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ZERO PRODUCT DETERMINED MATRIX ALGEBRAS

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Abstract. Let A be an algebra over a commutative unital ring C. We say that A is zero product determined if for every C-module X and every bilinear map $\{.,.\} : A \times A \to X$ the following holds: if $\{x, y\} = 0$ whenever $xy = 0$, then there exists a linear operator T such that $\{x, y\} = T(xy)$ for all $x, y \in A$. If we replace in this definition the ordinary product by the Lie (resp. Jordan) product, then we say that A is zero Lie (resp. Jordan) product determined. We show that the matrix algebra $M_n(B)$, $n \geq 2$, where B is any unital algebra, is always zero product determined, and under some technical restrictions it is also zero Jordan product determined. The bulk of the paper is devoted to the problem whether $M_n(B)$ is zero Lie product determined. We show that this does not hold true for all unital algebras B. However, if B is zero Lie product determined, then so is $M_n(B)$.

1. Introduction

Let C be a (fixed) commutative unital ring, and let A be an algebra over C. By A^2 we denote the C-linear span of all elements of the form xy where $x, y \in A$. Let X be a C-module and let $\{.,.\} : A \times A \to X$ be a C-bilinear map. Consider the following conditions:

(a) for all $x, y \in A$ such that $xy = 0$ we have $\{x, y\} = 0$;
(b) there exists a C-linear map $T : A^2 \to X$ such that $\{x, y\} = T(xy)$ for all $x, y \in A$.

Trivially, (b) implies (a). We shall say that A is a *zero product determined algebra* if for every C-module X and every C-bilinear map $\{.,.\} : A \times A \to X$, (a) implies (b).

So far A could be any nonassociative algebra. Assume now that A is associative. Recall that A becomes a Lie algebra, usually denoted by A^-, if we replace the original product by the so-called Lie product given by $[x, y] = xy - yx$. Similarly, A becomes a Jordan algebra, denoted by A^+, by replacing the original product by the Jordan product given by $x \circ y = xy + yx$. We shall say that A is a *zero Lie product determined algebra* if A^- is a zero product determined algebra. That is to say, for every C-bilinear map $\{.,.\} : A \times A \to X$, where X is any C-module, we have that $\{.,.\}$ must be of the form $\{x, y\} = T([x, y])$ for some C-linear map $T : [A, A] \to X$ provided that $[x, y] = 0$ implies $\{x, y\} = 0$. Analogously, we shall say that A is a *zero Jordan product determined algebra* if A^+ is a zero product determined algebra (that is, $\{.,.\}$ must be of the form $\{x, y\} = T(x \circ y)$ in case $x \circ y = 0$ implies $\{x, y\} = 0$).

There are various reasons for introducing these concepts. We shall not discuss all of them in this rather short paper; we refer the reader to [1] where one can find a variety of applications of the fact that certain Banach algebras are zero product determined (note, however, that the

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terminology and the setting in [1] are somewhat different than in the present paper). Let us mention only one motivation which can be most easily explained. This is the connection to the thoroughly studied problems of describing zero (associative, Lie, Jordan) product preserving linear maps. We say that a linear map S from an algebra A into an algebra B preserves zero products if for all $x, y \in A$, $xy = 0$ implies $S(x)S(y) = 0$. The standard goal is to show that, roughly speaking, S is "close" to a homomorphism. Defining $\{.\,,.\} : A \times A \to B$ by $\{x, y\} = S(x)S(y)$ we see that $\{.\,,.\}$ satisfies (a); now if A is zero product determined, then it follows that $S(x)S(y) = T(xy)$ for some linear map T, which brings us quite close to our goal (for example, if we further assume that A and B are unital and $S(1) = 1$, then it follows immediately that $S = T$ is a homomorphism; without this assumption the problem remains nontrivial). Similar remarks can be stated for zero Lie product preserving maps (also known as commutativity preserving maps) and zero Jordan product preserving maps. The approach that we have just outlined was used in recent papers [1] (for zero product preservers) and [3] (for zero Lie product preservers).

The goal of this paper is to examine whether the algebra $M_n(B)$ of $n \times n$ matrices over a unital algebra B is zero (Lie, Jordan) product determined. In Section 2 we show that for the ordinary product the answer is "yes" for every algebra B and every $n \geq 2$, and in Section 3 we show the same for the Jordan product - however, for $n \geq 3$ and additionally assuming that B contains the element $\frac{1}{2}$ (i.e., 2 is invertible in B). The Lie product case, treated in Section 4, is more entangled. We show that $M_n(B)$ is zero Lie product determined provided that B is such as well, and thereby extend [3, Theorem 2.1]. On the other hand, we give an example justifying imposing some assumption on B.

We conclude the introduction by recording two general remarks about the problem of showing that a bilinear map $\{.\,,.\} : A \times A \to X$ satisfies (b). Firstly, it is clear that the only possible way of defining $T : A^2 \to X$ is given by $T(\sum_t x_t y_t) = \sum_t \{x_t, y_t\}$. The problem, however, is to show that T is well-defined. Accordingly, (b) is equivalent to the condition

(b') if $x_t, y_t \in A$, $t = 1, \dots, m$, are such that $\sum_{t=1}^m x_t y_t = 0$, then $\sum_{t=1}^m \{x_t, y_t\} = 0$.

Secondly, if A is a unital algebra, then (b) is equivalent to

(b") if $x_t, y_t \in A$, $t = 1, 2$, are such that $\sum_{t=1}^2 x_t y_t = 0$, then $\sum_{t=1}^2 \{x_t, y_t\} = 0$.

Indeed, if (b") is fulfilled, then we infer from $x \cdot y - xy \cdot 1 = 0$ that $\{x, y\} - \{xy, 1\} = 0$. Thus $\{x, y\} = T(xy)$ where $T : A^2 \to X$ is defined by $T(z) = \{z, 1\}$. Incidentally, Lemma 4.5 below shows that the assumption that A is unital cannot be omitted. This lemma actually considers the case when A is a Lie algebra. Let us point out that the two remarks above hold for algebras that may be nonassociative. In what follows, however, by an *algebra* we will always mean an *associative algebra*.

2. Zero (associative) product determined matrix algebras

Throughout the paper we will consider the matrix algebra $M_n(B)$ where B is a unital algebra (associative, but not necessarily commutative). As usual, a matrix unit will be denoted by e_{ij}. By be_{ij}, where $b \in B$, we denote the matrix whose (i, j) entry is b and all other entries are 0.

Theorem 2.1. *If B is a unital algebra, then $M_n(B)$ is a zero product determined algebra for every $n \geq 2$.*

Proof. Set $A = M_n(B)$. Let X be a C-module and let $\{.\,,.\}: A \times A \to X$ be a bilinear map such that for all $x, y \in A$, $xy = 0$ implies $\{x, y\} = 0$. Throughout the proof, a and b will denote arbitrary elements in B and i, j, k, l will denote arbitrary indices.

We begin by noticing that

$$\{ae_{ij}, be_{kl}\} = 0 \quad \text{if } j \neq k, \tag{1}$$

since $ae_{ij}\, be_{kl} = 0$. Further, we claim that

$$\{ae_{ij}, be_{jl}\} = \{ab\, e_{ik}, e_{kl}\} \quad \text{if } j \neq k. \tag{2}$$

Indeed, as $k \neq j$ we have $(ae_{ij} + ab\, e_{ik})(be_{jl} - e_{kl}) = 0$, which implies $\{ae_{ij} + ab\, e_{ik}, be_{jl} - e_{kl}\} = 0$. Apply (1) and (2) follows.

Replacing a by ab and b by 1 in (2) we get

$$\{abe_{ij}, e_{jl}\} = \{ab\, e_{ik}, e_{kl}\}. \tag{3}$$

Together with (2) this yields

$$\{ae_{ij}, be_{jl}\} = \{ab\, e_{ij}, e_{jl}\}. \tag{4}$$

Let $x_t,\, y_t \in A$ be such that $\sum_{t=1}^m x_t\, y_t = 0$, and let us show that $\sum_{t=1}^m \{x_t, y_t\} = 0$ (as pointed out above, we could assume that $m = 2$, but this does not simplify our proof). Writing

$$x_t = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^t\, e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^n \sum_{l=1}^n b_{kl}^t\, e_{kl}$$

it follows, by examining the (i, l) entry of $x_t y_t$, that for all i and l we have

$$\sum_{t=1}^m \sum_{j=1}^n a_{ij}^t\, b_{jl}^t = 0. \tag{5}$$

Note that

$$\sum_{t=1}^m \{x_t, y_t\} = \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \{a_{ij}^t\, e_{ij}, b_{kl}^t\, e_{kl}\}.$$

By (1) this summation reduces to

$$\sum_{t=1}^m \{x_t, y_t\} = \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \{a_{ij}^t\, e_{ij}, b_{jl}^t\, e_{jl}\}.$$

Using first (4) and then (3) we see that

$$\{a_{ij}^t\, e_{ij}, b_{jl}^t\, e_{jl}\} = \{a_{ij}^t b_{jl}^t\, e_{ij}, e_{jl}\} = \{a_{ij}^t b_{jl}^t\, e_{i1}, e_{1l}\}.$$

Therefore

$$\sum_{t=1}^m \{x_t, y_t\} = \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \{a_{ij}^t b_{jl}^t\, e_{i1}, e_{1l}\} = \sum_{i=1}^n \sum_{l=1}^n \left\{ \Big(\sum_{t=1}^m \sum_{j=1}^n a_{ij}^t b_{jl}^t \Big) e_{i1}, e_{1l} \right\} = 0$$

by (5). $\square$

3. Zero Jordan product determined matrix algebras

In the recent paper [4] Chebotar et al. considered zero Jordan product preserving maps on matrix algebras. Fortunately, some arguments from this paper are almost directly applicable to the more general situation treated in the present paper. The proof of the next theorem is to a large extent just a straightforward modification of the proof of [4, Theorem 2.2] (see also [2, Lemma 7.19]). There is one problem, however, which we have to face: unlike in [4], where the map $\{x, y\} = S(x) \circ S(y)$ is studied, we cannot assume in advance that our map $\{.\,,.\}$ treated below is symmetric (in the sense that $\{x, y\} = \{y, x\}$ for all x and y). Because of this our proof is somewhat more involved than the one of [4, Theorem 2.2].

Theorem 3.1. *If B is a unital algebra containing the element $\frac{1}{2}$, then $M_n(B)$ is a zero Jordan product determined algebra for every $n \geq 3$.*

Proof. Let $A = M_n(B)$, let X be a C-module, and let $\{.\,,.\} : A \times A \to X$ be a bilinear map such that for all $x, y \in A$, $x \circ y = 0$ implies $\{x,\, y\} = 0$. Let a and b denote arbitrary elements from B and let i, j, k, l denote arbitrary indices.

First, since $ae_{ij} \circ be_{kl} = 0$ if $i \neq l$ and $j \neq k$, it is clear that

$$\{ae_{ij}, be_{kl}\} = 0 \text{ if } i \neq l \text{ and } j \neq k. \tag{6}$$

Let $i \neq k$. Then $ae_{ik} \circ (e_{kk} - e_{ii}) = 0$ and so

$$\{ae_{ik}, e_{kk}\} = \{ae_{ik}, e_{ii}\}. \tag{7}$$

Similarly,

$$\{e_{kk}, ae_{ik}\} = \{e_{ii}, ae_{ik}\}. \tag{8}$$

From $(ae_{ik} - e_{ii}) \circ (ae_{ik} + e_{kk}) = 0$, $i \neq k$, we derive $\{ae_{ik} - e_{ii}, ae_{ik} + e_{kk}\} = 0$. Since $\{ae_{ik}, ae_{ik}\} = 0$ and $\{e_{ii}, e_{kk}\} = 0$ by (6), it follows that $\{ae_{ik}, e_{kk}\} = \{e_{ii}, ae_{ik}\}$. This identity together with (7) and (8) yields

$$\{ae_{ik}, e_{ii}\} = \{ae_{ik}, e_{kk}\} = \{e_{ii}, ae_{ik}\} = \{e_{kk}, ae_{ik}\}. \tag{9}$$

Now let $i \neq k$ and $j \neq k$. Then $(ae_{ij} + abe_{ik}) \circ (be_{jk} - e_{kk}) = 0$, and hence $\{ae_{ij} + ab\, e_{ik}, be_{jk} - e_{kk}\} = 0$. By (6) this reduces to $\{ae_{ij}, be_{jk}\} = \{ab\, e_{ik}, e_{kk}\}$. On the other hand, we also have $(be_{jk} - e_{kk}) \circ (ae_{ij} + abe_{ik}) = 0$, and so $\{be_{jk} - e_{kk}, ae_{ij} + abe_{ik}\} = 0$. By (6) this reduces to $\{be_{jk}, ae_{ij}\} = \{e_{kk}, abe_{ik}\}$. Since $\{ab\, e_{ik}, e_{kk}\} = \{e_{kk}, abe_{ik}\}$ by (9), it follows that

$$\{ae_{ij}, be_{jk}\} = \{ab\, e_{ik}, e_{kk}\} = \{be_{jk}, ae_{ij}\} \text{ if } i \neq k \text{ and } j \neq k. \tag{10}$$

If $i \neq k$, then $(ae_{ik} - e_{ii}) \circ (abe_{ik} + be_{kk}) = 0$ and $(abe_{ik} + be_{kk}) \circ (ae_{ik} - e_{ii}) = 0$. By a similar argument as before this yields

$$\{ae_{ik}, be_{kk}\} = \{ab\, e_{ik}, e_{kk}\} = \{be_{kk}, ae_{ik}\} \text{ if } i \neq k. \tag{11}$$

Setting $i = j$ in (10) we get $\{ae_{ii}, be_{ik}\} = \{ab\, e_{ik}, e_{kk}\} = \{be_{ik}, ae_{ii}\}$ if $i \neq k$. Further, $\{ab\, e_{ik}, e_{kk}\} = \{ab\, e_{ik}, e_{ii}\}$ by (9), and so we have $\{ae_{ii}, be_{ik}\} = \{ab\, e_{ik}, e_{ii}\} = \{be_{ik}, ae_{ii}\}$. For our purposes it is more convenient to rewrite this identity so that the roles of i and k, and the roles of a and b are replaced. Hence we have

$$\{be_{kk}, ae_{ki}\} = \{ba\, e_{ki}, e_{kk}\} = \{ae_{ki}, be_{kk}\} \text{ if } i \neq k. \tag{12}$$

Further, we claim that

$$\{ae_{ij}, be_{ji}\} = \frac{1}{2}\left(\{abe_{ii}, e_{ii}\} + \{bae_{jj}, e_{jj}\}\right). \tag{13}$$

If $i \neq j$, then $\left(\frac{1}{2}abe_{ii} + ae_{ij} - \frac{1}{2}bae_{jj}\right) \circ (be_{ji} - e_{ii} + e_{jj}) = 0$ and consequently

$$\left\{\frac{1}{2}ab\, e_{ii} + ae_{ij} - \frac{1}{2}ba\, e_{jj},\, be_{ji} - e_{ii} + e_{jj}\right\} = 0.$$

Using (6), (9), (10), (11) and (12) this yields $\{ae_{ij}, be_{ji}\} = \frac{1}{2}\left(\{abe_{ii}, e_{ii}\} + \{bae_{jj}, e_{jj}\}\right)$. We still have to prove (13) for $i = j$.

Let $i \neq k$. Then $(ae_{ii} - be_{ik} + be_{ki} - ae_{kk}) \circ (be_{ii} - ae_{ik} + ae_{ki} - be_{kk}) = 0$ and this gives $\{ae_{ii} - be_{ik} + be_{ki} - ae_{kk},\, be_{ii} - ae_{ik} + ae_{ki} - be_{kk}\} = 0$. By (6), (9), (10), (11) and (12) this can be reduced to

$$\{ae_{ii}, be_{ii}\} + \{ae_{jj}, be_{jj}\} = \frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\} + \{(a \circ b)e_{jj}, e_{jj}\}\right). \tag{14}$$

Since $n \geq 3$, we can choose l such that $l \notin \{i, k\}$. Applying (14) we get

$$\begin{aligned}
& (\{ae_{ii}, be_{ii}\} + \{ae_{kk}, be_{kk}\}) + (\{ae_{ii}, be_{ii}\} + \{ae_{ll}, be_{ll}\}) \\
= \;& \frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\} + \{(a \circ b)e_{kk}, e_{kk}\}\right) + \frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\} + \{(a \circ b)e_{ll}, e_{ll}\}\right) \\
= \;& \{(a \circ b)e_{ii}, e_{ii}\} + \frac{1}{2}\left(\{(a \circ b)e_{kk}, e_{kk}\} + \{(a \circ b)e_{ll}, e_{ll}\}\right) \\
= \;& \{(a \circ b)e_{ii}, e_{ii}\} + \{ae_{kk}, be_{kk}\} + \{ae_{ll}, be_{ll}\}.
\end{aligned}$$

Consequently, $\{ae_{ii}, be_{ii}\} = \frac{1}{2}\{(a \circ b)e_{ii}, e_{ii}\}$ which proves the $i = j$ case of (13).

Let $x_t, y_t \in A$ be such that $\sum_{t=1}^m x_t \circ y_t = 0$. We have to prove that $\sum_{t=1}^m \{x_t, y_t\} = 0$. Writing

$$x_t = \sum_{i=1}^n \sum_{j=1}^n a_{ij}^t\, e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^n \sum_{l=1}^n b_{kl}^t\, e_{kl}$$

it follows that for all i and l we have

$$\sum_{t=1}^m \sum_{j=1}^n (a_{ij}^t b_{jl}^t + b_{ij}^t a_{jl}^t) = 0. \tag{15}$$

First notice that

$$\sum_{t=1}^m \{x_t, y_t\} = \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n \{a_{ij}^t\, e_{ij}, b_{kl}^t\, e_{kl}\}$$

and by (6) this summation reduces to

$$\sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{\substack{l=1 \\ l \neq i}}^n \{a_{ij}^t\, e_{ij}, b_{jl}^t\, e_{jl}\} + \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq j}}^n \{a_{ij}^t\, e_{ij}, b_{ki}^t\, e_{ki}\} + \sum_{t=1}^m \sum_{i=1}^n \sum_{j=1}^n \{a_{ij}^t\, e_{ij}, b_{ji}^t\, e_{ji}\}.$$

Using (10), (11) and (12) in the first two summations and (13) in the third summation, we see that this is further equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a_{ij}^t b_{jl}^t\, e_{il}, e_{ll}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}\{b_{ki}^t a_{ij}^t\, e_{kj},\, e_{jj}\}$$
$$+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}\left(\{a_{ij}^t b_{ji}^t\, e_{ii}, e_{ii}\} + \{b_{ji}^t a_{ij}^t\, e_{jj}, e_{jj}\}\right)\right).$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{b_{ij}^t a_{jl}^t\, e_{il}, e_{ll}\},$$

and the third summation as

$$\frac{1}{2}\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(\{a_{ij}^t b_{ji}^t\, e_{ii}, e_{ii}\} + \{b_{ij}^t a_{ji}^t\, e_{ii}, e_{ii}\}),$$

it follows that

$$\sum_{t=1}^{m}\{x_t, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{(a_{ij}^t b_{jl}^t + b_{ij}^t a_{jl}^t)\, e_{il}, e_{ll}\} + \frac{1}{2}\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(a_{ij}^t b_{ji}^t + b_{ij}^t a_{ji}^t)\, e_{ii}, e_{ii}\}$$
$$= \sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{jl}^t + b_{ij}^t a_{jl}^t)\, e_{il},\, e_{ll}\right\} + \frac{1}{2}\sum_{i=1}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{ji}^t + b_{ij}^t a_{ji}^t)\, e_{ii},\, e_{ii}\right\};$$

each of these two summations is 0 by (15). □

We were unable to find out whether or not Theorem 3.1 also holds for $n = 2$; therefore we leave this as an open problem.

4. Zero Lie product determined matrix algebras

Theorem 4.1. *If B is a zero Lie product determined unital algebra, then $M_n(B)$ is a zero Lie product determined algebra for every $n \geq 2$.*

Proof. Let $A = M_n(B)$, let X a C-module, and let $\{.\,,.\} : A \times A \to X$ be a bilinear map such that $\{x, y\} = 0$ whenever $x, y \in A$ are such that $[x, y] = 0$. First notice that $\{x, x\} = 0$ for all $x \in A$, and hence $\{x, y\} = -\{y, x\}$ for all $x, y \in A$. Further, the equality $\{x^2, x\} = 0$ holds for all $x \in A$, and linearizing it we get $\{x \circ y, z\} + \{z \circ x, y\} + \{y \circ z, x\} = 0$ for all $x, y, z \in A$. We shall use these identities without mention.

Our first goal is to derive various identities involving elements of the form ae_{ij}. In what follows a and b will be arbitrary elements in B and i, j, k, l will be arbitrary indices.

First, it is clear that

$$\{ae_{ij},\, be_{kl}\} = 0 \quad \text{if } j \neq k \text{ and } i \neq l \tag{16}$$

since $[ae_{ij},\, be_{kl}] = 0$. Similarly,

$$\{ae_{ii}, e_{ii}\} = 0. \tag{17}$$

Also, if $i \neq j$, then $[a\,e_{ij} + a\,e_{ji},\, e_{ij} + e_{ji}] = 0$, and so $\{a\,e_{ij} + a\,e_{ji},\, e_{ij} + e_{ji}\} = 0$. As $\{a\,e_{ij},\, e_{ij}\} = 0$ and $\{a\,e_{ji},\, e_{ji}\} = 0$ by (16), it follows that

$$\{a\,e_{ij},\, e_{ji}\} = -\{a\,e_{ji},\, e_{ij}\} \quad \text{if } i \neq j. \tag{18}$$

Next, we claim that

$$\{ae_{ij},\, be_{jk}\} = \{ab\,e_{ik},\, e_{kk}\} = -\{ab\,e_{ik},\, e_{ii}\} \quad \text{if } i \neq k. \tag{19}$$

Indeed, since $[ab\,e_{ik},\, e_{ii} + e_{kk}] = 0$ we have $\{ab\,e_{ik},\, e_{ii} + e_{kk}\} = 0$, and so $\{ab\,e_{ik},\, e_{kk}\} = -\{ab\,e_{ik},\, e_{ii}\}$. We now consider two cases, when $j \neq k$ and when $j = k$. In the first case we have, since also $i \neq k$, $[ae_{ij} + ab\,e_{ik},\, be_{jk} - e_{kk}] = 0$, and hence $\{ae_{ij} + ab\,e_{ik},\, be_{jk} - e_{kk}\} = 0$. From (16) it follows that $\{ae_{ij},\, e_{kk}\} = 0$ and $\{ab\,e_{ik},\, be_{jk}\} = 0$, and so the identity above reduces to $\{ae_{ij},\, be_{jk}\} = \{ab\,e_{ik},\, e_{kk}\}$. In the second case, when $j = k$, we have $[ae_{ik} - e_{ii},\, ab\,e_{ik} + be_{kk}] = 0$, which implies $\{ae_{ik} - e_{ii},\, ab\,e_{ik} + be_{kk}\} = 0$. Since $\{ae_{ik},\, ab\,e_{ik}\} = 0$ and $\{e_{ii}, be_{kk}\} = 0$ by (16), it follows that $\{ae_{ik},\, be_{kk}\} = \{e_{ii},\, ab\,e_{ik}\} = -\{ab\,e_{ik},\, e_{ii}\}$, and (19) is thereby proved.

Let us prove that

$$\{ae_{ij},\, be_{ji}\} = \{ab\,e_{ij},\, e_{ji}\} + \{a\,e_{jj},\, b\,e_{jj}\}. \tag{20}$$

In view of (17) we may assume that $i \neq j$. Then we have

$$\{ae_{ij}, be_{ji}\} = \{e_{ij} \circ ae_{jj},\, be_{ji}\} = -\{be_{ji} \circ e_{ij},\, ae_{jj}\} - \{a\,e_{jj} \circ be_{ji},\, e_{ij}\}.$$

Since $\{be_{ii},\, ae_{jj}\} = 0$ by (16) and $\{ab\,e_{ij},\, e_{ji}\} = -\{ab\,e_{ji},\, e_{ij}\}$ by (18), (20) follows.

Finally, we claim that

$$\{ae_{ij},\, be_{ji}\} = \{ab\,e_{ik},\, e_{ki}\} - \{ba\,e_{jk},\, e_{kj}\} + \{a\,e_{kk},\, b\,e_{kk}\}. \tag{21}$$

Assume first that $i \neq j$. Taking into account (17) and (20) we see that (21) holds if $k = i$ or $k = j$. If $k \neq i$ and $k \neq j$, then

$$\{a\,e_{ij},\, b\,e_{ji}\} = \{a\,e_{ik} \circ e_{kj},\, be_{ji}\} = -\{ba\,e_{jk},\, e_{kj}\} + \{a\,e_{ik},\, b\,e_{ki}\},$$

and so applying (20) we get (21). Now suppose that $i = j$. Then

$$\{a\,e_{ii},\, b\,e_{ii}\} = \{a\,e_{ik} \circ e_{ki},\, b\,e_{ii}\} = -\{ba\,e_{ik},\, e_{ki}\} + \{a\,e_{ik},\, b\,e_{ki}\}.$$

From (20) it follows that

$$\{a\,e_{ii},\, b\,e_{ii}\} = \{ab\,e_{ik},\, e_{ki}\} - \{ba\,e_{ik},\, e_{ki}\} + \{a\,e_{kk},\, b\,e_{kk}\},$$

and so (21) holds is this case as well.

Now pick $x_t,\, y_t \in A$ such that $\sum_{t=1}^{m} [x_t,\, y_t] = 0$. The theorem will be proved by showing that $\sum_{t=1}^{m} \{x_t,\, y_t\} = 0$.

Write

$$x_t = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{t} e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^{n} \sum_{l=1}^{n} b_{kl}^{t} e_{kl}$$

where $a^t_{ij},\, b^t_{kl} \in B$. Computing the (i,l) entry of $[x_t, y_t]$ we see that

$$\sum_{t=1}^{m}\sum_{j=1}^{n}(a^t_{ij}b^t_{jl} - b^t_{ij}a^t_{jl}) = 0 \text{ for all } i,\, l. \tag{22}$$

By (16) we have

$$\sum_{t=1}^{m}\{x_t,\, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\{a^t_{ij}e_{ij},\, b^t_{kl}e_{kl}\}$$

$$= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a^t_{ij}e_{ij},\, b^t_{jl}e_{jl}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}\{a^t_{ij}e_{ij},\, b^t_{ki}e_{ki}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a^t_{ij}e_{ij},\, b^t_{ji}e_{ji}\}.$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a^t_{jl}e_{jl},\, b^t_{ij}e_{ij}\} = -\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{b^t_{ij}e_{ij},\, a^t_{jl}e_{jl}\},$$

and using (19) we see that the sum of the first and the second summation is equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}(\{a^t_{ij}b^t_{jl}e_{il},\, e_{ll}\} - \{b^t_{ij}a^t_{jl}e_{il},\, e_{ll}\}) = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{(a^t_{ij}b^t_{jl} - b^t_{ij}a^t_{jl})e_{il},\, e_{ll}\}$$

$$= \sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\left\{\left(\sum_{t=1}^{m}\sum_{j=1}^{n}(a^t_{ij}b^t_{jl} - b^t_{ij}a^t_{jl})\right)e_{il},\, e_{ll}\right\} = 0$$

by (22). Hence

$$\sum_{t=1}^{m}\{x_t,\, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a^t_{ij}e_{ij},\, b^t_{ji}e_{ji}\}.$$

We claim that this sum is equal to zero. Applying (21) we have that

$$\{a^t_{ij}e_{ij},\, b^t_{ji}e_{ji}\} = \{a^t_{ij}b^t_{ji}e_{i1},\, e_{1i}\} - \{b^t_{ji}a^t_{ij}e_{j1},\, e_{1j}\} + \{a^t_{ij}\, e_{11},\, b^t_{ji}\, e_{11}\}.$$

Therefore

$$\sum_{t=1}^{m}\{x_t,\, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a^t_{ij}b^t_{ji}e_{i1},\, e_{1i}\} - \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{b^t_{ji}a^t_{ij}e_{j1},\, e_{1j}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a^t_{ij}\, e_{11},\, b^t_{ji}\, e_{11}\}.$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{b^t_{ij}a^t_{ji}e_{i1},\, e_{1i}\}$$

and applying (22), we obtain

$$\sum_{t=1}^{m}\{x_t,\, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(a^t_{ij}b^t_{ji} - b^t_{ij}a^t_{ji})e_{i1},\, e_{1i}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a^t_{ij}\, e_{11},\, b^t_{ji}\, e_{11}\}$$

$$= \sum_{i=1}^{n} \left\{ \left(\sum_{t=1}^{m} \sum_{j=1}^{n} (a_{ij}^{t} b_{ji}^{t} - b_{ij}^{t} a_{ji}^{t}) \right) e_{i1},\, e_{1i} \right\} + \sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \{a_{ij}^{t}\, e_{11},\, b_{ji}^{t}\, e_{11}\}$$

$$= \sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \{a_{ij}^{t}\, e_{11},\, b_{ji}^{t}\, e_{11}\}.$$

Thus, the proof will be complete by showing that

$$\sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \{a_{ij}^{t}\, e_{11},\, b_{ji}^{t}\, e_{11}\} = 0. \tag{23}$$

Consider the map $\langle\, .\, ,.\rangle : B \times B \to X$ defined by $\langle a,\, b\rangle = \{a\, e_{11},\, b\, e_{11}\}$ for all $a, b \in B$. It is clear that $\langle\, .\, ,.\rangle$ is bilinear and has the property that $[a, b] = 0$ implies $\langle a, b\rangle = 0$. Since B is a zero Lie product determined algebra, $\langle\, .\, ,.\rangle$ also satisfies the condition that $\sum_{t=1}^{m} [a_t,\, b_t] = 0$ implies $\sum_{t=1}^{m} \langle a_t,\, b_t\rangle = 0$.

Taking $l = i$ in (22) we have that

$$\sum_{t=1}^{m} \sum_{j=1}^{n} (a_{ij}^{t} b_{ji}^{t} - b_{ij}^{t} a_{ji}^{t}) = 0$$

for every i, and hence

$$\sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} [a_{ij}^{t},\, b_{ji}^{t}] = 0.$$

This implies

$$\sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle a_{ij}^{t},\, b_{ji}^{t}\rangle = 0,$$

which is of course equivalent to (23). □

Commutative algebras are trivially zero Lie product determined. Thus we have

Corollary 4.2. *If B is a commutative unital algebra, then $M_n(B)$ is a zero Lie product determined algebra for every $n \geq 2$.*

In the simplest case where $B = C$ this corollary was proved in [3]. In fact, for this case [3, Theorem 2.1] tells us more than Corollary 4.2. In particular it states that for a C-bilinear map $\{.\, ,.\, \} : A \times A \to X$, where $A = M_n(C)$ and X is a C-module, the following conditions are equivalent:

(a) if $x, y \in A$ are such that $[x, y] = 0$, then $\{x,\, y\} = 0$;
(b) there is a C-linear map $T : [A, A] \to X$ such that $\{x,\, y\} = T([x, y])$ for all $x, y \in A$;
(c) $\{x, x\} = \{x^2, x\} = 0$ for all $x \in A$;
(d) $\{x, x\} = \{xy, z\} + \{zx, y\} + \{yz, x\} = 0$ for all $x, y, z \in A$.

The condition (c) has proved to be important because of the applications to the commutativity preserving map problem. So it is tempting to try to show that these conditions are equivalent in some more general algebras A. We remark that trivially (b) implies (c) and (d), (a) implies (c), and also (d) implies (c) as long as A is 3-torsionfree (just set $x = y = z$ in (d)). In the next example we show that in the algebra $M_2(C[x,y])$ neither (c) nor (d) implies (a), and so [3, Theorem 2.1] cannot be generalized to matrix algebras over commutative algebras.

Example 4.3. Let $A = M_2(C[x,y])$. We define a C-bilinear map $\{.\,,.\} : A \times A \to C$ as follows:

$$\{xe_{11}, ye_{11}\} = \{xe_{22}, ye_{22}\} = 1, \quad \{ye_{11}, xe_{11}\} = \{ye_{22}, xe_{22}\} = -1,$$
$$\{xe_{12}, ye_{21}\} = \{xe_{21}, ye_{12}\} = 1, \quad \{ye_{21}, xe_{12}\} = \{ye_{12}, xe_{21}\} = -1,$$

and

$$\{ue_{ij}, ve_{kl}\} = 0$$

in all other cases, that is, for all remaining choices of monomials u and v and $i,j,k,l \in \{1,2\}$. Since $[xe_{11}, ye_{11}] = 0$ and $\{xe_{11}, ye_{11}\} = 1$, $\{.\,,.\}$ does not satisfy (a) (or (b)). However, one can check that $\{.\,,.\}$ satisfies (c) and (d). The proof is a straightforward but tedious verification, and we omit details.

Our final goal is to show that there exists a unital algebra B such that $M_n(B)$ is not a zero Lie product determined algebra, and thereby to show that indeed one has to impose some condition on B in Theorem 4.1. For this we need two preliminary results which are of independent interest. The first one, however, is not really surprising, and possibly it is already known. Anyway, the following proof which was suggested to us by Misha Chebotar, is very short.

Until the end of this section we assume that C is a field.

Lemma 4.4. *Let $A = C\langle x_1, x_2, \dots, x_{2n}\rangle$ be a free algebra in $2n$ noncommuting indeterminates. Then $[x_1,x_2]+[x_3,x_4]+\dots+[x_{2n-1},x_{2n}]$ cannot be written as a sum of less than n commutators of elements in A.*

Proof. Let $a_i, b_i \in A$, $i = 1,\dots,m$, be such that

$$[a_1,b_1] + [a_2,b_2] + \dots + [a_m,b_m] = [x_1,x_2] + [x_3,x_4] + \dots + [x_{2n-1},x_{2n}]. \tag{24}$$

We have to show that $m \geq n$. We proceed by induction on n. The case when $n = 1$ is trivial, so we may assume that $n > 1$. Considering the degrees of monomials appearing in (24) we see that we may assume that all a_i's and b_i's are linear combinations of the x_i's. In particular, $b_m = \sum_{j=1}^{2n} \mu_j x_j$ with $\mu_j \in C$. Without loss of generality we may assume that $\mu_{2n} \neq 0$. Of course, we may replace any indeterminate x_i by any element in A in the identity (24). So, let us substitute 0 for x_{2n-1} and $-\sum_{j=1}^{2n-2} \mu_{2n}^{-1}\mu_j x_j$ for x_{2n}. Then we get $[c_1,d_1] + \dots + [c_{m-1},d_{m-1}] = [x_1,x_2] + \dots + [x_{2n-3},x_{2n-2}]$ where all c_i's and d_i's are linear combinations of $x_1,\dots,x_{2n-2}$. By induction assumption we thus have $m-1 \geq n-1$, and so $m \geq n$. □

For any $n \geq 2$, let B_n denote the unital C-algebra generated by $1, u_1, \dots, u_{2n}$ with the relation $[u_1,u_2] + [u_3,u_4] + \dots + [u_{2n-1},u_{2n}] = 0$. That is, $B_n = C\langle x_1,x_2,\dots,x_{2n}\rangle/I$ where I is the ideal of $C\langle x_1,x_2,\dots,x_{2n}\rangle$ generated by $[x_1,x_2] + [x_3,x_4] + \dots + [x_{2n-1},x_{2n}]$, and $u_i = x_i + I$.

Lemma 4.5. *There exists a bilinear map $\langle .\,,.\rangle : B_n \times B_n \to C$ such that for all $v_t, w_t \in B_n$, $\sum_{t=1}^{n-1}[v_t,w_t] = 0$ implies $\sum_{t=1}^{n-1}\langle v_t,w_t\rangle = 0$, but $\langle u_1,u_2\rangle + \langle u_3,u_4\rangle + \dots + \langle u_{2n-1},u_{2n}\rangle \neq 0$ (and so there is no linear map $T : [B_n,B_n] \to C$ such that $\langle x,y\rangle = T([x,y])$).*

Proof. The set S consisting of 1 and all possible products $u_{i_1}\dots u_{i_k}$ of the u_i's spans the linear space B_n, and the elements u_1, u_2 are linearly independent. Therefore we can define a bilinear map $\langle .\,,.\rangle : B_n \times B_n \to C$ such that $\langle u_1, u_2\rangle = -\langle u_2, u_1\rangle = 1$ and $\langle s,t\rangle = 0$ for all other possible choices of $s,t \in S$. In particular, $\langle u_1,u_2\rangle + \langle u_3,u_4\rangle + \ldots + \langle u_{2n-1},u_{2n}\rangle = 1$.

Assume now that $v_t, w_t \in B_n$ are such that $\sum_{t=1}^{n-1}[v_t,w_t] = 0$. We can write $v_t = \lambda_t u_1 + \mu_t u_2 + p_t$ and $w_t = \alpha_t u_1 + \beta_t u_2 + q_t$, where $\lambda_t,\mu_t,\alpha_t,\beta_t \in C$ and p_t,q_t lie in the linear span of $S\setminus\{u_1,u_2\}$. Note that $\sum_{t=1}^{n-1}\langle v_t,w_t\rangle = \sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t)$. Thus, the lemma will be proved by showing that $\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t) = 0$.

Let us write $v_t = \lambda_t x_1 + \mu_t x_2 + l_t + f_t + I$, $w_t = \alpha_t x_1 + \beta_t x_2 + m_t + g_t + I$, where $\lambda_t,\mu_t,\alpha_t,\beta_t \in C$, l_t, m_t are linear combinations of $x_3,\ldots,x_{2n}$ and f_t, g_t are linear combinations of mononials of degrees 0 or at least 2. Since $\sum_{t=1}^{n-1}[v_t,w_t]=0$, it follows that

$$\sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t + f_t, \alpha_t x_1 + \beta_t x_2 + m_t + g_t] \in I.$$

Therefore,

$$\sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t + f_t, \alpha_t x_1 + \beta_t x_2 + m_t + g_t] = \omega\Big([x_1,x_2] + [x_3,x_4] + \ldots + [x_{2n-1},x_{2n}]\Big) + h,$$

where $\omega \in C$ and $h \in I$ is a linear combination of monomials of degree at least 3. Considering the degrees of monomials involved in this identity it clearly follows that

$$\sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t, \alpha_t x_1 + \beta_t x_2 + m_t] = \omega\Big([x_1,x_2] + [x_3,x_4] + \ldots + [x_{2n-1},x_{2n}]\Big).$$

We may now apply Lemma 4.4 and conclude that $\omega = 0$. Thus,

$$0 = \sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t, \alpha_t x_1 + \beta_t x_2 + m_t] = \Big(\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t)\Big)[x_1,x_2] + f,$$

where f is a linear combination of monomials different from x_1x_2 and x_2x_1. Consequently, $\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t) = 0$. □

Lemma 4.5 in particular shows that B_n is not a zero Lie product determined algebra for every $n \geq 2$. We remark in this context that it is very easy to find examples of algebras that are not zero product determined or zero Jordan product determined, simply because there are algebras without nonzero zero divisors (domains), as well as such that the Jordan product of any of their two nonzero elements is always nonzero. Finding algebras that are not zero Lie product determined is more difficult since in every algebra we have plenty of elements commuting with each other.

We are now in a position to show that matrix algebras are not always zero Lie product determined.

Theorem 4.6. *For every $n \geq 1$, the algebra $M_n(B_{n^2+1})$ is not zero Lie product determined.*

Proof. By Lemma 4.5 there exists a bilinear map $\langle .\,,.\rangle : B_{n^2+1} \times B_{n^2+1} \to C$ such that $\sum_{t=1}^{n^2}[v_t,w_t] = 0$ implies $\sum_{t=1}^{n^2}\langle v_t,w_t\rangle = 0$, but there are $u_t \in B_{n^2+1}$, $t = 1,\ldots,2n^2+2$, such that $\sum_{t=1}^{n^2+1}[u_{2t-1},u_{2t}] = 0$ while $\sum_{t=1}^{n^2+1}\langle u_{2t-1},u_{2t}\rangle \neq 0$.

Set $A = M_n(B_{n^2+1})$, and define $\{.\,,.\} : A \times A \to C$ according to

$$\{v, w\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_{ij}, w_{ji} \rangle,$$

where v_{ij} and w_{ij} are entries of the matrices v and w, respectively. We claim that $\{.\,,.\}$ satisfies the condition "$[v, w] = 0 \Longrightarrow \{v, w\} = 0$", but does not satisfy the condition "$\sum_t [v_t, w_t] = 0 \Longrightarrow \sum_t \{v_t, w_t\} = 0$". The latter is obvious, since we may take $v_t = u_{2t-1}e_{11}$ and $w_t = u_{2t}e_{11}$, $t = 1, \ldots, n^2+1$. Now pick v and w in A such that $[v, w] = 0$, i. e. $vw = wv$. Considering just the diagonal entries of matrices on both sides of this identity we see that $\sum_{j=1}^n v_{ij}w_{ji} = \sum_{j=1}^n w_{ij}v_{ji}$ for every $i = 1, \ldots, n$. Accordingly, $\sum_{i=1}^n \sum_{j=1}^n v_{ij}w_{ji} = \sum_{i=1}^n \sum_{j=1}^n w_{ij}v_{ji}$. Rewriting $\sum_{i=1}^n \sum_{j=1}^n w_{ij}v_{ji}$ as $\sum_{i=1}^n \sum_{j=1}^n w_{ji}v_{ij}$ we thus see that $\sum_{i=1}^n \sum_{j=1}^n [v_{ij}, w_{ji}] = 0$. However, this implies $\sum_{i=1}^n \sum_{j=1}^n \langle v_{ij}, w_{ji} \rangle = 0$, that is, $\{v, w\} = 0$. □

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