Matrices defining elliptic curves

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Outline I

Outline II

- Existence of indecomposable rank r bundles
- Generalise the definitions of det and Pf
- Generalise the construction of the rank 3 adjoint
- Group actions on tensor products of vector spaces

6. Weierstrass form
   - Canonical form
   - Inflection point
   - Algorithm

Abstract

For a given plane curve we review the explicit constructions of:

- the $1-1$ correspondence between linear determinantal representations and rank one (non-exceptional) bundles,
- the $1-1$ correspondence between pfaffian representations and rank two (non-exceptional with fixed determinant) bundles.

We try to generalize these results to construct determinantal representations which would encode rank 3 bundles as its cokernel.

Abstract

Elliptic curves are of tame representation type according to Atiyah. In particular, on a given cubic curve the number of indecomposable ACM bundles of rank r equals to the number of r-torsion points.
We explicitly construct all determinantal representations of size 3×3 and 6×6.

Background

Elliptic curves have profound influence in mathematics. Since ancient times they turn up in the most astonishing places, joining together algebra and geometry. Recently they have become popular in **number theory** (cryptography of elliptic curves), **optimization** (semidefinite programming SDP) and also in **theoretical physics** (mirror symmetry of elliptic curves).

The abundance of results is due to the following two classical facts for smooth plane cubics:

- It can be brought by a change of coordinates into the Weierstrass canonical form, or equivalently the Hesse canonical form.
- It can be equipped by a group law (induced by the Jacobian group variety).

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The abundance of results is due to the following two classical facts for smooth plane cubics:

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- It can be equipped by a group law (induced by the Jacobian group variety).

Notation

- we work over the field $\mathbb{C}$, sometimes we restrict to $\mathbb{R}$,
- $F(x, y, z)$ homogeneous polynomial of degree 3,
- $\mathcal{C}$ a smooth curve defined by $\{F(x, y, z) = 0\} \subset \mathbb{P}^2$.

Weierstrass form: $y^2 z = x^3 + p\, xz^2 + q\, z^3, \quad p, q \in \mathbb{C},$

or $y^2 z = x(x + \theta_1 z)(x + \theta_2 z), \quad \theta_1, \theta_2 \in \mathbb{C},$

Hesse form: $\lambda(x^3 + y^3 + z^3) = \mu\, xyz, \quad \lambda, \mu \in \mathbb{P}^1.$

Definition: Determinantal representation

It is very useful to represent F as a determinant of some matrix: Find a $3r \times 3r$ matrix with **linear** terms

$$M(x, y, z) = xA + yB + zC$$

such that

$$\det M(x, y, z) = c\, F^r(x, y, z), \quad \text{for some } c \neq 0.$$

Matrix M is a determinantal representation (of order r) of $\mathcal{C}$.

Clearly, multiplying a determinantal representation by invertible matrices preserves the underlying curve. Two determinantal representations M and M' are equivalent if

$$M' = XMY \quad \text{for some } X, Y \in \mathrm{GL}(3r, \mathbb{C}).$$

We consider determinantal representations up to equivalence.

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Pfaffian representation

Pfaffian representation is a representation of order 2 with all 6×6 matrices being skew-symmetric. Study of pfaffian representations is strongly related to and motivated by determinantal representations. Every 3×3 determinantal representation A induces *decomposable pfaffian representation*

$$\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix}.$$

Note that the equivalence relation is well defined since

$$\begin{bmatrix} 0 & XMY \\ -(XMY)^t & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^t \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} X^t & 0 \\ 0 & Y \end{bmatrix}.$$

Theorem (Beauville, 2000: plane cubics as determinants)

Let L be a line bundle of degree 0 on $\mathcal{C}$ with $H^0(\mathcal{C}, L) = 0$. Then there exists a 3×3 linear matrix M such that $F = \det M$ and an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to L \to 0.$$

Conversely, let M be a 3×3 linear matrix such that $F = \det M$. Then the cokernel of $M : \mathcal{O}_{\mathbb{P}^2}(-2)^3 \to \mathcal{O}_{\mathbb{P}^2}(-1)^3$ is a line bundle L on $\mathcal{C}$ of degree 0 with $H^0(\mathcal{C}, L) = 0$.

This is the famous $1-1$ correspondence between detreminantal representations of hypersurfaces and points on the Jacobian variety, first described in Cook and Thomas, Line bundles and homogeneous matrices, (1979).

Theorem (Beauville, 2000: plane cubics as pfaffians)

Let E be a rank 2 vector bundle on $\mathcal{C}$ with $\det E \cong \omega_{\mathcal{C}}$ and $H^0(\mathcal{C}, E) = 0$. Then E admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^6 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-1)^6 \to E \to 0,$$

where the matrix A is linear skew-symmetric and $F = \operatorname{Pf} A$.

Note that the condition $H^0(\mathcal{C}, E) = 0$ implies that E is semi-stable.

$\tilde{M} \cdot M = \det M \cdot \mathrm{Id}$

Theorem (Dolgachev's explicit $1-1$ correspondence)

Let $L \in \mathrm{Pic}(\mathcal{C})^0 \setminus W_0$, where $\mathrm{Pic}(\mathcal{C})^0$ is the Picard variety of degree 0 invertible sheaves (divisor classes) on $\mathcal{C}$ and W_0 its subset of effective divisors. Then L and L^{-1} define a unique regular map

$$\mathcal{C} \to |H^0(\mathcal{C}, L(1))^\vee \otimes H^0(\mathcal{C}, L^{-1}(1))|$$

which extends to a rational map on $\mathbb{P}^2$. In coordinates, this is the adjugate matrix of a determinantal representation of $\mathcal{C}$. Conversely, the kernel and cokernel (twisted by -1) of a given determinantal representation define L and L^{-1} with the above properties.

We generalize Dolgachev's construction to rank 2

Define the pfaffian adjoint of A to be the skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & & & \\ & \ddots & (-1)^{i+j}\,\mathrm{Pf}^{ij} A & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

By analogy with determinants the following holds

$$\tilde{A} \cdot A = \mathrm{Pf}\, A \cdot \mathrm{Id}.$$

Proposition

Let A be a pfaffian representation of a smooth plane curve C defined by a homogeneous polynomial F of degree d. Then $\mathcal{E} = \mathrm{Coker}\, A$ is a rank 2 vector bundle on C and

(i) $h^0(C, \mathcal{E}) = 2d$,

(ii) $H^0(C, \mathcal{E}(-1)) = H^1(C, \mathcal{E}(-1)) = 0$,

(iii) $\det \mathcal{E} = \bigwedge^2 \mathcal{E} = \mathcal{O}_C(d-1)$.

Proposition

Let C be a smooth plane curve of degree d. To every rank 2 vector bundle $\mathcal{E}$ on C with properties

(i) $h^0(C, \mathcal{E}) = 2d$,

(ii) $H^0(C, \mathcal{E}(-1)) = 0$,

(iii) $\det \mathcal{E} = \bigwedge^2 \mathcal{E} = \mathcal{O}_C(d-1)$

we can assign a pfaffian representation $A_{\mathcal{E}}$. In particular, isomorphic bundles induce equivalent representations.

We will define a map ψ from C to the space of $2d \times 2d$ skew-symmetric matrices with entries from the space of homogeneous polynomials of degree $d-1$, such that $\psi^{-1}(\mathcal{P}_d) = C$.
Let $U = H^0(C, \mathcal{E})$ be the $2d$ dimensional vector space of global sections of $\mathcal{E}$. Choose a basis $\{s_1, \ldots, s_{2d}\}$ for U and define

$$C \ni x \overset{\psi}{\mapsto} \sum_{1 \leq i < j \leq 2d} (s_i(x) \wedge s_j(x))(E_{ij} - E_{ji}) = \begin{bmatrix} 0 & & \\ & \ddots & s_i(x) \wedge s_j(x) \\ & \ddots & \\ & & 0 \end{bmatrix}.$$

Since $s_i \wedge s_j \in \bigwedge^2 U$, by property (iii) the map ψ extends to

$$\Psi : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\bigwedge^2 U)$$

given by a linear system of plane curves of degree $d - 1$. In coordinates it equals to a $2d \times 2d$ skew-symmetric matrix with entries from the space of homogeneous polynomials of degree $d - 1$. This is exactly the adjoint matrix of the pfaffian representation.

Theta characteristic

It is obvious that a 3×3 determinantal representation is **symmetric** if and only if $L \cong L^{-1}$. Such L is by definition a non-effective theta characteristic i.e., $L^{\otimes 2} \cong \omega_C$ and $H^0(C, L) = \{0\}$. Since every nonsingular cubic has exactly three even theta characteristics we get:

Corollary

A smooth cubic curve has three symmetric determinantal representations.

The following theorem constructs all three symmetric 3×3 representations:

Theorem (J. Harris, 1979, p. 696)

There exist precisely three points $(a, b) \in \mathbb{C}^2$ such that

$$aF = \mathrm{Hes}(bF + \mathrm{Hes}(F)),$$

where Hes *is the Hessian i.e., the determinant of the second partial derivatives matrix. The resulting three symmetric determinantal representations of F are inequivalent.*

Using elementary transformations [Vinnikov] we can obtain all 3×3 determinantal representations of F from a given one.

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Jacobian

One could naively generalise the following

- Let C be a smooth curve defined by a polynomial F of degree d in $\mathbb{P}^2$. All linear determinantal $d \times d$ representations of F (up to equivalence) can be parametrised by points on the Jacobian variety of C not on the exceptional subvariety W_{g-1}.
- When C is a cubic, its Jacobian equals the curve itself and $W_0 = \{\mathcal{O}_C\}$. Therefore determinantal representations can be parametrised by affine points on C.

Moduli space

In the rank 2 case, the role of the Jacobian is replaced by the moduli space $M_C(2, \mathcal{O}_C(d-3))$ of (semi-stable) rank 2 vector bundles on C with determinant $\mathcal{O}_C(d-3)$.
We would expect

- There is a one to one correspondence between linear pfaffian representations of F (up to equivalence) and rank 2 bundles (up to isomorphism) on C in the open set $M_C(2, \mathcal{O}_C(d-3)) - \Theta_{2,\mathcal{O}_C(d-3)}$.
- When C is a cubic, the space $M_C(2, \mathcal{O}_C)$ consists only of direct sums of line bundles. Therefore pfaffian representations can be parametrised by the Kummer variety $\{L \oplus L^{-1}; L \in JC\} - \{\mathcal{O}_C \oplus \mathcal{O}_C\}$.

Ravindra and Tripathi, 2014 predicted three indecomposable pfaffian representations induced by non-trivial extensions of even theta characteristics.

On a smooth cubic C consider the non-trivial extension:

$$0 \to \mathcal{O}_C \to H \to \mathcal{O}_C \to 0.$$

If $L \neq \mathcal{O}_C$ satisfies $L \otimes L = \mathcal{O}_C$, then $E = H \otimes L$ has $\det(H) = \mathcal{O}_C$, and E is indecomposable because H is. Also, we have $h^0(E) = 0$ and $h^0(E(1)) = 6$, so our construction gives an indecomposable (non-block-diagonal) pfaffian representation.

Vinnikov explicitly parametrised determinantal representations by points on the affine curve:

Lemma

Consider the cubic in Weierstrass form:

$$F(x, y, z) = -yz^2 + x^3 + \alpha xy^2 + \beta y^3.$$

A complete set of determinantal representations of F is

$$x\,\mathrm{Id} + z \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} \frac{t}{2} & s & \alpha + \frac{3}{4}t^2 \\ 0 & -t & -s \\ -1 & 0 & \frac{t}{2} \end{pmatrix},$$

where $s^2 = t^3 + \alpha t + \beta$. Note that the last equation is exactly the affine part $F(t, 1, s)$.

Main theorem

Let C be a smooth cubic in the Weierstrass form

$$F(x, y, z) = yz^2 - x(x - y)(x - \lambda y).$$

A complete set of pfaffian representations of F consists of three indecomposable representations and for the whole affine curve of decomposable representations:

$$x\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 & 0 & \frac{3t^2-2t(1+\lambda)-(1-\lambda)^2}{4} & 0 & \frac{t-1-\lambda}{2} \\ & 0 & 0 & 0 & -t & 0 \\ & & 0 & \frac{t-1-\lambda}{2} & 0 & -1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

for $t = 0, 1, \lambda$;

and

$$x\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 & 0 & \frac{3t^2-2t(1+\lambda)-(1-\lambda)^2}{4} & s & \frac{t-1-\lambda}{2} \\ & 0 & 0 & -s & -t & 0 \\ & & 0 & \frac{t-1-\lambda}{2} & 0 & -1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix},$$

where $s^2 = t(t-1)(t-\lambda)$. Note that the last equation is exactly the affine part $F(t, 1, s)$.

A. Ayyer: Determinants and Perfect Matchings, J. Combin. Theory (2013)

Ayyer gives a combinatorial interpretation of the determinant of a matrix as a generating function over Brauer diagrams. As a corollary he obtains Cayley's relation between determinants and Pfaffians.

Determinantal representations induced by higher order torsion points

Even theta characteristics are 2-torsion elements in $\mathrm{Pic}^0(C)$. Repeated extension of an r-torsion line bundle with itself gives

Theorem (Ravindra, Tripathi)

Let M be a minimal $3r \times 3r$ linear matrix such that the cokernel

$$\mathcal{O}_{\mathbb{P}^2}(-2)^{3r} \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{3r}$$

is an indecomposable rank r bundle E with $\det E = \mathcal{O}_C$. Then $\det M = F^r$. Furthermore, such E and M are in $1-1$ correspondence with nontrivial r-torsion points of C.

Question

What symmetries does the repeated extension of an r-torsion line bundle with itself impose on the $3r \times 3r$ linear matrix?

Determinant

Let A be a $d \times d$ matrix. By definition,

$$\det(A) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod_{i=1}^{d} a_{i,\sigma(i)}.$$

A permutation

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & \cdots & d \\ j_1 & j_2 & j_3 & \cdots & j_d \end{bmatrix}$$

can be written as $\{(1, j_1), (2, j_2), \ldots, (d, j_d)\}$.

$$\text{Then } \det(A) = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) \prod a_{1,j_1} a_{2,j_2} \cdots a_{d,j_d}.$$

Pfaffian

Let A be a $2d \times 2d$ skew-symmetric matrix.

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} a_{\sigma(2i-1),\sigma(2i)}, \text{ where we sum over}$$

$\Pi = \{\sigma \in S_d \; : \; \sigma(2i-1) < \sigma(2i)$ and $\sigma(2i-1) < \sigma(2i+1)$.

A partition of $\{1, 2, \ldots, 2d\}$ into pairs can be written as $\{(i_1, j_1), (i_2, j_2), \ldots, (i_d, j_d)\}$ with $i_k < j_k$ and $i_k < i_{k+1}$.

Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2d \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_d \end{bmatrix}$ be the corresponding permutation.

$$\text{Then } \mathrm{Pf}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod a_{i_1, j_1} a_{i_2, j_2} \cdots a_{i_d, j_d}.$$

Tri???an

A partition of $\{1, 2, \ldots, 3d\}$ into triplets can be written as $\{(i_1, j_1, k_1), (i_2, j_2, k_2), \ldots, (i_d, j_d, k_d)\}$ with $i_m < j_m < k_m$ and $i_m < i_{m+1}$.

Let $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 3d \\ i_1 & j_1 & k_1 & i_2 & \cdots & k_d \end{bmatrix}$ be the corresponding permutation.

We could define

$$\mathrm{Tri}(A) = \sum_{\sigma \in \Pi} \mathrm{sgn}(\sigma) \prod a_{i_1, j_1, k_1} a_{i_2, j_2, k_2} \cdots a_{i_d, j_d, k_d}.$$

In this case A is a three dimensional matrix representing a three dimensional tensor.

How can we generalise the identities involving the adjugate matrix: $\tilde{M} \cdot M = \det M \;\mathrm{Id}$ and $\tilde{A} \cdot A = \mathrm{Pf}\, A \;\mathrm{Id}$?

We wish

Let C be a smooth plane curve of degree d. To every rank 3 vector bundle $\mathcal{E}$ on C with properties

(i) $h^0(C, \mathcal{E}) = 3d$,

(ii) $H^0(C, \mathcal{E}(-1)) = 0$,

(iii) $\det \mathcal{E} = \bigwedge^3 \mathcal{E} = \mathcal{O}_C(d-1)$

we can assign a representation $A_{\mathcal{E}}$. In particular, isomorphic bundles induce equivalent representations.

We can define a map ψ from C to the space of $3d \times 3d \times 3d$ matrices with entries from the space of homogeneous polynomials of degree $d-1$, such that $\psi^{-1}(\mathcal{P}_d) = C$. Choose a basis $\{s_1, \ldots, s_{3d}\}$ for $H^0(C, \mathcal{E})$ and define

$$x \stackrel{\psi}{\mapsto} \sum_{1 \le i < j < k \le 3d} (s_i(x) \wedge s_j(x) \wedge s_k(x))(E_{ijk} - E_{jik} + E_{jki} + E_{kij} - E_{kji} - E_{ikj})$$

However, we would like A to be a $3d \times 3d$ matrix with ??? symmetry for two reasons:

- Ravindra Tripathi determinantal representations corresponding to 3-torsion line bundles are of such 9×9 form.
- Decomposable bundles can be represented by block matrices (like in the case of decomposable pfaffian representations).

Let V be a representation of G, $\rho : G \to \mathrm{Gl}(V)$. We have two groups acting on $V^{\otimes n}$. The symmetric group S_n acts by permuting the factors

$$\sigma \in S_n : \; u_1 \otimes \cdots \otimes u_n \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

and G acts diagonally

$$\rho^{\otimes n}(g) : \; u_1 \otimes \cdots \otimes u_n \mapsto \rho(g)u_1 \otimes \cdots \otimes \rho(g)u_n.$$

Proposition

The above actions of G and S_n commute. Moreover, every S_n-isotypical component of $V^{\otimes n}$ is a G-sub-representation.

Recall the familiar representations of S_n:

- the trivial 1-dim representation;
- the 1-dim sign representation $\epsilon : S_3 \to \pm 1$,
- S_3 also has the geometric (or standard) 2-dim representation.

The blocks of the trivial and the sign representation in $V^{\otimes n}$ are $\mathrm{Sym}^n V$ and $\wedge^n V$ respectively. In particular,

$$S_2 \text{ induces} \quad V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^n V$$

and

$$S_3 \text{ induces} \quad V \otimes V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^n V \oplus \text{two copies of } V.$$

S_3 naturally acts on $V \otimes V \otimes V$ as described above.

But our aim is to produce matrix determinantal representations.

Example

To a 3-torsion point on a cubic curve F, we would like to assign 9×9 linear matrix whose determinant equals F^3.
For this we need S_3 to act on 9-dimensional V as a subrepresentation of Gl(9). Then S_3 also acts diagonally on $V \otimes V$ and commutes with S_2.

Weierstrass canonical form

Theorem

By a projective change of coordinates, every irreducible curve can be brought into the Weierstrass form

$$y^2 z = x^3 + pxz^2 + qz^3, \quad p, q \in \mathbb{C}$$

or equivalently $y^2 z = x(x + \theta_1 z)(x + \theta_2 z), \quad \theta_1, \theta_2 \in \mathbb{C}.$

Moreover, every reduced curve is projectively equivalent to one of the

$$x^3,\ x^2y,\ xy(x+y),\ xyz \quad \text{or}$$
$$(\alpha x + \beta y + \gamma z)(x^2 - yz) \quad \text{for some} \ \ \alpha, \beta, \gamma \in \mathbb{C}.$$

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Why do we want the Weierstrass canonical form?

Corollary

Any coordinate independent statement that holds for a Weierstrass cubic, holds for any irreducible cubic curve.

This implies:

- Determinantal representations of any cubic curve $\mathcal{C}$ are in one to one correspondence with affine points on $\mathcal{C}$.

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Inflection point

Every irreducible cubic has inflection points:
$\{F = 0\} \cap \{\text{Hes } F = 0\} \subset \mathbb{P}^2$.

Proposition

If we find an inflection point on $\mathcal{C}$, we can put it into the Weierstrass form.

Change the coordinates so that the inflection point is $(0, 1, 0)$ and the inflection tangent is $z = 0$. Considering all possible monomials occurring in F yields the Weierstrass form.

Corollary

When the defining polynomial F is real, a real change of coordinates gives the Weierstrass form with $p, q \in \mathbb{R}$.

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Algorithm

- The enumerative problem of locating flexes of a plane cubic is solvable, since the corresponding Galois group is solvable [Harris, 1979].
- When $\mathcal{C}$ contains a rational point [Silverman and Tate, 1992] provided an algorithm that puts it into a Weierstrass form.

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