NEAR-DERIVATIONS IN LIE ALGEBRAS

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ABSTRACT. Let L be a Lie algebra. We call a linear map $f : L \to L$ a near-derivation if there exists a linear map $g : L \to L$ such that $(\operatorname{ad} x)f - g(\operatorname{ad} x)$ is a derivation for every $x \in L$. The paper is devoted to describing the structure of near-derivations in certain Lie algebras arising from associative ones.

1. INTRODUCTION

Let R be a nonassociative algebra. Recall that a linear map $\delta : R \to R$ is said to be a *derivation* if $\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y)$ for all $x, y \in R$. As it is well-known, $[\delta', \delta] = \delta'\delta - \delta\delta'$ is a derivation whenever δ and δ' are derivations. Is it possible to determine all linear maps $f : R \to R$ with the property that $[f, \delta]$ is a derivation whenever δ is a derivation? Besides derivations, the other obvious examples are scalar multiples of the identity. Can f be expressed through these basic examples, or are there some different ones?

We find this question a natural one, and therefore interesting in its own right. The work on this paper actually begun by addressing ourselves to this question. Searching the literature we have observed that the question is connected to the theory of generalized derivations on Lie algebras developed by Leger and Luks [5]. We have therefore appropriately reformulated and extended the question (see below), and restricted ourselves, in this paper, to the case where $R = L$ is a Lie algebra. One might of course consider the above question in other types of algebras R.

According to [5], a linear map $f : L \to L$ is a *generalized derivation* if there exist linear maps $g, h : L \to L$ such that

$$[f(x), y] = g([x, y]) - [x, h(y)] \quad \text{for all } x, y \in L.$$

Basic examples of generalized derivations are derivations (i.e., $f = g = h$) and maps from the centroid of L (i.e., $f = g$, $h = 0$). Leger and Luks have determined the form of generalized derivations on various finite-dimensional Lie algebras. In particular, under favorable conditions they have showed that generalized derivations can be expressed as sums of derivations and maps from the centroid of L. Moreover, the centroid sometimes consists only of scalar multiples of the identity. Let us point out that maps with the

range in the center of L are also generalized derivations (take $g = h = 0$). However, [5] mostly deals with Lie algebras with trivial center.

By $\operatorname{ad} x$ we denote the inner derivation induced by $x \in L$, i.e. $(\operatorname{ad} x)(y) = [x, y]$. We shall say that a linear map $f : L \to L$ is a *near-derivation* of L if there exists a linear map $g : L \to L$ such that $(\operatorname{ad} x)f - g(\operatorname{ad} x)$ is a derivation for every $x \in L$. Note that this is slightly more general than the concept of a generalized derivation. In fact, if f is a generalized derivation, then $(\operatorname{ad} x)f - g(\operatorname{ad} x)$ is an inner derivation for every $x \in L$. On the other hand, the problem of describing near-derivations is clearly more general than the problem mentioned in the first paragraph.

The main purpose of this paper is to describe near-derivations in certain Lie algebras that arise from associative ones. Unlike in [5], we are mostly interested in infinite dimensional algebras. Therefore this paper has only a small overlap with [5]. On the other hand, our conclusions on near-derivations are similar to those obtained by Leger and Luks for generalized derivations. Our typical result states that a near-derivation f of L is of the form $f = \delta + \gamma I + \tau$, where δ is a derivation, γ is an element in the center C of a certain associative algebra containing L (by γI we mean the map given by $x \mapsto \gamma x$), and τ is a linear map from L into C. Results of this kind will be proved in Section 3. The proofs rest heavily on the theory of functional identities [2]. In Section 2 we shall therefore give a brief fragmentary review of this theory.

Throughout the paper, $\mathbb{F}$ will denote a field with $\operatorname{char}(\mathbb{F}) \neq 2$. By an algebra, either Lie or associative, we shall always mean an algebra over $\mathbb{F}$.

2. FUNCTIONAL IDENTITIES PRELIMINARIES

The main concepts of the theory of functional identities are rather technical and so it does not seem appropriate to introduce them precisely in this short paper. We shall mention only a few facts which should make it possible for a non-specialist to follow the paper superficially. For a full account of the theory we refer the reader to the recent book [2].

Let Q be a unital associative ring with center C, and let S be a subset of Q. Let $x_1, \ldots, x_m \in S$. Given $1 \leq i \leq m$ we write

$$\overline{x}_m^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in S^{m-1} = S \times \ldots \times S,$$

and given $1 \leq 1 \leq i < j \leq m$ we write

$$\overline{x}_m^{ij} = \overline{x}_m^{ji} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \in S^{m-2}.$$

Roughly speaking, a functional identity on S is an identical relation holding for all elements in S which involves some (unknown) functions. Just for an illustration we mention one of the basic examples of such identities:

$$(1) \qquad \sum_{i=1}^{n} E_i(\overline{x}_m^i)x_i + \sum_{j=1}^{n} x_j F_j(\overline{x}_m^j) = 0 \quad \text{for all } x_i \in S.$$

Here, E_i and F_j are functions from S^{m-1} into Q. The goal is to describe the form of these functions, or, when this is not possible, to determine the structure of the ring admitting this identity. We say that S is a d-free subset of Q, where d is a positive integer, if certain functional identities (including such as (1)) in a certain number of variables (connected to d) have only "obvious" solutions, meaning that the involved functions are of the form which yields the identity in question for entirely formal reasons. In case of (1) this means that there exist functions $p_{ij} : S^{n-2} \to Q$, $i \neq j$, and $\lambda_i : S^{n-1} \to C$ such that

$$E_i(\overline{x}_m^i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j p_{ij}(\overline{x}_m^{ij}) + \lambda_i(\overline{x}_m^i), \quad F_j(\overline{x}_m^j) = -\sum_{\substack{i=1 \\ i \neq j}}^{n} p_{ij}(\overline{x}_m^{ij})x_i - \lambda_j(\overline{x}_m^j).$$

For the exact definition of d-freeness we refer the reader to [2]. For our purposes it is mainly important that there exist relevant examples of Lie algebras which are d-free subsets of associative algebras. They are described in Remark 2.1 below. But first we have to introduce some notation and recall a few facts.

Let A be a prime associative algebra. By $Q_{ml}(A)$ we denote the maximal left ring of quotients of A (see e.g. [2, Appendix A]). The center C of $Q_{ml}(A)$ is a field called the extended centroid of A. By $\deg(x)$ we denote the degree of the algebraicity of $x \in A$ over C. If x is not algebraic, then we write $\deg(x) = \infty$. Further, we set $\deg(A) = \sup\{\deg(x) \,|\, x \in A\}$. The condition that $\deg(A) = \infty$ is equivalent to the condition that A is not a PI-algebra, while the condition that $\deg(A) = n < \infty$ is equivalent to the condition that A is a PI-algebra satisfying the standard polynomial identity of degree $2n$, but not satisfying a polynomial identity of degree $< 2n$. If A is a central simple algebra (this means that A is simple and its center is equal to $\mathbb{F}1$), then $\deg(A) = \infty$ is the same as saying that A is infinite dimensional over $\mathbb{F}$, while $\deg(A) = n < \infty$ is equivalent to $\dim_{\mathbb{F}} A = n^2$. See [2, Appendix C] for more details.

Remark 2.1. Let A be a prime algebra.

- If A $\deg(A) \geq d + 1$, every noncommutative Lie ideal L of A is a d-free subset of $Q_{ml}(A)$ [2, Corollary 5.16];
- If A $\deg(A) \geq 2d + 3$, A has an involution and K is the set of skew elements in A, then every noncentral Lie ideal L of K is a d-free subset of $Q_{ml}(A)$ [2, Corollary 5.19].

(Recall that an element in an algebra x with involution is said to be skew if $x^* = -x$).

Let us emphasize that the case when $\deg(A) = \infty$ is not excluded in Remark 2.1. The same holds for results in the next section.

When considering near-derivations we will arrive at a special functional identity, which is examined in the next lemma.

Lemma 2.2. *Let $B : S \times S \to Q$ be a skew-symmetric map. Suppose that*

$$[B(x,y),z] + [B(z,x),y] + [B(y,z),x] \in C \quad \text{for all } x,y,z \in S.$$

If S is a 4-free subset of Q, then there exist $\lambda \in C$ and a skew-symmetric map $\nu : S \times S \to C$ such that $B(x,y) = \lambda[x,y] + \nu(x,y)$ for all $x,y \in S$.

Proof. First of all, from the definition of 4-freeness we infer that

$$(2) \quad [B(x,y),z] + [B(z,x),y] + [B(y,z),x] = 0 \quad \text{for all } x,y,z \in S.$$

Using [2, Theorem 4.13] it follows that B is a quasi-polynomial. This means that there exist $\lambda_1, \lambda_2 \in C$ and maps $\mu_1, \mu_2 : S \to C$, $\nu : S^2 \to C$ such that

$$B(x,y) = \lambda_1 xy + \lambda_2 yx + \mu_1(x)y + \mu_2(y)x + \nu(x,y).$$

Since $B(x,y) = -B(y,x)$ it follows that

$$(\lambda_1 + \lambda_2)(xy + yx) + (\mu_1 - \mu_2)(x)y - (\mu_1 - \mu_2)(y)x$$
$$+\nu(x,y) - \nu(y,x) = 0.$$

But then $\lambda_1 = -\lambda_2$, $\mu_1 = \mu_2$ and ν is skew-symmetric [2, Lemma 4.4]. Setting $\lambda = \lambda_1$ and $\nu = \nu_1$ we thus have

$$B(x,y) = \lambda[x,y] + \mu(x)y + \mu(y)x + \nu(x,y).$$

Using this expression back in (2) it follows that

$$2\mu(x)[y,z] + 2\mu(y)[z,x] + 2\mu(z)[x,y] = 0.$$

Again applying [2, Lemma 4.4] it follows that $2\mu(x) = 0$, and hence $\mu(x) = 0$ since $\text{char}(\mathbb{F}) \neq 2$ by assumption. □

3. Near-derivations

We begin with a crucial lemma, from which all other results will be derived.

Lemma 3.1. *Let L be a Lie algebra and let f be a near-derivation of L. Suppose there exists a unital associative algebra Q, containing L as its Lie subalgebra, such that L is a 4-free subset of Q. Then there exist $\gamma \in C$, the center of Q, and a skew-symmetric bilinear map $\beta : L \times L \to C$ such that*

$$(f + \gamma I)([x,y]) = [f(x),y] + [x,f(y)] + \beta(x,y) \quad \text{for all } x,y \in L.$$

Proof. Our assumption is that there exists a linear map $g : L \to L$ such that for every $x \in L$, the map $y \mapsto [x,f(y)] - g([x,y])$ is a derivation. This means that

$$(3) \quad \begin{aligned} &[x,f([y,z])] - g([x,[y,z]]) \\ =&[[x,f(y)] - g([x,y]),z] + [y,[x,f(z)] - g([x,z])] \end{aligned}$$

for all $x,y,z \in L$. In view of the Jacobi identity we have

$$g([x,[y,z]]) + g([z,[x,y]]) + g([y,[z,x]]) = 0;$$

according to (3) this can be rewritten as

$$[x, f([y, z])] - [[x, f(y)], z] + [g([x, y]), z] - [y, [x, f(z)]] + [y, g([x, z])]$$
$$+[z, f([x, y])] - [[z, f(x)], y] + [g([z, x]), y] - [x, [z, f(y)]] + [x, g([z, y])]$$
$$+[y, f([z, x])] - [[y, f(z)], x] + [g([y, z]), x] - [z, [y, f(x)]] + [z, g([y, x])]$$
$$=0.$$

Rearranging the terms we get

$$\Big[(2g - f)([x, y]) - [f(x), y] - [x, f(y)], \, z\Big]$$
$$+\Big[(2g - f)([z, x]) - [f(z), x] - [z, f(x)], \, y\Big]$$
$$+\Big[(2g - f)([y, z]) - [f(y), z] - [y, f(z)], \, x\Big] = 0$$

for all $x, y, z \in L$. We are now in a position to apply Lemma 2.2. Thus there exist $\lambda \in C$ and a skew-symmetric map $\nu : S \times S \to C$ such that

$$(2g - f)([x, y]) - [f(x), y] - [x, f(y)] = \lambda[x, y] + \nu(x, y)$$

for all $x, y \in L$. Thus, the map $h = 2g - f - \lambda I : L \to CL \subseteq Q$ satisfies

$$h([x, y]) = [f(x), y] + [x, f(y)] + \nu(x, y).$$

Since

$$h([x, [y, z]]) + h([z, [x, y]]) + h([y, [z, x]]) = 0$$

by the Jacobi identity, it follows that

$$[f(x), [y, z]] + [x, f([y, z])] + \nu(x, [y, z])$$
$$+[f(z), [x, y]] + [z, f([x, y])] + \nu(z, [x, y])$$
$$+[f(y), [z, x]] + [y, f([z, x])] + \nu(y, [z, x])$$
$$=0.$$

Note that we can rewrite this as

$$\Big[f([y, z]) - [f(y), z] - [y, f(z)], x\Big]$$
$$+\Big[f([x, y]) - [f(x), y] - [x, f(y)], z\Big]$$
$$+\Big[f([z, x]) - [f(z), x] - [z, f(x)], y\Big]$$
$$=\nu(x, [y, z]) + \nu(z, [x, y]) + \nu(y, [z, x]) \in C.$$

Again we are in a position to apply Lemma 2.2. Hence it follows that

$$f([x, y]) - [f(x), y] - [x, f(y)] = \alpha[x, y] + \beta(x, y)$$

for some $\alpha \in C$ and skew-symmetric $\beta : L \times L \to C$. It is clear that the linearity of f implies the bilinearity of β. Setting $\gamma = -\alpha$ we get the desired conclusion. $\qquad\square$

Recall that the second cohomology group $H^2(L, \mathbb{F})$ of a Lie algebra L is 0 in case the following holds true: If $\phi : L \times L \to \mathbb{F}$ is a skew-symmetric bilinear map such that

$$(4) \qquad \phi(x, [y, z]) + \phi(z, [x, y]) + \phi(y, [z, x]) = 0 \quad \text{for all } x, y \in L,$$

then there exists a linear functional $\tau : L \to \mathbb{F}$ such that

$$(5) \qquad\qquad\qquad \phi(x, y) = \tau([x, y]) \quad \text{for all } x, y \in L.$$

Instead of considering maps with the range in $\mathbb{F}$ one could take maps mapping into any linear space V over $\mathbb{F}$. That is, if $H^2(L, \mathbb{F}) = 0$, then a skew-symmetric bilinear map $\phi : L \times L \to V$ satisfying (4) must be of the form (5) for some linear map $\tau : L \to V$. Indeed, by composing ϕ by an arbitrary linear functional ξ on V one can use the condition $H^2(L, \mathbb{F}) = 0$ to conclude that $\xi(\phi(x, y)) = \tau_\xi([x, y])$ for some linear functional τ_ξ on L. Thus, if $x_i, y_i \in L$ are such that $\sum_i [x_i, y_i] = 0$, then $\xi(\sum_i \phi(x_i, y_i)) = \sum_i \xi(\phi(x_i, y_i)) = 0$. Since ξ is arbitrary, it follows that $\sum_i \phi(x_i, y_i) = 0$. This shows that a linear map $\tau : [L, L] \to V$ determined by $\tau([x, y]) = \phi(x, y)$ is well-defined. Now one extends τ to a linear map on L.

Theorem 3.2. *Assume the conditions of Lemma 3.1, and assume further that $H^2(L, \mathbb{F}) = 0$. Then there exist $\gamma \in C$, a derivation $\delta : L \to Q$ and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.*

Proof. By Lemma 3.1 the map $d = f - \gamma I : L \to CL \subseteq Q$ satisfies

$$d([x, y]) - [d(x), y] - [x, d(y)] = \beta(x, y) \in C$$

for all $x, y \in L$. Consequently,

$$\beta(x, [y, z]) = d([x, [y, z]]) - [d(x), [y, z]] - [x, [d(y), z]] - [x, [y, d(z)]],$$

since $[x, \beta(y, z)] = 0$. Using the Jacobi identity it readily follows that

$$\beta(x, [y, z]) + \beta(z, [x, y]) + \beta(y, [z, x]) = 0.$$

Since $H^2(L, \mathbb{F}) = 0$ there exists a linear map $\tau : L \to C$ such that $\beta(x, y) = \tau([x, y])$ for all $x, y \in L$. That is,

$$d([x, y]) - [d(x), y] - [x, d(y)] = \tau([x, y]).$$

It now follows immediately that $\delta = d - \tau$ is a derivation from L into Q. $\square$

Examples of 4-free Lie algebras can be extracted from Remark 2.1; however, for Lie algebras of these types we will get a detailed description in the sequel without using Theorem 3.2. We have to admit that it is not clear to us how to apply Theorem 3.2 to concrete Lie algebras in order to get some information that cannot be obtained by other means. But this is because the d-freeness of Lie algebras (viewed as subsets of some associative algebras) has been so far systematically investigated only in some specific situations. We believe that further investigation in this direction would be interesting, and would also give light to the meaning of Theorem 3.2.

Let us point out that the maps from the decomposition $f = \delta + \gamma I + \tau$ do not necessarily map L into itself. For example, it is possible that none of γI and τ leaves L invariant, but their sum $\gamma I + \tau$ does; see [2, Example 2.9]. Thus, in general one cannot avoid involving Q and C in the description of a near-derivation on a Lie algebra L. Let us consider one situation when this is possible.

Theorem 3.3. *Assume the conditions of Lemma 3.1, and assume further that L has trivial center and $[L, L] = L$. Then there exist a derivation $\delta : L \to L$ and ζ from the centroid of L such that $f = \delta + \zeta$.*

Proof. Lemma 3.1 implies that

$$\gamma[[x, y], z] = [[f(x), y], z] + [[x, f(y)], z] - [f([x, y]), z] \in L$$

for all $x, y, z \in L$. Since $[L, L] = L$, and hence also $[[L, L], L] = L$, it follows that $\gamma L \subseteq L$. That is, the map $\zeta : x \mapsto \gamma x$ maps L into L and so it lies in the centroid of L. Further,

$$\beta(x, y) = f([x, y]) + \gamma[x, y] - [f(x), y] - [x, f(y)]$$

then lies in $L \cap C$ which is zero since L has trivial center. But then $\delta = f - \zeta$ is a derivation. $\qquad \square$

The next two corollaries concern simple Lie algebras arising from a central simple associative algebra A. In both of them we will have to assume that the dimension of A is big enough, i.e. infinite or greater than a certain integer. When applying the theory of functional identities to concrete classes of algebras, it often happens that one must exclude algebras of low dimensions. It is not always the case that this exclusion is necessary, sometimes this is just a price that one has to pay for using powerful methods which, however, are not efficient in the low dimensional setting. Anyhow, we shall not consider separately these special cases in this paper.

Corollary 3.4. *Let A be a central simple algebra such that $\dim_{\mathbb{F}} A \geq 25$. Set $L = [A, A]$ and suppose that $1 \notin L$. Then every near-derivation f of L is of the form $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$.*

Proof. A well-known result by Herstein [3, Theorem 1.12] implies that L is a simple Lie algebra. Therefore the conditions that L has trivial center and $[L, L] = L$ are fulfilled. Further, by Remark 2.1 it follows from $\dim_{\mathbb{F}} A \geq 25$ that L is a 4-free subset of $Q_{ml}(A)$. The center C of Q is the extended centroid of A; it is well-known that in simple unital algebras it coincides with the center, so that in our case $C = \mathbb{F}1$. Thus the element γ from the proof of Theorem 3.3 is actually a scalar. $\qquad \square$

Recall that an involution $*$ on a central simple algebra A is said to be of the first kind if $(\lambda x)^* = \lambda x^*$ for all $\lambda \in \mathbb{F}$ and $x \in A$. In this case the set of skew elements K of A is a Lie algebra which does not contain nonzero elements from $\mathbb{F}1$.

Corollary 3.5. *Let A be a central simple algebra with involution of the first kind. Suppose that $\dim_{\mathbb{F}} A \geq 121$. Let K be the set of skew elements in A, and set $L = [K, K]$. Then every near-derivation f of L is of the form $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$.*

Proof. By another Herstein's theorem [3, Theorem 2.15] L is a simple Lie algebra. Since $\dim_{\mathbb{F}} A \geq 121$, L is a 4-free subset of $Q_{ml}(A)$ (Remark 2.1). Now just repeat the arguing from the preceding proof. $\square$

As we will now show, both corollaries can be extended to a considerably more general setting, and moreover, a more precise information on their form can be given. The only disadvantage is that we cannot avoid involving the presence of the extended centroid in this setting, and so the results do not have such simple forms as corollaries do. The proofs depend on the structure of Lie derivations in associative rings. A survey on this topic can be found in [2, Chapter 6]. However, we shall make use of some results from [1] which give a somewhat more precise information than those in [2].

Let S be a subset of an (associative) algebra A. By $\langle S \rangle$ we denote the subalgebra of A generated by S.

Theorem 3.6. *Let A be a prime algebra, let C be the extended centroid of A, and let L be a noncommutative Lie ideal of A. Suppose that $\deg(A) \geq 5$. If f is a near-derivation of L, then there exist an (associative) derivation $\delta : \langle L \rangle \to \langle L \rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.*

Proof. Remark 2.1 tells us that L is a 4-free subset of $Q = Q_{ml}(A)$. Therefore it follows from Lemma 3.1 that there is $\gamma \in C$ such that the map $d = f - \gamma I : L \to CL \subseteq Q$ satisfies

(6) $d([x, y]) - [d(x), y] - [x, d(y)] = \beta(x, y) \in C.$

We set $\overline{Q} = Q/C$, and for $x \in Q$ we write $\overline{x} = x + C \in \overline{Q}$. From (6) we see that the map $\overline{d} : L \to \overline{Q}$ defined by $\overline{d}(x) = \overline{d(x)}$ satisfies $\overline{d}([x, y]) = [\overline{d}(x), \overline{y}] + [\overline{x}, \overline{d}(y)]$. We are now in a position to apply [1, Theorem 1.3]: thus there exist a derivation $\delta : L \to \langle L \rangle C + C$ and a linear map $\tau : L \to C$ such that $d(x) = \delta(x) + \tau(x)$ for every $x \in L$. Hence $f(x) = d(x) + \gamma x = \delta(x) + \gamma x + \tau(x)$ for every $x \in L$. $\square$

Following the same pattern, just that using [1, Theorem 1.8] instead of [1, Theorem 1.3], one establishes the following theorem.

Theorem 3.7. *Let A be a prime algebra with involution, let C be the extended centroid of A, let K be the skew elements of A, and let L be a non-central Lie ideal of K. Suppose that $\deg(A) \geq 11$. If f is a near-derivation of L, then there exist an (associative) derivation $\delta : \langle L \rangle \to \langle L \rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \to C$ such that $f = \delta + \gamma I + \tau$.*

We conclude with a few comments concerning the notions discussed in the introduction.

Remark 3.8. At the very beginning of the paper the following question was posed: What is the form of a linear map $f : L \to L$ such that $[f, \delta]$ is a derivation for every derivation δ? Since f is in particular a near-derivation, let us assume that our typical conclusion $f = \delta + \gamma I + \tau$ holds. Clearly, f satisfies (3) with $g = f$, from which one readily infers that $\tau([L, [L, L]]) = 0$. Assuming that $[L, L] = L$ it thus follows that the central map $\tau = 0$.

Remark 3.9. All results in this section of course hold for generalized derivations, since the concept of a near-derivation is more general. Let us show that this generalization is not an empty one, that is, let us show that there exist near-derivations that are not generalized derivations.

Let L be a non-abelian Lie algebra such that $[L, L]$ is a proper subset of the center $Z(L)$ of L (concrete examples can be easily found). Pick $a \in L \setminus Z(L)$ and $b \in Z(L) \setminus [L, L]$. Let φ be a linear functional on L such that $\varphi(b) = 1$ and $\varphi([L, L]) = 0$. Now define $f : L \to L$ by $f(y) = \varphi(y)a$. It is easy to check that $(\operatorname{ad} x)f$ is a derivation for every $x \in L$, so that f is a near-derivation. However, f is not a generalized derivation. Indeed, if there were $g, h : L \to L$ such that $[f(x), y] = g([x, y]) - [x, h(y)]$, then it would follow by setting $x = b$ that $[a, y] = 0$ for every $y \in L$, contradicting the assumption that $a \notin Z(L)$.

Remark 3.10. The concept of a near-derivation in this paper is different from the notion of a near-derivation introduced in [4]. The author is thankful to the referee for pointing out this.

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