LIE GRADINGS ON ASSOCIATIVE ALGEBRAS

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ABSTRACT. In this paper we apply the method of functional identities to the study of group gradings by an abelian group G on simple Lie algebras, under very mild restrictions on the grading group or the base field of coefficients.

1. INTRODUCTION

In this paper two areas of active research come together: Lie maps of associative rings and group gradings of Lie algebras. The latest reference to the first area is [12]. A number of references to the latest research in the second area can be found in [7]. One of the main goals in both areas is to "reduce" Lie maps or Lie gradings to the associative ones. The classical results about Lie maps deal with simple or prime rings but the latest achievements reflected in [12] include much wider classes of rings sufficient to settle some questions about graded Lie algebras.

Specifically, the situation in the theory of graded algebras is the following. Suppose a Lie algebra L over a field F is graded by a group G. This is well known [15] to be equivalent to L being a (right) H-comodule Lie algebra over the group algebra $H = FG$, that is, to the existence of a Lie homomorphism $\rho : L \to L \otimes H$ such that

$$(\rho \otimes \mathrm{id}_H)\rho = (\mathrm{id}_L \otimes \Delta)\rho \tag{1}$$

and

$$(\mathrm{id}_L \otimes \varepsilon)\rho = \mathrm{id}_L. \tag{2}$$

In the case of a graded algebra, ρ is determined by $\rho(a_g) = a_g \otimes g$ where a_g is a homogeneous element of degree g. Here Δ and ε are the coproduct and the counit of H, respectively. If L is a Lie subalgebra generating an associative algebra A and ρ extends to an associative homomorphism $\rho : A \to A \otimes H$ (with (1),(2) preserved!) then A also becomes G-graded. Since both L_g and A_g are defined as the sets of elements x in L and A satisfying $\rho(x) = x \otimes g$ we have $L_g = L \cap A_g$.

In what follows we will use techniques of [12] to show the existence of such extension under certain natural restrictions on L and G. Notice, however, that the natural extension of a Lie homomorphism is not an associative homomorphism but rather the direct sum of a homomorphism and the negative of an antihomomorphism. The grading theory counterpart of this situation is the so called *involution grading* on an associative algebra with involution. In the case of matrix algebras such gradings have been completely described in [7]. In what follows we will show that this approach works in a more general situation considered in this paper.

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The paper is organized as follows. In Section 2 we recall some definitions, fix the notation, and indicate the main idea upon which this paper is based. Then, in Sections 3 - 5, we apply the techniques of [12] to the study of Lie maps of tensor products. In the remaining Sections 6 - 8 we give some applications of the results obtained to the grading theory. Our fundamental results, Theorems 6.8, 7.1 and 8.4, show that under certain technical conditions a grading of a Lie algebra L is induced by an associative or an involution grading of an associative algebra A generated by L. These abstract theorems are then applied to more concrete situations. In particular, we obtain (modulo some technicalities) new proofs and improvements of existing results concerning classical Lie algebras.

2. Preliminaries

Let A be a not necessarily associative algebra over a field F, and let G be a group. We say that A is *graded by* G if $A = \sum_{g\in G} A_g$ and $A_gA_h \subset A_{gh}$, for any $g, h \in G$. An element $a \in A_g$ is called *homogeneous of degree* g and we write $\deg a = g$. A subspace M is called *graded* if $M = \sum_{g\in G}(M \cap A_g)$. The set $\operatorname{Supp} A = \{g \in G \,|\, A_g \neq 0\}$ is called the *support* of the grading. Let H be the group algebra $H = FG$. This is a Hopf algebra with coproduct $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$ and antipode $S(g) = g^{-1}$, for any $g \in G$. As mentioned above, A becomes a right H-comodule algebra with a structure homomorphism $\rho : A \to A \otimes H$. If A is a simple Lie algebra, it is shown in [16] that the elements in $\operatorname{Supp} A$ commute, which enables one to restrict oneself to the gradings by abelian groups.

Suppose A is an associative algebra and $A^{(-)}$ is the Lie algebra attached to A. As mentioned above, a grading by an abelian group G on $A^{(-)}$ is equivalent to a comodule mapping $\rho : A^{(-)} \to A^{(-)} \otimes H$, $H = FG$. We remark that $A^{(-)} \otimes H$ is actually equal to $(A \otimes H)^{(-)}$. Indeed, this follows from

$$[a\otimes h, b\otimes k] = (a\otimes h)(b\otimes k)-(b\otimes k)(a\otimes h) = (ab)\otimes(hk)-(ba)\otimes(kh) = [a, b]\otimes(hk).$$

Thus, ρ is a Lie homomorphism between associative algebras A and $A\otimes H$. Its range is a rather "small" subset of $A \otimes H$, which makes the results and the methods from [12] more or less inapplicable. However, ρ can be extended to a Lie automorphism $\widetilde{\rho}$ of the algebra $A \otimes H$, defined as follows:

$$\widetilde{\rho}(a \otimes h) = \sum_{g\in G} a_g \otimes (gh), \quad \text{where } a = \textstyle\sum_{g\in G} a_g \quad \text{and } a_g \in A_g.$$

Since A is spanned by the elements of A_g, it follows that $A \otimes H$ is spanned by $a_g \otimes h = \widetilde{\rho}(a_g \otimes (g^{-1}h))$, $g \in G$, $h \in H$. Thus $\widetilde{\rho}$ is surjective. Now, this makes the theory exposed in [12] applicable. Actually, the results are not directly applicable, but the methods are. More precisely, the approach based on the concept of the fractional degree works, as we shall see.

The previous paragraph reveals the main idea of our approach. It will be used not only for Lie algebras $A^{(-)}$, A being an associative algebra, but also for some other types of Lie subalgebras of associative algebras.

Let us fix some notation and terminology. From now on, by an "algebra" we mean an associative algebra over a fixed field F. Let A be an algebra. We set $A^\sharp = A$ if A is unital, and $A^\sharp$ is the algebra obtained by adjoining a unity to A if A is not unital. If X is a subset of A, then by $\langle X\rangle$ we denote the subalgebra of A generated by X. By Z_A we denote the center of A. If A is an algebra with involution $*$, then by $K(A, *)$ or simply K_A we denote the Lie algebra of skew symmetric elements of A. If V is a subspace of A stable under the involution, then we denote by $K(V,*)$ the subspace of skew-symmetric elements of V and by $H(V,*)$ the subspace of symmetric elements of V under $*$. We will be mostly concerned with prime and simple algebras. By a *central simple* algebra we will mean a simple algebra such that its centroid is F. If such an algebra is unital, then its center Z_A consists of scalar multiples of 1; otherwise $Z_A = 0$. Recall that the *extended centroid* of a prime F-algebra A can be defined as the center of the maximal left (or right) algebra of quotients of A. It is well-known that the extended centroid of A is a field containing F as a subfield (see e.g. [10]). We say that A is a *centrally closed* prime algebra if its extended centroid coincides with F. The notion of a centrally closed simple algebra coincides with the notion of a central simple algebra.

In some of our main results we will assume that the dimension of the algebra in question is big enough, i.e. greater than some concrete positive integer. Let us point out that we are not dealing with finite dimensional algebras only, so all such results hold for infinite dimensional algebras.

3. The fractional degree

The fractional degree of an element in a ring was introduced in [8] as an auxiliary notion, primarily needed for extending the existing results on functional identities from the prime ring to the semiprime ring setting. As we shall see, this notion is also suitable for the purposes of this paper.

We do not intend to discuss the fractional degree and related concepts in detail in this section; for this we refer to the original paper [8] or to [12, Section 5.1]. Our main goal is to establish a result on the fractional degree of elements in tensor products of algebras (Proposition 3.6).

We begin by recalling the main definitions. Let A be a subalgebra of an algebra Q. We say that an element $a \in A$ is *fractionable* in Q if the following two conditions hold:

(i) If $\varphi : A \to Q$ is an additive map such that $\varphi(xay) = ax\varphi(y)$ for all $x, y \in A$, then there exists $q \in Q$ such that $\varphi(x) = axq$ for all $x \in A$;
(ii) If $q \in Q$ is such that $qAa = 0$ or $aAq = 0$, then $q = 0$.

For example, if A is unital and the identity element of A is also the identity element of Q, then every invertible element in A is fractionable in Q. Indeed, (ii) is trivial, while (i) follows by taking $y = a^{-1}$. Let us also mention a nontrivial example: every nonzero element in a prime algebra A is fractionable in the maximal left algebra of quotients Q of A [12, Lemma 5.8].

By $\mathcal{M}(A)$ we denote the multiplication algebra of A, i.e. the algebra of linear operators on A of the form $x \mapsto \sum_i a_i x b_i$ where $a_i, b_i \in A$. These operators can be extended to $A^\sharp$ in the obvious way. We say that the *fractional degree* of an element $t \in A$ is greater than n (in Q), where $n \geq 0$, if for every $i = 0, 1, \ldots, n$ there exists $\mathcal{E}_i \in \mathcal{M}(A)$ such that

$$\mathcal{E}_i(t^j) = 0 \text{ if } j \neq i, \text{ and } \mathcal{E}_i(t^i) \text{ is fractionable in } Q$$

(here, of course, is should be understood that $t^0 = 1 \in A^\sharp$). We write this as $f\text{-}\deg_{A,Q}(t) > n$. Of course, we define that $f\text{-}\deg_{A,Q}(t) = n$ if $f\text{-}\deg_{A,Q}(t) > n-1$ but $f\text{-}\deg_{A,Q}(t) \not> n$. If $f\text{-}\deg_{A,Q}(t) > n$ for every positive integer n, then we write $f\text{-}\deg_{A,Q}(t) = \infty$.

We remark that the standard notation for the fractional degree is $f\text{-}\deg_Q(t)$ [12], but below we shall arrive at situations where it might not be entirely obvious which algebra plays the role of A. Therefore we have decided to expand the notation. On the other hand, [12] deals with the situation where t does not necessarily lie in A, but in the idealizer of A in Q. But we do not need this level of generality here.

We continue by recording two lemmas which will be easily derived from the results in [12].

Lemma 3.1. *Let A be a centrally closed prime algebra, and let Q be its maximal left algebra of quotients. Let L be a noncentral Lie ideal of A. If $\dim_F A \geq d^2$ (possibly ∞), then L contains an element a with $f\text{-}\deg_{A,Q}(a) \geq d$.*

Proof. By [12, Theorems C.1 and C.2] A contains elements such that their degree of algebraicity over F is $\geq d$. But then [12, Lemma C.5] tells us that L contains such elements as well. Now use [12, Lemma 5.10]. $\square$

Lemma 3.2. *Let A be a centrally closed prime algebra with involution, and let Q be its maximal left algebra of quotients. Let L be a noncentral Lie ideal of K_A. If $\operatorname{char} F \neq 2$, $d \geq 5$ and $\dim_F A \geq d^2$, then L contains an element a with $f\text{-}\deg_{A,Q}(a) \geq d$.*

Proof. Just follow the proof of Lemma 3.1, except that instead of [12, Lemma C.5] use [12, Lemma C.6]. $\square$

We now proceed with treating the fractional degree in tensor products.

Lemma 3.3. *Let $A \subseteq Q$ be arbitrary algebras, and let H be a finite dimensional unital algebra. If $a \in A$ is fractionable in Q, then $a \otimes 1 \in A \otimes H$ is fractionable in $Q \otimes H$.*

Proof. We first remark that for each $r \in Q \otimes H$ we have

$$(a \otimes 1)(A \otimes 1)r = 0 \Longrightarrow r = 0 \tag{3}$$

Indeed, just pick a basis $\{h_1, \ldots, h_n\}$ of the linear space H, write $r = r_1 \otimes h_1 + \ldots + r_n \otimes h_n$ where $r_i \in Q$, and note that $(a \otimes 1)(A \otimes 1)r = 0$ yields $aAr_i = 0$. As a is fractionable, each $r_i = 0$ and hence $r = 0$. Similarly we see that $r(A \otimes 1)(a \otimes 1) = 0$ implies $r = 0$. In particular, each of the conditions $r(A \otimes H)(a \otimes 1) = 0$ and $(a \otimes 1)(A \otimes H)r = 0$ yields $r = 0$, meaning that $a \otimes 1$ satisfies the conditon (ii) of the definition of the fractionability.

Now consider an additive map $\Phi : A \otimes H \to Q \otimes H$ such that

$$\Phi\Big((x \otimes h)(a \otimes 1)(y \otimes k)\Big) = (a \otimes 1)(x \otimes h)\Phi(y \otimes k) \tag{4}$$

for all $x, y \in A$, $h, k \in H$. We can write

$$\Phi(x \otimes 1) = \varphi_1(x) \otimes h_1 + \ldots + \varphi_n(x) \otimes h_n,$$

where $\varphi_1, \ldots, \varphi_n : A \to Q$ are additive maps. By (4) we have

$$\begin{aligned}
& \varphi_1(xay) \otimes h_1 + \ldots + \varphi_n(xay) \otimes h_n \\
= \;& \Phi(xay \otimes 1) = \Phi\Big((x \otimes 1)(a \otimes 1)(y \otimes 1)\Big) \\
= \;& (a \otimes 1)(x \otimes 1)\Phi(y \otimes 1) = (ax \otimes 1)\Big(\varphi_1(y) \otimes h_1 + \ldots + \varphi_n(y) \otimes h_n\Big) \\
= \;& ax\varphi_1(y) \otimes h_1 + \ldots + ax\varphi_n(y) \otimes h_n
\end{aligned}$$

for all $x, y \in A$. Therefore $\varphi_i(xay) = ax\varphi_i(y)$ for each i and all $x, y \in A$. Since a is fractionable in Q it follows that $\varphi_i(x) = axq_i$ for some $q_i \in Q$. Accordingly, $\Phi(x \otimes 1) = (a \otimes 1)(x \otimes 1)q$ where $q = q_1 \otimes h_1 + \ldots + q_n \otimes h_n$. Using this together with (4) we get

$$\begin{aligned}
&(a \otimes 1)(x \otimes 1)\Phi(y \otimes h) = \Phi\Big((x \otimes 1)(a \otimes 1)(y \otimes h)\Big)\\
=&\Phi\Big((x \otimes h)(a \otimes 1)(y \otimes 1)\Big) = (a \otimes 1)(x \otimes h)\Phi(y \otimes 1)\\
=&(a \otimes 1)(x \otimes h)(a \otimes 1)(y \otimes 1)q = (a \otimes 1)(x \otimes 1)(a \otimes 1)(y \otimes h)q.
\end{aligned}$$

Thus

$$(a \otimes 1)(x \otimes 1)\Big(\Phi(y \otimes h) - (a \otimes 1)(y \otimes h)q\Big) = 0,$$

and so (3) implies $\Phi(y \otimes h) = (a \otimes 1)(y \otimes h)q$. This proves that $a \otimes 1$ satisfies the condition (i) of the definition of the fractionability. □

The following example shows that the assumption that H is finite dimensional is really necessary.

Example 3.4. Let A be the algebra of all infinite matrices that have only finitely many nonzero entries. Pick a nonzero matrix $a \in A$, and let $Q \supseteq A$ be an algebra such that a is fractionable in Q (such algebras exist by [12, Lemma 5.8]). Further, let H be any infinite dimensional unital algebra. We claim that $a \otimes 1 \in A \otimes H$ *is not* fractionable in $Q \otimes H$. Indeed, pick a linearly independent subset $\{h_i \mid i = 1, 2, \ldots\}$ of H, and consider $\Phi : A \otimes H \to A \otimes H$ given by

$$\Phi(r) = \sum_{i=1}^{\infty} (a \otimes 1)r(e_{ii} \otimes h_i), \tag{5}$$

where e_{ii} is a matrix unit. Note that this is well-defined since for each r this sum is actually finite (namely, $r(e_{ii} \otimes h_i)$ can be nonzero only for finitely many i). It is clear that Φ satisfies $\Phi(r(a \otimes 1)s) = (a \otimes 1)r\Phi(s)$ for all $r, s \in A \otimes H$. Thus, if $a \otimes 1$ was fractionable in $Q \otimes H$, there would exist $q = \sum_{j=1}^{s} q_j \otimes k_j \in Q \otimes H$ such that

$$\Phi(r) = (a \otimes 1)rq = \sum_{j=1}^{s} (a \otimes 1)r(q_j \otimes k_j) \tag{6}$$

for all $r \in A \otimes H$. Now let n be such that h_n does not lie in the linear span of $k_1, \ldots, k_s$, and let m be such that $b = ae_{mn} \neq 0$ (such m exists since $a \neq 0$). By (5) we have $\Phi(e_{mn} \otimes 1) = b \otimes h_n$. On the other hand, from (6) we see that $\Phi(e_{mn} \otimes 1) = \sum_{j=1}^{s} (ae_{mn}q_j) \otimes k_j$. Thus

$$b \otimes h_n \in Q \otimes k_1 + \ldots + Q \otimes k_s,$$

which is clearly a contradiction. This shows that $a \otimes 1$ does not satisfy the condition (i).

Having in mind applications to graded algebras, we are primarily interested in the situation when A is a simple algebra and $H = FG$ is a group algebra. Note that the algebra A from Example 3.4 is simple. We shall be therefore forced to confine ourselves to finite groups G in some of our main applications. The next lemma will make it possible for us to avoid this confinement in the case of unital algebras.

Lemma 3.5. *Let A be a simple unital algebra, let $Q \supseteq A$ be any algebra having the same identity element as A, and let H be an arbitrary unital algebra. Then $a \otimes 1 \in A \otimes H$ is fractionable in $Q \otimes H$ for every nonzero $a \in A$.*

Proof. Let $\Phi : A \otimes H \to Q \otimes H$ be an additive map satisfying (4). Since A is simple and unital, we have $\sum_{i=1}^{n} x_i a y_i = 1$ for some $x_i, y_i \in A$. We have

$$\begin{aligned}\Phi(x \otimes h) &= \Phi\Big((x \otimes h)\big(\sum_{i=1}^{n} x_i a y_i \otimes 1\big)\Big)\\ &= \sum_{i=1}^{n} \Phi\Big((x \otimes h)(x_i \otimes 1)(a \otimes 1)(y_i \otimes 1)\Big)\\ &= \sum_{i=1}^{n} (a \otimes 1)(x \otimes h)(x_i \otimes 1)\Phi(y_i \otimes 1).\end{aligned}$$

Thus $\Phi(x \otimes h) = (a \otimes 1)(x \otimes h)q$ where $q = \sum_{i=1}^{n}(x_i \otimes 1)\Phi(y_i \otimes 1)$. This proves that the condition (i) is fulfilled. The condition (ii) follows from the (implicitly already established) fact that $1 \otimes 1$ lies in the ideal of $A \otimes H$ generated by $a \otimes 1$. $\square$

Let us mention that in order to handle only *unital* simple algebras we could avoid using the fractional degree, and deal with a (somewhat simpler) concept of the *strong degree* [8, 12] instead. Still, the fractional degree approach works in a number of more general instances.

Proposition 3.6. *Let $A \subseteq Q$ be arbitrary algebras, and H a unital algebra. Then*

$$f\text{-}\deg_{A\otimes H, Q\otimes H}(t \otimes 1) \geq f\text{-}\deg_{A,Q}(t)$$

holds for every $t \in A$, provided that one of the following two conditions holds:

(a) *H is finite dimensional;*
(b) *A is a simple unital algebra and Q has the same identity element as A.*

Proof. This is an immediate consequence of Lemmas 3.3 and 3.5. Indeed, one can pick appropriate elements in $\mathcal{M}(A \otimes 1)$ (which is contained in $\mathcal{M}(A \otimes H)$), and then use Lemmas 3.3 and 3.5. $\square$

4. Lie maps on Lie ideals of algebras

We say that a map σ from an algebra B into a unital algebra R is a *direct sum of a homomorphism and the negative of an antihomomorphism* if there exists an idempotent $\epsilon \in Z_R$ such that $x \mapsto \epsilon\sigma(x)$ is a homomorphism and $x \mapsto (1-\epsilon)\sigma(x)$ is the negative of an antihomomorphism.

Proposition 4.1. *Let R be a unital algebra and let S be its subalgebra such that the centralizer of S in R is equal to Z_R. Let M be a Lie ideal of some associative algebra, let N be a Lie ideal of S, and let $\rho : M \to N$ be a surjective Lie homomorphism. Suppose there exists $t \in N$ such that $f\text{-}\deg_{S,R}(t) \geq 8$. Then there exist a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : \langle M \rangle \to R$ and a linear map $\tau : M \to Z_R$ such that $\rho(x) = \sigma(x) + \tau(x)$ for all $x \in M$ and $\tau([M, M]) = 0$.*

Proof. By [12, Theorem 5.6] S is a $(t; 8)$-free subset of R. Since N is a Lie ideal of S and $t \in N$, we have $[t, S] \subseteq N$. Therefore [12, Corollary 3.18] tells us that N is a 7-free subset of R (we note that the condition that the degree of algebraicity of t over Z_R is not ≤ 2 holds automatically since $f\text{-}\deg_{S,R}(t) > 2$; cf. the remark following the proof of [12, Lemma 5.4]).

Let $\overline{R}$ be the quotient Lie algebra $R^{(-)}/Z_R$. For every $r \in R$ we set $\overline{r} = r + Z_R \in \overline{R}$. Define $\alpha : M \to \overline{R}$ by $\alpha(x) = \overline{\rho(x)}$. We claim that α satisfies all conditions of [12, Theorem 6.19]. Firstly, as a linear space Z_R is trivially a direct summand of the additive group R. Secondly, N is a 7-free subset of R and $\alpha(M) = \overline{N}$. Therefore [12, Theorem 6.19] yields the existence of a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : \langle M \rangle \to R$ such that $\alpha(x) = \overline{\sigma(x)}$ for all $x \in M$ (remark: since [12] deals with rings, and not algebras over F, applying the results from [12] formally yields additive maps instead of linear ones; however, from the proofs it is clear that σ is linear if ρ is). That is, $\overline{\rho(x)} = \overline{\sigma(x)}$ for every $x \in M$, so that $\tau(x) = \sigma(x) - \rho(x)$ lies in Z_R. Finally, since both ρ and σ are Lie homomorphisms, we have

$$\begin{aligned}\tau([x,y]) &= \sigma([x,y]) - \rho([x,y]) = [\sigma(x), \sigma(y)] - [\rho(x), \rho(y)] \\ &= [\rho(x) + \tau(x), \rho(y) + \tau(y)] - [\rho(x), \rho(y)] \\ &= 0\end{aligned}$$

for all $x, y \in M$. □

Since $\sigma(x) = \rho(x) - \tau(x)$ for $x \in M$, σ actually maps $\langle M \rangle$ into the subalgebra of R generated by S and Z_R. This will be tacitly used in the proof of the next corollary.

Theorem 4.2. *Let A be a centrally closed prime algebra and H a unital commutative algebra. Assume that either H is finite dimensional or A is central simple unital. Further, let M be a Lie ideal of some associative algebra, L a noncentral Lie ideal of A, and $\rho : M \to L \otimes H$ a surjective Lie homomorphism. If $\dim_F A \geq 64$, then there exist a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : \langle M \rangle \to A^\sharp \otimes H$ and a linear map $\tau : M \to 1 \otimes H$ such that $\rho(x) = \sigma(x) + \tau(x)$ for all $x \in M$ and $\tau([M, M]) = 0$.*

Proof. Let Q be the maximal left algebra of quotients of A. Set $R = Q \otimes H$, $S = A \otimes H$ and $N = L \otimes H$. Since H is commutative, N is a Lie ideal of S.

Suppose that $r \in R$ is such that $[r, S] = 0$. Writing r as $\sum_i q_i \otimes h_i$, where $q_i \in Q$ and the h_i's are linearly independent elements in H, it follows from $[r, A \otimes 1] = 0$ that $[q_i, A] = 0$ for every i. Since A is a centrally closed prime F-algebra, it follows that each q_i is a scalar multiple of 1 (see [10, Remark 2.3.1]). Therefore $r \in 1 \otimes H = Z_R$. We have thereby showed that the centralizer of S in R is Z_R.

By Lemma 3.1 there exists $a \in L$ with $f\text{-}\deg_{A,Q}(a) \geq 8$, and so $t = a \otimes 1 \in N$ satisfies $f\text{-}\deg_{S,R}(t) \geq 8$ by Proposition 3.6. Using Proposition 4.1 one easily infers the desired conclusion. □

Incidentally, we remark that if $L = A$, then we can replace 64 by 9. The proof is more or less the same (yet slightly easier), just that one has to apply [12, Theorem 6.1] instead of [12, Theorem 6.19] at an appropriate place.

The structure of Lie ideals can be quite complicated in general (see e.g. [13]), but not in simple algebras: every Lie ideal L of a simple algebra A over any field F with $\operatorname{char} F \neq 2$ is either central or it contains $[A, A]$ [14, Theorem 1.5]. Furthermore, $[A, A]$ is a simple Lie algebra provided that is has trivial intersection with Z_A. (The prototype example is $A = M_n(F)$: its only Lie ideals are A, 0, $Z_A = F1$

and $[A,A] = \mathfrak{sl}_n(F)$.) These are the reasons that when considering Lie ideals of a simple algebra A one usually restricts the attention to $[A,A]$.

Corollary 4.3. *Let A be a central simple algebra such that $\dim_F A \geq 64$. Let H be a unital commutative algebra. If A is not unital, then assume that H is finite dimensional. Then every surjective Lie homomorphism $\rho : [A,A] \otimes H \to [A,A] \otimes H$ can be extended to a direct sum of a homomorphism and the negative of an antihomomorphism $\sigma : A \otimes H \to A \otimes H$.*

Proof. A well-known Herstein's result says that $\langle [A,A] \rangle = A$ [14, Corollary, p. 9]. This readily implies that

$$\langle [A,A] \otimes H \rangle = A \otimes H.$$

Another Herstein's theorem says that $[[A,A],[A,A]] = [A,A]$ [14, Theorem 1.8], and this yields

$$[[A,A] \otimes H, [A,A] \otimes H] = [A,A] \otimes H.$$

These two facts together with Theorem 4.2 give the desired conclusion; namely, in our situation $M = [A,A] \otimes H$ and so σ is defined on $A \otimes H = \langle M \rangle$, and $[M,M] = M$ so that $\tau = 0$. □

Remark 4.4. Let us point out that the range of σ indeed lies in $A \otimes H$, even when A is not unital. This follows from the fact that $[A,A] \otimes H$ generates the algebra $A \otimes H$. However, the idempotent ϵ yielding the decomposition of σ to a sum of a homomorphism and the negative of an antihomomorphism may lie in $A^\sharp \otimes H$.

5. Lie maps on Lie ideals of skew elements

The proofs in this section are slightly more involved than those in the previous section, but conceptually they are the same. Therefore we will occasionally omit some details.

Proposition 5.1. *Let $\operatorname{char} F \neq 2$, R a unital algebra and S a subalgebra of R. Assume that S has an involution and that the centralizer of S in R is equal to Z_R. Further, let B be an arbitrary algebra with involution, M a Lie ideal of K_B, N a Lie ideal of K_S, and $\rho : M \to N$ a surjective Lie homomorphism. Suppose there exists $t \in N$ such that $f\text{-}\deg_{S,R}(t) \geq 21$. Then there exist a homomorphism $\sigma : \langle M \rangle \to R$ and a linear map $\tau : M \to Z_R$ such that $\rho(x) = \sigma(x) + \tau(x)$ for all $x \in M$ and $\tau([M,M]) = 0$.*

Proof. By [12, Theorem 5.6] S is a $(t;21)$-free subset of R. Accordingly, K_S is a $(t;10)$-free subset of R by [12, Theorem 3.28]. Since N is a Lie ideal of K_S and $t \in N$, we have $[t, K_S] \subseteq N$, and so N is a 9-free subset of R by [12, Corollary 3.18].

Let $\overline{R}$ and $\overline{r}$ have the same meaning as in the proof of Proposition 4.1. Define $\alpha : M \to \overline{R}$ by $\alpha(x) = \overline{\rho(x)}$. Clearly, N is a 9-free subset of R satisfying $\alpha(M) = \overline{N}$. Therefore [12, Theorem 6.18] implies that there is a homomorphism $\sigma : \langle M \rangle \to R$ such that $\alpha(x) = \overline{\sigma(x)}$ for all $x \in M$. The rest of the proof is the same as the final part of the proof of Proposition 4.1. □

Theorem 5.2. *Let $\operatorname{char} F \neq 2$, A a centrally closed prime algebra with involution and L a noncentral Lie ideal of K_A. Let also B be an arbitrary algebra with involution and M a Lie ideal of K_B. Further, let H be a unital commutative algebra. Assume that either H is finite dimensional or A is a central simple unital algebra. If $\rho : M \to L \otimes H$ is a surjective Lie homomorphism and $\dim_F A \geq 441$, then there* *exists a homomorphism* $\sigma : \langle M\rangle \to A^\sharp \otimes H$ *and a linear map* $\tau : M \to 1 \otimes H$ *such that* $\rho(x) = \sigma(x) + \tau(x)$ *for all* $x \in M$ *and* $\tau([M, M]) = 0$.

Proof. As in the proof of Theorem 4.2, we set $R = Q \otimes H$, where Q is the maximal left algebra of quotients of A, $S = A \otimes H$ and $N = L \otimes H$. The proof of Theorem 4.2 then shows that the centralizer of S in R is $Z_R = 1 \otimes H$.

By Lemma 3.2 there exists $a \in L$ with $f\text{-}\deg_{A,Q}(a) \geq 21$. Therefore $t = a \otimes 1 \in N$ has $f\text{-}\deg_{S,R}(t) \geq 21$ by Proposition 3.6. Define an involution on S by $(x \otimes h)^* = x^* \otimes h$, and note that $L \otimes H$ is a Lie ideal of $K_S = K_A \otimes H$. Now use Proposition 5.1. □

A Lie ideal of K_A that is of special importance is $[K_A, K_A]$. Namely, if A is a simple algebra, then under certain mild conditions $[K_A, K_A]$ is a simple Lie algebra, see [14, Theorem 2.15].

Corollary 5.3. *Let* $\operatorname{char} F \neq 2$ *and let* A *be a central simple algebra such that* $\dim_F A \geq 441$. *Suppose that* A *has an involution and set* $K = K_A$. *Let* H *be a unital commutative algebra. If* A *is not unital, assume that* H *is finite dimensional. Then every surjective Lie homomorphism* $\rho : [K, K] \otimes H \to [K, K] \otimes H$ *can be extended to a homomorphism* $\sigma : A \otimes H \to A \otimes H$.

Proof. Herstein's theorem says that $\langle [K, K]\rangle = A$ [14, Theorem 2.13]. This yields

$$\langle [K, K] \otimes H\rangle = A \otimes H.$$

From Herstein's theory of Lie ideals of skew elements (see e.g. [14, Theorem 2.15]), one can derive that $[[K, K], [K, K]] = [K, K]$, and this implies

$$[[K, K] \otimes H, [K, K] \otimes H] = [K, K] \otimes H.$$

Applying Theorem 5.2 one easily completes the proof. □

6. Applications to graded algebras: Lie ideals

Let A be an associative algebra, L a Lie ideal of A such that $A = \langle L\rangle$ and G an abelian group. In this section we consider the possibility of a group grading of L being induced from a group grading of A. As suggested by the techniques of the previous sections, we have to assume that G finite if A is not simple and unital. We already know that the grading by an abelian group G on L is completely equivalent to a comodule map $\rho : L \to L \otimes H$, $H = FG$, which can be viewed as a Lie homomorphism from L to $A \otimes H$. We also know that ρ extends to a Lie homomorphism from $L\otimes H$ to itself which we denote by the same letter: $\rho(a\otimes h) = \rho(a)(1\otimes h)$. Now $L\otimes H$ is a Lie ideal of $A\otimes H$ and $A\otimes H = \langle L\otimes H\rangle$. As mentioned in Section 2, $\rho : L \otimes H \to L \otimes H$ is surjective. Now one can apply the results of Section 4 allowing to extend ρ to a map $\widetilde{\rho} : A\otimes H \to A\otimes H$ with certain properties depending on the properties of A and L.

If A is centrally closed prime with $\dim A \geq 64$ then by Theorem 4.2 there exists $\sigma : A \otimes H \to A \otimes H$ which is the sum of a homomorphism and a negative of an antihomomorphism and a linear map $\tau : L \otimes H \to 1 \otimes H$ with $\tau([L, L]) = 0$ such that for any $x \in L$ one has $\rho(x) = \sigma(x) + \tau(x)$. If, additionally, we assume $L = [L, L]$ then $\rho(x) = \sigma(x)$, for any $x \in L$. So in this case we may set $\widetilde{\rho} = \sigma$ to obtain an extension of ρ which is the sum of a homomorphism and the negative of an antihomomorphism. If A is central simple and $L = [A, A]$ then we can use Corollary 4.3 to derive the existence of such $\widetilde{\rho}$.

In what follows we would like to examine the properties of the restriction map $\widetilde{\rho} : A \to A \otimes H$ extending the Lie homomorphism $\rho : L \to L \otimes H$ in a more general setting where H is an arbitrary commutative bialgebra. In this case all we know is the validity of the axioms (1) and (2) for ρ.

Proposition 6.1. *Let A be a centrally closed prime algebra with $\dim_F A > 4$, L a noncentral Lie ideal of A such that $\langle L \rangle = A$. Suppose that L is a Lie H-comodule algebra for a commutative bialgebra H, and suppose that an H-comodule map $\rho : L \to L \otimes H$ extends to a direct sum of a homomorphism and the negative of an antihomomorphism $\widetilde{\rho} : A \to A \otimes H$. Then $(\mathrm{id}_A \otimes \varepsilon)\widetilde{\rho} = \mathrm{id}_A$. In particular, $\widetilde{\rho}$ is not the negative of an antihomomorphism. Further, suppose that $e \in H$ is an idempotent such that $a \mapsto (1 \otimes e)\widetilde{\rho}(a)$ is a homomorphism and $a \mapsto (1 \otimes f)\widetilde{\rho}(a)$, where $f = 1 - e$, is the negative of a homomorphism, and suppose that $\Delta(e) = e \otimes e + f \otimes f$ and $\Delta(f) = e \otimes f + f \otimes e$. Then $(\widetilde{\rho} \otimes \mathrm{id}_H)\widetilde{\rho} = (\mathrm{id}_A \otimes \Delta)\widetilde{\rho}$.*

Proof. We have $\widetilde{\rho} = \widetilde{\varphi} - \widetilde{\psi}$, where $\widetilde{\varphi} : A \to A \otimes H$ is a homomorphism, $\widetilde{\psi} : A \to A \otimes H$ is an antihomomorphism, and $\widetilde{\varphi}(A \otimes H)\widetilde{\psi}(A \otimes H) = \widetilde{\psi}(A \otimes H)\widetilde{\varphi}(A \otimes H) = 0$. Let us set $\rho' = (\mathrm{id}_A \otimes \varepsilon)\widetilde{\rho}$, $\varphi = (\mathrm{id}_A \otimes \varepsilon)\widetilde{\varphi}$, and $\psi = (\mathrm{id}_A \otimes \varepsilon)\widetilde{\psi}$. Since $\mathrm{id}_A \otimes \varepsilon$ is an algebra homomorphism from $A \otimes H$ into A, we have that ρ', ϕ, ψ are linear maps from A into A, $\rho' = \varphi - \psi$, φ is a homomorphism, ψ is an antihomomorphism, and $\varphi(A)\psi(A) = \psi(A)\varphi(A) = 0$. But then

$$\varphi(A)\rho'(A)\psi(A) = \varphi(A)\big((\varphi - \psi)(A)\big)\psi(A) = 0.$$

By the axiom (2) of the comodule, ρ' acts as the identity on L. Hence we have

$$\varphi(A)L\psi(A) = \varphi(A)\rho'(L)\psi(A) = 0.$$

By [11, Lemma 4] it follows that either $\varphi = 0$ or $\psi = 0$. The case $\varphi = 0$ will be dealt with in the following lemma, which we will also need in the future (Section 8).

Lemma 6.2. *Let A be a centrally closed prime algebra with $\dim_F A > 4$, and let L be a noncentral Lie ideal of A such that $\langle L \rangle = A$. Then the identity map of L cannot be induced from the negative of an antihomomorphism of A.*

Proof. To keep our notation closer to the preceding argument, we assume that ρ' is the negative of an antihomomorphism which induces th identity map on L. In this case, for $x \in L$ and $a \in A$ we will have $[x, a] = \rho'([x, a]) = -[\rho'(a), \rho'(x)] = [x, \rho'(a)]$, so that $[L, \rho'(a) - a] = 0$. Since $\langle L \rangle = A$ by assumption, this yields $[A, \rho'(a) - a] = 0$, and hence $\mu(a) = \rho'(a) - a \in Z_A$ for each $a \in A$. If also $b \in A$ then

$$ab + \mu(ab) = \rho'(ab) = -\rho'(b)\rho'(a) = -(b + \mu(b))(a + \mu(a)),$$

that is,

$$(a + \mu(a))b + (b + \mu(b))a = -\mu(ab) - \mu(a)\mu(b) \in Z_A.$$

This means that ρ' satisfies the condition

$$\rho'(a)b + \rho'(b)a \in Z_A$$

for all $a, b \in A$. We are now in a position to use the theory of functional identities. Since A is a 3-free subset of the maximal left algebra of quotients of A [12, Corollary 5.12, Theorem C.2], it follows by the very definition of 3-freeness that $\rho' = 0$ - a contradiction. □

In particular, this shows that $\widetilde{\varphi} \neq 0$ and hence $\widetilde{\rho}$ cannot be the negative of a homomorphism.

Therefore $\psi = 0$ and $\rho' = \varphi$ is a homomorphism. Since ρ' is the identity on L and $\langle L \rangle = A$, it follows that $\rho' = \mathrm{id}_A$. We have thereby proved that $(\mathrm{id}_A \otimes \varepsilon)\widetilde{\rho} = \mathrm{id}_A$.

Assume now that $e \in H$ is an idempotent such that $\widetilde{\phi}(a) = (1 \otimes e)\widetilde{\rho}(a)$ and $\widetilde{\psi}(a) = -(1 \otimes f)\widetilde{\rho}(a)$, $f = 1 - e$, and suppose that $\Delta(e) = e \otimes e + f \otimes f$, $\Delta(f) = e \otimes f + f \otimes e$. We want to prove that $(\widetilde{\rho} \otimes \mathrm{id}_H)\widetilde{\rho} = (\mathrm{id}_A \otimes \Delta)\widetilde{\rho}$.

We have

$$(\widetilde{\rho} \otimes \mathrm{id}_H)\widetilde{\rho} = (\widetilde{\varphi} \otimes \mathrm{id}_H)\widetilde{\varphi} - (\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\varphi} - (\widetilde{\varphi} \otimes \mathrm{id}_H)\widetilde{\psi} + (\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\psi}. \tag{7}$$

We will now consider separately each of the four terms on the right-hand side. Let us do this in detail for, say, the second term; the others can be considered similarly.

Let $a_1, \dots, a_n \in L$. We will compute $\big((\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\varphi}\big)(a_1 \dots a_n)$. Since $\widetilde{\varphi}$ is a homomorphism and $\widetilde{\psi}$ is an antihomomorphism, we have

$$\begin{aligned} &\big((\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\varphi}\big)(a_1 \dots a_n) = \big(\widetilde{\psi} \otimes \mathrm{id}_H\big)\big(\widetilde{\varphi}(a_1) \dots \widetilde{\varphi}(a_n)\big) \\ =&\Big(\big(\widetilde{\psi} \otimes \mathrm{id}_H\big)(\widetilde{\varphi}(a_n))\Big) \dots \Big(\big(\widetilde{\psi} \otimes \mathrm{id}_H\big)(\widetilde{\varphi}(a_1))\Big). \end{aligned} \tag{8}$$

One can check that

$$\big(\widetilde{\psi} \otimes \mathrm{id}_H\big)(\widetilde{\varphi}(a)) = -(1 \otimes f \otimes e)\Big(\big(\widetilde{\rho} \otimes \mathrm{id}_H\big)(\widetilde{\rho}(a))\Big)$$

for every $a \in L$. Of course, $\widetilde{\rho}$ coincides with ρ on L, and so in view of (1) we have

$$\big(\widetilde{\psi} \otimes \mathrm{id}_H\big)(\widetilde{\varphi}(a)) = -(1 \otimes f \otimes e)\Big(\big(\mathrm{id}_A \otimes \Delta\big)(\rho(a))\Big).$$

Therefore it follows from (8), together with the fact that $\mathrm{id}_A \otimes \Delta$ is an algebra homomorphism, that

$$\begin{aligned} &\big((\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\varphi}\big)(a_1 \dots a_n) = (-1)^n(1 \otimes f \otimes e)\Big((\mathrm{id}_A \otimes \Delta)\big(\rho(a_n) \dots \rho(a_1)\big)\Big) \\ =&(-1)^n(1 \otimes f \otimes e)\Big((\mathrm{id}_A \otimes \Delta)\big((1 \otimes e)\widetilde{\rho}(a_n \dots a_1) + (-1)^{n-1}(1 \otimes f)\widetilde{\rho}(a_1 \dots a_n)\big)\Big) \\ =&(-1)^n(1 \otimes f \otimes e)\big(1 \otimes \Delta(e)\big)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_n \dots a_1))\Big) \\ &- (1 \otimes f \otimes e)\big(1 \otimes \Delta(f)\big)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_1 \dots a_n))\Big). \end{aligned}$$

Using $\Delta(e) = e \otimes e + f \otimes f$ and $\Delta(f) = e \otimes f + f \otimes e$ it obviously follows that

$$\big((\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\varphi}\big)(a_1 \dots a_n) = -(1 \otimes f \otimes e)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_1 \dots a_n))\Big).$$

Following the same pattern one shows

$$\begin{aligned} \big((\widetilde{\varphi} \otimes \mathrm{id}_H)\widetilde{\varphi}\big)(a_1 \dots a_n) =&(1 \otimes e \otimes e)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_1 \dots a_n))\Big), \\ \big((\widetilde{\varphi} \otimes \mathrm{id}_H)\widetilde{\psi}\big)(a_1 \dots a_n) =&- (1 \otimes e \otimes f)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_1 \dots a_n))\Big), \\ \big((\widetilde{\psi} \otimes \mathrm{id}_H)\widetilde{\psi}\big)(a_1 \dots a_n) =&(1 \otimes f \otimes f)\Big((\mathrm{id}_A \otimes \Delta)(\widetilde{\rho}(a_1 \dots a_n))\Big). \end{aligned}$$

Since

$$1 \otimes e \otimes e + 1 \otimes e \otimes f + 1 \otimes f \otimes e + 1 \otimes f \otimes f = 1 \otimes 1 \otimes 1$$

it now follows from (7) that $(\widetilde{\rho} \otimes \mathrm{id}_H)\widetilde{\rho}$ agrees with $(\mathrm{id}_A \otimes \Delta)\widetilde{\rho}$ on all elements of the form $a_1 \dots a_n$ where $a_i \in L$. Since $\langle L \rangle = A$ this means that $(\widetilde{\rho} \otimes \mathrm{id}_H)\widetilde{\rho}$ and $(\mathrm{id}_A \otimes \Delta)\widetilde{\rho}$ agree on A. $\square$

Let us show by a simple example that the condition $\dim_F A > 4$ is not redundant.

Example 6.3. Let $A = \mathbb{H}$ be the algebra of quaternions. Then $[A, A]$ is the linear span of i, j, and k. The map $\widetilde{\rho}(x) = -\overline{x}$, where $\overline{x}$ denotes the conjugate of x, is the negative of an antihomomorphism of A, which, however, acts as the identity on $[A, A]$.

Example 6.4. Another example works over any field F. Let $A = M_2(F)$ be the matrix algebra of order 2. Consider $L = \mathfrak{sl}_2(F) = [A, A]$. Let $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then the map $\widetilde{\rho}(X) = -S^{-1}X^tS$ where X^t is the transpose of a matrix $X \in M_2$ is the negative of an antiautomorphism on A and $\rho = \widetilde{\rho}|_L$ acts as the identity map on L.

Proposition 6.1 shows that $\widetilde{\rho}$ satisfies both axioms (1) and (2), provided some conditions are satisfied. However, $\widetilde{\rho}$ is not always a homomorphism. Let us give an example, the simplest one of its kind, when it is not. The relevance of this example will become clear in Section 8.

Example 6.5. Let A be an algebra with involution, let $L = A^{(-)}$, and let $G = \mathbb{Z}_2$. Thus, we have $G = \{1, t\}$ and $t^2 = 1$. Suppose that $\operatorname{char} F \neq 2$. Setting $L_1 = K(A, *)$ and $L_t = H(A, *)$ we see that L becomes graded by G. Thus, $\widetilde{\rho} = \rho$ is given by $\widetilde{\rho}(a_1 + a_t) = a_1 \otimes 1 + a_t \otimes t$. Note that $\epsilon = 1 \otimes \frac{1+t}{2}$ is a nontrivial central idempotent in $A \otimes H$, $H = FG$, such that $a \mapsto \epsilon\widetilde{\rho}(a)$ is a homomorphism and $a \mapsto (1 - \epsilon)\widetilde{\rho}(a)$ is the negative of an antihomomorphism. In particular, $\widetilde{\rho}$ is not a homomorphism.

The assumptions concerning e in the last part of Proposition 6.1 might seem somewhat artificial. However, we shall see later that they actually occur in a situation in which we will be interested in. Anyway, the following simple result tells us that the first assumption that an idempotent yielding the decomposition of $\widetilde{\rho}$ is of the form $1 \otimes e$ is in fact automatically fulfilled.

Proposition 6.6. *Let A and H be algebras with $Z_{A^\sharp} = F$. Then all central idempotents of $A^\sharp \otimes H$ are of the form of $1 \otimes e$, e a central idempotent of H.*

Proof. Note that if $\epsilon = \sum_{i=1}^n r_i \otimes h_i$ is central, with $h_1, \ldots, h_n$ linearly independent, then each r_i is also central, that is, a scalar multiple of the identity element 1 of $A^\sharp$. Then $\epsilon = 1 \otimes e$, $e \in Z_H$ and $\epsilon^2 = \epsilon$ immediately gives $e^2 = e$, as claimed. □

In the rest of this section we restrict ourselves to the case where central idempotents are trivial. We need an easy result.

Proposition 6.7. *Let A be an algebra with $Z_{A^\sharp} = F$, and let $H = FG$ be the group algebra of an abelian group G. We define a subgroup T_1 of G as follows. Let T be the subgroup of elements of G of finite order. In the case* $\operatorname{char} F = 0$ *we set $T_1 = T$. In the case* $\operatorname{char} F = p > 0$ *we define T_1 as the subgroup of T consisting of all elements whose order is coprime to p. Then any central idempotent of $A^\sharp \otimes H$ is of the form $1 \otimes e$ where the idempotent e lies in the group algebra of T_1.*

Proof. Without loss of generality we may assume that $A = A^\sharp$. Any idempotent is an element of $A \otimes FG'$ where G' is a finitely generated subgroup of G. This allows us to assume from the very beginning that G is finitely generated. Let us decompose H as $H = H_0 \otimes K$ according to the group decomposition $G = G_0 \times T$

where G_0 is torsion-free and T is torsion. If F has characteristic $p > 0$ then also $T = T_0 \times T_1$ where T_0 is a Sylow p-subgroup and T_1 has no elements of order p. Then $H = H_0 \otimes K_0 \otimes K_1$ where H_0, K, K_0, and K_1 are the group algebras of G_0, T, T_0, and T_1, respectively. We will assume that $H = H_0 \otimes K_0 \otimes K_1$ for any characteristic, having in mind that each of the factors may be trivial. Let u be an idempotent of $A \otimes H$. If we prove our claim after an extension of the field then it is true even before. So we may assume F algebraically closed. In this case K_1 is the direct sum of the (mutually orthogonal) copies of F each spanned by some e_i, $i = 1, \dots, n$, for some n. If ϵ is a central idempotent in $A \otimes H$, then $\epsilon = \epsilon_1 + \cdots + \epsilon_n$ where now $\epsilon_i = \epsilon(1 \otimes e_i)$ may be viewed as idempotents in $A \otimes He_i \cong A \otimes (H_0 \otimes K_0)$.

First we show that the central idempotents of $R = A \otimes H_0$ are trivial. Using induction by the rank of G_0, we can assume G_0 being infinite cyclic with generator x. Let u be a central idempotent in R. Then $u = (a_0 \otimes 1 + a_1 \otimes x + \cdots + a_m \otimes x^m)(1 \otimes x^{-k})$, for some non-negative m, k. Here $a_0, \dots, a_m$ are central elements of A. Since $u^2 = u$, we immediately obtain $k = 0$ and then $a_0^2 = a_0$, that is, $a_0 = 1$ by our assumption on A. Comparing the x-powers, we obtain $a_1 = 0$, then $a_2 = 0$, etc.

Now we have to show that for any R without nontrivial central idempotents the central idempotents of $R \otimes K_0$ are trivial. It is well-known that the largest nilpotent ideal N of the group algebra K_0 of a p-group T_0 over a field F of characteristic $p > 0$ coincides with the augmentation ideal so that $K_0 = F \oplus N$ where N is a nilpotent ideal. Then we can write a central idempotent u as $u = a_0 \otimes 1 + a_1 \otimes v_1 + \cdots a_m \otimes v_m$ where $a_i \neq 0$ are central elements in $R \otimes K_0$ while v_j are linearly independent elements, a part of a filtered basis of N. Considering $u^2 = u$ we easily obtain $a_0^2 = a_0$ or, by our assumption, $a_0^2 = 1$. We will also have

$$0 = a_1 \otimes v_1 + \cdots + a_m \otimes v_m + \sum_{i,j=1}^{m} a_i a_j \otimes v_i v_j.$$

If v_1 is the element with the least filtration then we must have $a_1 = 0$, which is a contradiction. So, we must have $u = 1 \otimes 1$ in $A \otimes (H_0 \otimes K_0)$, considering that we have no nontrivial central idempotents in $R = A \otimes H_0$.

Accordingly, each ϵ_i is either 0 or $1 \otimes e_i$. This readily implies that $\epsilon = 1 \otimes e$ where e is an idempotent in H. □

Some easy but important consequences of this result are as follows. We recall that a Lie algebra is called *perfect* if $L = [L, L]$.

Theorem 6.8. *Let a perfect Lie algebra L be a Lie ideal generating a centrally closed prime algebra A over a field F, $\dim_F A \geq 64$, and G an abelian group which is finite if A is not simple and unital. Suppose G has no periodic elements, if* $\operatorname{char} F = 0$ *or no periodic elements whose order is coprime to p if* $\operatorname{char} F = p > 0$. *If L is graded by G, then there is an associative grading $A = \bigoplus_{g\in G} A_g$ such that $L_g = A_g \cap L$, for all $g \in G$. If, additionally, A is simple and $L = [A, A]$ then every Lie grading of L is induced from an associative grading of A.*

Proof. We use Proposition 6.7 to obtain that any central idempotent of $A^\sharp \otimes H$ is trivial. Given a Lie grading of L we consider the comodule map $\rho : L \to L \otimes H$. As pointed out at the beginning of the section, Theorem 4.2 makes it possible for us to extend ρ to the sum of a homomorphism and the negative of an antihomomorphism $\widetilde{\rho} : A \to A \otimes H$. The central idempotent ϵ yielding the decomposition of $\widetilde{\rho}$ lies in

$A^\sharp \otimes H$ (see Remark 4.4). However, Proposition 6.7 tells us that $A^\sharp \otimes H$ has only trivial central idempotents, so that $\epsilon = 0$ or $\epsilon = 1$. The case $\epsilon = 0$ is impossible since $\widetilde{\rho}$ is not the negative of an antihomomorphism by Proposition 6.1. Thus $\epsilon = 1$ and so $\widetilde{\rho}$ is a homomorphism. This, together with $\langle L \rangle = A$, implies that $\widetilde{\rho}$ also satisfies the axioms (1) and (2). Now since both L_g and A_g are defined in the same way, using ρ, say, $A_g = \{x \,|\, \rho(x) = x \otimes g\}$, we obviously obtain $L_g = L \cap A_g$, for all $g \in G$.

The last claim easily follows if we apply Corollary 4.3. □

Now let $A = M_n(F)$ be a matrix algebra of order n over a field F, $\overline{A} = M_n(\overline{F})$ where $\overline{F}$ is the algebraic closure of F. Any grading of A by a group G naturally induces a grading of $\overline{A}$ by G if one sets $\overline{A}_g = A_g \otimes \overline{F}$. We say that a grading of the matrix algebra $A = M_n(F)$ is *elementary* if there is an n-tuple of elements $(g_1, \ldots, g_n) \in G^n$ such that for a certain choice of matrix units E_{ij} of $\overline{A}$ one has $\overline{A}_g = \mathrm{Span}\,\{E_{ij} \mid g = g_i^{-1} g_j\}$.

Theorem 6.9. *Let $L = \mathfrak{sl}_n(F)$ be graded by an abelian p-group G. If $\mathrm{char} F = p$ and $n \geq 8$ then any grading of L is induced from an (elementary) grading of $A = M_n(F)$.*

Proof. By [5] all gradings of a matrix algebra over a field of characteristic $p > 0$ by a finite abelian p-group are elementary. Applying Theorem 6.8 we easily derive our result. □

The following result is proved in [3] by entirely different methods.

Theorem 6.10. *Let $A = M_n(F)$, $\mathrm{char}\, F = p > 0$, $p \neq 2$ and $p \not| \, n$. Let G be a finite abelian p-group and $L = \mathfrak{sl}_n(F)$. Suppose $L = \sum_{g \in G} L_g$ is a grading on L. Then there exists an elementary grading $A = \sum_{g \in G} A_g$ such that $L_g = A_g \cap L$.*

Thus, in [3] the restriction that $n \geq 8$ is not required; such restrictions typically appear when functional identities are used. On the other hand, the advantage of the approach used in this paper is that we do not need the assumption that $p \not| \, n$.

7. Applications to graded algebras: Lie algebras of skew-symmetric elements

Let A be an algebra with involution $*$, let $\mathrm{char}\, F \neq 2$, let K_A be the Lie algebra of skew-symmetric elements under $*$, and set $L = [K_A, K_A]$. Again, given a grading of L by an abelian group G, we have a comodule Lie homomorphism $\rho : L \to L \otimes H$. As before, we can extend ρ to a surjective Lie homomorphism $\widetilde{\rho} : L \otimes H \to L \otimes H$. Applying Theorem 5.2 as above we can easily prove the following.

Theorem 7.1. *Let A be a centrally closed prime algebra with involution. Assume that $\mathrm{char}\, F \neq 2$ and $\dim_F(A) \geq 441$. Let L a noncentral Lie ideal of K_A such that $\langle L \rangle = A$ and L is a perfect Lie algebra. Let L be graded by an abelian group G, which should be assumed finite if A is not simple and unital. Then there is an associative grading $A = \bigoplus_{g \in G} A_g$ such that $L_g = A_g \cap L$, for all $g \in G$.*

An important special case is $L = [K_A, K_A]$ with A simple (see Corollary 5.3).

Theorem 7.2. *Let A be a central simple algebra with involution. Assume that $\mathrm{char}\, F \neq 2$ and $\dim_F(A) \geq 441$. Let $L = [K_A, K_A]$ be graded by an abelian group G, which should be assumed finite if A is not unital. Then there is an associative grading $A = \bigoplus_{g \in G} A_g$ such that $L_g = A_g \cap L$, for all $g \in G$.*

A direct consequence is the following theorem, which is an extension of results in [4] and [7].

Theorem 7.3. *Let $L = \mathfrak{so}(n)$ with $n \geq 21$, or $L = \mathfrak{sp}(2m)$ with $m \geq 11$, G be a finite abelian group G, and let* $\operatorname{char} F \neq 2$. *Then the G-gradings of L are induced from the involutions gradings of $M_n(F)$.*

The involution gradings of $M_n(F)$ are completely described in [7] provided that $\operatorname{char} F \neq 2$ and F has "sufficiently many" roots of 1 (for example, when F is algebraically closed).

There is no technical restriction on n in [4] in the case of zero characteristic.

The following theorem is proved in [3] by entirely different methods.

Theorem 7.4. *Let L be one of $\mathfrak{so}(n)$, $n \geq 5$, $n \neq 8$, and $\mathfrak{sp}(n)$, $n \geq 6$, n even, where F is an algebraically closed field,* $\operatorname{char} F \neq 2$. *Let G be a finite abelian group. Then any G-grading on L is the restriction of a G-grading of $M_n(F)$. Moreover, if G is a p-group then any G-grading on L is the restriction of an elementary G-grading of $M_n(F)$.*

8. Applications to graded algebras: two types of gradings on $[A, A]$

In this section we consider the case where a Lie grading on $L = [A, A]$ is given by an abelian group G with periodic elements over a field F of characteristic 0 or with periodic elements of order coprime to $p = \operatorname{char} F$. As noted at the beginning of Section 6 and in Proposition 6.7, the comodule Lie homomorphism $\rho : L \to L \otimes H$ is induced by a map $\widetilde{\rho} : A \to A \otimes H$ which is the the sum of an associative homomorphism $a \to (1 \otimes e)\widetilde{\rho}(a)$ and the negative of an antihomomorphism $a \to (1 \otimes f)\widetilde{\rho}(a)$ with e and $f = 1 - e$ being central idempotents of the group algebra $K_1 = FT_1$ where, in the same way as in Proposition 6.7, T_1 is the subgroup of all periodic elements of G in the case where $\operatorname{char} F = 0$ or of all periodic elements of G of order coprime to p if $p = \operatorname{char} F$. Note that A is not assumed to be unital, so that $1 \otimes e, 1 \otimes f \in A^\sharp \otimes H$.

To study the precise form of the idempotents e and f we will temporarily assume that F is algebraically closed. In this case, $K_1 = Fe_1 \oplus \cdots \oplus Fe_m$ where $e_1, \ldots, e_m$ are pairwise orthogonal indecomposable idempotents of K_1, $m = \dim K_1 = |T_1|$. Also, it is well-known [1] that in this case $T_1 \cong \widehat{T_1}$ where $\widehat{T_1}$ is the group of multiplicative characters $\chi : T_1 \to F^*$. Also, the idempotents in the above decomposition of K_1 take the form of

$$e_\chi = \frac{1}{m} \sum_{t \in T_1} \chi(t)^{-1} t. \tag{9}$$

Note that if $(\psi \mid \chi)_{T_1}$ stands for the scalar product of the characters ψ and χ of a group T_1 then

$$\chi(e_\psi) = \frac{1}{m} \sum_{t \in T_1} \psi(t)^{-1} \chi(t) = (\psi \mid \chi)_{T_1} = 1 \text{ if } \psi = \chi \text{ and } 0 \text{ otherwise}. \tag{10}$$

One more important formula, a direct consequence of (9), is this. For any $t \in T_1$ and $\chi \in \widehat{T_1}$ one has $te_\chi = \chi(t)e_\chi$. Further, given a subset Ω of $\widehat{T_1}$, one can define an idempotent $e_\Omega = \sum_{\chi \in \Omega} e_\chi$ in FT_1, and conversely, every idempotent in FT_1

is of such form. In particular, there is $\Lambda \subset \widehat{T_1}$ such that $e = e_\Lambda$. Note that $f = 1 - e = e_{\Lambda'}$ where $\Lambda \cup \Lambda' = \widehat{T_1}$.

Before we formulate our next result we introduce the group $\overline{\operatorname{Aut}} A$ of automorphisms and the negatives of antiautomorphisms of the algebra A. This has a subgroup of automorphisms $\operatorname{Aut} A$ of index at most 2.

Now given $\chi \in \widehat{T_1}$ we set $\alpha(\chi) = (\mathrm{id}_A \otimes \chi)\rho$. For example, if $\rho(x) = x \otimes g$ then $\alpha(\chi)(x) = \chi(g)x$. Our goal will be to prove the following.

Proposition 8.1. *Let A be a centrally closed prime algebra with $\dim_F A > 4$, and L a noncentral Lie ideal of A such that $\langle L\rangle = A$. Let L be G-graded, for an abelian group G and let $\rho : L \to L \otimes H$ be the Lie comodule map, where H is the group algebra of G. Adopting the notation and assumptions preceding this proposition, we write $e = e_\Lambda$ and $f = e_{\Lambda'}$ for the central idempotents of H such that given the extension $\widetilde{\rho} : A \to A \otimes H$ we have that $a \to (1 \otimes e)\widetilde{\rho}(a)$ is an associative homomorphism while $a \to (1 \otimes f)\widetilde{\rho}(a)$ is the negative of an antihomomorphism. Then the mapping $\alpha : \widehat{T_1} \to \overline{\operatorname{Aut}} A$ is a group homomorphism and hence Λ is a subgroup of index at most 2 in $\widehat{T_1}$.*

Proof. Let us assume $\chi \in \widehat{T_1}$ and $x, x' \in A$. We will first show that if $\chi \in \Lambda$ then $\alpha(\chi)$ is a homomorphism while if $\chi \notin \Lambda$ then $\alpha(\chi)$ is the negative of an antihomomorphism. Indeed, we have

$$\begin{aligned}
\alpha(\chi)(xx') &= (\mathrm{id}_A \otimes \chi)(\rho(xx')(1 \otimes e_\Lambda) + \rho(xx')(1 \otimes e_{\Lambda'}))\\
&= (\mathrm{id}_A \otimes \chi)(\rho(x)(1 \otimes e_\Lambda)\rho(x')(1 \otimes e_\Lambda) - \rho(x')(1 \otimes e_{\Lambda'})\rho(x)(1 \otimes e_{\Lambda'}))\\
&= (\mathrm{id}_A \otimes \chi)(\rho(x))(\mathrm{id}_A \otimes \chi)(\rho(x'))(\mathrm{id}_A \otimes \chi)(1 \otimes e_\Lambda)\\
&- (\mathrm{id}_A \otimes \chi)(\rho(x'))(\mathrm{id}_A \otimes \chi)(\rho(x))(\mathrm{id}_A \otimes \chi)(1 \otimes e_{\Lambda'})\\
&= \alpha(\chi)(x)\alpha(\chi)(x') \sum_{\lambda \in \Lambda} \chi(e_\lambda) - \alpha(\chi)(x')\alpha(\chi)(x) \sum_{\mu \in \Lambda'} \chi(e_\mu).
\end{aligned}$$

If $\chi \in \Lambda$ then only one term, in the first summand of the latter term survives, and we have $\alpha(\chi)(xx') = \alpha(\chi)(x)\alpha(\chi)(x')$, that is, $\alpha(\chi)$ is a homomorphism. Otherwise, only one term of the second summand survives and one has $\alpha(\chi)(xx') = -\alpha(\chi)(x')\alpha(\chi)(x)$, that is, $\alpha(\chi)$ is the negative of a antihomomorphism.

Now we would like to show that $\alpha(\chi\psi) = \alpha(\psi)\alpha(\chi)$ for any two $\chi, \psi \in \widehat{T_1}$. By the comodule axiom (2) the elements x with $\rho(x) = x \otimes g$ span L. The restriction $\bar{\alpha}(\chi)$ of each $\alpha(\chi)$ to L is a Lie algebra automorphism. For any x as just above we have

$$\alpha(\chi\psi)(x) = (\chi\psi)(g)x = \chi(g)\psi(g)x = \alpha(\psi)(\alpha(\chi)(x)).$$

Thus $\bar{\alpha} : \widehat{T_1} \to \operatorname{Aut} L$ is a group homomorphism.

Let us show that each $\alpha(\chi)$ is bijective. We will prove that $\alpha(\chi^{-1})$ is an inverse of $\alpha(\chi)$. Indeed, for any $x \in L$, we have

$$(\alpha(\chi)\alpha(\chi^{-1}))(x) = \alpha(\chi)(\alpha(\chi^{-1})(x)) = \bar{\alpha}(\chi)(\bar{\alpha}(\chi^{-1})(x)) = \bar{\alpha}(\chi\chi^{-1})(x) = x.$$

Thus $\alpha(\chi)\alpha(\chi^{-1})$ is a homomorphism or the negative of a homomorphism of A extending the identity map of L. By Lemma 6.2 this cannot be an antihomomorphism. But since $A = \langle L\rangle$ we must have $\alpha(\chi)\alpha(\chi^{-1}) = \mathrm{id}_A$, as claimed.

Now choose any $\chi, \psi \in \widehat{T_1}$. Then all the mappings $\alpha(\chi)$, $\alpha(\psi)$ and $\alpha(\chi\psi)$ are bijective which allows us to form the product $\alpha(\chi)\alpha(\psi)\alpha(\chi\psi)^{-1}$. The same calculation as before shows that this map is identical on L. Again, we must have $\alpha(\chi)\alpha(\psi) = \alpha(\chi\psi)$. Thus α is a group homomorphism from $\widehat{T_1}$ to $\overline{\mathrm{Aut}}\, A$. $\square$

If we do not impose restrictions on the field of coefficients F then we obtain the following.

Proposition 8.2. *Let A be a centrally closed prime algebra with $\dim_F A > 4$, and L a noncentral Lie ideal of A such that $\langle L\rangle = A$. Let L be G-graded, for an abelian group G and $\rho : L \to L \otimes H$ the Lie comodule map, where H is the group algebra of G. Suppose e and f are the nontrivial central idempotents of H such that for an extension $\widetilde{\rho} : A \to A \otimes H$ we have that $a \to (1 \otimes e)\widetilde{\rho}(a)$ is an associative homomorphism while $a \to (1 \otimes f)\widetilde{\rho}(a)$ is the negative of an antihomomorphism. Then $t = e - f$ is an element of G of order 2.*

Proof. Let us assume, for the time being, that our field is algebraically closed. By [6, Proposition 5.2] we have $T_1 = \langle a\rangle \times T_1'$, $\widehat{T_1} = \langle \chi_0\rangle \times \Delta$ where the order of a and χ_0 is 2^k, for some $k > 0$, $\Lambda = \langle \chi_0^2\rangle \times \Delta$. We also have $\chi_0(T_1') = 1$ and $\delta(a) = 1$, for all $\delta \in \Delta$. Now each $\chi \in \widehat{T_1}$ is of the form $\chi = \mu\delta$ where $\mu(T_1') = 1$ and $\delta \in \Delta$. In this case the idempotent e_χ can be transformed as follows:

$$\begin{aligned} e_\chi &= \frac{1}{m}\sum_{t\in T_1}\chi(t)^{-1}t = \frac{1}{m}\sum_{u\in\langle a_0\rangle, v\in T_1'}\mu\delta(uv)^{-1}uv \\ &= \left(\frac{1}{2^k}\sum_{u\in\langle a_0\rangle}\mu(u)^{-1}u\right)\left(\frac{1}{|T_1'|}\sum_{v\in T_1'}\delta(v)^{-1}v\right). \end{aligned}$$

If we fix μ with $\mu(T_1') = 1$ then the sum of all $e_{\mu\delta}$ with $\delta \in \Delta$ by the previous calculation will be equal to the idempotent e_μ of the group algebra $F\langle a_0\rangle$ because the remaining factor

$$\sum_{\delta\in\Delta}\frac{1}{|T_1'|}\sum_{v\in T_1'}\delta(v)^{-1}v$$

equals 1 as the sum of all indecomposable idempotents of the group algebra FT_1'. Now each term of either e_Λ or $e_{\Lambda'}$ is of that form, which allows us to restrict to the case where T_1' is trivial. So we need an explicit computation only in the case where T_1 is a 2-group generated by a single element a of order $m = 2^k$. We have $\chi_0(a) = \xi^{-1}$, where ξ is a primitive 2^kth root of 1. Further, we have

$$e = \sum_{i=0}^{2^{k-1}-1} e_{\chi_0^{2i}} \text{ and } f = \sum_{i=0}^{2^{k-1}-1} e_{\chi_0^{2i+1}}.$$

To compute $e - f$, we need to use (9), which we will rewrite as follows:

$$e_{\chi_0^s} = \frac{1}{2^k}\sum_{r=0}^{2^k-1}\xi^{sr}a^r.$$

Let ζ_r be a primitive $2r$th root of 1. Then one can write $e - f$ as follows

$$e - f = \sum_{r=0}^{2^k-1}\left(\sum_{s=0}^{2^k-1}(\xi\zeta_r)^{sr}\right)a^r. \tag{11}$$

Now
$$\sum_{s=0}^{2^k-1}(\xi\zeta_r)^{sr}=\begin{cases}2^k \text{ if } (\xi\zeta_r)^r=1\\ \frac{(\xi\zeta_r)^{r2^k}-1}{(\xi\zeta_r)^r-1}=0 \text{ otherwise}\end{cases}$$
Thus a^r enters the right hand side of (11) with nonzero coefficient only if $\xi^r(-1)=(\xi\zeta_r)^r=1$, that is $\xi^r=-1$. Obviously, then we must have $r=2^{k-1}$ and $e-f=a^{2^{k-1}}$, a group element of order 2, as claimed.

At this time we can go back to the original field F because both e and f are defined over F. From what we have proved, it also follows that $\Delta(t)=t\otimes t$ and so t is a group-like element, hence $t\in G$. Obviously, $o(t)=2$. □

Corollary 8.3. *Under the same conditions, as in Proposition 8.2, one has $\Delta(e)=e\otimes e+f\otimes f$ and $\Delta(f)=e\otimes f+f\otimes e$. Therefore, in this case, the axioms (1) and (2) always hold for the extension map $\widetilde{\rho}:A\to A\otimes H$.*

Proof. Applying Δ to both sides of the equations $e+f=1$ and $e-f=t$ and considering that t is an element of a group, we obtain $\Delta(e)+\Delta(f)=1\otimes 1=(e+f)\otimes(e+f)$, $\Delta(e)-\Delta(f)=t\otimes t=(e-f)\otimes(e-f)$. Adding and subtracting the sides of these equations, we easily obtain the desired. For the last claim use Proposition 6.1. □

As shown by example earlier, the map $\widetilde{\rho}:A\to A\otimes H$ we have obtained before cannot serve as the comodule map making A into a G-graded algebra. So we have to make an additional assumption that A has an involution compatible with $\widetilde{\rho}$. Namely, if A has an involution $*$ then we can extend it to $A\otimes H$ by setting $(a\otimes h)^*=a^*\otimes h$, for any $a\in A$ and $h\in H$. Then we require the following

(i) $\widetilde{\rho}(x^*)=\widetilde{\rho}(x)^*$, for any $x\in A$;

(ii) $L^*=L$.

Notice that (ii) will be automatically satisfied in the case $L=[A,A]$. Both conditions are satisfied when $A=M_n(F)$ and $L=\mathfrak{sl}_n(F)$ (see Theorem 8.6)

From these conditions it follows easily that $(L_g)^*=L_g$, for any $g\in G$. Indeed, for any $x\in L_g$ one has $x^*\in L$ and

$$\rho(x^*)=\widetilde{\rho}(x^*)=(\widetilde{\rho}(x))^*=(\rho(x))^*=(x\otimes g)^*=x^*\otimes g,$$

as needed.

Thus each L_g splits into the sum of the space of symmetric and skew-symmetric elements so that L is spanned by homogeneous symmetric and skew-symmetric elements.

We set
$$\rho^*(x)=\widetilde{\rho}(x)(1\otimes e)-\widetilde{\rho}(x^*)(1\otimes f).$$
Next we check that ρ^* is now an associative homomorphism. Given $x,y\in A$, we have

$$\begin{aligned}\rho^*(xy) &= \widetilde{\rho}(xy)(1\otimes e)-\widetilde{\rho}((xy)^*)(1\otimes f)\\ &= (\widetilde{\rho}(x)(1\otimes e))(\widetilde{\rho}(y)(1\otimes e))-\widetilde{\rho}(y^*x^*)(1\otimes f)\\ &= (\widetilde{\rho}(x)(1\otimes e))(\widetilde{\rho}(y)(1\otimes e))+(\widetilde{\rho}(x^*)(1\otimes f))(\widetilde{\rho}(y^*)(1\otimes f))\\ &= (\widetilde{\rho}(x)(1\otimes e)-\widetilde{\rho}(x^*)(1\otimes f))(\widetilde{\rho}(y)(1\otimes e)-\widetilde{\rho}(y^*)(1\otimes f))\\ &= \rho^*(x)\rho^*(y).\end{aligned}$$

Now for any skew-symmetric $x \in L_g$ one has

$$\begin{aligned}\rho^*(x) &= \rho(x)(1\otimes e) - \rho(x^*)(1\otimes f)\\ &= \rho(x)(1\otimes e) - (\rho(x)(1\otimes f))^*\\ &= (x\otimes g)(1\otimes e) - ((x\otimes g)(1\otimes f))^*\\ &= (x\otimes g)(1\otimes e) - (x^*\otimes g)(1\otimes f)\\ &= (x\otimes g)(1\otimes e) + (x\otimes g)(1\otimes f)) = (x\otimes g)(1\otimes(e+f)) = x\otimes g.\end{aligned}$$

Similar computation in the case where x is symmetric gives

$$\begin{aligned}\rho^*(x) &= \rho(x)(1\otimes e) - \rho(x^*)(1\otimes f)\\ &= (x\otimes g)(1\otimes e) - (x^*\otimes g)(1\otimes f)\\ &= (x\otimes g)(1\otimes e) - (x\otimes g)(1\otimes f) = (x\otimes g)(1\otimes(e-f)) = x\otimes(gt).\end{aligned}$$

On skew-symmetric and symmetric elements of L, therefore, the conditions (1) and (2) are satisfied. For instance, both sides of (1) on a symmetric element x of L_g will give $x\otimes(gt)\otimes(gt)$. Checking (2) is even simpler. Now since all maps on both sides of (1) and (2) for ρ^* are associative homomorphisms, it follows that both conditions are satisfied on the whole of A provided that $A = \langle L\rangle$. In this case ρ^* induces a G-grading on A. From the above arguments we have the following result.

Theorem 8.4. *Let a perfect Lie algebra L be a Lie ideal generating a centrally closed prime algebra A, $\dim_F A \geq 64$, L given a grading $L = \bigoplus_{g\in G} L_g$ by an abelian group G, and $\operatorname{char} F \neq 2$. Additionally we assume that G is finite if A is not simple and unital. Let the following condition be satisfied. If we do not have $L_g = A_g \cap L$ for an associative grading of A, then we have an involution $*$ on A satisfying* (i) *and* (ii) *above. Then any grading of L is of one of two types*

(1) *There exists an associative G-grading of A such that $L_g = A_g \cap L$, for each $g \in G$.*
(2) *There exist an element t of order 2 in G, an involution $*$ on A and an involution grading $A = \bigoplus_{g\in G} A_g$ such that*

$$L_g = K(A_g,*)\cap L \oplus H(A_{gt},*)\cap L.$$

The same conclusion holds if A is a central simple algebra and $L = [A,A]$ where we need only to require (i) *satisfied.*

Proof. First of all, under these conditions, ρ extends to $\widetilde{\rho}$ which is the sum of a homomorphism and the negative of an antihomomorphism. As previously, $\widetilde{\rho}$ cannot be an antihomomorphism. If $\widetilde{\rho}$ is a homomorphism then $\widetilde{\rho}$ makes A into a G-graded algebra and the first case occurs.

Otherwise, by our assumption, there is an involution satisfying (i) and (ii). Then the map ρ^*, as shown just before the statement of this theorem, makes A into a G-graded algebra. We set $t = e - f$. By Proposition 8.2 t is an element of order 2 in G. The computation preceding the theorem shows that the skew-symmetric elements in L_g have degree g in the grading of A induced by ρ^* while the symmetric elements have degree gt. This proves that, indeed, $L_g = K(A_g,*)\cap L \oplus H(A_{gt},*)\cap L$, for each $g \in G$.

Checking that ρ^* is compatible with $*$ is a simple exercise, which we leave to the reader. □

In conclusion, we will consider two important cases where the conditions of the above theorem hold.

Theorem 8.5. *Let a perfect Lie algebra L be a Lie ideal generating a centrally closed prime algebra A, $\dim_F A \geq 64$, L given a grading $L = \bigoplus_{g\in G} L_g$ by an abelian group G, and* $\operatorname{char} F \neq 2$. *Suppose that G is an abelian group such that the order of each 2-element is actually 2, and that G is finite if A is not unital and simple. If $L = [A, A]$ is G-graded, then one of the two cases occur:*

(1) *There exists an associative G-grading of A such that $L_g = A_g \cap L$, for each $g \in G$.*
(2) *There exist an element t of order 2 in G and an involution $*$ on A such that $L_g = K(A_g, *) \cap L \oplus H(A_{gt}, *) \cap L$.*

The same conclusion holds if A is a central simple algebra and $L = [A, A]$.

Proof. Using the same argument as before, we may assume that the associative extension $\widetilde{\rho} : A \to A \otimes H$ is an actual sum of a homomorphism and the negative of an antihomomorphism. This can be done because otherwise we would have the first case in our theorem. Extending the field of coefficients to the algebraically closed $\bar{F}$, and applying Proposition 8.2, we will find the subgroup $\Lambda \subset \widehat{T_1}$ of index 2. Therefore, there is an element $\chi \notin \Lambda$. By our assumption $o(\chi) = 2q$ where q is odd. In this case χ^q is still outside of Λ, and we may assume that from the very beginning we have χ of order 2 being outside of Λ. By the argument preceding Theorem 8.4 we should have that $\alpha(\chi)$ is the negative of an involution on A. Thus we may set $x^* = -\alpha(\chi)(x)$. Notice that because $o(\chi) = 2$ our involution is defined over the original field F. To apply Theorem 8.4, we only need to check that $\widetilde{\rho}(a^*) = \widetilde{\rho}(a)^*$ and this is enough to do for the homogeneous elements of L. If $a \in L_g$ then

$$\begin{aligned} \widetilde{\rho}(a^*) &= \widetilde{\rho}(-\alpha(\chi)(a)) = -\chi(g)a \otimes g = (-\alpha(\chi)(a) \otimes 1)(a \otimes g) \\ &= (-\alpha(\chi) \otimes 1)\widetilde{\rho}(a) = \widetilde{\rho}(a)^*, \end{aligned}$$

as claimed. □

Finally, we apply our results to the classical Lie algebra $L = \mathfrak{sl}_n(F)$.

Theorem 8.6. *Let $L = \mathfrak{sl}_n(F)$ be given a grading $L = \bigoplus_{g\in G} L_g$ by a finite abelian group G, F an algebraically closed field of characteristic $\neq 2$, $n \geq 8$. Set $A = M_n(F)$. Then one of the two cases occur:*

(1) *$L_g = A_g$ is an associative grading of $M_n(F)$*
(2) *There is an involution grading $A = \bigoplus_{g\in G} A_g$ on $A = M_n(F)$ and an element t of order 2 in G such that for all $g \in G$ one has $L_g = K(A_g, *) \oplus H(A_{gt}, *) \cap L$.*

Proof. The existence of the involution in the hypotheses of Theorem 8.4 is proven in [6, Theorem 5.5, Proposition 6.4]. □

Remark 8.7. (i) The difference between the conclusion in Theorem 8.5 and 8.6 is explained by the fact that in $L = \mathfrak{sl}_n(F)$ we always have $K(A, *) \subset L$.
(ii) Theorem 8.6 was proved in [6] in the case of the fields of characteristic zero. There is no technical restriction of $n \geq 8$ in [6] in that paper.
(iii) The following theorem was proved by attracting the techniques of formal groups in [3].

Theorem 8.8. *Let $L = \mathfrak{sl}_n(F)$ where F is an algebraically closed field of prime characteristic $p \neq 2$ such that $p \nmid n$. Let G be a finite abelian group. Then any G-grading on L is either of type (1) or of type (2) (as described above).*

(iv) The following theorem was proved in [2] by a nontrivial adaptation of the proof of [9, Theorem 3.3].

Theorem 8.9. *Let $R = M_n(F)$, $n \neq 2$, where F is an algebraically closed field of prime characteristic $p \neq 2$, and, in the case $n = 3$, also $p \neq 3$. Let $Z = [R, R] \cap Z(R)$ and $L = [R, R]/Z$. Let G be a finite abelian group. Then any G-grading on L is either of type (1) or of type (2) above. Moreover, if G is a p-group then any G-grading on L is of type (1), i.e., is induced by an elementary G-grading of R.*

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