Positive semidefinite quadratic determinantal representations

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Outline I

1. Symmetric quadratic determinantal representations
   - Positive maps
   - Self-dual sheaves
2. Linear matrix inequalities
   - Spectrahedral cone $\leftrightarrow$ hyperbolic polynomial
   - Cubic symmetroid
   - Criteria for P to be positive
3. Polynomial nonnegativity
   - PSD and SOS polynomials
   - PSD and SOS matrices
   - Šivic biquadratic form

Warm up question

Given a homogeneous ***nonnegative*** *polynomial* $p(x, y, z)$ *of degree 6, does there exist a* ***positive*** *linear map* $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *such that*

$$\det P(\mathbf{x}\mathbf{x}^T) = p(x, y, z) \ \textit{ for all } \ \mathbf{x} = [x,\ y,\ z]^T?$$

We will “tackle” this question from three sides:

- symmetric quadratic determinantal representations and the associated sheaves (kernels);
- semidefinite linear determinantal representations (LMI representations of hyperbolic polynomials);
- polynomial algebra (SOS and PSD polynomials and matrices).

Positive maps

Definition

A linear map $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ is positive if it sends positive semidefinite matrices to positive semidefinite matrices.

Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.

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Positive maps $\leftrightarrow$ determinantal representations

Clearly it is enough to check the positivity of P on rank 1 matrices. In coordinates our question then becomes

Question

Given a nonnegative plane sextic $\mathcal{C}$ in $\mathbb{P}^2$, does there exist a symmetric quadratic determinantal representation of $\mathcal{C}$ which is semidefinite for all $(x, y, z) \in \mathbb{P}^2$?

Indeed, $P\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix}$,

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$.

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Quadratic determinantal representations

Without the nonnegativity and semidefiniteness conditions this is a classical case of determinantal hypersurfaces

Question

Given a plane curve $\mathcal{C}$ of degree $2d$, does there exist a $d \times d$ symmetric quadratic determinantal representation of $\mathcal{C}$?

Beauville: Determinantal Hypersurfaces, 2000

On an integral curve C, a coherent torsion-free rank 1 (arithmetically Cohen-Macaulay, ACM) sheaf $\mathcal{F}$ that is

generated by its global sections and $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$

admits a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^d \to \mathcal{F} \to 0,$$

where M is a symmetric quadratic matrix with $\det M = p$.

Remark: The above $\mathcal{F}$ is non-exceptional. This is equivalent to $H^0(C, \mathcal{F}(-1)) = H^1(C, \mathcal{F}) = 0$. Then $h^0(C, \mathcal{F}) = d$ and its global sections yield M.

Beauville: Determinantal Hypersurfaces, 2000

Actually, any $\mathcal{F}$ that is self-dual

$$\boxed{\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))}$$

admits a resolution $0 \to \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(-2-d_i) \xrightarrow{M} \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0,$

where $M = [m_{ij}]$ is symmetric with m_{ij} of degree $d_i + d_i - 2$.

Remark: We are only interested in non-exceptional $\mathcal{F}$, for which $d_i = 0$ for $i = 1, \ldots, d$. The set of such pairs $(C, \mathcal{F})$ is Zariski dense in the universal Jacobian $\mathcal{J}_{2d}^{2d(d-1)}$.

Beauville: Determinantal Hypersurfaces, 2000

Define the moduli space $\mathcal{R}_{2d}$ of pairs (C, α), where C is a smooth plane curve of degree $2d$ (over a field of char 0), and α is a half-period, i.e. a 2-torsion divisor on $\mathrm{Jac}(C)$, i.e. a nontrivial line bundle on C satisfying $\alpha^{\otimes 2} \cong \mathcal{O}_C$.

Proposition

For (C, α) general in $\mathcal{R}_{2d}$, the half-period α admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d-1)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-d+1)^d \to \alpha \to 0,$$

where M is a symmetric quadratic matrix with $\det M = p$.

Note, $\mathcal{F}$ is obtained from the half-period α by $\mathcal{F} = \alpha \otimes \mathcal{O}_C(d-1)$.

Simple singularities

When $C/\mathbb{C}$ has only simple (this means AED) singularities, there are **finitely many** ACM sheaves with the following self-duality

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2)).$$

It is possible to **explicitly count** them using methods in [Piontkowski, 2007]. Their number depends on the genus of the curve and the local type of $\mathcal{F}$: $(\mathcal{F}_s)_{s \in \mathrm{Sing}\, C}$ is a collection of self-dual modules $\mathcal{F}_s \cong \mathrm{Hom}(\mathcal{F}_s, \mathcal{O}_{C,s}(2d-2))$. For a simple singularity there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring.

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Summary

Question

Given a plane curve C of degree $2d$ with only simple singularities, does there exist a $d \times d$ symmetric quadratic determinantal representation of C?

Answer

There are finitely many symmetric quadratic determinantal representations of C corresponding to non-exceptional torsion-free rk 1 sheaves on C with self-duality $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$.

A smooth C has $\mathbf{2^{2g}}$ self-dual $\mathcal{F}$; the number decreases rapidly with the number and order of singularities A_n, D_m, E_l. When all $\mathcal{F}$ are exceptional, C has no quadratic representations.

$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$

Minimal resolutions $M = [m_{ij}]$ with $\deg m_{ij} = d_i + d_j + 2$:

	$\mathcal{F}$ non-except.	$\mathcal{F}$ exceptional
quartic:	$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ SINGULAR, $\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ SQUARE
sextic:	$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$, square

spectrahedra $\leftrightarrow$ real zero polynomials

A Hermitian / symmetric linear matrix polynomial M is a pencil

$$M_0 + x_1 M_1 + \cdots + x_n M_n, \text{ where } M_i \in \mathrm{H}_d(\mathbb{C}) \ / \ \mathrm{Sym}_d(\mathbb{R}).$$

Spectrahedron $S(M)$ is the set of points where M is positive semidefinite

$$S(M) := \{a \in \mathbb{R}^n \ : \ M(a) \succeq 0\}.$$

Spectrahedra are precisely the sets on which

semidefinite programming SDP can be performed.

spectrahedra $\leftrightarrow$ real zero polynomials

We can always assume that 0 belongs to the interior of $S(M)$, and after conjugation with a unitary / orthogonal matrix we can take $M_0 = \mathrm{Id}$.
Now consider

$$p = \det M.$$

Note that $S(M)$ can be retrieved from the polynomial p only. It consists of those points a for which p has NO zeros between the origin and a. Indeed, for each $a \in \mathbb{R}^n$, the nonzero eigenvalues of $a_1 M_1 + \cdots + a_n M_n$ are in 1-1 correspondence with the zeros of the univariate polynomial $p_a(t) := p(t\, a)$, by the following rule $\lambda \mapsto -1/\lambda$.

spectrahedra $\leftrightarrow$ real zero polynomials

Conversely, take a real zero polynomial or RZ-polynomial p,

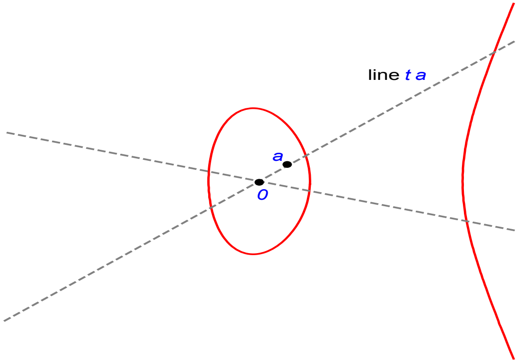
$$p(0) = 1 \text{ and } \forall a \in \mathbb{R}^n \;\; p(t\,a) = 0 \Rightarrow t \in \mathbb{R}.$$

It is natural to ask whether the rigidly convex set defined by p

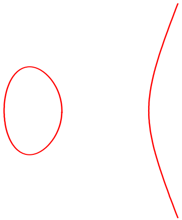
$$\{a \in \mathbb{R}^n \;:\; p_a(t) = p(t\,a) \text{ has no roots in } [0,1)\}$$

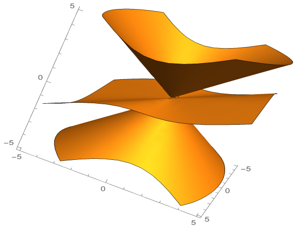
is a spectrahedron of some linear matrix M. When M exists, it is called a linear matrix inequality or LMI for p.

Cubic curve $(x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1 x_2^2 = 0$



Terminology $\mathbb{A}^n \leftrightarrow \mathbb{P}^n$





$\mathbb{A}^n$:	spectrahedron	RZ-polynomial	rigidly convex set	PSD LMI
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{P}^n$:	spectrahedral cone	hyperbolic poly.	hyperbolicity cone	SD LMI

Cubic curve $(x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1 x_2^2 = 0$

has three symmetric determinantal representations:

- two are definite determinantal representations

$$\begin{bmatrix} x_0 & -\frac{x_1}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ -\frac{x_1}{\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{x_1}{8} \\ \frac{x_1}{\sqrt{2}} & -\frac{x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8} \end{bmatrix}, \begin{bmatrix} x_0 & -\frac{x_1}{2\sqrt{2}} & \frac{x_1}{2\sqrt{2}} \\ -\frac{x_1}{2\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{7x_1}{8} \\ \frac{x_1}{2\sqrt{2}} & -\frac{7x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8} \end{bmatrix}$$

- one is nondefinite

$$x_0 \,\mathrm{Id} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} + x_1 \begin{bmatrix} 0 & \frac{i}{2\sqrt{2}} & \frac{-i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & -\frac{1}{8} & \frac{9}{8} \\ \frac{-i}{2\sqrt{2}} & \frac{9}{8} & -\frac{1}{8} \end{bmatrix}$$

Veronese surface

In coordinates write $\mathbb{P}^5$ as a symmetric matrix $\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix}$,

and consider the Veronese embedding

$$\begin{array}{cccc} \nu_2 : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ & (x, y, z) & \mapsto & \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}. \end{array}$$

Cubic symmetroid as LMI

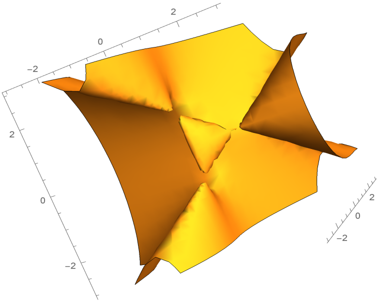
Cubic symmetroid in $\mathbb{P}^5$ is the hypersurface defined by

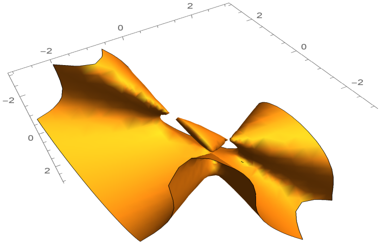
$$\det \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} = 0.$$

It is singular along the Veronese surface. Semidefinite matrices lie in the spectrahedral cone

$$\left\{ (z_0, \ldots, z_5) \in \mathbb{P}^5 \ : \ \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \succeq 0 \right\}.$$

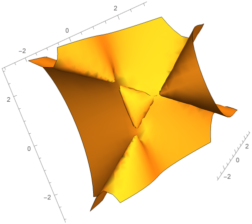
Cubic symmetroid as LMI





Cubic symmetroid as LMI

"Being" a spectrahedral cone is much more than a convex cone!



SD matrix		**symmetroid**
rk 1	$\leftrightarrow$	points on the Veronese
rk 2	$\leftrightarrow$	∂ spectrahedral cone $\lambda aa^T + (1-\lambda)bb^T, \lambda \in [0,1]$
rk ≤ 3	$\leftrightarrow$	spectrahedral cone $\alpha aa^T + \beta bb^T + \gamma cc^T, \alpha, \beta, \gamma \geq 0$

Hyperplane sections of symmetroid

The Veronese map ν_2 is given by the complete linear system. Thus the preimage of a hyperplane H in $\mathbb{P}^5$ is a conic Q_H in $\mathbb{P}^2$.

Conic Q_H is singular $\iff$ H is tangent to the Veronese surface

More precisely,

- if Q_H is a line pair, then H is a tangent to the Veronese surface at a single point,
- if Q_H is a double line, then H is tangent to the Veronese surface along the curve that is the image of Q_H^{red} under the restriction of ν_2.

Back to our problem

Recall that $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ is positive if and only if

$$\begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix} \succeq 0 \text{ for all } \mathbf{x} = (x, y, z) \in \mathbb{P}^2,$$

where

$$p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$$

and $P = [p_{ij}]_{0 \le i,j \le 5} : \mathbb{P}^5 \to \mathbb{P}^5$ in the corresponding basis.

Map ν_P

This way $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ induces

$$\begin{array}{cccc} \nu_P : & \mathbb{P}^2 & \to & \mathbb{P}^5 \\ & (x, y, z) & \mapsto & (p_0, \ldots, p_5), \end{array}$$

Lemma

P is positive if and only if the image of ν_P lies in the spectrahedral cone of the symmetroid.

Therefore, classifying positive maps is the same as classifying linear maps that preserve the spectrahedral cone of the symmetroid.

The convex hull of Im (ν_P)

When P is invertible, ν_P is given by the complete linear system. In this case the image of ν_P is the singular locus of the hypersurface

$$\det P^{-1}\left(\begin{bmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x_3 & x_4 & x_5 \end{bmatrix}\right) = 0.$$

In other words, the convex hull of Im ν_P equals to the spectrahedral cone of the above hypersurface.
Else, the convex hull of Im (ν_p) is a projection of a spectrahedral cone shadow (inside the spectrahedral cone of the symmetroid).

The convex hull of $\mathrm{Im}\,(\nu_{P^{-1}})$

On the other side, the hypersurface

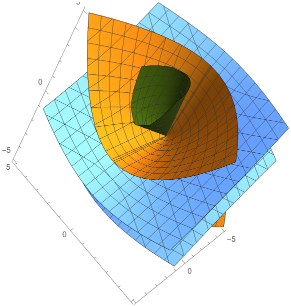
$$\det P\left(\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix}\right) = 0$$

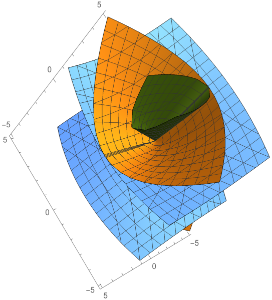
contains the Veronese surface in its spectrahedral cone. When P is invertible, this spectrahedral cone equals to the convex hull of the image of

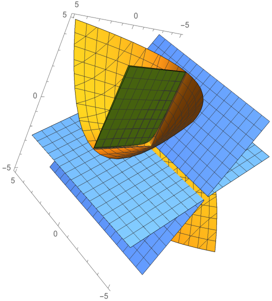
$$\begin{array}{cccc} \nu_{P^{-1}}: & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ & (x,y,z) & \mapsto & P^{-1}\left(\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}\right) \end{array}$$

Spectrahedral cones of:

$\det P([z_{ij}]) = 0$, $\det[z_{ij}] = 0$, $\det P^{-1}([z_{ij}]) = 0$







Convex algebraic geometry

$$P_{n,2d} = \{\text{non-negative (PSD) forms in } \mathbb{R}[x_0, \ldots, x_{n-1}] \text{ of degree 2d}\}$$

$$\cup$$

$$\Sigma_{n,2d} = \{\text{sums of squares (SOS-polynomials)}\}$$

det $\Uparrow$

$$P^M_{n,2d} = \{\text{positive semidefinite } d \times d \text{ matrix quadratic polynomials}\}$$

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Polynomial non-negativity

. Given a form $p \in \mathbb{R}_{2d}[x_0, \dots, x_{n-1}]$, does there exist $x \in \mathbb{R}^n$ such that $p(x) < 0$?

- if not, p is called positive semidefinite or PSD

$$p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n$$

- the problem is NP-hard, but decidable.

To verify the

answer **yes** is easy: find x such that $p(x) < 0$;

answer **no** is hard: we need a certificate, that is proof that there is no feasible point.

SOS polynomials

If there exist polynomials $g_1, \ldots, g_r$ such that

$$p = \sum_{i=1}^{r} g_i^2,$$

then p is called a sum-of-squares (SOS) polynomial.

Clearly, an SOS polynomial is PSD.

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Sum of squares and semidefinite programming

Lemma

A homogeneous polynomial $p \in \mathbb{R}[x_0, \ldots, x_{n-1}]$ of degree $2d$ is an SOS polynomial if and only if there exists a positive semidefinite Gram matrix

$$Q \succeq 0 \text{ such that } p = z^T Q z,$$

where z denotes a vector of all monomials of degree d.

- this is an SDP problem in standard primal form;
- the number of components of z is $\binom{n+d-1}{d}$.

Proof

Factorize $Q = V\,V^T$ and write $V = [v_1 \cdots v_r]$ so that

$$p = z^T V\,V^T z = \|V^T z\|^2 = \sum_{i=1}^{r} (v_i^T z)^2.$$

Convex cone: $p, q \in C \Rightarrow \lambda p + \mu q \in C$ for all $\lambda, \mu > 0$

$$P_{n,2d} = \{\text{PSD polynomials of degree } 2d\}$$

$$\Sigma_{n,2d} = \{\text{SOS polynomials of degree } 2d\}$$

are both convex cones in $\mathbb{R}^N$ where $N = \binom{n+2d-1}{2d}$.

We know since Hilbert that

$$\Sigma_{n,2d} \subset P_{n,2d};$$

- testing if $p \in P_{n,2d}$ is NP-hard,
- but testing if $p \in \Sigma_{n,2d}$ is an SDP.

Hilbert's 17th problem

Hilbert in 1888 showed that $\Sigma_{n,2d} = P_{n,2d}$ in the following cases:

$n\backslash 2d$	2	4	6	8	$\cdots$
2	$=$	$=$	$=$	$=$	
3	$=$	$=$	$\subset$	$\subset$	
4	$=$	$\subset$	$\subset$	$\subset$	
5	$=$	$\subset$	$\subset$	$\subset$	
$\vdots$					$\ddots$

- $2d = 2$, quadratic polynomial forms
- $n = 2$, homogeneous polynomials in two variables
- $2d = 4$, $n = 3$, quartic forms in three variables

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Hilbert's 17th problem

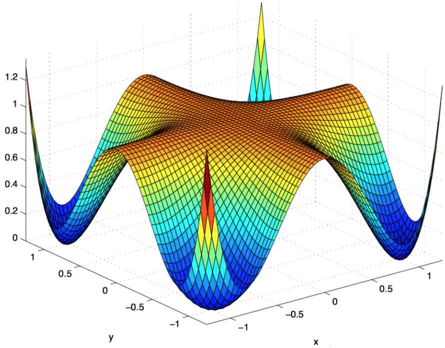
Artin in 1927 showed that every PSD polynomial is an SOS of **rational functions.**

A constructive solution was found in 1984 by Delzell.

Hilbert's 17th problem

In 1967 Motzkin constructed the first example of a positive semidefinite polynomial, that is not a sum of squares:

$$p(x,y,z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2$$



Examples of $P_{3,6} \backslash \Sigma_{3,6}$

Blekherman in 2012 provided a geometric explanation for the containment $\Sigma_{3,6} \subset P_{3,6}$. The difference lies in fulfillment of certain linear relations (Cayley-Bacharach relations) from Hilbert's proof.

- Robinson's polynomial with 10 zeros (1973):

$$x^6+y^6+z^6-x^4y^2-x^4z^2-y^4x^2-y^4z^2-z^4x^2-z^4y^2+3x^2y^2z^2;$$

- lots of examples from Reznick's construction (2007).

The geometry of $P_{3,6} \backslash \Sigma_{3,6}$ remains puzzling!

An algebraic boundary of a cone is the hypersurface that arises as Zariski closure of its topological boundary.

- Nie, 2011: The algebraic boundary of the cone $P_{n,2d}$ is the **discriminant** of degree $n(2d-1)^{n-1}$.
- Blekherman, Sturmfels, et al., 2011: Discriminant is also a component in the algebraic boundary of $\Sigma_{3,6}$. Besides, $\partial \Sigma_{3,6}$ has another unique non-discriminant component of degree 83200 which consists of forms that are **sums of three squares of cubics.**

Remark: A sextic C that is a sum of three squares of cubics coincides with an ACM rk 1sheaf $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(3))$ that is globaly generated; this is exactly an effective even theta characteristic.

PSD and SOS matrices

Definition

A symmetric polynomial matrix $P(x)$ is an SOS-matrix if

$$P(x) = M(x)\,M(x)^T$$

for a possibly non-square polynomial matrix $M(x)$.

Definition

A matrix polynomial $P(x)$ is positive semidefinite if $P(x)$ is positive semidefinite for all $x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$.

Connection to biquadratic forms

Recall that positive linear maps $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ are in one-to-one correspondence with PSD quadratic ternary matrices $P(\mathbf{x}\,\mathbf{x}^T)$. Moreover, they are in one-to-one correspondence with non-negative biquadratic forms

$$\mathbf{u}^t P(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u},$$

where $\mathbf{x} = [x, y, z]^T$ and $\mathbf{u} = [u, v, w]^T$.

Lemma

P is positive $\Leftrightarrow$ $\mathbf{u}^T P(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ *is a PSD polynomial*

$\Leftrightarrow$ $P(\mathbf{x}\,\mathbf{x}^T)$ *is a PSD quadratic matrix .*

Choi matrix (analogy to the Gram matrix for SOS polynomials)

Choi map: A linear map $\phi : \mathrm{M}_3 \to \mathrm{M}_3$ induces a linear map $\Phi : \mathrm{M}_9 \to \mathrm{M}_9$ by the following rule

$$\Phi\left([X_{ij}]_{i,j=1,2,3}\right) = [\phi(X_{ij})]_{i,j=1,2,3}.$$

Theorem (Choi, 1974)

Choi matrix

$$[\phi(E_{ij})]_{i,j=1,2,3} \quad \textit{is positive semidefinite}$$

if and only if the restriction $\phi : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *induces an*

$$\textit{SOS quadratic matrix } \phi(\mathbf{x}\,\mathbf{x}^T).$$

This is equivalent to $\mathbf{u}^T P(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ *being a biquadratic SOS form.*

Such ϕ are called completely positive, in optimization they are called SOS.

SOS matrices

The third equivalent definition of quadratic SOS matrices is the following:

Lemma

Quadratic matrix $P(\mathbf{x}\,\mathbf{x}^T)$ is an SOS matrix if and only if there exist $A_j \in \mathbb{R}^{3,3}$ such that

$$P(x, y, z) = \sum_{j=1}^{r} A_j X A_j^T, \text{ where } X = \mathbf{x}\,\mathbf{x}^T = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.$$

SOS matrices

Indeed, for the $3 \times r$ linear matrix $M = [m_1 \cdots m_r]$ write

$$P(x,y,z) = M\,M^T = \sum_{j=1}^{r} m_j\,m_j^T = \sum_{j=1}^{r} \begin{bmatrix} m_{1j} \\ m_{2j} \\ m_{3j} \end{bmatrix} \cdot [m_{1j}\; m_{2j}\; m_{3j}]$$

$$= \sum_{j=1}^{r} A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot [x\; y\; z]\, A_j^T .$$

Here the linear forms m_{ij} determine A_j.

We need examples!

Like in the polynomial case (Hilbert,1888 $\rightarrow$ Motzkin, 1967 $\rightarrow$ Reznick, 2007) we need lots of examples to understand the difference between the convex cones $P^M_{3,6}$ and $\Sigma^M_{3,6}$. Until recently, the only examples have been derived from **Choi's quadratic matrix**:

$$\det \begin{bmatrix} x^2+z^2 & -xy & -xz \\ -xy & x^2+y^2 & -yz \\ -xz & -yx & y^2+z^2 \end{bmatrix} = x^4y^2+y^4z^2+z^4x^2-3x^2y^2z^2.$$

Nonnegative biquadratic form with 10 zeros (**max**!)

Theorem (Šivic)

The map $P_t : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *defined by*

$$\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \mapsto \begin{bmatrix} (t^2-1)^2 z_0 + z_2 + t^4 z_5 & -(t^4-t^2+1)z_1 & -(t^4-t^2+1)z_3 \\ -(t^4-t^2+1)z_1 & t^4 z_0 + (t^2-1)^2 z_1 + z_5 & -(t^4-t^2+1)z_4 \\ -(t^4-t^2+1)z_3 & -(t^4-t^2+1)z_4 & z_0 + t^4 z_2 + (t^2-1)^2 z_5 \end{bmatrix}$$

is positive for all $t \in \mathbb{R}$. *When* $t \notin \{1, 0, -1\}$, *the associated biquadratic form* $\mathbf{u}^T P_t(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ *has 10 zeros:*

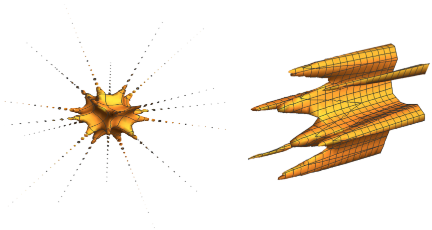
$$\{[1,1,1;1,1,1], [-1,1,1;-1,1,1], [1,-1,1;1,-1,1], [1,1,-1;1,1,-1],$$
$$[1,\pm t,0;\pm t,1,0], [0,1,\pm t;0,\pm t,1], [\pm t,0,1;1,0,\pm t]\}.$$

Nonnegative biquadratic form with 10 zeros (**max**!)

In particular, for

$$P_t(\mathbf{x}\,\mathbf{x}^T) = \begin{bmatrix} (t^2-1)^2x^2+y^2+t^4z^2 & -(t^4-t^2+1)xy & -(t^4-t^2+1)xz \\ -(t^4-t^2+1)xy & t^4x^2+(t^2-1)^2y^2+z^2 & -(t^4-t^2+1)yz \\ -(t^4-t^2+1)xz & -(t^4-t^2+1)yz & x^2+t^4y^2+(t^2-1)^2z^2 \end{bmatrix}$$

det $P_t(\mathbf{x}\,\mathbf{x}^T)$ is nonnegative with 10 singularities of type A_1.

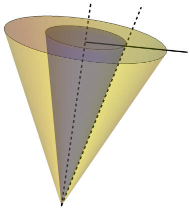


Hyperbolicity

The polynomial

$$p(z_0, \ldots, z_5) = \det P_t([z_{ij}])$$

is hyperbolic with respect to $\mathrm{Id}_3 \equiv (1, 0, 1, 0, 0, 1)$.



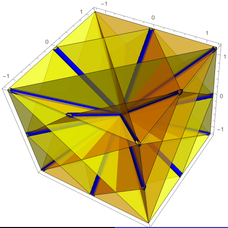
It is straightforward to verify that the univariate polynomial $p(t\,\mathrm{Id} + \mathbf{x}\mathbf{x}^T)$ has no zero that is strictly positive.

"TO DO LIST"

- Find examples of non-negative polynomials that have NO positive semidefinite quadratic determinantal representation.
  This proves that $\det : P^M_{3,6} \longrightarrow P_{3,6}$ is not surjective.

We believe that Robinson's polynomial is such, due to its particular configuration of 10 zeros

$$\{[1,1,1], [-1,1,1], [1,-1,1], [1,1,-1], [1,\pm 1,0], [0,1,\pm 1], [\pm 1,0,1]\}.$$



"TO DO LIST"

- Find geometric explanation for the containment $\Sigma^M_{3,6} \subset P^M_{3,6}$.

Follow Blekherman's explanation of the difference between the two cones in the polynomial case. The proof of Hilbert's 17th theorem for matrices is more constructive than for polynomials (because of the Cayley-Hamilton theorem).

What are the Cayley-Bacharach relations for matrix polynomials?

"TO DO LIST"

- Find algebraic boundaries $\partial P^M_{3,6}$ and $\partial \Sigma^M_{3,6}$.

We proved that $\partial P^M_{3,6}$ is the discriminant for biquadratic ternary forms. It is an ireducible hypersurface in $\mathbb{P}^{35}$ of degree 1328.

Recall that the non-discriminant boundary for $\Sigma_{3,6}$ consists of polynomials that are sums of three squares. Our "guess" is that the non-discriminant boundary $\partial \Sigma^M_{3,6} = \left\{ \sum_{j=1}^{5} A_j X A_j^T \right\}$: Take $P \in \Sigma^M_{3,6}$ that is a sum of 4 squares. This means that $P = M\, M^T$ for a linear 3×4 matrix M. By the Cauchy-Binet formula $\det P = \det M_{123}^2 + \det M_{124}^2 + \det M_{134}^2 + \det M_{234}^2$. Therefore the set of real zeros equals to the determinantal variety $\text{rank} M \leq 2$ which consists of 6 points.

"TO DO LIST"

- Prove that that det : $P^M_{3,6} \longrightarrow P_{3,6}$ is not a convex map.

Clearly, determinant of an SOS quadratic matrix is an SOS sextic polynomial. On the other hand, Quarez's example

$$\det \begin{bmatrix} x^2+z^2 & 0 & -xz \\ 0 & x^2+y^2 & -yz \\ -xz & -yx & y^2+z^2 \end{bmatrix} = x^4y^2 + y^4x^2 + z^4x^2 + y^4z^2$$

is a positive semidefinite quadratic matrix that is not SOS, but its determinant is an SOS sextic polynomial.

"TO DO LIST"

- Find an example of a ternary quartic whose Hessian is positive semidefinite but not an SOS matrix.

This is another problem from optimization. Namely, a multivariate polynomial is convex if and only if its Hessian matrix of second partial derivatives is positive semidefinite. Ahmadi and Parrilo (2013) described the difference between convexity and SOS-convexity of polynomials.

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