ON TRACTABILITY AND IDEAL PROBLEM IN NON-ASSOCIATIVE OPERATOR ALGEBRAS

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Abstract. The question of the existence of nontrivial ideals of Lie algebras of compact operators is considered from different points of view. One of the approaches is based on the concept of a tractable Lie algebra, which can be of interest independently of the main theme of the paper. Among other results it is shown that an infinite-dimensional closed Lie or Jordan algebra of compact operators cannot be simple. Several partial answers to Wojtyński's problem on the topological simplicity of Lie algebras of compact quasinilpotent operators are also given.

1. Introduction

This paper centers around the following problem by Wojtyński [18, Question 3]: Does every closed Lie algebra of compact quasinilpotent operators on a Banach space contain a non-trivial closed Lie ideal? At the moment we are not able to give a complete answer. We do give, however, several partial answers and, approaching the problem from different directions, we find certain tools and introduce some concepts that might be of independent interest.

The paper begins by introducing the concept of a *tractable* Lie algebra. It is defined through a certain property of the associative algebra generated by all inner derivations of a Lie algebra in question. After considering this notion in Section 2 in a pure algebraic context and just from the point of view that is interesting in its own right, we present its connection to Wojtyński's problem in Section 3. This section then contains some (partial) answers to this and to some related problems. In particular, it is shown that the Jordan algebra version of Wojtyński's problem has a positive answer. In Section 4 we consider the situation where the algebra in question contains a non-zero finite rank operator; in this case the triangularization technique turns out to be useful. In Section 5 we consider the question on the existence of a not necessarily closed ideal. In particular we solve an algebraic version of Wojtyński's problem: A Lie algebra of compact quasinilpotent operators is not simple, even without the assumption of closedness. Moreover, we prove that any infinite-dimensional closed Lie or Jordan algebra of compact operators is not simple.

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2 2. Tractable Lie algebras

Throughout, A will be an associative algebra over a field F. For every $a \in A$ we define $\mathrm{L}_a, \mathrm{R}_a : A \to A$ by $\mathrm{L}_a x = ax$ and $\mathrm{R}_a x = xa$. Further, we write $\mathrm{ad}(a) = \mathrm{L}_a - \mathrm{R}_a$. Let $\mathcal{E}\ell(A)$ denote the algebra generated by all L_a and R_b, $a, b \in A$. The elements in $\mathcal{E}\ell(A)$ are called *elementary operators* on A (in pure algebra $\mathcal{E}\ell(A)$ is usually called the multiplication algebra of A, but having applications to the normed context in mind we will use a different terminology and notation). Thus, the elements in $\mathcal{E}\ell(A)$ are of the form $E = \mathrm{L}_a + \mathrm{R}_b + \sum_i \mathrm{L}_{a_i}\mathrm{R}_{b_i}$, $a, b, a_i, b_i \in A$. By $\mathcal{E}\ell^\circ(A)$ we denote the ideal of $\mathcal{E}\ell(A)$ consisting of elements of the form $E = \sum_i \mathrm{L}_{a_i}\mathrm{R}_{b_i}$, $a_i, b_i \in A$. Of course, if A is unital, then $\mathcal{E}\ell^\circ(A) = \mathcal{E}\ell(A)$. But we are more interested in algebras without unity.

Let L be a Lie subalgebra of A. We shall say that $E \in \mathcal{E}\ell(A)$ is a *Lie elementary operator on* L if the restriction of E to L is a sum of products of operators of the form $\mathrm{ad}(a)$, $a \in L$. Clearly, L is invariant under E. We shall say that L is a *tractable Lie algebra* if there exists a Lie elementary operator on L which is not zero on L and coincides on L with some operator from $\mathcal{E}\ell^\circ(A)$. Of course, the tractability of L depends not only on L but also on A. But it shall always be clear from the context which algebra A we have in mind.

We need some more notation. If L is a Lie subalgebra of A, then by $\langle L\rangle$ we denote the associative subalgebra of A generated by L. Further, we set

$$\mathrm{ann}(L) = \{x \in \langle L\rangle : xL = Lx = 0\}.$$

Note that $\mathrm{ann}(L)$ is an ideal of $\langle L\rangle$.

Let us first record two simple observations.

r2 **Remark 2.1.** *If L is noncommutative and A is unital, then L is automatically tractable.* Indeed, taking any $a \in L$ that does not lie in the center of L, we have $\mathrm{ad}(a) = \mathrm{L}_a\mathrm{R}_1 - \mathrm{L}_1\mathrm{R}_a \in \mathcal{E}\ell^\circ(A)$ and $\mathrm{ad}(a)L \neq 0$.

Using this one can show that *each simple finite-dimensional complex Lie algebra of operators is tractable.* More generally, *a semisimple Lie algebras of operators L on a finite-dimensional complex space X is tractable with respect to $A = \langle L\rangle$* (hence with respect to each algebra of operators that contains L).

Indeed it suffices to show that $\langle L\rangle$ is a unital algebra. Decomposing L into direct sum of simple algebras L_i we have that $\langle L\rangle = \oplus_i\langle L_i\rangle$ so it suffices to assume that L is simple. Since X decomposes into the sum of subspaces invariant under L, we may consider an irreducible subspace Y of X. The restriction map $a \to a|Y$ is injective on L, so the Lie algebra $L|Y$ of operators on Y is isomorphic to L and $\langle L|Y\rangle$ is isomorphic to $\langle L\rangle$. Thus we may assume that L is irreducible. Hence $\langle L\rangle$ is irreducible; by Burnside's Theorem, $\langle L\rangle$ is the algebra of all operators, so it is unital.

r1 **Remark 2.2.** Let E be a Lie elementary operator on L. Thus we have

e1 $$E = \sum ad(a_{i1})\, ad(a_{i2}) \dots ad(a_{in_i}), \quad \text{where } a_{ij} \in L. \tag{2.1}$$

We can rewrite E as

e2 $$E = L_u + R_v + \sum L_{u_i} R_{v_i}, \tag{2.2}$$

where

e3 $$u = \sum a_{i1}a_{i2}\dots a_{in_i}, \quad v = \sum (-1)^{n_i} a_{in_i}\dots a_{i2}a_{i1}, \tag{2.3}$$

and u_i, v_i are elements in A that can also be expressed through the a_{ij}'s. Hence we see that the following is true: If there exists an elementary Lie operator E on L of the form (2.1) such that $EL \neq 0$ and u and v from (2.3) lie in $\mathrm{ann}(L)$ (e.g., if both u and v are 0), then L is tractable.

Let us show that the concept we introduced is not an empty one.

p2 **Proposition 2.3.** *Let $X = \{x_1, x_2, \ldots\}$ be an infinite set, let $A = F_0\langle X\rangle$ be the non-unital free algebra on X, and let L be a Lie subalgebra of A generated by X. Then L is not tractable.*

Proof. We may regard elements in A as polynomials in noncommuting indeterminates x_i, so we can define their degrees in the standard way. The Lie algebra L consists of polynomials that can be written as sums of what we will call Lie monomials - by these we mean scalar multiples of elements such as x_i, $[x_i, x_j], [[x_i, x_j], x_k], [[[x_i, x_j], x_k], x_l], [[x_i, x_j], [x_k, x_l]]$, etc. If b_1, b_2 are Lie monomials, so is $[b_1, b_2]$; conversely, if b is a Lie monomial of degree ≥ 2 (i.e. $b \neq \lambda x_i$), then b can be written as $[b_1, b_2]$ where b_1 and b_2 are Lie monomials whose degrees are smaller than the degree of b.

Let E be a Lie elementary operator on L of the form (2.1). We can rewrite E according to (2.2). Suppose there exist $w_j, z_j \in A$ such that

$$E = \mathrm{L}_u + \mathrm{R}_v + \sum \mathrm{L}_{u_i}\mathrm{R}_{v_i} = \sum \mathrm{L}_{w_j}\mathrm{R}_{z_j}.$$

The set X is an infinite one, so there exists r such that x_r does not appear in the polynomials v, v_i, z_j. Considering Ex_r we thus have

$$ux_r = \sum u_i x_r v_i + \sum w_j x_r z_j - x_r v.$$

But this is possible only if $ux_r = 0$, and so $u = \sum a_{i1}a_{i2}\ldots a_{in_i} = 0$.

Therefore, the proposition will be proved by showing that the following is true for all $a_{ij} \in L$:

e4 (2.4) $$\sum a_{i1}a_{i2}\ldots a_{in_i} = 0 \implies \sum \mathrm{ad}(a_{i1})\,\mathrm{ad}(a_{i2})\ldots \mathrm{ad}(a_{in_i}) = 0.$$

Note that there is no loss of generality in assuming that all the a_{ij}'s are Lie monomials; this is simply because every element in L is a sum of Lie monomials. We shall prove (2.4) by induction on the maximal degree d of the Lie monomials a_{ij} appearing in (2.4).

If $d = 1$, that is, if each $a_{ij} = \lambda_{ij}x_{ij}$ for some $\lambda_{ij} \in \mathbb{C}$ and $x_{ij} \in X$, then (2.4) trivially holds. Namely, different elements of the form $x_{i1}x_{i2}\ldots x_{in}$ are linearly independent, and so the left hand side of (2.4) is 0 only in the trivial situation when the sum of the coefficients at each $x_{i1}x_{i2}\ldots x_{in}$ is equal to zero. This clearly forces the right hand side to be 0 too.

Let $d > 1$. Each a_{ij} of degree d can be written as $[b_{ij}, c_{ij}]$ where b_{ij} and c_{ij} are Lie monomials of degree $< d$. Let $a_{ir_1}, \ldots, a_{ir_{k_i}}$, $r_1 < \ldots < r_{k_i}$, be these elements.

Then we have

$$\begin{aligned} 0 &= \sum a_{i1} \dots a_{ir_1} \dots a_{ir_{k_i}} \dots a_{in_i} \\ &= \sum a_{i1} \dots [b_{ir_1}, c_{ir_1}] \dots [b_{ir_{k_i}}, c_{ir_{k_i}}] \dots a_{in_i} \\ &= \sum \Big(a_{i1} \dots b_{ir_1} c_{ir_1} \dots b_{ir_{k_i}} c_{ir_{k_i}} \dots a_{in_i} - a_{i1} \dots c_{ir_1} b_{ir_1} \dots b_{ir_{k_i}} c_{ir_{k_i}} \dots a_{in_i} \pm \\ &\dots + (-1)^{k_i} a_{i1} \dots c_{ir_1} b_{ir_1} \dots c_{ir_{k_i}} b_{ir_{k_i}} \dots a_{in_i} \Big). \end{aligned}$$

We are now in a position to apply the induction assumption. Accordingly,

$$\begin{aligned} 0 &= \sum \Big(\mathrm{ad}(a_{i1}) \dots \mathrm{ad}(b_{ir_1}) \mathrm{ad}(c_{ir_1}) \dots \mathrm{ad}(b_{ir_{k_i}}) \mathrm{ad}(c_{ir_{k_i}}) \dots \mathrm{ad}(a_{in_i}) - \\ &- \mathrm{ad}(a_{i1}) \dots \mathrm{ad}(c_{ir_1}) \mathrm{ad}(b_{ir_1}) \dots \mathrm{ad}(b_{ir_{k_i}}) \mathrm{ad}(c_{ir_{k_i}}) \dots \mathrm{ad}(a_{in_i}) \pm \\ &\dots + (-1)^{k_i} \mathrm{ad}(a_{i1}) \dots \mathrm{ad}(c_{ir_1}) \mathrm{ad}(b_{ir_1}) \dots \mathrm{ad}(c_{ir_{k_i}}) \mathrm{ad}(b_{ir_{k_i}}) \dots \mathrm{ad}(a_{in_i}) \Big) \\ &= \sum \mathrm{ad}(a_{i1}) \dots [\mathrm{ad}(b_{ir_1}), \mathrm{ad}(c_{ir_1})] \dots [\mathrm{ad}(b_{ir_{k_i}}), \mathrm{ad}(c_{ir_{k_i}})] \dots \mathrm{ad}(a_{in_i}). \end{aligned}$$

However, $[\mathrm{ad}(b_{ij}), \mathrm{ad}(c_{ij})] = \mathrm{ad}(a_{ij})$ and so we have $\sum \mathrm{ad}(a_{i1}) \, \mathrm{ad}(a_{i2}) \dots \mathrm{ad}(a_{in_i}) = 0$. $\square$

In the two propositions that follow we will present two types of examples of tractable Lie algebras. The first one deals with Lie algebras that are in some sense close to associative ones. We say that a Lie subalgebra L of an associative algebra A is *closed under triads* if $aba \in L$ for all $a, b \in L$. Besides the obvious example where L itself is an associative algebra, this also covers an important case where L is a Lie algebra of all skew-symmetric elements of an associative algebra A with involution $*$: $L = \{a \in A \,|\, a^* = -a\}$.

p0 **Proposition 2.4.** *Let L be a Lie subalgebra of an associative algebra A. If L is closed under triads, then L is tractable, unless $acbab + babca = abacb + bcaba$ for all $a, b, c \in L$.*

Proof. Pick $a, b \in L$, and set

$$E = \mathrm{ad}(aba)\, \mathrm{ad}(b) - \mathrm{ad}(a)\, \mathrm{ad}(bab).$$

A straightforward computation shows that E can be represented as

$$E = \mathrm{L}_a \mathrm{R}_{bab} + \mathrm{L}_{bab} \mathrm{R}_a - \mathrm{L}_{aba} \mathrm{R}_b - \mathrm{L}_b \mathrm{R}_{aba}.$$

Accordingly, E is a Lie elementary operator on L which belongs to $\mathcal{E}\ell_0(A)$. Thus L is tractable, unless $Ec = acbab + babca - abacb - bcaba = 0$ for every $c \in L$. $\square$

The conditon $acbab + babca = abacb + bcaba$ for all $a, b, c \in L$ means that L satisfies a very special polynomial identity. Using the theory of polynomial identities it would be possible to analyse this situation for some classes of Lie algebras; for example, if $L = A$ is a prime algebra or if L is the set of skew-symmetric elements of a prime algebra A with involution, then using standard methods one could show that this polynomial identity can hold on L only in some exceptional cases. But this analysis would lead us to far from the scope of this paper.

For the proof of the second proposition we need some auxiliary results. The first one might be interesting in its own right. In particular it shows that every simple Lie algebra which is also a Lie ideal of an associative algebra, is, up to an isomorphism, equal to $[B, B]$ where B is a simple associative algebra. This is kind of a converse to Herstein's theorem stating that $[B, B]$ is a simple Lie algebra if B is a simple associative algebra and $[B, B]$ has trivial intersection with the center of R [3, Theorem 1.12]. In the proof we shall need another theorem by Herstein saying that a Lie ideal of a simple algebra B either contains $[B, B]$ or is contained in the center of B [3, Theorem 1.3]. All these is true under the assumption that $\mathrm{char}(F) \neq 2$.

L1 **Lemma 2.5.** *Let L be a Lie ideal of an associative algebra A. If L is simple (as a Lie algebra) and char$(F) \neq 2$, then $B = \langle L\rangle/\mathrm{ann}(L)$ is a simple associative algebra and $L \cong L/\mathrm{ann}(L) = [B, B]$.*

Proof. Pick $a \in \langle L\rangle \setminus \mathrm{ann}(L)$. We will prove the simplicity of B by showing that the ideal of $\langle L\rangle$ generated by a is equal to $\langle L\rangle$.

If a does not lie in the center Z of $\langle L\rangle$, that is, if $[a, \langle L\rangle] \neq 0$, then $0 \neq [a, l] \in L$ for some $l \in L$. Therefore the ideal of the Lie algebra L generated by $[a, l]$ is equal to L. Consequently, L is contained in the ideal of $\langle L\rangle$ generated by a. Since the algebra $\langle L\rangle$ is generated by L, this ideal is equal to $\langle L\rangle$.

Assume therefore that $a \in Z$. We claim that there is $x \in \langle L\rangle$ such that $ax \notin Z$. Indeed, if ax was in Z for every $x \in \langle L\rangle$, then we would have $0 = [ax, l]$ for every $l \in L$. Since a commutes with l, this implies that $a[x, l] = 0$. That is, $a[\langle L\rangle, L] = 0$. However, $[\langle L\rangle, L]$ is an ideal of L, and certainly $[\langle L\rangle, L] \neq 0$ since L is noncommutative. Therefore $[\langle L\rangle, L] = L$. Thus, $aL = 0$, which clearly yields $a\langle L\rangle = 0$, and so also $\langle L\rangle a = 0$. But this contradicts $a \notin \mathrm{ann}(L)$. This proves that $ax \notin Z$ for some $x \in \langle L\rangle$. But then, by what was proved in the preceding paragraph, the ideal of $\langle L\rangle$ generated by ax is equal to $\langle L\rangle$. But then the ideal generated by a is also equal to $\langle L\rangle$.

Note that $L \cap \mathrm{ann}(L) = 0$ in view of the simplicity of L. Therefore the restriction of the quotient map $x \mapsto x + \mathrm{ann}(L)$ to L is an isomorphism between Lie algebras L and $M = L/\mathrm{ann}(L)$. Of course, M is a Lie ideal of B. It is clear that M is not contained in the center of B (since otherwise $[L, \langle L\rangle]$ would be contained in $\mathrm{ann}(L)$). Therefore $M \supseteq [B, B]$ by Herstein's theorem [3, Theorem 1.3]. Since M is simple and $[B, B]$ is obviously an ideal of M, it follows that $M = [B, B]$. $\square$

The next lemma must be known, but we give the proof since it is rather short. We shall need it only for the case when B is a simple algebra, but in view of the proof it is more convenient to state it for prime algebras.

L2 **Lemma 2.6.** *Let B be a noncommutative prime algebra. If $b, c \in B$ are such that $b[B, B]c = 0$, then $b = 0$ or $c = 0$.*

Proof. Since non-zero elements in the center of a prime ring are not zero divisors, we may assume that at least one of b, c does not lie in the center. Without loss of generality we may assume that b is not in the center.

We have $b[x, y]c = 0$ for all $x, y \in B$, that is, $bxyc = byxc$. In particular, $bx(by)c = b^2yxc = b^2xyc$. Thus, $(bxb - b^2x)Bc = 0$ for every $x \in B$. Sice B is prime, this yields $c = 0$ or $bxb = b^2x$. If the latter occurs, then we have $bxyb = b^2xy = bxby$ for all $x, y \in B$, i. e., $bB[b, B] = 0$. Therefore $[b, B] = 0$, that is, b lies in the center of B. Hence $c = 0$. $\square$

r3 **Lemma 2.7.** *Let L be a Lie subalgebra of an associative algebra A. Suppose there exists $a \in L$ such that $a^n \in \mathrm{ann}(L)$, $a^{n-1}La^{n-1} \neq 0$, and n does not divide char(F). Then L is tractable.*

Proof. Note that the Lie elementary operator $E = (\mathrm{ad}(a))^n$ coincides with an operator in $\mathcal{E}\ell^\circ(A)$ on L; namely, u and v from (2.3) lie in $\mathrm{ann}(L)$. Since $\mathrm{R}_{a^{n-2}}E = -n\mathrm{L}_{a^{n-1}}\mathrm{R}_{a^{n-1}}$, E is not 0 on L. □

We now have enough information to prove our final proposition in this section.

p1 **Proposition 2.8.** *Let L be a Lie ideal of an associative algebra A over a field F with char$(F) = 0$. Suppose that L is a simple Lie algebra and that L contains a non-zero nilpotent element. Then L is tractable.*

Proof. Since L contains non-zero nilpotents and since $L \cap \mathrm{ann}(L) = 0$ by simplicity of L, there exists $a \in L$ and $n \geq 2$ such that $a^n \in \mathrm{ann}(L)$ and $a^{n-1} \notin \mathrm{ann}(L)$. Using the notation of Lemma 2.5 we set $b = a^{n-1} + \mathrm{ann}(L) \in B$. Since B is simple, we have $b[B,B]b \neq 0$ by Lemma 2.6. In view of Lemma 2.5 we then also have $a^{n-1}La^{n-1} \neq 0$. Therefore L is tractable by Lemma 2.7. □

3. Weakly tractable subspaces of an algebra and topological
3 non-simplicity

In what follows normed spaces and algebras are assumed to be complex. By an operator on a normed space we mean a bounded linear operator.

Let L be a Lie algebra of compact quasinilpotent operators on a Banach space X, $\dim L > 1$. The Wojtyński's problem (see the Introduction) is to show that L has a non-trivial closed Lie ideal. The relation with the previous material is that if L is tractable then the answer is positive, as we shall see. Let us consider a more general situation.

Let A be an algebra. If V is a subspace of A, let us denote by $\mathcal{E}\ell_V(A)$ the set of all operators $T \in \mathcal{E}\ell(A)$ that preserve V: $TV \subset V$. Then $\mathcal{E}\ell_V(A)$ is a subalgebra of $\mathcal{E}\ell(A)$. For example, if V is a Lie (Jordan) subalgebra of A then all operators $\mathrm{ad}(a) : x \mapsto ax - xa$ (respectively $p_a : x \mapsto ax + xa$), for $a \in V$, belong to $\mathcal{E}\ell_V(A)$. If V is a Lie (Jordan) ideal of A, then the same is true for all $a \in A$.

Let V be a subspace of A. A subspace W of V is called *El-stable* with respect to V if all operators in $\mathcal{E}\ell_V(A)$ preserve W. In other words, W is *El*-stable if $\mathcal{E}\ell_V(A) \subseteq \mathcal{E}\ell_W(A)$. We say that V is *El-simple* (with respect to A) if it does not have non-trivial *El*-stable subspaces. If A is normed, V is called *topologically El-simple* if it does not have non-trivial *El*-stable closed subspaces.

Returning to the above examples we see that if V is a Lie subalgebra, a Jordan subalgebra or a Lie ideal of A, then all its *El*-stable subspaces with respect to A are, respectively, Lie ideals of V, Jordan ideals of V or Lie ideals of A.

The converse is not true. The simplest example is the following. Let A be the algebra of all operators on a finite-dimensional space X over $\mathbb{C}$. Then, as it is well known and easy to see, $\mathcal{E}\ell(A)$ is the set of all operators on A. So $\mathcal{E}\ell_V(A)$ is the set of all operators leaving V invariant, whence V has no non-trivial *El*-stable subspaces. Now, if V is a non-simple Lie subalgebra of A, then V has non-trivial Lie ideals that are not *El*-stable.

Therefore the statement that a Lie subalgebra, Jordan subalgebra, or Lie ideal V of a (normed) algebra A is not (topologically) *El*-simple, is a priori stronger than the statement that V is not a (topologically) simple Lie algebra, (topologically) simple Jordan algebra, or a minimal (closed) Lie ideal of A.

Let $\mathcal{E}\ell^\circ_V(A) = \mathcal{E}\ell^\circ(A) \cap \mathcal{E}\ell_V(A)$. A subspace V of an algebra A will be called *weakly tractable* (with respect to A) if $\mathcal{E}\ell^\circ_V(A)V \neq 0$. It is clear that every tractable

Lie algebra is weakly tractable (with respect to the same A). Let us show that Proposition 2.3 can be extended to weak tractability.

p2w **Proposition 3.1.** *Let L and A be as in Proposition 2.3. Then L is not weakly tractable.*

Proof. In view of Proposition 2.3 it suffices to show that every $E \in \mathcal{E\ell}_L(A)$ is a Lie elementary operator. Set $E = \sum_k L_{w_k} R_{z_k}$ where $w_k, z_k \in A \cup \{1\}$. We are assuming that Ef is a Lie polynomial (i.e., a sum of Lie monomials) whenever $f \in A$ is a Lie polynomial. Without loss of generality we may assume that none of the polynomials w_k, z_k involves x_1. Since x_1 is a Lie polynomial, it follows that $Ex_1 = \sum w_k x_1 z_k$ is a Lie polynomial, which is linear in x_1. Therefore there exists a Lie elementary operator $D = \sum \mathrm{ad}(x_{i1})\mathrm{ad}(x_{i2})\ldots\mathrm{ad}(x_{in_i})$, $x_{ij} \neq x_1$, such that $Ex_1 = Dx_1$; indeed, this follows easily from $\mathrm{ad}([u,v]) = [\mathrm{ad}(u), \mathrm{ad}(v)]$. So we have

$$\sum w_k x_1 z_k = \sum [x_{i1}, [x_{i2}, [\ldots, [x_{in_i}, x_1]\ldots]]].$$

Now, this is the identity in the free algebra, and so we may replace x_1 by any other element. Therefore,

$$\sum w_k g z_k = \sum_l [x_{i1}, [x_{i2}, [\ldots, [x_{in_i}, g]\ldots]]]$$

holds for every $g \in A$. That is, $Eg = Dg$ for every $g \in A$, i.e., $E = D$. □

Note that if A is an algebra of compact operators, then the ideal $\mathcal{E\ell}^\circ(A)$ of $\mathcal{E\ell}(A)$ consists of compact operators [17].

El-simple **Proposition 3.2.** *Let V be a closed subspace in a closed algebra A of compact quasinilpotent operators, and let* $\dim(V) > 1$. *If V is weakly tractable with respect to A, then V is not topologically El-simple with respect to A.*

Proof. By [15, Lemma 5.10], all elementary operators on A are quasinilpotent. Let $\mathcal{M}$ be the algebra of the restrictions of all operators in $\mathcal{E\ell}_V(A)$ to V. Since $\mathcal{E\ell}(A)$ consists of quasinilpotents, the same is true for $\mathcal{M}$. Since all operators in $\mathcal{E\ell}^\circ(A)$ are compact, it follows from our assumptions that $\mathcal{M}$ contains a non-zero compact operator. It can be easily seen from Lomonosov's Lemma [9] that if an algebra of quasinilpotent operators contains a non-zero compact operator then it has a non-trivial invariant subspace. Clearly a subspace invariant under $\mathcal{M}$ is an *El*-stable subspace of V. □

Lie-simple **Corollary 3.3.** *Let L be a closed non-one-dimensional Lie algebra L of compact quasinilpotent operators on a Banach space X. If L is weakly tractable with respect to the closure of $\langle L\rangle$, then L is not topologically simple.*

Proof. Let A be the closed subalgebra in $B(X)$ generated by L. By [14, Corollary 11.6], it consists of compact quasinilpotent operators. Since L is weakly tractable, the condition $\mathcal{E\ell}_L^\circ(A)L \neq 0$ holds. By Proposition 3.2, L is not a topologically *El*-simple subspace of A. Since each *El*-stable subspace of L is an ideal of L, we conclude that L is not topologically simple as a Lie algebra. □

Jordan-simple **Corollary 3.4.** *A closed non-one-dimensional Jordan algebra J of compact quasinilpotent operators on a Banach space X is not topologically simple.*

Proof. Let A be the closed algebra generated by J. It follows easily from [5, Corollary 11.1], that A consists of compact quasinilpotent operators. For each $a \in A$, let p_a be the operator on A defined as above. Then $p_a \in \mathcal{E}\ell_J(A)$ for each $a \in J$. It follows that $w_a = \frac{1}{2}(p_a^2 - p_{a^2})$ belongs to $\mathcal{E}\ell_J(A)$ for $a \in J$. Since $w_a(x) = axa$, we have that $w_a \in \mathcal{E}\ell^\circ(A)$. If $w_a(J) \neq 0$ for some $a \in J$, then, by Proposition 3.2, J has a non-trivial *El*-stable closed subspace, which is clearly a Jordan ideal.

Suppose that $w_a(J) = 0$ for all $a \in J$. Then

$$(\mathrm{L}_a\mathrm{R}_b + \mathrm{L}_b\mathrm{R}_a)J = (w_{a+b} - w_a - w_b)(J) = 0$$

for every $a, b \in J$. Hence $(ab + ba)c + c(ab + ba) = 0$ for all $a, b, c \in J$. Let I_b be the closed ideal of J generated by an element b in J. If $ab + ba \neq 0$, for some $a, b \in J$, then I_{ab+ba} is a required ideal. If, however, $ab + ba = 0$ for all $a, b \in J$, then $I_b = \mathbb{C}b$ is a required ideal for any non-zero b. □

For each algebra A, set $Z_2(A) = \{x \in A : a^2xa = axa^2 \text{ for all } a \in A\}$.

Lie-ideals **Corollary 3.5.** *Let A be a closed algebra of compact quasinilpotent operators. Then each minimal closed Jordan ideal of A is one-dimensional, and each minimal closed Lie ideal of A is either one-dimensional or contained in $Z_2(A)$.*

Proof. Suppose that J is a minimal closed Jordan ideal of A with $\dim(J) > 1$. Then $\mathcal{E}\ell_J(A) = \mathcal{E}\ell(A)$, and we obtain as above, in terms of the proof of Corollary 3.4, that $w_a(J) = 0$ for all $a \in A$. If $ab + ba \neq 0$, for some $a \in A, b \in J$, then the Jordan ideal I_{ab+ba} of A is one-dimensional, a contradiction. If, however, $ab+ba = 0$ for all $a \in A, b \in J$, then $I_b = \mathbb{C}b$ is a Jordan ideal of A, for any non-zero $b \in J$, a contradiction.

Suppose that V is a minimal closed Lie ideal in A with $\dim(V) > 1$. Since V is a Lie ideal, each operator $T_a = (\mathrm{ad}(a))^3 - \mathrm{ad}(a^3)$ leaves it invariant: $T_a \in \mathcal{E}\ell_V(A)$ for $a \in A$. If $T_aV = 0$ then $a^2xa = axa^2$ for all $x \in V$. Thus if V is not contained in $Z_2(A)$ then there is $a \in A$ with $T_aV \neq 0$. Since $T_a \in \mathcal{E}\ell^\circ(A)$, it follows from Proposition 3.2 that V is not topologically *El*-simple. Hence V is not a minimal closed Lie ideal of A. □

Let us call a normed algebra *bicompact* if all operators $\mathrm{L}_a\mathrm{R}_b$, $a, b \in A$, are compact. Proposition 3.2 and Corollary 3.5 can be easily extended to the case that A is an arbitrary bicompact Jacobson-radical Banach algebra. The following example shows that a bicompact radical Banach algebra can have a minimal closed Lie ideal of infinite dimension.

Example 3.6. It was shown in [2] that for any quasinilpotent operator $T \in B(X)$ there is an algebraic norm $|||\cdot|||$ majorizing the operator norm on the algebra $A(T)$ generated by T, such that the completion B of $A(T)$ with respect to $|||\cdot|||$ is a bicompact radical Banach algebra. Choosing for T a quasinilpotent operator without invariant subspaces (its existence was proved in [11]) we obtain that there is a bicompact radical Banach algebra B with a non-trivial topologically irreducible representation π on a Banach space X. *Let* $A = B \oplus X$ with multiplication $(a \oplus x)(b \oplus y) = ab \oplus \pi(a)y$. Then A is a bicompact radical Banach algebra. The subspace $J = \{0\} \oplus X$ is a Lie ideal of A and it is easy to check that it does not contain smaller closed Lie ideals.

Let WOT denote the weak operator topology on $B(X)$. Let us say that a set $\mathcal{U}$ of operators is *weakly finitely generated* if there is a finite subset $\mathcal{W}$ of $\mathcal{U}$ such that the WOT-closed algebra generated by $\mathcal{W}$ contains $\mathcal{U}$.

For a closed Lie subalgebra L of a Banach algebra A, let $K_2(L)$ be the set of all $a \in L$ such that $\mathrm{L}_{a^2} + \mathrm{R}_{a^2}$ is compact as an operator from L into $\langle L\rangle$. *Then $K_2(L)$ is invariant under the set* $\exp(\mathrm{ad}(L))$ *of inner automorphisms of* L. Indeed, if Q is the unit ball of L, $\varphi_t = \exp(t\,\mathrm{ad}(b))$ for $b \in L$ and every $t \in \mathbb{R}$, then φ_t is the restriction to L of the automorphism $\mathrm{L}_{\exp(tb)}\mathrm{R}_{\exp(-tb)} = \exp(\mathrm{L}_{tb})\exp(\mathrm{R}_{-tb})$ of the closed algebra generated by L, and

$$\left(\mathrm{L}_{\varphi_t(a)^2} + \mathrm{R}_{\varphi_t(a)^2}\right) Q = \mathrm{L}_{\exp(tb)}\mathrm{R}_{\exp(-tb)}\left(\mathrm{L}_{a^2} + \mathrm{R}_{a^2}\right)\varphi_{-t}(Q).$$

for each $a \in K_2(L)$. As $\varphi_{-t}(Q)$ is a bounded subset of L, the set $\left(\mathrm{L}_{a^2} + \mathrm{R}_{a^2}\right)\varphi_{-t}(Q)$ is precompact, whence $\mathrm{L}_{\varphi_t(a)^2} + \mathrm{R}_{\varphi_t(a)^2}$ is compact as an operator from L into $\langle L\rangle$.

zel **Theorem 3.7.** *Let L be a closed non-one-dimensional Lie algebra of compact quasinilpotent operators, and let N be a set in $K_2(L)$ invariant under* $\exp(\mathrm{ad}(L))$. *If N is non-zero and weakly finitely generated, then L is not topologically simple.*

Proof. Let A be the closure of $\langle L\rangle$. By [14, Corolary 11.6], A consists of compact quasinilpotent operators. By [15, Lemma 5.10], $\mathcal{E}\ell(A)$ consists of quasinilpotents. For $a \in N$, clearly $\mathrm{ad}(a)^2 = \mathrm{L}_{a^2} + \mathrm{R}_{a^2} - 2\mathrm{L}_a\mathrm{R}_a$ is a compact operator from L into $\langle L\rangle$, therefore it is a compact operator on L. If $\mathrm{ad}(a)^2$ is non-zero on L then $\mathcal{E}\ell_L(A)$ has in L a non-trivial closed invariant subspace by Lomonosov's Lemma [9]. This subspace is a closed ideal of L.

So it suffices to consider the case that $\mathrm{ad}(a)^2 = 0$ for all $a \in N$. Let K be a finite subset of N weakly generating N. By Zelmanov's theorem [20, Theorem 1], the elements with the property $\mathrm{ad}(a)^2 = 0$ generate a locally nilpotent subalgebra of L. Thus the Lie algebra generated by K is nilpotent, hence its center is non-zero. So there is a non-zero element $b \in L$ commuting with K. It follows that b commutes with the WOT-closed algebra generated by K, hence with N.

The closed linear span I of N is an ideal of L. Indeed, for each $b \in L$, the automorphisms $\varphi_t = \exp(t\,\mathrm{ad}(b))$ of L preserve N. So these automorphisms preserve I, whence

$$\mathrm{ad}(b)(x) = \lim_{t\to 0}\frac{1}{t}(\varphi_t(x) - x) \in I$$

for each $x \in I$.

Hence if L is topologically simple then $I = L$. It follows that b commutes with all elements of L, and so $\mathbb{C}b$ is a closed ideal of L, a contradiction. □

An element a of a normed algebra A is called *completely continuous* if L_a and R_a are compact operators on A.

zel1 **Corollary 3.8.** *Let L be a closed non-one-dimensional Lie algebra of compact quasinilpotent operators, $N_1 = \left\{a \in L : a^2 = 0\right\}$, $N_2 = \left\{a \in L : a^2 \in \mathrm{ann}(L)\right\}$, let N_3 be the set of all $a \in L$ such that a^2 commutes with each element of L, N_4 the set of all $a \in L$ such that a^2 is completely continuous element of $\langle L\rangle$, and let N_5 be the set of all $a \in L$ such that L_{a^2} and R_{a^2} are compact as operators from L into $\langle L\rangle$. If N_i is non-zero and weakly finitely generated for some i, then L is not topologically simple.*

Proof. All sets are invariant under $\exp(\mathrm{ad}(L))$. Clearly $N_1 \subset N_2 \subset N_4 \subset N_5 \subset K_2(L)$, and $N_3 \subset N_4$ by [1, Theorem 1]. So the statement follows by Theorem 3.7. □

The last result of the section establishes the topological non-simplicity in a quite special situation. In the next section it will be used to establish the algebraic non-simplicity in a very general case (see Theorem 5.2).

Recall that for each subspace $Y \subset X$ and for each operator a leaving Y invariant, one can consider the restriction $a|Y$ of a to Y and the quotient operator $a|(X/Y)$ on X/Y which sends $x+Y$ to $ax+Y$. We denote $a|Y$ by $\pi_Y(a)$ and $a|(X/Y)$ by $\pi_{X/Y}(a)$. The maps $a \mapsto \pi_Y(a)$ and $a \mapsto \pi_{X/Y}(a)$ are representations of the algebra of all operators leaving Y invariant; they are called the *restriction representation* and *quotient representation* defined by Y.

i44 **Theorem 3.9.** *Let L be an infinite-dimensional Banach Lie algebra. Assume that there is a bounded non-zero representation h of L by finite rank operators on a normed space X. Then L is not topologically simple. If moreover h is injective, then L has a proper closed ideal of finite codimension.*

Proof. Without loss of generality, one may assume that X is complete and h is injective. Every $x \in X$ defines an operator $U_x : L \longrightarrow X$ by $U_x(a) = h(a)\,x$. The space $\mathcal{U}$ of operators U_x is locally finite-dimensional. Indeed, $\mathcal{U}a = h(a)\,X$ is finite-dimensional for every $a \in L$.

By theorem of Livshits [8], there are a finite-dimensional space $\mathcal{U}_1$ of operators from L to X and a finite-dimensional subspace W of X such that $\mathcal{U} \subset \mathcal{U}_1 + \mathcal{U}_W$, where $\mathcal{U}_W$ is the space of all (bounded) operators from L to X with ranges in W. It follows that $\dim(\mathcal{U}/\mathcal{U} \cap \mathcal{U}_W) < \infty$. Let $X_0 = \{x \in X : U_x \in \mathcal{U}_W\}$. Then X_0 is closed in X and has finite codimension in X. It is clear that $h(L)\,X_0 \subset W$.

Let $Y = \{x \in X : \dim(h(L)\,x) < \infty\}$. It is a subspace of X. As $h(L)\,h(a)\,x \subset h(L)\,x + h(a)\,h(L)\,x$ for every $a \in L$ and $x \in X$, we see that $h(L)\,h(a)\,x$ is of finite dimension for every $x \in Y$, i.e., Y is invariant for $h(L)$. As Y contains X_0, it is a closed subspace of finite codimension in X. The quotient representation $\pi_{X/Y}$ of $h(L)$ is of finite rank, so its kernel is of finite codimension. If $\pi_{X/Y} \neq 0$, then $\{a \in L : \pi_{X/Y}(h(a)) = 0\}$ is a proper closed ideal of finite codimension in L.

Assume that $\pi_{X/Y} = 0$. Since $\pi_Y(h(L))$ is locally finite-dimensional, it follows as above from the theorem of Livshits that there is a finite-dimensional subspace Z of Y such that $\dim(\pi_Y(h(L))/\pi_Y(h(L_Z))) < \infty$, where L_Z is the space of all $a \in L$ such that $h(a)\,Y \subset Z$.

Clearly L_Z is a closed subalgebra of finite codimension in L. By [7, Theorem 7.1], if $L_Z \neq L$ then L has a proper closed ideal of finite codimension. So we may assume that $L_Z = L$, that is $\pi_{Y/Z} = 0$. Then Z is an invariant subspace for $h(L)$. The restriction representation π_Z of $h(L)$ is of finite rank, so its kernel is of finite codimension. If $\pi_Z \neq 0$, then $\{a \in L : \pi_Z(h(a)) = 0\}$ is a proper closed ideal of finite codimension in L.

Assume that $\pi_Z = 0$. As $\pi_{X/Y} = \pi_{Y/Z} = \pi_Z = 0$, then $h(a)\,h(b)\,h(c) = 0$ for every $a, b, c \in L$. Hence $[[L, L], L] = 0$ and any closed subspace of L containing $[L, L]$ is an ideal of L. If $[L, L]$ is dense in L, then L is commutative, and every closed subspace of L is an ideal of L. In any case, L has a proper closed ideal of finite codimension. □

4

4. Around the triangularizability

A set Γ of closed subspaces of a normed space X is called a *closed subspace chain* if it is linearly ordered by inclusion. A closed subspace chain is called *complete* if it contains the intersection and the closure of the sum of subspaces of an arbitrary its subchain.

The *gaps* of a complete closed subspace chain Γ are defined as pairs $(Z,Y) \in \Gamma\times\Gamma$ with $Z \subsetneqq Y$ such that there are no subspaces in Γ intermediate between Z and Y. The quotients Y/Z for such pairs are called the *gap-quotients* of Γ. If Γ consists of invariant subspaces for a set M of operators on X, every gap (Z,Y) induces a representation $\pi_{Y/Z}$ of M by operators on Y/Z given by

$$\pi_{Y/Z}(a)(y+Z) = ay + Z,$$

which is called a *gap-representation* of M with respect to Γ.

A closed subspace chain is called a *maximal closed subspace chain* if it is not a subchain of a larger chain. This is equivalent to the condition that all its gap-quotients (if they exist) are one-dimensional. Complete chains without gap-quotients are called *continuous*.

A set M of operators on X is called *triangularizable* if there is a maximal closed subspace chain consisting of invariant subspaces for M.

It was proved by Ringrose [12] (see also [10, Theorem 5.12]) that if a complete closed subspace chain Γ in a Banach space X consists of subspaces invariant for a compact operator a then $\mathrm{Sp}_*(a) = \cup \mathrm{Sp}_*(\pi_{Z/Y}(a))$ where the union is over all gaps $Y \subset Z$ of Γ, and Sp_* denotes the non-zero part of spectrum: $\mathrm{Sp}_*(a) = \mathrm{Sp}(a) \setminus \{0\}$. In particular, if Γ is maximal then a is quasinilpotent if and only if $\pi_{Z/Y}(a) = 0$ for all gaps. It follows that if a Lie or Jordan algebra J is triangularizable then the set of all quasinilpotent compact operators in J is a closed ideal of J.

We will apply the following analogue of the Ringrose Theorem:

ring-anal **Lemma 4.1.** *Let Γ be a maximal closed subspace chain in a normed space X. A finite rank operator a that leaves invariant all subspaces in Γ is nilpotent if and only if it belongs to the kernels of all gap-representations of Γ.*

The non-trivial part of the proof is contained in the proof of Lemma 5.6 below, and we omit it here.

iid03 **Theorem 4.2.** *Let J be a non-one-dimensional, Lie or Jordan, algebra of operators on a normed space X. If J contains a non-zero finite rank operator and if J is triangularizable, then J is not topologically simple.*

Proof. Assuming the contrary, we need only to consider the case when the ideal I of all finite rank operators in J is dense in J.

Let Γ be a maximal closed subspace chain consisting of invariant subspaces for J, and let, as usual, π_Y be the restriction representation of J on Y, for every $Y \in \Gamma$. Assume firstly that π_Y is not zero for every non-zero $Y \in \Gamma$.

Let $\Gamma_a = \{Y \cap aX : Y \in \Gamma\}$ for any $a \in J$. If a is a finite rank operator in J, then Γ_a consists of a finite number of finite-dimensional subspaces invariant for a. Let $(0,Z)$ be the gap of Γ_a, and let Y be a subspace in Γ such that $Y \cap aX = 0$. As $aY = aY \cap aX \subset Y \cap aX = 0$, we obtain that $\pi_Y(a) = 0$. If $Y \neq 0$ then $\pi_Y \neq 0$ and the kernel of π_Y is a non-trivial ideal of J, a contradiction. If there are no non-zero $Y \in \Gamma$ with $Y \cap aX = 0$, then every non-zero subspace in Γ contains Z.

This implies that there is a gap-quotient of Γ containing Z. As gap-quotients of Γ are one-dimensional, $Z \in \Gamma$ and Z is one-dimensional.

When J is a Lie algebra, $\pi_Z([b,c]) = 0$ for every $b, c \in J$. As the kernel of π_Z is zero, J is commutative and is a sum of one-dimensional ideals, a contradiction.

Consider the case when J is a Jordan algebra. If I contains a non-zero nilpotent operator, one may assume, using Lemma 4.1, that J has a dense ideal I_0 of nilpotent finite rank operators. Then π_Z vanishes on I_0 and therefore on J, a contradiction. Therefore I has no nilpotent operators. As the kernel of π_Z is zero and $\pi_Z([[b,c],d]) = 0$, we obtain that

tozh

$$[[b,c],d] = 0, \text{ for every } b, c, d \in J. \tag{4.1}$$

Now it remains to prove that if a Jordan algebra I of finite-rank operators on a normed space X does not contain non-zero nilpotent operators and satisfies (4.1) then it is not simple. As the algebra generated by $a \in I$ is a commutative semisimple finite-dimensional algebra, it is a finite direct sum of simple algebras each of which is isomorphic to the field. So a is a finite linear combination $\sum \lambda_i p_i$ of orthogonal projections that also belong to I. It is easy to see from $[[p_i, b], p_i] = 0$ that each p_i commutes with every $b \in J$. Hence J is a semisimple commutative associative algebra. If I contains at least two orthogonal non-zero projections p and q, then J has non-trivial ideals pJ and qJ, otherwise I is one-dimensional and $J = I$. In any case, we have a contradiction.

Now consider the general case when $\pi_Y = 0$ for some non-zero $Y \in \Gamma$. Then there is a largest subspace $W \in \Gamma$ such that $\pi_W = 0$ and $\pi_Y \neq 0$ for every $Y \in \Gamma$ properly contaning W. As $\pi_X \neq 0$, then $W \neq X$. Let $\pi_{X/W}$ be the quotient representation of L on X/W. Assume first that $\pi \neq 0$. Note that $\pi_{X/W}(L)$ is a strictly triangularizable set of operators on X/W, $\Lambda = \{Y/W : W \subset Y \in \Gamma\}$ is a maximal subspace chain consisting of invariant subspaces for $\pi_{X/W}(L)$.

Let $\pi_{Y/W}$ be the quotient representation of $\pi(J)$ on Y/W for every $Y/W \in \Lambda$. If $\pi_{Y/W} \neq 0$ for every non-zero $Y/W \in \Lambda$, then the above argument for $\pi_{Y/W}$ instead of π_Y leads to a contradiction. So there is a non-zero $Y/W \in \Lambda$ such that $\pi_{Y/W} = 0$. Then $aY \subset W$ and $aW = 0$ for every $a \in J$, whence $\pi_Y(a)\pi_Y(b) = 0$ for every $a, b \in J$. This implies that $\pi_Y(ab \pm ba) = 0$ for all $a, b \in J$. As $\pi_Y \neq 0$ and its kernel is trivial, every one-dimensional subspace of J is an ideal of J, a contradiction.

At last, assume that $\pi = 0$. The above argument for $\pi_{X/W}$ instead of $\pi_{Y/W}$ shows that J is a sum of one-dimensional ideals of J, a contradiction.

This shows that L is not topologically simple. $\square$

For an operator a acting on a non-complete normed space we use the term *quasinilpotent* if the equality $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$ holds.

corJ13

Corollary 4.3. *Let J be a non-one-dimensional Jordan algebra of quasinilpotent operators on a normed space X. If J contains a non-zero finite rank operator, then J is not topologically simple.*

Proof. Assume that J is topologically simple. Without loss of generality, one may assume that X is complete and that J consists of quasinilpotent compact operators. By [5, Corollary 11.1], J is triangularizable. Therefore J is not topologically simple by Theorem 4.2, a contradiction. $\square$

The result cannot be considered as a consequence of Theorem 3.4 because we do not assume that J is closed.

The following result was proved in [19, Theorem 6] under additional conditions of separability and closedness for Lie algebras of quasinilpotent compact operators. A normed Lie algebra L is called *Engel* if $\mathrm{ad}(a)$ is a quasinilpotent operator on L for every $a \in L$.

iid3 **Corollary 4.4.** *Let L be a non-one-dimensional Engel Lie algebra of operators on a normed space. If L contains a non-zero finite rank operator then L is not topologically simple.*

Proof. Assuming the contrary, we have that the ideal of finite rank operators in L is dense in L. Without loss of generality, one can assume that X is a Banach space. Since L consists of compact operators, it is triangularizable by [14, Corollary 11.5]. By Theorem 4.2, L is not topologically simple, a contradiction. □

Let $h : L \longrightarrow M$ be a homomorphism of Lie algebras. Then

im
$$h \cdot \mathrm{ad}_L(a) = \mathrm{ad}_M(h(a)) \cdot h \tag{4.2}$$

for every $a \in L$. Suppose that M is normed and $h(M)$ is dense in M. Then $\mathrm{ad}_M(h(a)) = 0$ if and only if $a \in \ker(h \cdot \mathrm{ad}_L)$. If $\mathrm{ad}_L(a)$ is of finite rank, then, under the same conditions to h, $\mathrm{ad}_M(h(a))$ is of finite rank.

We say that a normed Lie algebra L is *semi-Engel* if there are an Engel normed Lie algebra M and a bounded injective homomorphism $h : L \longrightarrow M$ with dense range. It is clear that every Engel normed Lie algebra is semi-Engel. Note also that every Lie algebra of quasinilpotent operators on a normed space is an Engel normed algebra.

33 **Corollary 4.5.** *Let L be a non-one-dimensional semi-Engel normed Lie algebra. If $ad(a)$ is of finite rank for some non-zero $a \in L$, then L has a non-trivial closed ideal.*

Proof. Let M be an Engel normed Lie algebra and $h : L \longrightarrow M$ be a bounded injective homomorphism with dense range. Let $N = \mathrm{ad}(M)$. It is a normed Lie algebra of quasinilpotent operators on the completion of M which contains a finite rank operator $\mathrm{ad}_M(h(a))$.

If $\mathrm{ad}_L = 0$ then L is commutative and is a sum of one-dimensional ideals. If $\mathrm{ad}_L \neq 0$ and either $\dim(N) < 1$ or $\mathrm{ad}_M(h(a)) = 0$, then $\ker(\mathrm{ad}_L)$ is a non-trivial ideal of L. Assume now that $\ker(\mathrm{ad}_L) = 0$. Then $\dim(N) > 1$ and $\mathrm{ad}_M(h(a)) \neq 0$, and N has a non-trivial closed ideal I by Corollary 4.4. Then $\{b \in L : \mathrm{ad}_M(h(b)) \in I\}$ is a non-trivial closed ideal of L. □

Let M be a set of operators. We say that M is *almost triangularizable* if there is a complete chain of closed subspaces invariant for M that admits only finite-dimensional gap-quotients (if any exist of course).

iid13 **Corollary 4.6.** *Let J be an infinite-dimensional, Lie or Jordan, algebra of operators on a normed space X. If J contains a non-zero finite rank operator and if J is almost triangularizable, then J is not topologically simple.*

Proof. Assume, to the contrary, that J is topologically simple and let Γ be a chain of closed J-invariant subspaces with only finite-dimensional gaps. Since kernels of non-zero finite rank representations of J are non-trivial ideals, one may assume that all gap-representations of J with respect to Γ are zero. So gap-quotients of Γ are one-dimensional, and J is in fact triangularizable. By Theorem 4.2, J is not topologically simple. $\square$

A normed space X is called an *operator range* if there is a bounded operator T from a Banach space Y onto X.

i44at **Theorem 4.7.** *Let J be a normed, Lie or Jordan, algebra of finite rank operators on an infinite-dimensional Banach space X. If J is an operator range then J has a non-trivial invariant closed subspace of finite dimension or codimension.*

Proof. Let $J = TV$ for some bounded operator from a Banach space V. Assume that J has no non-trivial invariant closed subspaces of finite dimension or codimension. Every $x \in X$ defines a bounded operator $S_x : V \longrightarrow X$ by $S_x(v) = (Tv)\,x$. The space $\mathcal{U}$ of operators S_x is locally finite-dimensional. Indeed, $\mathcal{U}v = (Tv)\,X$ is finite-dimensional for every $v \in V$.

By theorem of Livshits [8], there are a finite-dimensional space $\mathcal{U}_1$ of operators from V to X and a finite-dimensional subspace W of X such that $\mathcal{U} \subset \mathcal{U}_1 + \mathcal{U}_W$, where $\mathcal{U}_W$ is the space of all bounded operators from V to X with ranges in W. It follows that $\dim(\mathcal{U}/\mathcal{U} \cap \mathcal{U}_W) < \infty$. Let $X_0 = \{x \in X : S_x \in \mathcal{U}_W\}$. Then X_0 is closed in X and has finite codimension in X. If $a = Tv$ and $x \in X_0$ then $ax = (Tv)\,x = S_x(v) \in W$, so that $JX_0 \subset W$.

Let $Y = \{x \in X : \dim(Jx) < \infty\}$. It is a subspace of X. As $Jax \subset Jx + aJx$ for every $a \in J$ and $x \in X$, we see that Jax is of finite dimension for every $x \in Y$, i.e., Y is invariant for J. As Y contains X_0, it is a closed subspace of finite codimension in X. As J has no non-trivial invariant closed subspaces of finite codimension, $Y = X$.

Then J is locally finite-dimensional. It follows as above from the theorem of Livshits that there is a finite-dimensional subspace Z of X such that $\dim(J/J_Z) < \infty$, where J_Z is the space of all $a \in J$ such that $aX \subset Z$.

Clearly J_Z is a closed subalgebra of finite codimension in J. Then there is a finite-dimensional subspace J_0 such that $J = J_0 + J_Z$. As J_0 consists of finite rank operators, J_0X is finite-dimensional. As $JX \subset J_0X + J_ZX \subset J_0X + Z$, we obtain that $J_0X + Z$ is a finite-dimensional subspace invariant for J. As J has no non-trivial invariant closed subspaces of finite dimension, $JX = 0$, a contradiction. $\square$

i44at1 **Corollary 4.8.** *Let J be a normed, Lie or Jordan, algebra of finite rank operators on an infinite-dimensional Banach space X. If J is an operator range then J is almost triangularizable.*

Proof. Let Γ be a maximal closed subspace chain consisting of invariant subspaces for J. Assume, to the contrary, that there is an infinite-dimensional gap-quotient Y/Z of Γ. Let π be the quotient representation of J corresponding to Y/Z. Then $\pi(J) = \pi T(V)$ is an operator range. By Theorem 4.7, $\pi(J)$ is reducible. This implies that there is an invariant closed subspace for J between Z and Y, a contradiction. So J is almost triangularizable. $\square$

The following result is a Jordan algebra analogue of Theorem 3.9.

i44j **Corollary 4.9.** *Let J be an infinite-dimensional Banach Jordan algebra. Assume that there is a bounded non-zero representation h of J by finite rank operators on a normed space X. Then J is not topologically simple.*

Proof. Without loss of generality, one may assume that X is complete. Assume, to the contrary, that J is topologically simple. Then h is injective. As $h(J)$ is an infinite-dimensional Jordan algebra of finite operators and an operator range, then it is almost triangularizable by Corollary 4.8. Then $h(J)$ is not topologically simple by Corollary 4.6. Therefore J is not topologically simple, a contradiction. □

For Lie algebras of operators, Theorem 4.7 allows us to obtain the following statement having an independent interest.

i44l **Corollary 4.10.** *Let L be a Lie algebra of operators on an infinite-dimensional Banach space X, and let J be the set of all finite rank operators in L. If J is a non-zero operator range, then L has a non-trivial invariant closed subspace.*

Proof. If J is finite-dimensional, then the statement follows from [15, Theorem 4.33]. If J is infinite-dimensional, then, taking into account that J has a non-trivial invariant subspace of finite dimension or codimension by Theorem 4.7, the statement follows by [6, Theorems 6.4 and 7.1]. □

If L consists of compact operators then the condition $J = 0$ also implies by [13, Theorem 2] that L has a non-trivial invariant closed subspace.

5 5. Non-simplicity of Lie and Jordan algebras

In this section we present results on the existence of ideals (non-necessarily closed) in Lie or Jordan algebras of operators which act on normed spaces or on vector spaces without topology. For the sake of simplicity, vector spaces and algebras are considered over algebraically closed fields of characteristic 0.

A natural algebraic analog of Wojtyński's problem would be the question: is every Lie algebra of nilpotent operators non-simple? But this question has a negative answer, as the following example shows.

smok **Example 5.1.** Let A be a simple nil-algebra. Such exists by the celebrated result of Smoktunowicz [16]. By [3, Theorem 1.12], $L = [A, A]$ is a simple Lie algebra. If $a \in L$, then a is nilpotent, $a^n = 0$, which yields $ad(a)^{2n-1} = 0$. Thus, there exist simple Lie algebras L such that $ad(a)$ is nilpotent for all $a \in L$. The adjoint representation $a \mapsto ad(a)$ of such an algebra is injective, so $ad(L)$ is a simple Lie algebra of nilpotent operators.

To obtain positive results we impose the conditions that some operators in algebras in question are of finite rank or compact.

alg-gen **Theorem 5.2.** *Let L be an infinite-dimensional Banach Lie algebra such that $ad(a)$ has at most countable spectrum for each $a \in L$. If $h(L)$ contains a non-zero compact operator, for some bounded representation h on a normed space, then L is not simple.*

Proof. Assuming the contrary, we need only to consider the case when $h(L)$ consists of compact operators on a Banach space and h is injective. Let M be the closure of $h(L)$.

Assume first that there is an element $a \in L$ whose $\mathrm{ad}(a)$ has a non-zero spectrum. Then there is an isolated non-zero point λ in the spectrum. As $h \cdot \mathrm{ad}_L(b) =$

$\mathrm{ad}_M(h(b)) \cdot h$ for every $b \in L$, we obtain, for Riesz projections $p_\lambda(\mathrm{ad}_L(a))$ and $p_\lambda(\mathrm{ad}_M(h(a)))$ corresponding to λ, that

ip

$$
\begin{aligned}
h \cdot p_\lambda(\mathrm{ad}_L(a)) &= (2\pi i)^{-1} h \cdot \int_\Omega (\mu - \mathrm{ad}_L(a))^{-1} d\mu \\
&= (2\pi i)^{-1} \left(\int_\Omega (\mu - \mathrm{ad}_M(h(a)))^{-1} d\mu \right) \cdot h \\
&= p_\lambda(\mathrm{ad}_M(h(a))) \cdot h,
\end{aligned} \tag{5.1}
$$

where Ω is an admissible contour in $\mathbb{C}$ enclosing the point λ. As $p_\lambda(\mathrm{ad}_L(a)) \neq 0$, $p_\lambda(\mathrm{ad}_M(h(a)))$ is not zero on $h(L)$. Since the range of $p_\lambda(\mathrm{ad}_M(h(a)))$ consists of finite rank operators by [15, Lemma 3.12], $h(L)$ contains non-zero finite rank operators. Then L is not simple by Theorem 3.9, a contradiction.

Therefore one may assume that L is Engel. If $\mathrm{ad}_M(h(a))$ is not quasinilpotent, it follows from (5.1) that $p_\lambda(\mathrm{ad}_M(h(a))) = 0$ on $h(L)$ for each isolated point $\lambda \neq 0$ in the spectrum of $\mathrm{ad}_M(h(a))$. Since $h(L)$ is dense in M, we obtain that $p_\lambda(\mathrm{ad}_M(h(a))) = 0$, a contradiction. So $h(L)$ is also Engel and is not simple by Theorem 5.4. This implies that L is also not simple, a contradiction. □

alg-gen1 **Corollary 5.3.** *Let J be an infinite-dimensional closed, Lie or Jordan, algebra of operators on a Banach space. If J contains a non-zero compact operator, then J is not simple.*

Proof. The case of Lie algebras immediately follows from Theorem 5.2. Let J be Jordan and, without loss of generality, consist of compact operators. If J has finite rank operators then the assertion follows from Corollary 4.9. Otherwise J consists of quasinilpotent compact operators, and the assertion follows by Corollary 3.4. □

i43 **Theorem 5.4.** *Let L be a non-one-dimensional Engel Lie algebra of operators on a normed space X. If L contains a non-zero compact operator, then L is not simple.*

Proof. Assume, to the contrary, that L is simple. Without loss of generality, one may of course assume that X is complete. As the set $I = \{a \in L : a \text{ is compact}\}$ is an ideal of L, it coincides with L. So L is an Engel Lie algebra of compact operators on a Banach space. Let A be the closed algebra generated by L. By [14, Corollary 11.5], A is commutative modulo the Jacobson radical. Then $J = \{a \in L : a \text{ is quasinilpotent}\}$ is an ideal of L which contains $[L, L]$. If $J = 0$ then L is commutative and is a sum of one-dimensional ideals. So $L = J$ is a Lie algebra of quasinilpotent compact operators. By [15, Lemma 5.10], $\mathcal{E}\ell(A)$ consists of quasinilpotent operators. Then the algebra B generated by $\mathrm{ad}(L)$ consists of quasinilpotent operators. As Ba does not contain a for every non-zero $a \in L$, either Ba or the one-dimensional subspace generated by a is a non-trivial ideal of L, a contradiction. □

Note that this result cannot be considered as a special case of Corollary 5.3 because we do not assume the completeness of L.

We will need an algebraic version of triangularizability. For this we introduce the notions of *complete subspace chain, maximal subspace chain, continuous subspace chain* in vector spaces which differ from the corresponding notions for normed spaces only by the absence of the word "closed" in the definitions. Gaps and gap-representations are defined as above. A set M of operators on a vector space X

is called *strictly triangularizable* if there is a maximal subspace chain consisting of invariant subspaces for M.

The following theorem supplies us with examples of strictly triangularizable sets of operators.

iat **Theorem 5.5.** *Any Lie or Jordan algebra J of nilpotent finite rank operators generates the algebra of nilpotent operators, and every algebra of nilpotent operators is strictly triangularizable.*

Proof. Let A be the algebra generated by J. Every element a of A is a polynomial in some finite set K of elements of L. As every finite set of finite rank operators is *locally finite* in the sense that it generates a finite-dimensional algebra, the algebra B generates by K is finite-dimensional. Note that the Lie or Jordan algebra I generated by K is in $B \cap J$ and consists therefore of nilpotents. Taking a regular representation $\pi_l : b \mapsto bx$ of B on the vector space $X = B$, we have that $\pi_l(I)$ consists of nilpotent operators on a finite-dimensional space. By [4, Theorem 2.2.1], $\pi_l(B)$ consists of nilpotent operators on X. Then

$$a^{n+1} = \pi_l(a)^n a = 0$$

for some n, i.e. a is a nilpotent operator. This means that A is an algebra of nilpotent operators.

Let Γ be a maximal chain of invariant subspaces for an algebra A of nilpotent operators. If Γ is not a maximal subspace chain, then there is a non-one-dimensional gap-quotient $W = Y/Z$. It is clear that $B = \pi_{Y/Z}(A)$ is an non-zero algebra of nilpotent operators. As $x \notin Bx$ for every non-zero $x \in W$, either Bx or $\{y \in W : By = 0\}$ is a non-trivial invariant subspace for B. This means that there is an intermediate invariant subspace for A between Z and Y, a contradiction. $\square$

This theorem shows that if A is a simple nil-algebra then $\mathcal{E}\ell(A)$ is not a nil-algebra since it has no invariant subspaces. On the other hand, considering the regular representations, it follows from the theorem that every simple nil-algebra has a maximal subspace chain consisting of left ideals and a maximal subspace chain consisting of right ideals.

The following lemma works in the pure algebraic setting as well as, with a minor modification, for operators on a normed space (we have mentioned this in the previous section).

imur **Lemma 5.6.** *Let Γ be a complete subspace chain containing 0 and X, and a be a finite rank operator on X leaving subspaces of Γ invariant. If all gap-representations with respect to Γ vanish on a, then a is nilpotent.*

Proof. Let $X_0 = aX$ and let $\Lambda = \{Y \cap X_0 : Y \in \Gamma\}$. Since X_0 is finite-dimensional, Λ is a finite chain of subspaces $0 = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_m = X_0$ that are invariant for a.

Let $W_{2j} = \cap\{Y \in \Gamma : Z_j \subset Y\}$ for $j > 0$. Then

$$Z_j = \cap\{Y \cap X_0 : Z_j \subset Y \in \Gamma\} = W_{2j} \cap X_0.$$

For every $Y \in \Gamma$ with $Y \subsetneqq W_{2j}$, we have that $Y \cap X_0 \subsetneqq Z_j$. If $Z_{j-1} \subsetneqq Y \cap X_0$ then there is an intermediate subspace of Λ between Z_{j-1} and Z_j, a contradiction. So $Y \cap X_0 \subset Z_{j-1}$. Thus for every $Y \in \Gamma$ with $W_{2j-2} \subset Y \subsetneqq W_{2j}$ for $j > 1$, we have that

zvonok
$$aY \subset Y \cap aX = Y \cap X_0 = Z_{j-1} = W_{2j-2} \cap X_0 \subset W_{2j-2}. \tag{5.2}$$

Let W_{2j-1} be $\sum\{Y \in \Gamma : Y \subsetneqq W_{2j}\}$ for $j > 0$, and let $W_0 = 0$ and $W_{2m+1} = X$. Then $W_{2j-1} \in \Gamma$ and it follows from (5.2) that $aW_{2j-1} \subset W_{2j-2}$. If $W_{2j-1} = W_{2j}$ then $aW_{2j} \subset W_{2j-2}$, otherwise (W_{2j-1}, W_{2j}) is a gap of Γ, whence $aW_{2j} \subset W_{2j-1}$ by assumption. Taking into account that $aW_{2m+1} \subset W_{2m}$, we obtain that $aW_j \subset W_{j-1}$ for every $j > 0$. Hence

$$a^{2m+1}X = a^{2m+1}W_{2m+1} \subset a^{2m}W_{2m} \subset \cdots \subset aW_1 \subset W_0 = 0.$$

□

In particular, Lemma 5.6 says that if Γ is continuous, a is automatically a nilpotent operator. On the other hand, it is clear that if Γ is maximal then every its gap-representation (being of rank one) vanishes on nilpotent operators leaving subspaces of Γ invariant. The following reflects this fact and is a sort of a converse assertion to Theorem 5.5.

ii **Theorem 5.7.** *Let J be a strictly triangularizable, Lie or Jordan, algebra of operators. Then the set of all nilpotent finite rank operators in J is an ideal of J.*

Proof. Apply Lemma 5.6. □

The restriction on ranges in the theorem is essential. It is not difficult to construct two nilpotent operators a, b in a (strictly) triangularizable algebra such that $a + b$ is not nilpotent. The simplest example is following. Let Γ be a maximal continuous subspace chain in a vector space X. Then $\Gamma^{(2)} = \{Y \oplus Y : Y \in \Gamma\}$ is a continuous chain of subspaces for $X \oplus X$. Let a, b be defined by $a(x, y) = (0, x)$ and $b(x, y) = (y, 0)$, for all $(x, y) \in X \oplus X$. Then a and b are nilpotents, subspaces of $\Gamma^{(2)}$ are invariant for a and b, but $(a + b)^2$ is the identity operator.

In fact the Ringrose result can be transferred to the algebraic setting in full generality: for every finite rank operator a leaving the subspaces of a maximal (subspace or closed subspace) chain Γ invariant, the set $\{\pi(a)\}$ for $\dim X < \infty$ or $\{\pi(a)\} \cup \{0\}$ for $\dim X = \infty$ is the spectrum of a, where π runs over gap-representations with respect to Γ. This is probably well known but we could not find a reference.

A Lie algebra L is called *nil-Engel* if $\mathrm{ad}(a)$ is a nilpotent operator, for every $a \in L$.

i42 **Corollary 5.8.** *Let L be a non-one-dimensional nil-Engel Lie algebra. If $\mathrm{ad}(a)$ is of finite rank for some non-zero $a \in L$, then L is not simple.*

Proof. Let $M = \mathrm{ad}(L)$. The set $I = \{b \in L : \mathrm{ad}(b) \text{ is of finite rank}\}$ is a non-zero ideal of L. So, one may assume that $I = L$. Then M is a Lie algebra of nilpotent finite rank operators. By Theorem 5.5, it has a non-trivial invariant subspace which is an ideal of L. □

Now we list several results which are algebraic analogs of the results in Section 4.

iid0 **Theorem 5.9.** *Let J be a non-one-dimensional, Lie or Jordan, algebra of operators on a vector space X. If J contains a non-zero finite rank operator and if J is strictly triangularizable, then J is not simple.*

Proof. Similar to the proof of Theorem 4.2 with using Theorem 5.7 instead of Ringrose's result. □

corJ1 **Corollary 5.10.** *Let J be a non-one-dimensional Jordan algebra of nilpotent operators. If J contains a non-zero finite rank operator, then J is not simple.*

Proof. Similar to the proof of Corollary 4.3. □

Let M be a set of operators. We say that M is *strictly almost triangularizable* if there is a maximal chain of subspaces invariant for M that admits only finite-dimensional gap-quotients.

iid1 **Corollary 5.11.** *Let J be a infinite-dimensional, Lie or Jordan, algebra of operators on a vector space X. If J contains a non-zero finite rank operator and if J is strictly almost triangularizable, then J is not simple.*

Proof. Similar to the proof of Corollary 4.6. □

References

Bon [1] F. F. Bonsall, Compact linear operators from an algebraic standpoint, *Glasgow Math. J.* **8** (1967), 41-49.

Bonsall [2] F. F. Bonsall, Operators that act compactly on an algebra of operators, *Bull. London Math. Soc.* **1** (1969), 163-170.

H [3] I. N. Herstein, *Topics in ring theory*, The University of Chicago Press, Chicago, 1969.

Jac62 [4] N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

KenST [5] M. Kennedy, V. Shulman, Yu. Turovskii, Invariant subspaces of subgraded Lie algebras of compact operators, *Integr. Eq. Oper. Theory* **63** (2009), 47-93.

KST1 [6] E. Kissin, V. Shulman, Yu. Turovskii, Banach Lie algebras with Lie subalgebras of finite codimension: their invariant subspaces and Lie ideals, *Journal of Functional Analysis* **256** (2009), 323-351.

KST [7] E. Kissin, V. Shulman, Yu. Turovskii, Banach Lie algebras with Lie subalgebras of finite codimension have Lie ideals, *Proc. London Math. Soc.* (to appear).

Livschits [8] L. Livshits, Locally finite-dimensional sets of operators, *Proc. Amer. Math. Soc.* **119** (1993), 165-169.

lom [9] V. I. Lomonosov, On invariant subspaces of operators commuting with compact operators, *Functional analysis and its applications* **7** (1973), 213-214.

RR [10] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin Heidelberg New York, 1973.

read [11] C. J. Read, Quasinilpotent operators and the invariant subspace problem, *J. London Math. Soc.* **56** (1997), 595-606.

Rin [12] J. R. Ringrose, Superdiagonal forms for compact linear operators, *Proc. London Math. Soc.* **12** (1962), 367-384.

KRYM [13] V. Shulman, Yu. Turovskii, Solvable Lie algebras of operators have invariant subspaces, *Spectral and Evolutionary Problems (Simferopol, Crimea, Ukraine)*, **9** (1999), 38-44.

ST2000 [14] V. Shulman, Yu. Turovskii, Joint spectral radius, operator semigroups and a problem of W. Wojtyński, *J. Functional Analysis* **177** (2000), 383-441.

ST [15] V. Shulman, Yu. Turovskii, Invariant subspaces of operator Lie algebras and Lie algebras with compact adjoint action, *J. Functional Analysis* **223** (2005), 425-508.

S [16] A. Smoktunowicz, A simple nil ring exists, *Comm. Algebra* **30** (2002), 27-59.

Vala [17] K. Vala, On compact sets of compact operators, *Ann. Acad. Sci. Fenn. Ser. A I* **351** (1964), 1-8.

W [18] W. Wojtyński, Engel's theorem for nilpotent Lie algebras of Hilbert-Schmidt operators, *Bull. Acad. Polon. Sci.* **24** (1976), 797-801.

W1 [19] W. Wojtyński, Banach-Lie algebras of compact operators, *Studia Math.* **59** (1977), 263-273.

Zel [20] E. Zelmanov, Absolute divisors of zero in Jordan pairs and Lie algebras, *Matem. Sbornik* **112** (1980), 611-629 (in Russian).

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