ORTHOGONALLY ADDITIVE POLYNOMIALS AND ORTHOSYMMETRIC MAPS IN BANACH ALGEBRAS WITH PROPERTIES $\mathbb{A}$ AND $\mathbb{B}$

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Abstract. The paper considers Banach algebras with properties $\mathbb{A}$ or $\mathbb{B}$, introduced recently in [1]. The class of Banach algebras satisfying any of these two properties is quite large; in particular it includes C^*-algebras and group algebras on locally compact groups. Our first main result states that a continuous orthogonally additive n-homogeneous polynomial on a commutative Banach algebra with property $\mathbb{A}$ and having a bounded approximate identity is of a standard form. The other main results describe Banach algebras A with property $\mathbb{B}$ and having a bounded approximate identity which admit nonzero continuous symmetric orthosymmetric n-linear maps from A^n into $\mathbb{C}$.

1. Introduction

The basic purpose of this paper is to show that the pattern established in [1], based on the so-called properties $\mathbb{A}$ and $\mathbb{B}$, provides a way to treat two seemingly quite different classes of maps, namely *orthogonally additive polynomials* and *orthosymmetric maps*.

We say that a complex Banach algebra A has *property* $\mathbb{B}$ if every continuous bilinear map $f\colon A \times A \to X$, where X is an arbitrary Banach space, with the property that for all $x, y \in A$,

$$xy = 0 \implies f(x, y) = 0,$$

necessarily satisfies

$$f(xy, z) = f(x, yz) \quad \text{for all } x, y, z \in A.$$

The definition of property $\mathbb{A}$ is slightly more technical and will be recalled in the next section, but only for the case needed; i.e., for commutative algebras. The point of the paper [1] is that every (not necessarily commutative) Banach algebra with property $\mathbb{A}$ has also property $\mathbb{B}$, and that the class of Banach algebras with property $\mathbb{A}$ (and hence also that of algebras with property $\mathbb{B}$) is fairly large, it includes for example C^*-algebras, group

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algebras on arbitrary locally compact groups, Banach algebras generated by idempotents, topologically simple Banach algebras containing nontrivial idempotents, etc. These assertions have turned out to be, sometimes rather unexpectedly, applicable to a variety of topics, see for example [1, 2, 3, 7, 13]. The present paper continues this line of investigation by presenting new applications.

Let A be a commutative Banach algebra. A map $P : A \to \mathbb{C}$ is said to be *orthogonally additive* if $P(x+y) = P(x) + P(y)$ whenever $x, y \in A$ are such that $xy = 0$. We will be interested in the case where P is a continuous n-homogeneous polynomial; i.e., P is of the form $P(x) = \varphi(x, \dots, x)$, $x \in A$, for some continuous n-linear map $\varphi\colon A^n \to \mathbb{C}$. Orthogonally additive polynomials have been widely discussed in the context of Banach lattices (see [10] and the references therein). However, we were primarily motivated by the results stating that every continuous orthogonally additive n-homogeneous polynomial P from A into $\mathbb{C}$ can be represented in the form $P(x) = \omega(x^n)$, $x \in A$, for some $\omega \in A^*$ in the case where A is a commutative C^*-algebra [5, 8, 12] or A is the Fourier algebra $A(G)$ of a locally compact group G having an abelian subgroup of finite index [4] (some restriction on G is inevitable). In Section 2 we will show, by making use of the result from [4], that the same representation theorem holds if A is a commutative Banach algebra with property $\mathbb{A}$ and having a bounded approximate identity (recall that C^*-algebras and group algebras on locally compact groups have approximate bounded identities).

Now let A be a not necessarily commutative Banach algebra. A continuous multilinear map $\varphi\colon A^n \to \mathbb{C}$ is said to be *orthosymmetric* if $\varphi(x_1, \dots, x_n) = 0$ whenever $x_i x_j = x_j x_i = 0$ for some $1 \le i < j \le n$. This notion has also origin in lattice theory [6], and may be considered as a variation of the notion of orthogonally additive polynomials (cf. [4, Lemma 2.3]). If A is commutative, then a simple example of an orthosymmetric map is $\varphi(x_1, \dots, x_n) = \omega(x_1 \cdots x_n)$ where $\omega \in A^*$. Note that this map is symmetric. A natural problem in certain situations is actually to show that orthosymmetric maps are necessarily symmetric (cf. [6]). In Section 3 we will consider symmetric orthosymmetric maps in Banach algebras with property $\mathbb{B}$ (and having a bounded approximate identity) and show that their existence for $n \ge 3$ has an effect on the structure of the algebra. More precisely, the existence of a nonzero symmetric orthosymmetric map for $n = 3$ turns out to be equivalent to the existence of a "trace-like" map on the algebra, and for $n \ge 4$ to the existence of a multiplicative functional.

2. Orthogonal additivity in commutative Banach algebras with property $\mathbb{A}$

Let X and Y be Banach spaces. A map $P\colon X \to Y$ is said to be a continuous n-homogeneous polynomial if there exists a continuous n-linear map $\varphi\colon X^n \to Y$ such that $P(x) = \varphi(x, \dots, x)$ for each $x \in X$. Such a map

φ is unique if it is required to be symmetric. In this case it can be obtained through the *polarization formula*:

$$\varphi(x_1, \dots, x_n) = \frac{1}{n!2^n} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P(\epsilon_1 x_1 + \cdots + \epsilon_n x_n).$$

We are interested in the situation where $X = A$ is a commutative Banach algebra, $Y = \mathbb{C}$, and P is orthogonally additive. Let us first recall some definitions and fix the notation.

Let A be a commutative Banach algebra. Suppose that A is *faithful*; i.e., for every $x \in A$, $xA = \{0\}$ implies $x = 0$. A *multiplier* on A is a linear map $L\colon A \to A$ such that $L(xy) = L(x)y$ for all $x, y \in A$. Such a map is necessarily continuous and the set $\mathcal{M}(A)$ of all multipliers on A is a unital closed subalgebra of $\mathcal{B}(A)$, the Banach algebra of all continuous linear operators on A. Further, the map $x \mapsto L_x$ continuously embeds the algebra A into the algebra $\mathcal{M}(A)$, where $L_x(y) = xy$ for all $x, y \in A$. Note that A can be therefore thought of as an ideal of $\mathcal{M}(A)$. An invertible element $u \in \mathcal{M}(A)$ is said to be *doubly power-bounded* if $\sup_{k\in\mathbb{Z}} \|u^k\| < \infty$. Let $\mathcal{D}(A)$ stands for the linear span of all doubly power-bounded elements in $\mathcal{M}(A)$. Since the set of doubly power-bounded elements is closed under multiplication, $\mathcal{D}(A)$ is in fact equal to the algebra generated by all doubly power-bounded elements in $\mathcal{M}(A)$.

Lemma 2.1. *Let A be a faithful commutative Banach algebra. Let $P\colon A \to \mathbb{C}$ be a continuous orthogonally additive n-homogeneous polynomial, and let $\varphi\colon A^n \to \mathbb{C}$ be the symmetric n-linear map associated with P. Then*

$$\varphi(xy_1, \dots, xy_{n-1}, xy_n) = \varphi(x, \dots, x, xy_1 \cdots y_n), \tag{1}$$

for all $x \in A$ and $y_1, \dots, y_n \in \overline{\mathcal{D}(A)}^{\mathrm{so}}$.

Proof. We temporarily fix $x \in A$ and doubly power-bounded elements $y_1,\dots, y_n$ in $\mathcal{M}(A)$.

Let $\mathbb{T}$ be the circle group and let $A(\mathbb{T}^n)$ stand for the Fourier algebra on $\mathbb{T}^n$. Then $A(\mathbb{T}^n)$ consists of the functions $f \in C(\mathbb{T}^n)$ such that

$$\|f\|_{A(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} \left|\widehat{f}(k)\right| < \infty.$$

Here, $\widehat{f}(k)$ stands for the kth Fourier coefficient of f. For every $f \in A(\mathbb{T}^n)$ we can define $f(y_1, \dots, y_n) \in \mathcal{M}(A)$ by

$$f(y_1, \dots, y_n) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k) y_1^{k_1} \cdots y_n^{k_n}.$$

Furthermore,

$$\|f(y_1, \dots, y_n)\| \le \sum_{k\in\mathbb{Z}^n} \left|\widehat{f}(k)\right| \|y_1^{k_1}\| \dots \|y_n^{k_n}\| \le C_1 \cdots C_n \|f\|_{A(\mathbb{T}^n)},$$

where $C_i = \sup_{k\in\mathbb{Z}} \|y_i^k\|$ for each $i \in \{1, \dots, n\}$. Hence, the map $f \mapsto f(y_1, \dots, y_n)$ gives a continuous homomorphism from $A(\mathbb{T}^n)$ into $\mathcal{M}(A)$.

We now define $Q\colon A(\mathbb{T}^n) \to \mathbb{C}$ by $Q(f) = P(xf(y_1,\dots,y_n))$ for each $f \in A(\mathbb{T}^n)$. It is easily seen that Q is a continuous orthogonally additive n-homogeneous polynomial on $A(\mathbb{T}^n)$. By [4, Corollary 2.6] there exists $\xi \in A(\mathbb{T}^n)^*$ such that $Q(f) = \xi(f^n)$ for each $f \in A(\mathbb{T}^n)$. The polarization formula then yields

$$\varphi\big(xf_1(y_1,\dots,y_n),\dots,xf_n(y_1,\dots,y_n)\big) = \xi(f_1\cdots f_n) \tag{2}$$

for all $f_1,\dots,f_n \in A(\mathbb{T}^n)$. Let $f_0,f_1,\dots,f_n \in A(\mathbb{T}^n)$ be defined by $f_0(u_1,\dots,u_n) = 1$ and $f_i(u_1,\dots,u_n) = u_i$ for all $(u_1,\dots,u_n) \in \mathbb{T}^n$ and $i \in \{1,\dots,n\}$. On account of (2), we have

$$\begin{aligned}
\varphi(xy_1,\dots,xy_n) &= \varphi\big(xf_1(y_1,\dots,y_n),\dots,xf_n(y_1,\dots,y_n)\big)\\
&= \xi(f_1\cdots f_n)\\
&= \xi\big(f_0 \overset{n-1}{\cdots} f_0(f_1\cdots f_n)\big)\\
&= \varphi\big(xf_0(y_1,\dots,y_n),\dots,xf_0((y_1,\dots,y_n),x(f_1\cdots f_n)(y_1,\dots,y_n)\big)\\
&= \varphi(x,\dots,x,xy_1\cdots y_n),
\end{aligned}$$

which proves (1) in the case where $y_1,\dots,y_n$ are doubly power-bounded elements in $\mathcal{M}(A)$. By linearity, (1) holds for all $y_1,\dots,y_n \in \mathcal{D}(A)$. The separate continuity of the n-linear functionals $(y_1,\dots,y_n) \mapsto \varphi(xy_1,\dots,xy_n)$ and $(y_1,\dots,y_n) \mapsto \varphi(x,\dots,x,xy_1\cdots y_n)$ with respect to the strong operator topology on $\mathcal{M}(A)$ now implies that (1) holds for all $y_1,\dots, y_n \in \overline{\mathcal{D}(A)}^{\mathrm{so}}$, as required. ☐

A commutative Banach algebra A is said to have *property* $\mathbb{A}$ if it is contained in the closure, $\overline{\mathcal{D}(A)}^{\mathrm{so}}$, of $\mathcal{D}(A)$ in $\mathcal{M}(A)$ with respect to the strong operator topology; i.e., for every $x \in A$ there exists a net $(u_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{D}(A)$ such that $\lim_{\lambda\in\Lambda} u_\lambda y = xy$ with respect to the norm for each $y \in A$. For examples of algebras with property $\mathbb{A}$ we refer the reader to [1, Sections 1.2 and 1.3].

Theorem 2.2. *Let A be a commutative Banach algebra with property $\mathbb{A}$ and having a bounded approximate identity. Let $P\colon A \to \mathbb{C}$ be a continuous orthogonally additive n-homogeneous polynomial. Then there exists $\omega \in A^*$ such that $P(x) = \omega(x^n)$ for each $x \in A$.*

Proof. Let $\varphi\colon A^n \to \mathbb{C}$ be the symmetric n-linear map associated with P. Let $(\rho_\lambda)_{\lambda\in\Lambda}$ be an approximate identity of A of bound C. Since A has property $\mathbb{A}$, Lemma 2.1 yields

$$\varphi\left(\rho_\lambda y_1,\dots,\rho_\lambda y_{n-1},\rho_\lambda y_n\right) = \varphi\left(\rho_\lambda,\dots,\rho_\lambda,\rho_\lambda y_1\cdots y_n\right)$$

for all $y_1,\dots, y_n \in A$ and $\lambda \in \Lambda$. Consequently,

$$\begin{aligned}
\varphi\left(y_1,\dots,y_n\right) &= \lim_{\lambda\in\Lambda} \varphi\left(\rho_\lambda y_1,\dots,\rho_\lambda y_{n-1},\rho_\lambda y_n\right)\\
&= \lim_{\lambda\in\Lambda} \varphi\left(\rho_\lambda,\dots,\rho_\lambda,\rho_\lambda y_1\cdots y_n\right)
\end{aligned}$$

for all $y_1,\dots,\ y_n \in A$. By Cohen's factorization theorem, each $y \in A$ can be written in the form $y = y_1 \cdots y_n$ with $y_1,\dots,\ y_n \in A$. Hence the net $\big(\varphi(\rho_\lambda,\dots,\rho_\lambda,\rho_\lambda y)\big)_{\lambda\in\Lambda}$ is convergent. We may define a linear functional ω on A by $\omega(y) = \lim_{\lambda\in\Lambda}\varphi(\rho_\lambda,\dots,\rho_\lambda,\rho_\lambda y)$ for each $y \in A$. Since $|\varphi(\rho_\lambda,\dots,\rho_\lambda,\rho_\lambda y)| \leq \|\varphi\|C^n\|y\|$ for all $y \in A$ and $\lambda \in \Lambda$, it follows that $|\omega(y)| \leq \|\varphi\|C^n\|y\|$ for each $y \in A$, which implies that ω is continuous. Further $\varphi(y_1,\dots,y_n) = \omega(y_1 \cdots y_n)$ for all $y_1,\dots,\ y_n \in A$. In particular, $P(x) = \omega(x^n)$ for each $x \in A$. $\square$

Remark 2.3. It seems plausible that Theorem 2.2 remains true if one replaces $\mathbb{A}$ with $\mathbb{B}$ in the statement. Unfortunately we are unable to show that this is true in general. To give some indication of plausibility, we now give a proof for the special case where $n \leq 3$ and A is unital.

Thus, let P be a continuous orthogonally additive n-homogeneous polynomial, $n \leq 3$, and let φ be the symmetric n-linear map associated with P. We must show that $P(x) = \omega(x^n)$, $x \in A$, for some $\omega \in A^*$. There is nothing to prove if $n = 1$. If $n = 2$, then the polarization formula immediately shows that φ satisfies the condition that $xy = 0$ implies $\varphi(x,y) = 0$. Since A has property $\mathbb{B}$ it follows that $P(x) = \varphi(x,x) = \varphi(1,x^2)$ for every $x \in A$, which is the desired conclusion. Now let $n = 3$. From

$$2\varphi(x,y,z) = \varphi(x+z,y,x+z) - \varphi(x,y,x) - \varphi(z,y,z)$$

we easily infer that $xy = zy = 0$ implies $\varphi(x,y,z) = 0$ (cf. [4, Lemma 2.3]). Pick $u,\ v \in A$ such that $uv = 0$. The map $\psi_1\colon A^2 \to \mathbb{C}$, $\psi_1(x,y) = \varphi(u,vx,y)$, $x,\ y \in A$, satisfies the condition that $xy = 0$ implies $\psi_1(x,y) = 0$. Therefore $\psi_1(xy,z) = \psi_1(x,yz)$ for all $x,\ y,\ z$ in A. In particular, $\psi_1(y,z) = \psi_1(1,yz)$. This means that the map $\psi_2\colon A^2 \to \mathbb{C}$ defined by $\psi_2(u,v) = \varphi(u,vy,z) - \varphi(u,v,yz)$, where $y,\ z \in A$ are fixed (but arbitrary) elements in A, satisfies $uv = 0$ implies $\psi_2(u,v) = 0$. Hence $\psi_2(uw,v) = \psi_2(u,wv)$ for all $u,\ v,\ w \in A$; i.e.,

$$\varphi(uw,vy,z) - \varphi(uw,v,yz) = \varphi(u,wvy,z) - \varphi(u,wv,yz)$$

for all $u,\ w,\ v,\ y,\ z \in A$. Setting $u = v = 1$ and $w = y = z$, and using the symmetry of φ we get $\varphi(w,w,w) = \varphi(w,1,w^2)$. On the other hand, setting $u = z = 1$ and $w = v = y$ we infer $2\varphi(w,1,w^2) = \varphi(w,w,w) + \varphi(1,w^3,1)$. Comparing both relations we arrive at the desired conclusion $P(w) = \varphi(w,w,w) = \varphi(1,w^3,1)$.

3. Orthosymmetric maps in Banach algebras with property $\mathbb{B}$

In this section we study a symmetric orthosymmetric n-linear map $\varphi\colon A^n \to \mathbb{C}$ in a Banach algebra A with property $\mathbb{B}$. First we will describe the form of φ, and then discuss conditions under which such a map exists.

For $x,y \in A$ we write $x \circ y = xy + yx$ and $[x,y] = xy - yx$. We remark that

$$(x \circ y) \circ z - (x \circ z) \circ y = [[z,y],x].$$

By $A \circ A$ we denote the linear span of all $x \circ y$, x, $y \in A$. Similarly we introduce $[A, A]$, $[[A, A], A]$, etc. The ideal of A generated by the set S will be denoted by $\mathrm{Id}(S)$.

Theorem 3.1. *Let A be a Banach algebra with property $\mathbb{B}$ and having a bounded approximate identity. If $\varphi\colon A^n \to \mathbb{C}$, $n \geq 2$, is a continuous symmetric orthosymmetric n-linear map, then there exists $\omega \in A^*$ such that*

$$\varphi(x_1, \ldots, x_n) = \omega\big((\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_n\big) \tag{3}$$

for all $x_1, \ldots, x_n \in A$.

Proof. We proceed by induction on n. For $n = 2$ the result is basically already known. More precisely, [2, Theorem 1.2] covers the case where A is a C^*-algebra, but fortunately the same proof works for Banach algebras with property $\mathbb{B}$ and having a bounded approximate identity. Therefore we present only the basic idea of the proof, and refer to [2] for details. For any pair of elements u, $v \in A$ such that $uv = 0$ define a bilinear map $\psi_1\colon A^2 \to \mathbb{C}$ by $\psi_1(x, y) = \varphi(vx, yu)$. Clearly, $xy = 0$ implies $\psi_1(x, y) = 0$ and so $\psi_1(xz, y) = \psi_1(x, zy)$ for all $x, y, z \in A$. That is to say, the bilinear map $\psi_2(u, v) = \varphi(vxz, yu) - \varphi(vx, zyu)$ has the property that $uv = 0$ implies $\psi_2(u, v) = 0$. Using property $\mathbb{B}$ one then obtains a functional equation for φ involving six elements from which the desired conclusion; i.e., $\varphi(x_1, x_2) = \omega(x_1 \circ x_2)$ for some $\omega \in A^*$, can be derived by making use of a bounded approximate identity.

Thus, let $n > 2$ and assume that the theorem holds for a smaller number of variables. Note that by fixing any $z \in A$, the induction hypothesis can be applied to the map $(x_1, \ldots, x_{n-1}) \mapsto \varphi(x_1, \ldots, x_{n-1}, z)$. Thus, for each $z \in A$ there exists $\psi(\,\cdot\,, z) \in A^*$ such that

$$\varphi(x_1, \ldots, x_{n-1}, z) = \psi\big((\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_{n-1}, z\big)$$

for all $x_1, \ldots, x_{n-1} \in A$. Let $(\rho_\lambda)_{\lambda\in\Lambda}$ be an approximate identity of A of bound C. We now observe that

$$\begin{aligned}\psi(x, z) &= \frac{1}{2^{n-2}} \lim_{\lambda_{n-2}\in\Lambda} \cdots \lim_{\lambda_1\in\Lambda} \psi\big((\ldots(x \circ \rho_{\lambda_1}) \ldots) \circ \rho_{\lambda_{n-2}}, z\big)\\ &= \frac{1}{2^{n-2}} \lim_{\lambda_{n-2}\in\Lambda} \cdots \lim_{\lambda_1\in\Lambda} \varphi\left(x, \rho_{\lambda_1}, \ldots, \rho_{\lambda_{n-2}}, z\right),\end{aligned}$$

which clearly shows that ψ is a symmetric orthosymmetric bilinear map on A. Further, since $|\varphi(x, \rho_{\lambda_1}, \ldots, \rho_{\lambda_{n-2}}, z)| \leq C^{n-2}\|\varphi\|\|x\|\|z\|$ for all x, $z \in A$ and $\lambda \in \Lambda$, it follows that $|\psi(x, z)| \leq C^{n-2}\|\varphi\|\|x\|\|z\|$ for all x, $z \in A$ and, consequently, ψ is continuous. From what has already been proved it may be concluded that there exists $\omega \in A^*$ such that $\psi(x, z) = \omega(x \circ z)$ for all $x, z \in A$. We thus get

$$\begin{aligned}\varphi(x_1, \ldots, x_{n-1}, x_n) &= \psi\big((\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_{n-1}, x_n\big)\\ &= \omega\big((\ldots((x_1 \circ x_2) \circ x_3) \ldots) \circ x_{n-1}) \circ x_n\big)\end{aligned}$$

for all $x_1, \ldots, x_{n-1}, x_n \in A$, as required. $\square$

Remark 3.2. If A is commutative then φ gets a simpler form: $\varphi(x_1, \dots, x_n) = \omega'(x_1 \dots x_n)$ where $\omega' = 2^{n-1}\omega \in A^*$.

For $n = 2$ Theorem 3.1 tells us that φ arises from the Jordan product $\circ$ and a continuous linear functional, and this is all that can be said. The problem is of a different nature if $n \geq 3$. Moreover, the $n = 3$ case is different from the $n \geq 4$ case. We first treat the former.

Theorem 3.3. *Let A be a Banach algebra with property $\mathbb{B}$ and having a bounded approximate identity. Then there exists a nonzero continuous symmetric orthosymmetric 3-linear map $\varphi\colon A^3 \to \mathbb{C}$ if and only if $\overline{[A, A]} \neq A$.*

Proof. Suppose there exists a nonzero continuous symmetric orthosymmetric 3-linear map $\varphi\colon A^3 \to \mathbb{C}$. Theorem 3.1 tells us that there is $\omega \in A^*$ such that $\varphi(x, y, z) = \omega((x \circ y) \circ z)$ for all x, y, $z \in A$. Since φ is symmetric it follows that $\omega((x \circ y) \circ z) = \omega((x \circ z) \circ y)$ for all x, y, $z \in A$. Note that this can be rewritten as

$$\omega\left([[z, y], x]\right) = 0 \quad \text{for all } x, y, z \in A. \tag{4}$$

Now if $\overline{[A, A]}$ was equal to A, then $\overline{[[A, A], A]}$ would also be equal to A, and hence $\omega(u)$ would be 0 for every $u \in A$ as ω is continuous. However, this is impossible for $\varphi \neq 0$.

To prove the converse, assume that $\overline{[A, A]} \neq A$. Let $\omega \in A^*$ be such that $\omega \neq 0$ and ω vanishes on all commutators. Define $\varphi\colon A^3 \to \mathbb{C}$ by

$$\varphi(x, y, z) = \omega(xyz + zyx). \tag{5}$$

Since $\omega([A, A]) = 0$, we have

$$\varphi(x, y, z) = \omega\left((xy)z + z(yx)\right) = \omega(zxy + yxz),$$

and similarly

$$\varphi(x, y, z) = \omega(xzy + yzx).$$

Hence we see that φ is symmetric and orthosymmetric. Let $(\rho_\lambda)_{\lambda\in\Lambda}$ be a bounded approximate identity of A. From

$$\lim_{\lambda_2\in\Lambda} \lim_{\lambda_1\in\Lambda} \varphi\left(x, \rho_{\lambda_1}, \rho_{\lambda_2}\right) = \lim_{\lambda_2\in\Lambda} \lim_{\lambda_1\in\Lambda} \omega\left(x\rho_{\lambda_2}\rho_{\lambda_1} + \rho_{\lambda_1}\rho_{\lambda_2}x\right) = 2\omega(x)$$

we see that $\varphi \neq 0$. $\square$

Remark 3.4. Assume the conditions of Theorem 3.3 and add the assumption that $[[A, A], A] = [A, A]$. Then it can be concluded that φ is necessarily of the form (5) where $\omega \in A^*$ is such that $\omega([A, A]) = 0$. Indeed, from (3) and (4) we see that there exists $\omega' \in A^*$ such that $\varphi(x, y, z) = \omega'(xyz + zyx + yxz + zxy)$, x, y, $z \in A$, and ω' vanishes on commutators. Since $\omega'((yx)z) = \omega'(z(yx))$ and $\omega'(z(xy)) = \omega'((xy)z)$ it follows that $\varphi(x, y, z) = 2\omega'(xyz + zyx)$. Thus we take $\omega = 2\omega'$.

The simplest example of an algebra satisfying the conditions of Remark 3.4 is the matrix algebra $M_n = M_n(\mathbb{C})$. Since every linear functional on M_n

that vanishes on commutators is a scalar multiple of the trace, the following corollary holds.

Corollary 3.5. *If $\varphi\colon M_n^3 \to \mathbb{C}$ is a symmetric orthosymmetric 3-linear map, then there exists $\alpha \in \mathbb{C}$ such that $\varphi(x, y, z) = \alpha \mathrm{tr}(xyz + zyx)$ for all x, y, $z \in M_n$.*

It remains to consider the case where $n \geq 4$. For this purpose, we recall that a Banach algebra A is said to have the *weakly Wiener property* if spectral analysis holds for A; i.e., each proper closed ideal of A is contained in a primitive ideal.

Example 3.6. We list some examples of algebras having the weakly Wiener property.

(1) If A is a unital Banach algebra, then every proper ideal of A is of course contained in a maximal ideal of A, and so A has the weakly Wiener property.
(2) Every C^*-algebra has the weakly Wiener property [11, Theorem 11.5.4(e)].
(3) A locally compact group G is *weakly Wiener* if the group algebra $L^1(G)$ has the weakly Wiener property. For examples of this class of groups we refer the reader to [11, Section 12.6.36]. Among them are the locally compact abelian groups and the compact groups.
(4) By [9, Proposition 4.1.24], a regular Banach function algebra A has the weakly Wiener property if and only if the set of all $f \in A$ such that the support of f is compact is dense in A.

Theorem 3.7. *Let A be a Banach algebra with property $\mathbb{B}$, having a bounded approximate identity and the weakly Wiener property. If $n \geq 4$, then there exists a nonzero continuous symmetric orthosymmetric n-linear map $\varphi\colon A^n \to \mathbb{C}$ if and only if $\overline{\mathrm{Id}([A, A])} \neq A$.*

Proof. To prove the "only if" part it suffices to treat the case where $n = 4$. Namely, if $n \geq 5$ then by fixing $a_5, \ldots, a_n$ and considering $(x_1, x_2, x_3, x_4) \mapsto \varphi(x_1, x_2, x_3, x_4, a_5, \ldots, a_n)$ we arrive at the $n = 4$ case. Thus, let $\varphi\colon A^4 \to \mathbb{C}$ be a nonzero continuous symmetric orthosymmetric 4-linear map. By Theorem 3.1 there exists $\omega \in A^*$ such that

$$\varphi(x, y, z, w) = \omega(((x \circ z) \circ y) \circ w) \tag{6}$$

for all $x, y, z, w \in A$. Using the symmetry of φ we obtain

$$\omega(((x \circ z) \circ y) \circ w) = \omega(((x \circ y) \circ z) \circ w),$$

which shows that

$$[[y, z], x] \circ w \in \ker \omega \tag{7}$$

for all x, y, z, $w \in A$. Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be a bounded approximate identity of A. By (7), we have $[[y, z], x] \circ \rho_\lambda \in \ker \omega$ for all x, y, $z \in A$ and $\lambda \in \Lambda$. Taking the limit we get $[[A,A],A] \subseteq \ker\omega$, and so in particular $[[[y,z],x],w] \in \ker\omega$. Together with (7) this gives

$$A[[A,A],A] + [[A,A],A]A \subseteq \ker\omega.$$

Moreover, since

$$u[[x,y],z]w = u[[x,y],zw] - uz[[x,y],w]$$

this implies that $\ker\omega$ contains $\mathrm{Id}([[A,A],A])$. As $\omega \neq 0$ by (6), $I = \overline{\mathrm{Id}([[A,A],A])}$ is a proper closed ideal of A. Then I is contained in a primitive ideal P of A. The quotient Banach algebra $B = A/P$ is therefore primitive and satisfies $[[B,B],B] = \{0\}$. In other words, every commutator in B lies in the center Z of B. Thus, $[s,t]s = [s,ts] \in Z$ for any s, $t \in B$, and so in particular $[[s,t]s,t] = 0$. Since $[s,t] \in Z$, this can be written as $[s,t]^2 = 0$. However, the center of a primitive Banach algebra is either zero or is isomorphic to $\mathbb{C}$ and therefore cannot contain nonzero nilpotent elements. Thus $[s,t] = 0$ for all s, $t \in B$; i.e., B is commutative. That is, P contains $[A,A]$, and hence also $\overline{\mathrm{Id}([A,A])}$. Accordingly, $\overline{\mathrm{Id}([A,A])} \neq A$.

Conversely, assume that $\overline{\mathrm{Id}([A,A])} \neq A$. Let P be a primitive ideal of A containing $\overline{\mathrm{Id}([A,A])}$. Then A/P is a commutative primitive Banach algebra, and as such isomorphic to $\mathbb{C}$. The quotient homomorphism $\theta\colon A \to A/P$ can be therefore identified by a multiplicative functional. The map $\varphi(x_1,\dots,x_n) = \theta(x_1\dots x_n)$ is clearly symmetric and orthosymmetric, and is not zero. □

Remark 3.8. Note that the condition that $\overline{\mathrm{Id}([A,A])} \neq A$ is equivalent to the existence of a multiplicative linear functional on A. On the other hand, the condition from Theorem 3.3 that $\overline{[A,A]} \neq A$ is equivalent to the existence of a nonzero continuous linear functional τ on A that satisfies $\tau(xy) = \tau(yx)$ for all x, $y \in A$.

References

[1] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, Maps preserving zero products, *Studia Math.* **193** (2009), 131–159.

[2] J. Alaminos, M. Brešar, J. Extremera, A. R. Villena, Characterizing Jordan maps on C*-algebras through zero products, *Proc. Edinb. Math. Soc.* **53** (2010), 543–555.

[3] J. Alaminos, M. Brešar, Š. Špenko, A. R. Villena, Derivations preserving quasinilpotent elements, *Bull. Lond. Math. Soc.* **46** (2014), 379–384.

[4] J. Alaminos, J. Extremera, A. R. Villena, Orthogonal additive polynomials on Fourier algebras, preprint.

[5] Y. Benyamini, S. Lassalle, J. G. Llavona, Homogeneous orthogonally additive polynomials on Banach lattices, *Bull. London Math. Soc.* **38** (2006), 459–469.

[6] K. Boulabiar, G. Buskes, Vector lattice powers: f-algebras and functional calculus, *Comm. Algebra* **34** (2006), 1435–1442.

[7] M. Brešar, Š. Špenko, Determining elements in Banach algebras through spectral properties, *J. Math. Anal. Appl.* **393** (2012), 144–150.

[8] D. Carando, S. Lassalle, I. Zalduendo, Orthogonally additive polynomials over $C(K)$ are measures — a short proof, *Integral Equations Operator Theory* **56** (2006), 597–602.

[9] H.G. Dales, *Banach algebras and automatic continuity,* London Mathematical Society Monographs, New Series, 24, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[10] A. Ibort, P. Linares, J. G. Llavona, A representation theorem for orthogonally additive polynomials on Riesz spaces, *Rev. Mat. Complut.* **25** (2012), 21–30.

[11] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras. Vol. 2. $*$-algebras,* Encyclopedia of Mathematics and its Applications, 79, Cambridge University Press, Cambridge, 2001.

[12] D. Pérez-García, I. Villanueva, Orthogonally additive polynomials on spaces of continuous functions, *J. Math. Anal. Appl.* **306** (2005), 97–105.

[13] E. Samei, Reflexivity and hyperreflexivity of bounded n-cocycles from group algebras, *Proc. Amer. Math. Soc.* **139** (2011), 163–176.

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