FINITE DIMENSIONAL ZERO PRODUCT DETERMINED ALGEBRAS ARE GENERATED BY IDEMPOTENTS

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ABSTRACT. An algebra A is said to be zero product determined if every bilinear map f from $A \times A$ into an arbitrary vector space X with the property that $f(x, y) = 0$ whenever $xy = 0$ is of the form $f(x, y) = \Phi(xy)$ for some linear map $\Phi : A \to X$. It is known, and easy to see, that an algebra generated by idempotents is zero product determined. The main new result of this partially expository paper states that for finite dimensional (unital) algebras the converse is also true. Thus, if such an algebra is zero product determined, then it is generated by idempotents.

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1. Introduction

We say that an algebra A over a field F is *zero product determined* if for every bilinear map $f : A \times A \to X$, where X is an arbitrary vector space over F, with the property that for all $x, y \in A$,

$$xy = 0 \Longrightarrow f(x, y) = 0, \tag{1.1}$$

there exists a linear map $\Phi : A \to X$ such that

$$f(x, y) = \Phi(xy) \quad \text{for all } x, y \in A. \tag{1.2}$$

This concept was introduced in [12]. At about the same time, its variation in the functional-analytic context was introduced in [1], under the (somewhat unfortunately entirely different) name "a Banach algebra with property $\mathbb{B}$" (see Section 2 for definition). The original motivation for both concepts were problems related to zero product preserving linear maps, but later several other applications have been found, especially in the Banach algebra theory.

It seems natural to conjecture that an algebra should have plenty of zero divisors in order to be zero product determined. As nontrivial idempotents are zero divisors, it is perhaps not too surprising that every algebra which is generated by its idempotents is zero product determined [10]. Proving this is actually very easy, see below. There are, of course, algebras without any nontrivial idempotent but with many zero divisors. When asking ourselves about an example of a zero product determined unital algebra that is not generated by idempotents we realized, much to our surprise, that none of the papers following [1] and [12] contains such an example, and, moreover, that we are unable to construct one (although Banach algebras with property $\mathbb{B}$ that have no nontrivial idempotents exist in abundance). We have accordingly changed the perspective, which led us to prove that every finite dimensional zero product determined unital algebra is generated by idempotents. This is the main result of the paper. Finding a counterexample to this statement in infinite dimensions – if there is one, of course – is left as an open problem.

The main result will be proved in Section 3. Section 2 gives a general elementary introduction to zero product determined algebras and is partially expository; although most of its results have not been explicitly stated elsewhere, some of them are known to specialists. Since zero product determined algebras have been studied in quite a few papers since their introduction in 2009, we believe that it may be useful to gather together all of their basic properties at one place.

We end the introduction with a terminological convention: by an *algebra* we will mean an *associative unital algebra* over a fixed field F. It should be mentioned that nonassociative zero product determined algebras have also been studied by several authors, and that some important Banach algebras with property $\mathbb{B}$ are not unital. In this paper, however, it seems more relevant to restrict ourselves to associative algebras with unity. Incidentally, the non-unital situation is really different. For example, an algebra with trivial multiplication (the product of any elements is 0) is clearly zero product determined, but has no nonzero idempotents.

2. The class of zero product determined algebras

We will start this section by recasting the definition of a zero product determined algebra, after that study the preservation of the zero product determined property under some standard constructions, continue with examples of algebras that are or are not zero product determined, and finally mention some of the basic applications.

2.1. **Alternative definitions.** We begin by remarking that (1.2) is equivalent to

$$f(xy,z) = f(x,yz) \quad \text{for all } x,y,z \in A, \tag{2.1}$$

as well as to

$$f(x,y) = f(xy,1) \quad \text{for all } x,y \in A. \tag{2.2}$$

Indeed, (1.2) clearly implies (2.1), setting $z = 1$ in (2.1) we see that (2.1) implies (2.2), and (2.2) obviously implies (1.2). In the sequel we will use (2.1) or (2.2) instead of (1.2) without comment. Let us also point an important special case of (2.2):

$$f(1,y) = f(y,1) \quad \text{for all } y \in A. \tag{2.3}$$

The role of the space X in the definition of a zero product determined algebra is entirely formal, in fact it can be replaced by F. Namely, if (1.1) implies (2.1) for bilinear maps into F and $f : A \times A \to X$ satisfies (1.1) with X an arbitrary space, then by composing f by an arbitrary linear functional α on X we conclude that $\alpha(f(xy,z)) = \alpha(f(x,yz))$ for all $x,y,z \in A$, which of course yields (2.1).

The definition of property $\mathbb{B}$ for Banach algebras is based on identity (2.1): A Banach algebra A is said to have *property* $\mathbb{B}$ if for every *continuous* bilinear map $f : A \times A \to X$, where X is an arbitrary Banach space, (1.1) implies (2.1). The class of Banach algebras with property $\mathbb{B}$ turns out to be quite large, in particular it includes C^*-algebras and group algebras of arbitrary locally compact groups. The continuity of f thus plays an important role for many of these algebras are not zero product determined [6].

The following simple proposition was used as an essential tool in the seminal paper [12]. We remark that it obviously holds also for nonassociative algebras.

Proposition 2.1. *An algebra A is zero product determined if and only if for every bilinear map $f : A \times A \to X$, where X is an arbitrary vector space,* (1.1) *implies that $\sum_i f(x_i,y_i) = 0$ whenever $\sum_i x_i y_i = 0$.*

Proof. The "only if" part is clear. To prove the "if" part, note that if $f : A \times A \to X$ is a bilinear map such that $\sum_i f(x_i,y_i) = 0$ whenever $\sum_i x_i y_i = 0$, then

$$\Phi\Big(\sum_i x_i y_i\Big) := \sum_i f(x_i,y_i)$$

gives a well-defined linear map from A into X which obviously satisfies (1.2). $\square$

Let us introduce some notation. By A^{o} we denote the *opposite* algebra of the algebra A, i.e., A^{o} is the vector space A endowed with multiplication $x \cdot y = yx$ where yx is the product of y and x in A. We will deal with $A \otimes A^{\mathrm{o}}$, the tensor product of algebras A and A^{o}, which is of course a standard setting for studying the algebra A. For a subset S of an algebra we denote by $\operatorname{span} S$ the linear span of S. The next characterization of zero product determined algebras is also very simple, but, to the best of our knowledge, new (apparently similar characterizations in [15] and [21] also based on tensor products are somewhat different).

Proposition 2.2. *An algebra A is zero product determined if and only if*

$$y \otimes 1 - 1 \otimes y \in \operatorname{span}\{u \otimes v \in A \otimes A^{\mathrm{o}} \,|\, uv = 0\}$$

for every $y \in A$.

Proof. Note that $\mathcal{Z} := \operatorname{span}\{u \otimes v \in A \otimes A^{\mathrm{o}} \,|\, uv = 0\}$ is a left ideal of $A \otimes A^{\mathrm{o}}$. Thus, assuming that $y \otimes 1 - 1 \otimes y \in \mathcal{Z}$ for all $y \in A$ we also have

$$xy \otimes 1 - x \otimes y = (x \otimes 1)(y \otimes 1 - 1 \otimes y) \in \mathcal{Z}$$

for all $x, y \in A$. Take a bilinear map $f : A \times A \to X$ satisfying (1.1). The linear map $\beta : A \otimes A^{\mathrm{o}} \to X$ given by $\beta(x \otimes y) = f(x, y)$ then vanishes on $\mathcal{Z}$, hence it satisfies $\beta(xy \otimes 1 - x \otimes y) = 0$ for all $x, y \in A$, meaning that (2.2) holds. Thus A is zero product determined.

To prove the converse, assume that there exists $a \in A$ such that $a \otimes 1 - 1 \otimes a \notin \mathcal{Z}$. Take a linear functional α on $A \otimes A^{\mathrm{o}}$ such that $\alpha(a \otimes 1 - 1 \otimes a) \neq 0$ and $\alpha(\mathcal{Z}) = \{0\}$. A bilinear map $f : A \times A \to F$ given by $f(x, y) = \alpha(x \otimes y)$ thus satisfies (1.1) but not (2.3) for $f(a, 1) \neq f(1, a)$. Hence A is not zero product determined. $\square$

For simple rings viewed as algebras over their centers, and, more generally, for centrally closed prime algebras, Proposition 2.2 can be reformulated in terms of inner derivations. Recall that an inner derivation on the algebra A is a map of the form $d = L_y - R_y$ for some $y \in A$, where $L_y, R_y : A \to A$ are defined by $L_y(x) = yx$ and $R_y(x) = xy$. For an introduction to centrally closed prime algebras see, for example, [11, Section 7.5].

Corollary 2.3. *A necessary condition for an algebra A to be zero product determined is that every inner derivation d of A can be written as $d = \sum_i L_{u_i} R_{v_i}$ where $u_i v_i = 0$. If A is a centrally closed prime algebra, then this condition is also sufficient.*

Proof. The map $x \otimes y \mapsto L_x R_y$ is an algebra homomorphism from $A \otimes A^{\mathrm{o}}$ onto the multiplication algebra of A (i.e., the algebra of linear operators of A generated by all L_x and R_y), which is an isomorphism if A is prime and centrally closed [11, Theorem 7.44]. The desired conclusion thus readily follows from Proposition 2.2. $\square$

The notion of a zero product determined algebra has turned out to be applicable to some problems concerning derivations [1, 4, 8, 20]. We now see that it is naturally connected to derivations.

2.2. Stability under some algebra constructions.

The next four propositions are algebraic versions of results concerning Banach algebras with property $\mathbb{B}$ from [1].

Proposition 2.4. *A homomorphic image of a zero product determined algebra is a zero product determined algebra.*

Proof. Let $\varphi : A \to B$ be a surjective algebra homomorphism. If A satisfies the condition of Proposition 2.2, then we see by making use of the homomorphism $\varphi \otimes \varphi$ that so does B. $\square$

The following simple result was already observed in some other papers.

Proposition 2.5. *Algebras $A_1, \dots, A_n$ are zero product determined if and only if their direct product $A_1 \times \cdots \times A_n$ is zero product determined.*

Proof. The "if" part follows from Proposition 2.4. For proving the "only if" part it is enough to consider the case where $n = 2$. Thus, let A_1 and A_2 be zero product determined, set $A := A_1 \times A_2$, and take a bilinear map $f : A \times A \to X$ satisfying (1.1). Then in particular $f((x_1, 0),(y_1, 0)) = 0$ whenever $x_1y_1 = 0$, implying that

$$f\big((x_1, 0),(y_1, 0)\big) = f\big((x_1y_1, 0),(1, 0)\big) \quad \text{for all } x_1, y_1 \in A_1.$$

Similarly,

$$f\big((0, x_2),(0, y_2)\big) = f\big((0, x_2y_2),(0, 1)\big) \quad \text{for all } x_2, y_2 \in A_2.$$

Since (1.1) obviously implies

$$f\big((x_1, 0),(0, y_2)\big) = f\big((0, x_2),(y_1, 0)\big) = f\big((x_1y_1, 0),(0, 1)\big) = f\big((0, x_2y_2),(1, 0)\big) = 0,$$

we see that f satisfies (2.2). □

Proposition 2.12 below shows that Proposition 2.5 cannot be extended to the direct product of an infinite family of algebras.

Proposition 2.6. *The tensor product of two zero product determined algebras is a zero product determined algebra.*

Proof. Let A_1, A_2 be zero product determined, and set $A := A_1 \otimes A_2$. Let $f : A \times A \to X$ be a bilinear map satisfying (1.1). Take any pair $x_2, y_2 \in A_2$ and consider the map $(x_1, y_1) \mapsto f(x_1 \otimes x_2, y_1 \otimes y_2)$. Since A_1 is zero product determined it follows that

$$f(x_1 \otimes x_2, y_1 \otimes y_2) = f(x_1y_1 \otimes x_2, 1 \otimes y_2).$$

Similarly we see that

$$f(x_1 \otimes x_2, y_1 \otimes y_2) = f(x_1 \otimes x_2y_2, y_1 \otimes 1).$$

Since these two identities hold for arbitrary $x_1, y_1 \in A_1$, $x_2, y_2 \in A_2$, it follows that

$$f(x_1 \otimes x_2, y_1 \otimes y_2) = f(x_1y_1 \otimes x_2, 1 \otimes y_2) = f(x_1y_1 \otimes x_2y_2, 1 \otimes 1) = f((x_1 \otimes x_2)(y_1 \otimes y_2), 1 \otimes 1),$$

which implies (2.2). □

On the other hand, the assumption that $A_1 \otimes A_2$ is zero product determined does not imply that A_1 and A_2 are zero product determined. For example, the algebra $A \otimes M_2(F) \cong M_2(A)$ is zero product determined even if A is not – see Proposition 2.18 below. This also shows that a subalgebra of a zero product determined algebra may not be zero product determined. In fact, even a corner algebra eAe is not always zero product determined if A is. However, there is something we can say about ideals of zero product determined algebras.

Proposition 2.7. *Let I be an ideal of a zero product determined algebra A. If f is a bilinear map from $I \times I$ into a vector space X such that $f(s, t) = 0$ whenever $st = 0$, then $f(sy, t) = f(s, yt)$ for all $s, t \in I$ and $y \in A$.*

Proof. Pick $s, t \in I$ and consider the map $(x, y) \mapsto f(sx, yt)$ from $A \times A$ into X. As A is zero product determined it follows that $f(sx, yt) = f(sxy, t)$ for all $x, y \in A$. Setting $x = 1$ we get the desired conclusion. □

2.3. **Some examples and non-examples.** Let us first recall an elementary fact on tensor products which will be used repeatedly in this subsection: If $\sum_i a_i \otimes b_i = \sum_j c_j \otimes d_j$ and the a_i's are linearly independent, then each b_i lies in the linear span of the d_j's (see, e.g., [11, Lemma 4.9]).

Algebras that are of interest to us in this paper should of course have zero divisors. Nevertheless, the following straightforward observation deserves to be recorded.

Proposition 2.8. *If a zero product determined algebra A is a domain, then $A = F$.*

Proof. Proposition 2.2 shows that $y \otimes 1 = 1 \otimes y$ for every $y \in A$, implying that y is a scalar multiple of 1. □

We also record the following slightly more general result.

Proposition 2.9. *If A is a zero product determined algebra different from F, then every element in A is a sum of right (resp. left) zero divisors.*

Proof. It is enough to show that 1 is a sum of right zero divisors. By Corollary 2.8 we can pick a right zero divisor $y \in A$. Proposition 2.2 tells us that there exist $u_i, v_i \in A$ such that $y \otimes 1 = 1 \otimes y + \sum_{i=1}^n u_i \otimes v_i$ and $u_i v_i = 0$. Hence 1 is a linear combination of $y, v_1, \dots, v_n$, which readily yields the desired conclusion. □

We continue with a simple application of this proposition, which will be used in the proof of the main theorem.

Proposition 2.10. *If the Jacobson radical of the algebra A has codimension 1 in A and is nonzero, then A is not zero product determined.*

Proof. Every element in A can be written as $\lambda + r$ where $\lambda \in F$ and r is from the Jacobson radical R. If $\lambda \neq 0$ then this element is invertible, so only elements from R can be left or right zero divisors. Hence A is not zero product determined by Proposition 2.9. □

The next proposition is similar, but the proof is different.

Proposition 2.11. *If an algebra A has an ideal I of codimension 1 such that $I^2 \neq I$, then A is not zero product determined.*

Proof. We have $A = F \oplus I$. Define $f : A \times A \to A/I^2$ by $f(\lambda + s, \mu + t) = \lambda t + I^2$ where $\lambda, \mu \in F$, $s, t \in I$. One immediately checks that f satisfies (1.1). Since $f(1, t) = t + I^2 \neq 0$ for $t \in I \setminus I^2$, while $f(t, 1) = 0$, f does not satisfy (2.3). □

Each of the last two propositions shows that the algebra obtained by adjoining a unity to an algebra with trivial multiplication is not zero product determined. This indicates that the abundance of zero divisors is not sufficient for an algebra to be zero product determined.

The mere existence of an ideal of codimension 1 in A is not enough to conclude that A is not zero product determined. For example, the direct product $F \times \cdots \times F$ of finitely many copies of F is zero product determined by Proposition 2.5, but has ideals of codimension 1. The situation is different if we take infinitely many copies of F, as the next proposition shows. The idea of the proof is taken from [6].

Proposition 2.12. *If F is an infinite field, then the direct product $F \times F \times \dots$ of countably infinitely many copies of F is not zero product determined.*

Proof. Set $A := F \times F \times \dots$. Since F is infinite, there exists $y = (\eta_1, \eta_2, \dots) \in A$ such that $\eta_i \neq \eta_j$ for all $i \neq j$. Suppose A is zero product determined. By Proposition 2.2 we then have

$$y \otimes 1 - 1 \otimes y = \sum_{k=1}^{n} u_k \otimes v_k$$

for some $u_k, v_k \in A$ such that $u_k v_k = 0$. Let us write U_k (resp. V_k) for the set of all indices $i \in \mathbb{N}$ such that the i-th term of u_k (resp. v_k) is 0. Note that $u_u v_k = 0$ implies $U_k \cup V_k = \mathbb{N}$. Hence

$$\mathbb{N} = (U_1 \cup V_1) \cap (U_2 \cup V_2) \cap \dots \cap (U_n \cup V_n),$$

which can be rewritten as

$$\mathbb{N} = \big(U_1 \cap \dots \cap U_n\big) \cup \big(U_1 \cap \dots \cap U_{n-1} \cap V_n\big) \cup \dots \cup \big(V_1 \cap \dots \cap V_n\big).$$

Therefore at least one of the sets $U_1 \cap \dots \cap U_n$, $U_1 \cap \dots \cap U_{n-1} \cap V_n$, etc., is infinite. Let $W_1 \cap \dots \cap W_n$, $W_k \in \{U_k, V_k\}$, be such a set. Denote by I the ideal of A consisting of all $(\xi_1, \xi_2, \dots)$ such that $\xi_i = 0$ whenever $i \in W_1 \cap \dots \cap W_n$. Note that for each k either $u_k \in I$ or $v_k \in I$. Therefore $y \otimes 1 - 1 \otimes y \in I \otimes A + A \otimes I$, so there exist $s_i, t_j \in I$ and $z_i, w_j \in A$ such that

$$y \otimes 1 - 1 \otimes y + \sum_{i=1}^{m} s_i \otimes z_i = \sum_{j=1}^{n} w_j \otimes t_j. \tag{2.4}$$

We may assume that $s_1, \dots, s_m$ are linearly independent. If $y, 1, s_1, \dots, s_m$ were linearly dependent then I would contain a nonzero element in $\operatorname{span}\{y, 1\}$. But this is impossible for all the terms of such an element are different from each other. Hence $y, 1, s_1, \dots, s_m$ are linearly independent and so we infer from (2.4) that $1 \in \operatorname{span}\{t_1, \dots, t_n\} \subseteq I$, which is a contradiction. □

As already mentioned, Proposition 2.12 shows that Proposition 2.5 does not hold for infinite families of algebras. On the other hand, this proposition together with its proof gives some evidence that constructing zero product determined algebras different from those described in the sequel may not be an easy task.

Let us turn to positive examples. We begin with two elementary lemmas. The first one is basically known, see [10, Theorem 4.1]; we remark that the idea of the proof can be traced back to [18].

Lemma 2.13. *Let A be an algebra and X be a vector space. If a bilinear map $f : A \times A \to X$ satisfies* (1.1)*, then the set*

$$\{s \in A \,|\, f(xs, y) = f(x, sy) \text{ for all } x, y \in A\}$$

is a subalgebra of A which contains all idempotents in A.

Proof. One immediately checks that this set is a subalgebra. Given an idempotent $e \in A$ we infer from $xe \cdot (1 - e)y = 0$ and $x(1 - e) \cdot ey = 0$ that $f(xe, y - ey) = 0$ and $f(x - xe, ey) = 0$, and hence $f(xe, y) = f(xe, ey) = f(x, ey)$. □

With Proposition 2.2 in mind we now prove a similar lemma in the setting of tensor products.

Lemma 2.14. *Let A be an algebra and let $\mathcal{Z} := \operatorname{span}\{u \otimes v \in A \otimes A^{\mathrm{o}} \,|\, uv = 0\}$. The set*
$$\{s \in A \,|\, s \otimes 1 - 1 \otimes s \in \mathcal{Z}\}$$
is a subalgebra of A which contains all idempotents in A.

Proof. From the identity
$$(s \otimes 1)(t \otimes 1 - 1 \otimes t) + (1 \otimes t)(s \otimes 1 - 1 \otimes s) = st \otimes 1 - 1 \otimes st$$
and the fact that $\mathcal{Z}$ is a left ideal of A it follows that this set is a subalgebra. If $e \in A$ is an idempotent, then $e \otimes 1 - 1 \otimes e = e \otimes (1 - e) - (1 - e) \otimes e \in \mathcal{Z}$. □

Remark 2.15. Consider the following condition for an element a in an algebra A:

($\star$) There exist linearly independent $u_1, u_2 \in A$ and linearly independent $v_1, v_2 \in A$ such that $a \otimes 1 - 1 \otimes a = u_1 \otimes v_1 + u_2 \otimes v_2$ and $u_1 v_1 = u_2 v_2 = 0$.

From the end of the proof of Lemma 2.14 it is evident that every nontrivial idempotent e (i.e., an idempotent different from 0 and 1) satisfies ($\star$). Note that the same is true for every element of the form

($\star\star$) $a = \lambda 1 + \mu e$ where $\lambda, \mu \in F$, $\mu \neq 0$, and e is a nontrivial idempotent.

Thus, ($\star\star$) implies ($\star$). Let us show that the converse is also true. Assume that ($\star$) holds. Note that in this case each u_k, v_k lies in $\operatorname{span}\{1, a\}$.Thus there exist $\lambda, \mu, \omega, \tau \in F$ such that $u_1 = \lambda 1 + \mu a$ and $v_1 = \omega 1 + \tau a$. Since $u_1 v_1 = 0$ and $u_1 \neq 0$, $v_1 \neq 0$, it follows that $\mu \neq 0$ and $\tau \neq 0$. A simple computation shows that this implies ($\star\star$), unless u_1 and v_1 are linearly dependent so that $u_1^2 = 0$. Similarly examining u_2 and v_2 we see that it remains to consider the situation where $u_1^2 = u_2^2 = 0$ and $u_1, u_2 \in \operatorname{span}\{1, a\}$. But this easily implies that u_1 and u_2 are linearly dependent, contrary to the assumption. Thus ($\star\star$) holds.

An algebra thus contains a nontrivial idempotent if and only if it contains an element a satisfying ($\star$). This observation is perhaps of some interest in its own right; on the other hand, it could be of some use in further investigations of zero product determined algebras.

Note that each of Lemmas 2.13 and 2.14 immediately yields the following fundamental result.

Proposition 2.16. *An algebra generated by idempotents is zero product determined.*

Remark 2.17. If the field F is finite, then every element in $F \times F \times \dots$ is a linear combination of idempotents. The assumption in Proposition 2.12 that F must be infinite is thus really necessary.

The next proposition points out the case of special interest for us in light of our goal in the next section. As usual, by $M_n(B)$ we denote the algebra of all $n \times n$ matrices with entries from the algebra B.

Proposition 2.18. *The matrix algebra $M_n(B)$ is generated by idempotents, and is therefore zero product determined, for every algebra B and every $n \geq 2$.*

Proof. Let e_{ij} denote matrix units in $A := M_n(B)$. By be_{ij} we denote the matrix whose (i, j) entry is $b \in B$ and all other entries are 0. If $i \neq j$, then be_{ij} is a difference of two idempotents: $be_{ij} = (e_{ii} + be_{ij}) - e_{ii}$. From $be_{ii} = be_{ij} \cdot e_{ji}$ with $i \neq j$ it thus follows that be_{ii} lies in the subalgebra generated by idempotents. Since A is linearly spanned by matrices of the form be_{ij} this proves that A is generated by idempotents. □

The fact that $M_n(B)$ is zero product determined was first proved in [12], but with a different, somewhat longer proof.

We conclude with a simple but indicative example.

Example 2.19. The algebra $T_n(F)$ of all upper triangular matrices with entries in F is easily seen to be linearly spanned by idempotents (cf. the preceding proof), and is therefore zero product determined. However, its subalgebra consisting of matrices in which all diagonal entries are equal is not zero product determined by Proposition 2.10 (or Proposition 2.11).

2.4. **A word on applications.** The purpose of this section is to give some evidence about the applicability of the notion of a zero product determined algebra. We start with linear maps preserving zero product, i.e., maps φ between algebras with the property that $xy = 0$ implies $\varphi(x)\varphi(y) = 0$. The study of such maps has a long and rich history, see [1, 18] for details.

Proposition 2.20. *Let φ be a linear zero product preserving map from an algebra A into an algebra B. If A zero product determined and $\varphi(1) = 1$, then φ is a homomorphism.*

Proof. The map $f(x, y) = \varphi(x)\varphi(y)$ satisfies (1.1), and hence (2.2), i.e., $\varphi(x)\varphi(y) = \varphi(xy)$. □

This proposition gives only a basic idea on what results can be obtained. The problem becomes interesting if we do not assume $\varphi(1) = 1$, see [1, 17, 18, 31]. Let us also mention that one can similarly characterize derivations and related maps, see [8] and references given therein.

To the best of our knowledge, the problem whether (1.1) implies (1.2) for bilinear maps was first studied in [13], but with respect to the Lie product, not the ordinary product. The result obtained has turned out to be crucial for obtaining a definitive description of commutativity preserving (= zero Lie product preserving) linear maps on finite dimensional central simple algebras. Zero product determined algebras have been accordingly extensively studied in Lie and also Jordan algebras [12, 22, 23, 24, 32]. We do not want to enter the area of general nonassociative algebras in this paper. Our next result, however, concerns the Jordan product in (associative) zero product determined algebras. Its proof is an adaptation of the proof of [2, Theorem 1.2].

Proposition 2.21. *Let A be a zero product determined algebra and X be a vector space. If a symmetric bilinear map $f : A \times A \to X$ is such that for all $x, y \in A$,*

$$xy = yx = 0 \Longrightarrow f(x, y) = 0,$$

then $2f(x, y) = f(xy + yx, 1)$ for all $x, y \in A$.

Proof. Take $z, w \in A$ such that $zw = 0$. Define $f_{z,w} : A \times A \to X$ by $f_{z,w}(x, y) = f(wx, yz)$. Note that $xy = 0$ implies $f_{z,w}(x, y) = 0$. Therefore $f_{z,w}$ satisfies $f_{z,w}(x, y) = f_{z,w}(xy, 1)$ for all $x, y \in A$. That is, $f(wx, yz) = f(wxy, z)$ holds whenever $zw = 0$. We may now use the assumption that A is zero product determined for the map $(z, w) \mapsto f(wx, yz) - f(wxy, z)$, which results in

$$f(wx, yz) - f(wxy, z) = f(x, yzw) - f(xy, zw)$$

for all $x, y, z, w \in A$. Setting $x = z = 1$ we get the desired conclusion. □

Without assuming that f is symmetric the problem is much more complicated and solutions are known only in some special cases [3, 27]; see also an application of [3] to a problem with a different origin [26].

Recall that a linear map φ from an algebra A into an algebra B is called a Jordan homomorphism if $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for all $x, y \in A$.

Corollary 2.22. *Let A and B be algebras over a field F with characteristic different from 2, and let $\varphi : A \to B$ be a linear map such that $\varphi(x)\varphi(y) = 0$ whenever $x, y \in A$ satisfy $xy = yx = 0$. If A is zero product determined and $\varphi(1) = 1$, then φ is a Jordan homomorphism.*

Proof. Apply Proposition 2.21 for the map $(x, y) \mapsto \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$. □

The next result is of a different nature.

Proposition 2.23. *Every commutator in a zero product determined algebra is a sum of square-zero elements.*

Proof. Let $N := \text{span}\{a \in A \,|\, a^2 = 0\}$. Note that $xy = 0$ implies $(yx)^2 = 0$. Accordingly, the map $f : A \times A \to A/N$, $f(x, y) = yx + N$ satisfies (1.1). Hence $f(x, y) = f(xy, 1)$ for all $x, y \in A$, meaning that $[x, y] \in N$. □

The paper [6] contains a more thorough analysis of the problem of expressing commutators as sums of square-zero elements, based on property $\mathbb{B}$ and zero product determined algebras. For some comments on the history of this problem we refer to the paper [19] in which the authors proved Proposition 2.23 for simple rings containing a nontrivial idempotent (such rings are necessarily generated by idempotents, see Remark 3.2 below), and gave an answer to an old problem of Herstein by showing that there exists a simple ring which is not a domain and in which not every commutator is a sum of square-zero elements. Thus, this also shows that there exist simple rings with zero divisors that are not zero product determined.

This was just to give some justification that the concept of a zero product determined algebra can be useful. More (and not so straightforward) applications can be found in the papers cited. We also remark that the related property $\mathbb{B}$ has some somewhat surprising applications to problems of different kinds [1, 2, 4, 5, 7, 14, 16, 29, 30].

3. Finite dimensional zero product determined algebras

In the first subsection we will state auxiliary results needed in the proof of the main result, and the second subsection is devoted to this proof.

3.1. **Tools.**

The following lemma was implicitly proved by Herstein [25], and is explicitly stated (for rings) in [9] as Lemma 2.1. We will give a proof since it is very short.

Lemma 3.1. *The subalgebra generated by all idempotents in an algebra A contains the ideal generated by all commutators of idempotents in A with arbitrary elements in A.*

Proof. If e is an idempotent, then so are $e + ex(1 - e)$ and $e + (1 - e)xe$ for every $x \in A$. Their difference is the commutator $[e, x]$. Noticing that

$$z[e, x]w = [e, z[e, [e, x]]w] - [e, z][e, [e, x]w] - [e, z[e, x]][e, w] + 2[e, z][e, x][e, w] \tag{3.1}$$

for all $z, w \in A$ it thus follows that the ideal generated by $[e, x]$ is contained in the subalgebra generated by idempotents. □

Although elementary, this proof is perhaps intuitively unclear. One actually naturally derives (3.1) by considering Lie ideals [25].

Remark 3.2. Note that a nontrivial idempotent in a simple algebra A cannot be central. Lemma 3.1 thus shows that the existence of a nontrivial idempotent in A already implies that A is generated by idempotents.

The second tool is lifting idempotents modulo ideals in finite dimensional algebras.

Lemma 3.3. *Let I be an ideal of a finite dimensional algebra A. Every idempotent in the factor algebra A/I is of the form $e + I$ where e is an idempotent in A.*

The classical version of this lemma concerns nil ideals in arbitary algebras (or rings). However, in the class of exchange rings [28], which includes finite dimensional algebras as special examples, idempotents can be lifted modulo every ideal. The author is thankful to Janez Šter for pointing out this fact to him.

The only remaining tool that we need is the classical Wedderburn's structure theory.

3.2. **Main result.** Let A now denote a *finite dimensional zero product determined algebra.* By R we denote its (Jacobson) radical, i.e., its unique maximal nilpotent ideal.

Lemma 3.4. *If A is semisimple, then*

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r) \times F \times \cdots \times F$$

where D_i are division algebras and each $n_i \geq 2$. Accordingly, A is generated by idempotents.

Proof. The classical Wedderburn's theorem tells us that A is isomorphic to the direct product of algebras of the form $M_n(D)$ with D a division algebra and $n \geq 1$. Each of these algebras is also zero product determined by Proposition 2.5. If $n = 1$ then $D = F$ by Proposition 2.8. From Proposition 2.18 we see that A is generated by idempotents. □

Lemma 3.5. *If A has no nontrivial idempotents, then $A = F$.*

Proof. Lemma 3.3 implies that A/R also does not have nontrivial idempotents. Since A/R is semisimple and, by Proposition 2.4, zero product determined, we infer from Lemma 3.4 that $A/R = F$. Each of Propositions 2.10 and 2.11 now implies that $R = 0$, and hence $A = F$. □

Lemma 3.6. *If every idempotent in A is central, then $A \cong F \times \cdots \times F$.*

Proof. We claim that $A \cong A_1 \times \cdots \times A_r$ where each A_i has no nontrivial idempotents. The proof is a straightforward induction on $n := [A : F]$. The $n = 1$ case is trivial. Let $n > 1$. We may assume that A has a nontrivial idempotent e. Since e is central, eA and $(1-e)A$ are algebras of smaller dimension than A and they both satisfy the condition that their idempotents are central. We may now use the induction assumption for eA and $(1-e)A$, which clearly implies the desired conclusion for $A = eA \oplus (1-e)A$. As A is zero product determined, so is each A_i by Lemma 2.5. Lemma 3.5 now tells us that $A_i = F$. □

Theorem 3.7. *A finite dimensional algebra is zero product determined if and only if it is generated by idempotents.*

Proof. Of course, we only have to prove the "only if" part. Assume thus that A is zero product determined. Let us denote by I the ideal generated by all commutators of idempotents with arbitrary elements in A. Lemma 3.3 tells us that an idempotent in A/I can be written as $e+I$ with e an idempotent in A. Since $[e,A]\subseteq I$ it follows that every idempotent in A/I is central. As A/I is zero product determined by Proposition 2.4 we infer from Lemma 3.6 that $A/I\cong F\times\cdots\times F$. In particular, A/I is a semisimple algebra. Since $(I+R)/I$ is its nilpotent ideal we must have $I+R=I$, i.e., $R\subseteq I$. Lemma 3.1 thus implies that R is contained in the subalgebra generated by all idempotents, which we denote by E. We have to show that actually $A=E$. Given $a\in A$ we have that $a+R\in A/R$ is a linear combination of products of idempotents in A/R by Proposition 2.4 and Lemma 3.4. From Lemma 3.3 we thus infer that there exists $u\in E$ such that $a-u\in R$. Since $R\subseteq E$ the desired conclusion $a\in E$ follows. $\square$

Remark 3.8. We may replace "finite dimensional" by "artinian" in the statement of Theorem 3.7. The proof is basically the same, only the simple argument in the proof of Lemma 3.6 must be replaced by the well-known fact that an artinian ring can be written as a direct product of a finite number of indecomposable rings. We have decided to work in a narrower class of finite dimensional algebra in order to make the paper more easily accessible to a wide audience.

Concluding remarks. The work on this paper begun by an attempt to find new examples of zero product determined algebras, but eventually resulted in an unexpected characterization of finite dimensional algebras that are generated by idempotents. The initial problem of finding new examples remains entirely open – this paper only shows where not to look for them. A new problem that now arises is finding other classes of algebras for which the characterization from Theorem 3.7 holds.

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