem 5.3 as above we can easily infer the following.

Theorem 7.1. *Let A be a centrally closed prime algebra with involution. Assume that $\mathrm{char}\, F \neq 2$ and $\dim_F(A) > 16$. Let $J = S_A$ be the Jordan algebra of symmetric elements of A. Suppose J is graded by an abelian group G, which should be assumed finite if A is not simple and unital. If $\langle J \rangle = A$, then there is an associative grading $A = \bigoplus_{g \in G} A_g$ such that $J_g = A_g \cap J$, for all $g \in G$.*

An important special case is when A is simple (see Corollary 5.4).

Theorem 7.2. *Let A be a central simple algebra with involution. Assume that $\mathrm{char}\, F \neq 2$ and $\dim_F(A) > 16$. Let $J = S_A$ be graded by an abelian group G, which should be assumed finite if A is not unital. Then there is an associative grading $A = \bigoplus_{g \in G} A_g$ such that $J_g = A_g \cap J$, for all $g \in G$.*

A direct consequence is the following theorem, which is related to the results in [4] and [8]; they concern only the case of zero characteristic fields, but without any restrictions on the order of the matrix algebras.

Theorem 7.3. *Let $A = M_n(F)$, $n > 4$, $\mathrm{char}\, F \neq 2$, $*$ an involution on A, $J = S_A$ the simple Jordan algebra of symmetric elements on A. Let G be an abelian group. Then the G-gradings of J are induced from the gradings of $M_n(F)$.*

In [4] it was also stated that the gradings of J are induced from so called involution gradings of A. The involution gradings of $M_n(F)$ have been completely described in [8] provided that $\operatorname{char} F \neq 2$ and F has “sufficiently many” roots of 1 (for example, when F is algebraically closed).

8. Applications to graded algebras: two types of gradings on $A^{(+)}$

In this section we consider the case where a Jordan grading on $J = A^{(+)}$ is given by an abelian group G with periodic elements over a field F with $\operatorname{char} F = 0$ or with periodic elements of order coprime to $p = \operatorname{char} F$. As noted at the beginning of Section 6 and in Proposition 6.2, the comodule Jordan homomorphism $\rho : J \to J \otimes H$ equals to the sum of an associative homomorphism $a \to (1 \otimes e)\rho(a)$ and an antihomomorphism $a \to (1\otimes f)\rho(a)$ with e and $f = 1-e$ being central idempotents of the group algebra $K_1 = FT_1$ where, in the same way as in Proposition 6.2, T_1 is the subgroup of all periodic elements of G in the case where $\operatorname{char} F = 0$ or of all periodic elements of G of order coprime to p if $p = \operatorname{char} F$. Note that A is not assumed to be unital, so that $1 \otimes e, 1 \otimes f \in A^\sharp \otimes H$.

To study the precise form of the idempotents e and f, we will temporarily assume that F is algebraically closed. In this case, $K_1 = Fe_1 \oplus \cdots \oplus Fe_m$ where $e_1, \ldots, e_m$ are pairwise orthogonal indecomposable idempotents of K_1, $m = \dim K_1 = |T_1|$. Also, it is well-known [1] that in this case $T_1 \cong \widehat{T_1}$ where $\widehat{T_1}$ is the group of multiplicative characters $\chi : T_1 \to F^*$. The idempotents in the above decomposition of K_1 take the form of

$$e_\chi = \frac{1}{m} \sum_{t\in T_1} \chi(t)^{-1} t. \tag{4}$$

Note that if $(\psi \mid \chi)_{T_1}$ stands for the scalar product of the characters ψ and χ of a group T_1 then

$$\chi(e_\psi) = \frac{1}{m} \sum_{t\in T_1} \psi(t)^{-1}\chi(t) = (\psi \mid \chi)_{T_1} = 1 \text{ if } \psi = \chi \text{ and } 0 \text{ otherwise.} \tag{5}$$

One more important formula, a direct consequence of (4), is this. For any $t \in T_1$ and $\chi \in \widehat{T_1}$ one has $te_\chi = \chi(t)e_\chi$. Further, given a subset Ω of $\widehat{T_1}$, one can define an idempotent $e_\Omega = \sum_{\chi\in\Omega} e_\chi$ in FT_1, and conversely, every idempotent in FT_1 is of such form. In particular, there is $\Lambda \subset \widehat{T_1}$ such that $e = e_\Lambda$. Note that $f = 1 - e = e_{\Lambda'}$ where $\Lambda \cup \Lambda' = \widehat{T_1}$.

Before we formulate our next result, we introduce the group $\overline{\operatorname{Aut}}\, A$ of automorphisms and the antiautomorphisms of the algebra A. This has a subgroup of automorphisms $\operatorname{Aut} A$ of index at most 2.

Now given $\chi \in \widehat{T_1}$ we set $\alpha(\chi) = (\mathrm{id}_A \otimes \chi)\rho$. For example, if $x \in J_g$ then $\rho(x) = x \otimes g$ and so $\alpha(\chi)(x) = \chi(g)x$. Given also $y \in J_h$, we have $x \circ y \in J_{gh}$ and then

$$\alpha(\chi)(x \circ y) = \chi(gh)(x \circ y) = (\chi(g)x) \circ (\chi(h)y) = \alpha(\chi)(x) \circ \alpha(\chi)(y)$$

proving that $\alpha(\chi)$ is an automorphism of Jordan algebras. Also, given $\chi, \psi \in \widehat{T_1}$, we have

$$\alpha(\chi\psi)(x) = (\chi\psi)(g)x = \chi(g)\psi(g)x = \alpha(\chi)(\alpha(\psi)(x)).$$

Therefore, $\alpha : \widehat{T_1} \to \operatorname{Aut} J$ is a homomorphism of groups.

Our next goal will be to prove the following.

Proposition 8.1. *Let A be a centrally closed prime algebra, and let $J = A^{(+)}$. Let J be G-graded, for an abelian group G, and let $\rho : J \to J \otimes H$ be the Jordan comodule map, where $H = FG$. Adopting the notation and assumptions preceding this proposition, we write $e = e_\Lambda$ and $f = e_{\Lambda'}$ for the central idempotents of H such that $a \to (1 \otimes e)\rho(a)$ is an associative homomorphism while $a \to (1 \otimes f)\rho(a)$ is an antihomomorphism. Then the mapping $\alpha : \widehat{T_1} \to \overline{\mathrm{Aut}}\, A$ is a group homomorphism and hence Λ is a subgroup of index at most 2 in $\widehat{T_1}$.*

Proof. Since $J = A^{(+)}$ we have that $\overline{\mathrm{Aut}}\,(A)$ is a subgroup of $\mathrm{Aut}\,(J)$. Then, from our remarks preceding the statement of the proposition, we only need to show that $\alpha(\widehat{T_1}) \subset \overline{\mathrm{Aut}}\,(A)$ and $\alpha(\Lambda) \subset \mathrm{Aut}\,(A)$.

Let us assume $\chi \in \widehat{T_1}$ and $x, x' \in A$. Then we have the following.

$$
\begin{aligned}
\alpha(\chi)(xx') &= (\mathrm{id}_A \otimes \chi)(\rho(xx')(1 \otimes e_\Lambda) + \rho(xx')(1 \otimes e_{\Lambda'})) \\
&= (\mathrm{id}_A \otimes \chi)(\rho(x)(1 \otimes e_\Lambda)\rho(x')(1 \otimes e_\Lambda) + \rho(x')(1 \otimes e_{\Lambda'})\rho(x)(1 \otimes e_{\Lambda'})) \\
&= (\mathrm{id}_A \otimes \chi)(\rho(x))(\mathrm{id}_A \otimes \chi)(\rho(x'))(\mathrm{id}_A \otimes \chi)(1 \otimes e_\Lambda) \\
&+ (\mathrm{id}_A \otimes \chi)(\rho(x'))(\mathrm{id}_A \otimes \chi)(\rho(x))(\mathrm{id}_A \otimes \chi)(1 \otimes e_{\Lambda'}) \\
&= \alpha(\chi)(x)\alpha(\chi)(x') \sum_{\lambda \in \Lambda} \chi(e_\lambda) + \alpha(\chi)(x')\alpha(\chi)(x) \sum_{\mu \in \Lambda'} \chi(e_\mu).
\end{aligned}
$$

If $\chi \in \Lambda$ then only one summand, in the first sum, survives and we have $\alpha(\chi)(xx') = \alpha(\chi)(x)\alpha(\chi)(x')$, that is, $\alpha(\chi)$ is a homomorphism. Otherwise, only one summand, in the second sum, survives and we have $\alpha(\chi)(xx') = \alpha(\chi)(x')\alpha(\chi)(x)$, that is, $\alpha(\chi)$ is an antihomomorphism. So we have shown that α maps $\widehat{T_1}$ into $\overline{\mathrm{Aut}}\, A$, in such a way that $\alpha(\widehat{T_1}) \cap \mathrm{Aut}\,(A) = \alpha(\Lambda)$, as needed. □

If we do not impose restrictions on the field of coefficients F then we obtain the following.

Proposition 8.2. *Let A be a centrally closed prime algebra, and $J = A^{(+)}$. Suppose J is graded by an abelian group G and $\rho : J \to J \otimes H$ is the respective Jordan comodule map, where $H = FG$. Let e and f be nontrivial central idempotents of H such that if we view ρ as the mapping of associative algebras $\rho : A \to A \otimes H$, then $a \to (1 \otimes e)\rho(a)$ is a homomorphism while $a \to (1 \otimes f)\rho(a)$ is an antihomomorphism. Then $t = e - f$ is an element of G of order 2.*

Proof. Let us assume, for the time being, that our field is algebraically closed. By [7, Proposition 5.2] we have $T_1 = \langle a \rangle \times T_1'$, $\widehat{T_1} = \langle \chi_0 \rangle \times \Delta$ where the order of a and χ_0 is 2^k, for some $k > 0$, $\Lambda = \langle \chi_0^2 \rangle \times \Delta$. We also have $\chi_0(T_1') = 1$ and $\delta(a) = 1$, for all $\delta \in \Delta$. Now each $\chi \in \widehat{T_1}$ is of the form $\chi = \mu\delta$ where $\mu(T_1') = 1$ and $\delta \in \Delta$. In this case the idempotent e_χ can be transformed as follows:

$$
\begin{aligned}
e_\chi &= \frac{1}{m} \sum_{t \in T_1} \chi(t)^{-1} t = \frac{1}{m} \sum_{u \in \langle a_0 \rangle, v \in T_1'} \mu\delta(uv)^{-1} uv \\
&= \left(\frac{1}{2^k} \sum_{u \in \langle a_0 \rangle} \mu(u)^{-1} u \right) \left(\frac{1}{|T_1'|} \sum_{v \in T_1'} \delta(v)^{-1} v \right).
\end{aligned}
$$

If we fix μ with $\mu(T_1') = 1$ then the sum of all $e_{\mu\delta}$ with $\delta \in \Delta$ by the previous calculation will be equal to the idempotent e_μ of the group algebra $F\langle a_0\rangle$ because the remaining factor

$$\sum_{\delta\in\Delta} \frac{1}{|T_1'|} \sum_{v\in T_1'} \delta(v)^{-1}v$$

equals 1 as the sum of all indecomposable idempotents of the group algebra FT_1'. Now each term of either e_Λ or $e_{\Lambda'}$ is of that form, which allows us to restrict to the case where T_1' is trivial. So we need an explicit computation only in the case where T_1 is a 2-group generated by a single element a of order $m = 2^k$. We have $\chi_0(a) = \xi^{-1}$, where ξ is a primitive 2^kth root of 1. Further, we have

$$e = \sum_{i=0}^{2^{k-1}-1} e_{\chi_0^{2i}} \text{ and } f = \sum_{i=0}^{2^{k-1}-1} e_{\chi_0^{2i+1}}.$$

To compute $e - f$, we need to use (4), which we will rewrite as follows:

$$e_{\chi_0^s} = \frac{1}{2^k} \sum_{r=0}^{2^k-1} \xi^{sr} a^r.$$

Let ζ_r be a primitive $2r$th root of 1. Then one can write $e - f$ as follows

$$e - f = \sum_{r=0}^{2^k-1} \left(\sum_{s=0}^{2^k-1} (\xi\zeta_r)^{sr} \right) a^r. \tag{6}$$

Now

$$\sum_{s=0}^{2^k-1} (\xi\zeta_r)^{sr} = \begin{cases} 2^k \text{ if } (\xi\zeta_r)^r = 1 \\ \frac{(\xi\zeta_r)^{r2^k}-1}{(\xi\zeta_r)^r-1} = 0 \text{ otherwise} \end{cases}$$

Thus a^r enters the right hand side of (6) with nonzero coefficient only if $\xi^r(-1) = (\xi\zeta_r)^r = 1$, that is $\xi^r = -1$. Obviously, then we must have $r = 2^{k-1}$ and $e - f = a^{2^{k-1}}$, a group element of order 2, as claimed.

At this time we can go back to the original field F because both e and f are defined over F. From what we have proved, it also follows that $\Delta(t) = t \otimes t$ and so t is a group-like element, hence $t \in G$. Obviously, $o(t) = 2$. □

As shown by example earlier, the map $\rho : A \to A \otimes H$ we have obtained before cannot serve as the comodule map making A into a G-graded algebra. So we have to make an additional assumption that A has an involution compatible with ρ. Namely, if A has an involution $*$ then we can extend it to $A \otimes H$ by setting $(a \otimes h)^* = a^* \otimes h$, for any $a \in A$ and $h \in H$. Then we require the following

$$\rho(x^*) = \rho(x)^* \text{ for any } x \in A. \tag{7}$$

Also it follows easily from (7) that $(J_g)^* = J_g$, for any $g \in G$. Indeed, for any $x \in J_g$ one has $x^* \in J$ and

$$\rho(x^*) = (\rho(x))^* = (x \otimes g)^* = x^* \otimes g,$$

as needed.

Thus each J_g splits into the sum of the space of symmetric and skew-symmetric elements so that J is spanned by homogeneous symmetric and skew-symmetric elements.

We set

$$\rho^*(x) = \rho(x)(1 \otimes e) + \rho(x^*)(1 \otimes f).$$

Next we check that ρ^* is now an associative homomorphism. Given $x, y \in A$, we have

$$\begin{aligned}
\rho^*(xy) &= \rho(xy)(1 \otimes e) + \rho((xy)^*)(1 \otimes f) \\
&= (\rho(x)(1 \otimes e))(\rho(y)(1 \otimes e)) + \rho(y^* x^*)(1 \otimes f) \\
&= (\rho(x)(1 \otimes e))(\rho(y)(1 \otimes e)) + (\rho(x^*)(1 \otimes f))(\rho(y^*)(1 \otimes f)) \\
&= (\rho(x)(1 \otimes e) + \rho(x^*)(1 \otimes f))(\rho(y)(1 \otimes e) + \rho(y^*)(1 \otimes f)) \\
&= \rho^*(x)\rho^*(y).
\end{aligned}$$

Now for any symmetric $x \in J_g$ one has

$$\begin{aligned}
\rho^*(x) &= \rho(x)(1 \otimes e) + \rho(x^*)(1 \otimes f) \\
&= (x \otimes g)(1 \otimes e) + (x^* \otimes g)(1 \otimes f) \\
&= (x \otimes g)(1 \otimes e) + (x \otimes g)(1 \otimes f)) = (x \otimes g)(1 \otimes (e + f)) = x \otimes g.
\end{aligned}$$

Similar computation in the case where x is skew-symmetric gives

$$\begin{aligned}
\rho^*(x) &= \rho(x)(1 \otimes e) + \rho(x^*)(1 \otimes f) \\
&= (x \otimes g)(1 \otimes e) + (x^* \otimes g)(1 \otimes f) \\
&= (x \otimes g)(1 \otimes e) - (x \otimes g)(1 \otimes f) = (x \otimes g)(1 \otimes (e - f)) = x \otimes (gt).
\end{aligned}$$

On skew-symmetric and symmetric elements of J, therefore, the conditions (1) and (2) are satisfied. For instance, both sides of (1) on a skew-symmetric element x of J_g will give $x \otimes (gt) \otimes (gt)$. Checking (2) is even simpler. Now since all maps on both sides of (1) and (2) for ρ^* are associative homomorphisms, it follows that both conditions are satisfied on the whole of A. In this case ρ^* induces a G-grading on A. From the above argument we have the following result.

Theorem 8.3. *Let J be a Jordan algebra $J = A^{(+)}$ where A is a centrally closed prime algebra over a field F with* $\operatorname{char} F \neq 2$, *$G$ be an abelian group which is finite if A is not simple and unital. Let J is given a G-grading $J = \bigoplus_{g \in G} J_g$ which is not associative. Suppose there is an involution $*$ on A such that $J_g^* = J_g$. Then there are an associative grading $A = \bigoplus_{g \in G} A_g$ with $A_g^* = A_g$ and an element t of order 2 in G, such that*

$$J_g = (S_A \cap A_g) \oplus (K_A \cap A_{gt}).$$

Proof. Consider the structure map $\rho : J \to J \otimes H$. First of all, under these conditions, ρ is the sum of a homomorphism and an antihomomorphism. As previously, ρ cannot be an antihomomorphism. Neither ρ is a homomorphism because then, contrary to our assumption, our grading would be associative.

An easy check shows that the condition on the involution in the statement of the theorem is equivalent to (7). Then the map ρ^*, as shown just before the statement of this theorem, makes A into a G-graded algebra. We set $t = e - f$. By Proposition 8.2, t is an element of order 2 in G. The computation preceding the theorem shows that the symmetric elements in J_g have degree g in the grading of A induced by ρ^* while the skew-symmetric elements have degree gt. This proves that, indeed, $J_g = (S_A \cap A_g) \oplus (K_A \cap A_{gt})$, for each $g \in G$.

Checking that ρ^* is compatible with $*$ is a simple exercise, which we leave to the reader. So we have $A_g^* = A_g$, as claimed. $\square$

In conclusion, we will consider an important case where the above theorem applies.

Theorem 8.4. *Let J be a Jordan algebra $J = A^{(+)}$ where A is a centrally closed prime algebra over a field F with $\operatorname{char} F \neq 2$. Suppose J be given a grading $J = \bigoplus_{g\in G} J_g$ by an abelian group G. Suppose that G has the property that the order of each 2-element is actually 2, and that G is finite if A is not unital and simple. Then one of the two cases occurs:*

(1) *There exists an associative G-grading of A such that $J_g = A_g$, for each $g \in G$.*
(2) *There exist an element t of order 2 in G, an involution $*$ on A and an involution grading $A = \bigoplus_{g\in G} A_g$ such that*

$$J_g = (S_A \cap A_g) \oplus (K_A \cap A_{gt}).$$

Proof. Using the same argument as before, we may assume that $\rho : A \to A \otimes H$ is an actual sum of a homomorphism and an antihomomorphism. This can be done because otherwise we would have the first case in our theorem. Extending the field of coefficients to the algebraically closed $\bar{F}$, and applying Proposition 8.2, we will find the subgroup $\Lambda \subset \widehat{T_1}$ of index 2. Therefore, there is an element $\chi \notin \Lambda$. By our assumption $o(\chi) = 2q$ where q is odd. In this case χ^q is still outside of Λ, and we may assume that from the very beginning we have χ of order 2 being outside of Λ. By the argument preceding Theorem 8.3 we should have that $\alpha(\chi)$ is an involution on A. Thus we may set $x^* = \alpha(\chi)(x)$. Notice that because $o(\chi) = 2$ our involution is defined over the original field F. To apply Theorem 8.3, we only need to check that $\rho(a^*) = \rho(a)^*$ and this is enough to do for the homogeneous elements of J. If $a \in J_g$ then

$$\begin{aligned} \rho(a^*) &= \rho(\alpha(\chi)(a)) = \chi(g)a \otimes g = (\alpha(\chi)(a) \otimes 1)(a \otimes g) \\ &= (\alpha(\chi) \otimes 1)\rho(a) = \rho(a)^*, \end{aligned}$$

as claimed. □

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