Positive semidefinite quadratic determinantal representations

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Outline I

1. Symmetric quadratic determinantal representations
   - Positive maps
   - Self-dual sheaves
2. Linear matrix inequalities
   - Spectrahedral cone $\leftrightarrow$ hyperbolic polynomial
   - Cubic symmetroid
   - Criteria for P to be positive
3. Polynomial nonnegativity
   - PSD and SOS polynomials
   - PSD and SOS matrices
   - Šivic biquadratic form

Warm up question

Given a homogeneous ***nonnegative*** *polynomial* $p(x, y, z)$ *of degree 6, does there exist a* ***positive*** *linear map* $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *such that*

$$\det P(\mathbf{x}\mathbf{x}^T) = p(x, y, z) \ \textit{ for all } \ \mathbf{x} = [x,\ y,\ z]^T ?$$

We will “tackle” this question from three sides:

- symmetric quadratic determinantal representations and the associated sheaves (kernels);
- semidefinite linear determinantal representations (LMI representations of hyperbolic polynomials);
- polynomial algebra (SOS and PSD polynomials and matrices).

Positive maps

Definition

A linear map $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ is positive if it sends positive semidefinite matrices to positive semidefinite matrices.

Positive maps were popular in the 70s as they describe various quantum states in quantum physics. In the last decade there is again very active and fertile research in this area due to its connection to optimization.

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Positive maps $\leftrightarrow$ determinantal representations

Clearly it is enough to check the positivity of P on rank 1 matrices. In coordinates our question then becomes

Question

Given a nonnegative plane sextic $\mathcal{C}$ in $\mathbb{P}^2$, does there exist a symmetric quadratic determinantal representation of $\mathcal{C}$ which is semidefinite for all $(x, y, z) \in \mathbb{P}^2$?

Indeed, $P\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix}$,

where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$.

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where $p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}x^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$.

C. Scheiderer. *Hilbert's theorem on positive ternary quartics: A refined analysis*, JAG, 2010

This is exactly the question Scheiderer asked and thoroughly answered in the case of **plane quartics**.

- Quadratic determinantal representations $\begin{bmatrix} p_0 & p_1 \\ p_1 & p_2 \end{bmatrix} \stackrel{1-1}{\longleftrightarrow}$ globally generated (i.e., non-exceptional) ACM rank 1 sheaves with selfduality $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2))$. The number of such representations depends only on the singularities of C.
- Determine which and how many of the above determinantal representations are semidefinite.

Quadratic determinantal representations

Without the nonnegativity and semidefiniteness conditions this is a classical case of determinantal hypersurfaces

Question

Given a plane curve $\mathcal{C}$ of degree $2d$, does there exist a $d \times d$ symmetric quadratic determinantal representation of $\mathcal{C}$?

Beauville: Determinantal Hypersurfaces, 2000

On an integral curve C, a coherent torsion-free rank 1 (arithmetically Cohen-Macaulay, ACM) sheaf $\mathcal{F}$ that is

generated by its global sections and $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$

admits a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}^d \to \mathcal{F} \to 0,$$

where M is a symmetric quadratic matrix with $\det M = p$.

Remark: The above $\mathcal{F}$ is non-exceptional. This is equivalent to $H^0(C, \mathcal{F}(-1)) = H^1(C, \mathcal{F}) = 0$. Then $h^0(C, \mathcal{F}) = d$ and its global sections yield M.

Beauville: Determinantal Hypersurfaces, 2000

Actually, any $\mathcal{F}$ that is self-dual

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$$

admits a resolution $0 \to \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(-2-d_i) \xrightarrow{M} \bigoplus_{i=1}^{l} \mathcal{O}_{\mathbb{P}^2}(d_i) \to \mathcal{F} \to 0,$

where $M = [m_{ij}]$ is symmetric with m_{ij} of degree $d_i + d_i - 2$.

Remark: We are only interested in non-exceptional $\mathcal{F}$, for which $d_i = 0$ for $i = 1, \ldots, d$. The set of such pairs $(C, \mathcal{F})$ is Zariski dense in the universal Jacobian $\mathcal{J}_{2d}^{2d(d-1)}$.

Beauville: Determinantal Hypersurfaces, 2000

Define the moduli space $\mathcal{R}_{2d}$ of pairs (C, α), where C is a smooth plane curve of degree $2d$ (over a field of char 0), and α is a half-period, i.e. a 2-torsion divisor on $\mathrm{Jac}(C)$, i.e. a nontrivial line bundle on C satisfying $\alpha^{\otimes 2} \cong \mathcal{O}_C$.

Proposition

For (C, α) general in $\mathcal{R}_{2d}$, the half-period α admits a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d-1)^d \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-d+1)^d \to \alpha \to 0,$$

where M is a symmetric quadratic matrix with $\det M = p$.

Note, $\mathcal{F}$ is obtained from the half-period α by $\mathcal{F} = \alpha \otimes \mathcal{O}_C(d-1)$.

Simple singularities

When $C/\mathbb{C}$ has only simple (this means AED) singularities, there are **finitely many** ACM sheaves with the following self-duality

$$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2)).$$

It is possible to **explicitly count** them using methods in [Piontkowski, 2007]. Their number depends on the genus of the curve and the local type of $\mathcal{F}$: $(\mathcal{F}_s)_{s \in \operatorname{Sing} C}$ is a collection of self-dual modules $\mathcal{F}_s \cong \operatorname{Hom}(\mathcal{F}_s, \mathcal{O}_{C,s}(2d-2))$. For a simple singularity there are only finitely many isomorphism classes of indecomposable torsion-free modules over its local ring.

A smooth C has $\mathbf{2^{2g}}$ self-dual $\mathcal{F}$; the number decreases rapidly with the number and order of singularities A_n, D_m, E_l. When all $\mathcal{F}$ are exceptional, C has no quadratic representations.

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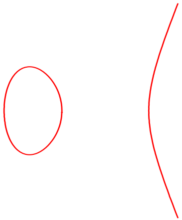
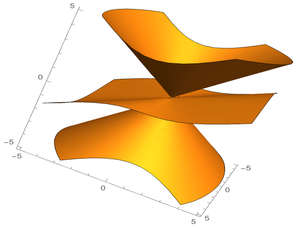
$\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(2d-2))$

Minimal resolutions $M = [m_{ij}]$ with $\deg m_{ij} = d_i + d_j + 2$:

	$\mathcal{F}$ non-except.	$\mathcal{F}$ exceptional
quartic:	$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ SINGULAR, $\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ SQUARE
sextic:	$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$, square

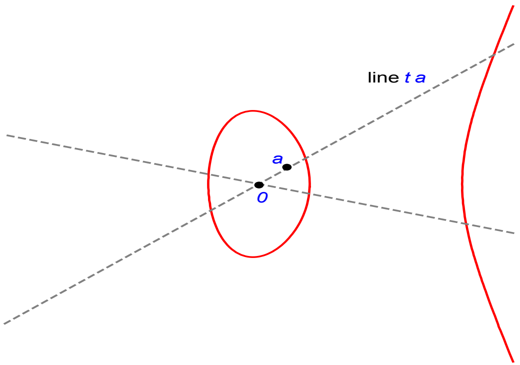
Terminology $\mathbb{A}^n \leftrightarrow \mathbb{P}^n$





$\mathbb{A}^n$:	spectrahedron	RZ-polynomial	rigidly convex set	PSD LMI
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\mathbb{P}^n$:	spectrahedral cone	hyperbolic poly.	hyperbolicity cone	SD LMI

Cubic curve $(x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1 x_2^2 = 0$



Cubic curve $(x_0 - x_1)(x_0 + x_1)(x_0 - 4x_1) - x_1 x_2^2 = 0$

has three symmetric determinantal representations:

- two are definite determinantal representations

$$\begin{bmatrix} x_0 & -\frac{x_1}{\sqrt{2}} & \frac{x_1}{\sqrt{2}} \\ -\frac{x_1}{\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{x_1}{8} \\ \frac{x_1}{\sqrt{2}} & -\frac{x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8} \end{bmatrix}, \begin{bmatrix} x_0 & -\frac{x_1}{2\sqrt{2}} & \frac{x_1}{2\sqrt{2}} \\ -\frac{x_1}{2\sqrt{2}} & \frac{8x_0 - x_1 + 4x_2}{8} & -\frac{7x_1}{8} \\ \frac{x_1}{2\sqrt{2}} & -\frac{7x_1}{8} & \frac{8x_0 - x_1 - 4x_2}{8} \end{bmatrix}$$

- one is nondefinite

$$x_0 \,\mathrm{Id} + x_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} + x_1 \begin{bmatrix} 0 & \frac{i}{2\sqrt{2}} & \frac{-i}{2\sqrt{2}} \\ \frac{i}{2\sqrt{2}} & -\frac{1}{8} & \frac{9}{8} \\ \frac{-i}{2\sqrt{2}} & \frac{9}{8} & -\frac{1}{8} \end{bmatrix}$$

Veronese surface

In coordinates write $\mathbb{P}^5$ as a symmetric matrix $\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix}$,

and consider the Veronese embedding

$$\begin{array}{rccc} \nu_2 : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ & (x, y, z) & \mapsto & \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}. \end{array}$$

Cubic symmetroid as LMI

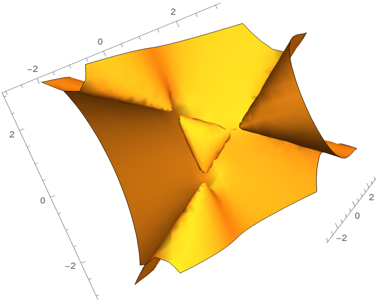
Cubic symmetroid in $\mathbb{P}^5$ is the hypersurface defined by

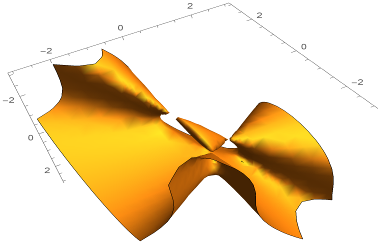
$$\det \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} = 0.$$

It is singular along the Veronese surface. Semidefinite matrices lie in the spectrahedral cone

$$\left\{ (z_0, \ldots, z_5) \in \mathbb{P}^5 \ : \ \begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \succeq 0 \right\}.$$

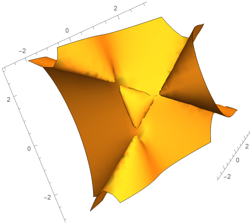
Cubic symmetroid as LMI





Cubic symmetroid as LMI

"Being" a spectrahedral cone is much more than a convex cone!



SD matrix		**symmetroid**
rk 1	$\leftrightarrow$	points on the Veronese
rk 2	$\leftrightarrow$	∂ spectrahedral cone $\lambda aa^T + (1-\lambda)bb^T, \lambda \in [0,1]$
rk ≤ 3	$\leftrightarrow$	spectrahedral cone $\alpha aa^T + \beta bb^T + \gamma cc^T, \alpha, \beta, \gamma \geq 0$

Hyperplane sections of symmetroid

The Veronese map ν_2 is given by the complete linear system. Thus the preimage of a hyperplane H in $\mathbb{P}^5$ is a conic Q_H in $\mathbb{P}^2$.

Conic Q_H is singular $\iff$ H is tangent to the Veronese surface

More precisely,

- if Q_H is a line pair, then H is a tangent to the Veronese surface at a single point,
- if Q_H is a double line, then H is tangent to the Veronese surface along the curve that is the image of Q_H^{red} under the restriction of ν_2.

Back to our problem

Recall that $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ is positive if and only if

$$\begin{bmatrix} p_0 & p_1 & p_3 \\ p_1 & p_2 & p_4 \\ p_3 & p_4 & p_5 \end{bmatrix} \succeq 0 \text{ for all } \mathbf{x} = (x, y, z) \in \mathbb{P}^2,$$

where

$$p_i = p_{0i}x^2 + p_{1i}xy + p_{2i}y^2 + p_{3i}xz + p_{4i}yz + p_{5i}z^2$$

and $P = [p_{ij}]_{0 \le i,j \le 5} : \mathbb{P}^5 \to \mathbb{P}^5$ in the corresponding basis.

Map ν_P

This way $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ induces

$$\begin{array}{cccc} \nu_P : & \mathbb{P}^2 & \to & \mathbb{P}^5 \\ & (x, y, z) & \mapsto & (p_0, \dots, p_5), \end{array}$$

Lemma

P is positive if and only if the image of ν_P lies in the spectrahedral cone of the cubic symmetroid.

Therefore, classifying positive maps is the same as classifying linear maps $P : \mathbb{P}^5 \to \mathbb{P}^5$ that **preserve** the spectrahedral cone of the cubic symmetroid.

The convex hull of Im (ν_P)

When P is invertible, ν_P is given by the complete linear system. In this case the image of ν_P is the singular locus of the hypersurface

$$\det P^{-1}\left(\begin{bmatrix} x_0 & x_1 & x_3 \\ x_1 & x_2 & x_4 \\ x_3 & x_4 & x_5 \end{bmatrix}\right) = 0.$$

In other words, the convex hull of Im ν_P equals to the spectrahedral cone of the above hypersurface.
Else, the convex hull of Im (ν_p) is a projection of a spectrahedral cone shadow (inside the spectrahedral cone of the symmetroid).

The convex hull of $\mathrm{Im}\,(\nu_{P^{-1}})$

On the other side, the hypersurface

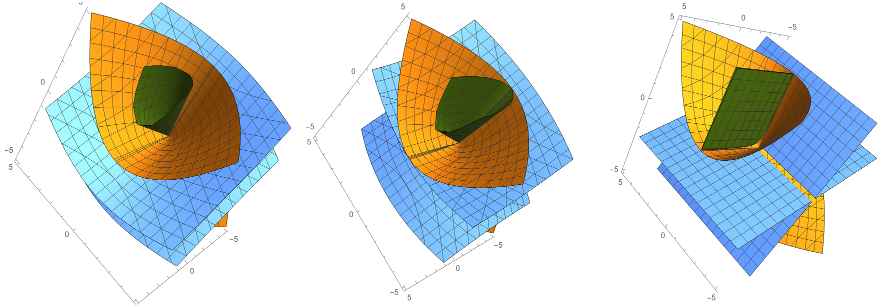
$$\det P\left(\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix}\right) = 0$$

contains the Veronese surface in its spectrahedral cone. When P is invertible, this spectrahedral cone equals to the convex hull of the image of

$$\begin{array}{cccc} \nu_{P^{-1}} : & \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ & (x, y, z) & \mapsto & P^{-1}\left(\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}\right) \end{array}$$

Spectrahedral cones of:

$\det P([z_{ij}]) = 0$, $\det[z_{ij}] = 0$, $\det P^{-1}([z_{ij}]) = 0$



Choi's example

Matrix

$$\begin{bmatrix} x^2+z^2 & -xy & -xz \\ -xy & x^2+y^2 & -yz \\ -xz & -yz & y^2+z^2 \end{bmatrix}$$

is positive definite for all $(x, y, z) \in \mathbb{P}^2$ except at the 7 points: $(1,1,1), (-1,1,1), (1,-1,1), (1,1,-1), (1,0,0), (0,1,0), (0,0,1)$.
The Veronese surface therefore lies inside the spectrahedral cone of

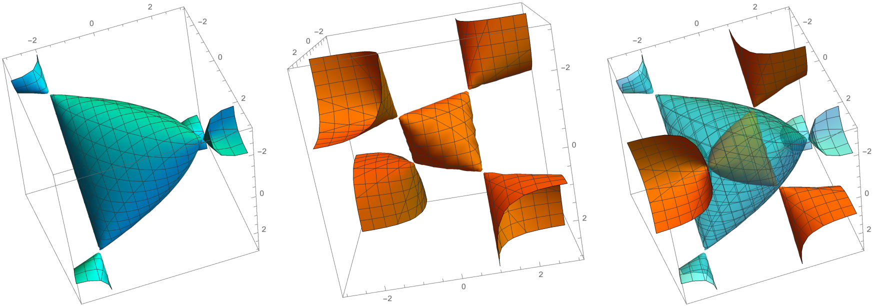
$$\det \begin{bmatrix} z_0+z_5 & -z_1 & -z_3 \\ -z_1 & z_0+z_2 & -z_4 \\ -z_3 & -z_4 & z_2+z_5 \end{bmatrix} = 0$$

and intersects its boundary in
$(1,1,1,1,1,1), (1,-1,1,-1,1,1), (1,-1,1,1,-1,1), (1,1,1,-1,-1,1),$
$(1,0,0,0,0,0), (0,0,1,0,0,0), (0,0,0,0,0,1)$

Choi's example and the cubic symmetroid

$\det P([z_{ij}]) = 0$, $\det[z_{ij}] = 0$;

intersected with $z_0 = z_1 = z_2 = 1$, thus containing $(1,1,1,1,1,1)$, $(1,-1,1,-1,1,1)$, $(1,-1,1,1,-1,1)$, $(1,1,1,-1,-1,1)$:



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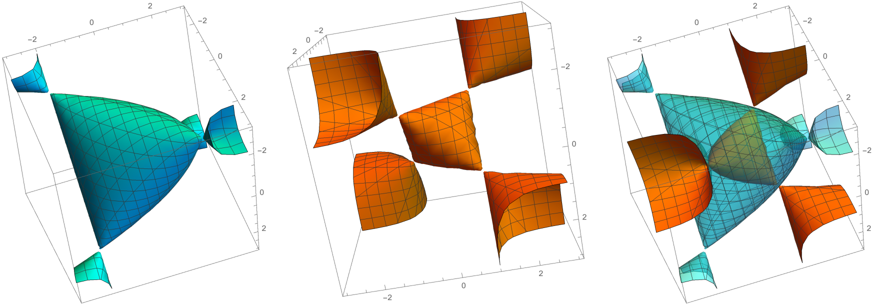

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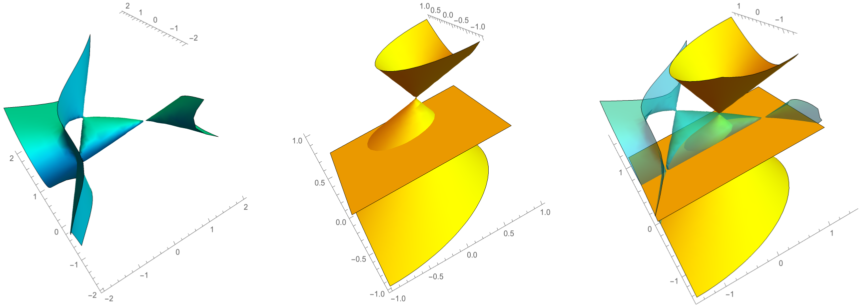
$\det P([z_{ij}]) = 0$, $\det[z_{ij}] = 0$;

intersected with $z_3 = z_4 = z_5 = 1 - z_0 - z_2$, thus containing $(1,1,1,1,1,1), (1,0,0,0,0,0), (0,0,1,0,0,0)$.



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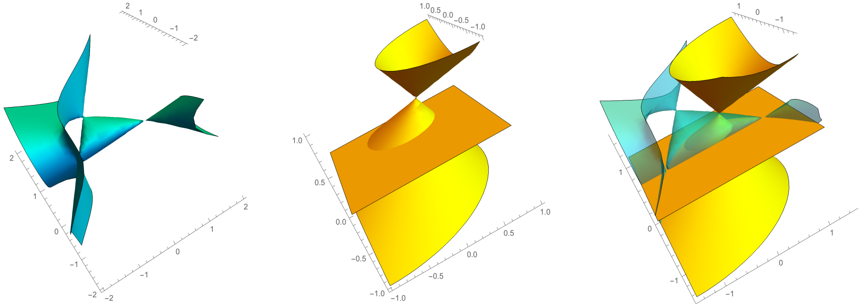

Convex algebraic geometry

$P_{n,2d} = \{$non-negative (PSD) forms in $\mathbb{R}[x_0, \ldots, x_{n-1}]$ of degree 2d$\}$

$\cup$

$\Sigma_{n,2d} = \{$sums of squares (SOS-polynomials)$\}$

det $\Uparrow$

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Convex cone: $p, q \in C \Rightarrow \lambda p + \mu q \in C$ for all $\lambda, \mu > 0$

$$P_{n,2d} = \{\text{PSD polynomials of degree } 2d\}$$

$$\Sigma_{n,2d} = \{\text{SOS polynomials of degree } 2d\}$$

are both convex cones in $\mathbb{R}^N$ where $N = \binom{n+2d-1}{2d}$.

We know since Hilbert that

$$\Sigma_{n,2d} \subset P_{n,2d};$$

- testing if $p \in P_{n,2d}$ is NP-hard,
- but testing if $p \in \Sigma_{n,2d}$ is an SDP (using **Gram matrix**).

Hilbert's 17th problem

Hilbert in 1888 showed that $\Sigma_{n,2d} = P_{n,2d}$ in the following cases:

$n \backslash 2d$	2	4	6	8	$\cdots$
2	$=$	$=$	$=$	$=$	
3	$=$	$=$	$\subset$	$\subset$	
4	$=$	$\subset$	$\subset$	$\subset$	
5	$=$	$\subset$	$\subset$	$\subset$	
$\vdots$					$\ddots$

- $2d = 2$, quadratic polynomial forms
- $n = 2$, homogeneous polynomials in two variables
- $2d = 4, n = 3$, quartic forms in three variables

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- $2d = 4, n = 3$, quartic forms in three variables

Hilbert's 17th problem

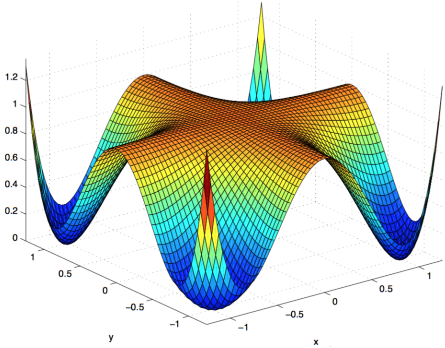
Artin in 1927 showed that every PSD polynomial is an SOS of **rational functions.**

A constructive solution was found in 1984 by Delzell.

Hilbert's 17th problem

In 1967 Motzkin constructed the first example of a positive semidefinite polynomial, that is not a sum of squares:

$$p(x, y, z) = x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2$$



Examples of $P_{3,6} \backslash \Sigma_{3,6}$

Blekherman in 2012 provided a geometric explanation for the containment $\Sigma_{3,6} \subset P_{3,6}$. The difference lies in fulfillment of certain linear relations (Cayley-Bacharach relations) from Hilbert's proof.

- Robinson's polynomial with 10 zeros (1973):

$$x^6+y^6+z^6-x^4y^2-x^4z^2-y^4x^2-y^4z^2-z^4x^2-z^4y^2+3x^2y^2z^2;$$

- lots of examples from Reznick's construction (2007).

The geometry of $P_{3,6} \backslash \Sigma_{3,6}$ remains puzzling!

An algebraic boundary of a cone is the hypersurface that arises as Zariski closure of its topological boundary.

- Nie, 2011: The algebraic boundary of the cone $P_{n,2d}$ is the **discriminant** of degree $n(2d-1)^{n-1}$.
- Blekherman, Sturmfels, et al., 2011: Discriminant is also a component in the algebraic boundary of $\Sigma_{3,6}$. Besides, $\partial \Sigma_{3,6}$ has another unique non-discriminant component of degree 83200 which consists of forms that are **sums of three squares of cubics.**

Remark: A sextic C that is a sum of three squares of cubics coincides with an ACM rk 1sheaf $\mathcal{F} \cong \mathcal{H}om(\mathcal{F}, \mathcal{O}_C(3))$ that is globaly generated; this is exactly an effective even theta characteristic.

PSD and SOS matrices

Definition

A symmetric polynomial matrix $P(x)$ is an SOS-matrix if

$$P(x) = M(x)\, M(x)^T$$

for a possibly non-square polynomial matrix $M(x)$.

Definition

A matrix polynomial $P(x)$ is positive semidefinite if $P(x)$ is positive semidefinite for all $x = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$.

Connection to biquadratic forms

Recall the natural $1-1$ correspondence between:

- positive linear maps $P : \mathrm{Sym}_3 \to \mathrm{Sym}_3$;
- PSD quadratic ternary matrices $P(\mathbf{x}\mathbf{x}^T)$;
- non-negative biquadratic forms $\mathbf{u}^t P(\mathbf{x}\mathbf{x}^T)\mathbf{u}$, where $\mathbf{x} = [x, y, z]^T$ and $\mathbf{u} = [u, v, w]^T$.

Lemma

$$\begin{aligned} P \textit{ is positive} \quad &\Leftrightarrow \quad \mathbf{u}^T P(\mathbf{x}\mathbf{x}^T)\mathbf{u} \textit{ is a PSD polynomial} \\ &\Leftrightarrow \quad P(\mathbf{x}\mathbf{x}^T) \textit{ is a PSD quadratic matrix .} \end{aligned}$$

Choi matrix (analogy to the Gram matrix for SOS polynomials)

Choi map: A linear map $\phi : \mathrm{M}_3 \to \mathrm{M}_3$ induces a linear map $\Phi : \mathrm{M}_9 \to \mathrm{M}_9$ by the following rule

$$\Phi\left([X_{ij}]_{i,j=1,2,3}\right) = [\phi(X_{ij})]_{i,j=1,2,3}.$$

Theorem (Choi, 1974)

Choi matrix

$$[\phi(E_{ij})]_{i,j=1,2,3} \text{ is positive semidefinite}$$

if and only if the restriction $\phi : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *induces an*

$$\text{SOS quadratic matrix } \phi(\mathbf{x}\,\mathbf{x}^T).$$

This is equivalent to $\mathbf{u}^T P(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ *being a biquadratic SOS form.*

Such ϕ are called completely positive, in optimization they are called SOS.

SOS matrices

The third equivalent definition of quadratic SOS matrices is the following:

Lemma

Quadratic matrix $P(\mathbf{x}\,\mathbf{x}^T)$ is an SOS matrix if and only if there exist $A_j \in \mathbb{R}^{3,3}$ such that

$$P(x,y,z) = \sum_{j=1}^{r} A_j X A_j^T, \text{ where } X = \mathbf{x}\,\mathbf{x}^T = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.$$

SOS matrices

Indeed, for the $3 \times r$ linear matrix $M = [m_1 \cdots m_r]$ write

$$
\begin{aligned}
P(x,y,z) &= M\,M^T = \sum_{j=1}^{r} m_j\,m_j^T = \sum_{j=1}^{r} \begin{bmatrix} m_{1j} \\ m_{2j} \\ m_{3j} \end{bmatrix} \cdot [m_{1j}\; m_{2j}\; m_{3j}] \\
&= \sum_{j=1}^{r} A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot [x\; y\; z]\, A_j^T .
\end{aligned}
$$

Here the linear forms m_{ij} determine A_j.

We need examples!

Like in the polynomial case,
Hilbert,1888 $\rightarrow$ Motzkin, 1967 $\rightarrow$ Reznick, 2007,
we need lots of examples to understand the difference between the convex cones $P^M_{3,6}$ and $\Sigma^M_{3,6}$.

We need examples!

Until recently, the only examples have been derived from **Choi's quadratic matrix**:

$$\det \begin{bmatrix} x^2+z^2 & -xy & -xz \\ -xy & x^2+y^2 & -yz \\ -xz & -yx & y^2+z^2 \end{bmatrix} = x^4y^2+y^4z^2+z^4x^2-3x^2y^2z^2.$$

The corresponding biquadratic form has 7 zeros

$$(1,1,1;1,1,1),\ (-1,1,1;-1,1,1),\ (1,-1,1;1,-1,1),\ (1,1,-1;1,1,-1),$$
$$(1,0,0;0,0,1),\quad (0,1,0;1,0,0),\quad (0,0,1;0,1,0).$$

Nonnegative biquadratic form with 10 zeros (**max**!)

Theorem (Šivic)

The map $P_t : \mathrm{Sym}_3 \to \mathrm{Sym}_3$ *defined by*

$$\begin{bmatrix} z_0 & z_1 & z_3 \\ z_1 & z_2 & z_4 \\ z_3 & z_4 & z_5 \end{bmatrix} \mapsto \begin{bmatrix} (t^2-1)^2 z_0 + z_2 + t^4 z_5 & -(t^4-t^2+1)z_1 & -(t^4-t^2+1)z_3 \\ -(t^4-t^2+1)z_1 & t^4 z_0 + (t^2-1)^2 z_1 + z_5 & -(t^4-t^2+1)z_4 \\ -(t^4-t^2+1)z_3 & -(t^4-t^2+1)z_4 & z_0 + t^4 z_2 + (t^2-1)^2 z_5 \end{bmatrix}$$

is positive for all $t \in \mathbb{R}$. *When* $t \notin \{1, 0, -1\}$, *the associated biquadratic form* $\mathbf{u}^T P_t(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ *has 10 zeros:*

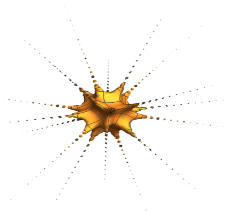
$$\{[1,1,1;1,1,1], [-1,1,1;-1,1,1], [1,-1,1;1,-1,1], [1,1,-1;1,1,-1], \\ [1,\pm t,0;\pm t,1,0], [0,1,\pm t;0,\pm t,1], [\pm t,0,1;1,0,\pm t]\}.$$

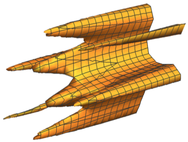
Nonnegative biquadratic form with 10 zeros (**max**!)

In particular, for

$$P_t(\mathbf{x}\,\mathbf{x}^T) = \begin{bmatrix} (t^2-1)^2x^2+y^2+t^4z^2 & -(t^4-t^2+1)xy & -(t^4-t^2+1)xz \\ -(t^4-t^2+1)xy & t^4x^2+(t^2-1)^2y^2+z^2 & -(t^4-t^2+1)yz \\ -(t^4-t^2+1)xz & -(t^4-t^2+1)yz & x^2+t^4y^2+(t^2-1)^2z^2 \end{bmatrix}$$

$\det P_t(\mathbf{x}\,\mathbf{x}^T)/(t^2-1)^2 = t^4(x^6+y^6+z^6)+$
$(t^8-2t^2)(x^4y^2+y^4z^2+z^4x^2)+(1-2t^6)(x^2y^4+y^2z^4+z^2x^4)-3(t^8-2t^6+t^4-2t^2+1)x^2y^2z^2$
is the generalized Robinson's polynomial with 10 singularities of type A_1.





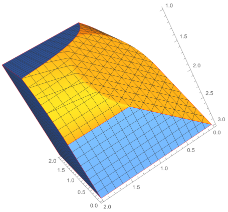
Extremal nonnegative biquadratic forms

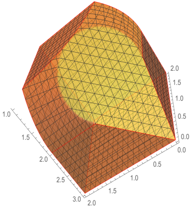
Our example is a parametrization of the **extremal** PSD quadratic matrices in the family:

$$P_{a,b,c}(\mathbf{x}\,\mathbf{x}^T) = \begin{bmatrix} (-1+a)x^2+by^2+cz^2 & -xy & -xz \\ -xy & cx^2+(-1+a)y^2+bz^2 & -yz \\ -xz & -yz & bx^2+cy^2+(-1+a)z^2 \end{bmatrix}.$$

Cho, Kye and Lee (Generalized Choi maps, LAA 1992) proved that $P_{a,b,c}$ is positive if and only if:

$a \geq 1$,
$a+b+c \geq 3$,
$bc \geq (2-a)^2$ if $1 \leq a \leq 2$.





Nonnegative biquadratic form with 8 zeros

The family of biquadratic forms with 8 zeros:

$$(1,1,1;1,1,1),\ (-1,1,1;-1,1,1),\ (1,-1,1;1,-1,1),\ (1,1,-1;1,1,-1),$$
$$(1,0,0;0,0,1),\quad (0,1,0;1,0,0),\quad (0,0,1;0,1,\mu),\quad (0,\nu,1;0,1,0)$$

is given by a linear combination of

$$a\begin{bmatrix} x^2+z^2 & -xy & -xz \\ -xy & x^2 & 0 \\ -xz & 0 & y^2 \end{bmatrix} + \begin{bmatrix} (\mu+\nu)^2x^2 & \mu(\mu+\nu)x(-y+\nu z) & -\nu(\mu+\nu)x(\mu y+z) \\ \mu(\mu+\nu)x(-y+\nu z) & \mu^2(y-\nu z)^2 & \mu\nu(\mu y+z)(y-\nu z) \\ -\nu(\mu+\nu)x(\mu y+z) & \mu\nu(\mu y+z)(y-\nu z) & \nu^2(\mu y+z)^2 \end{bmatrix}.$$

Nonnegative biquadratic form with 8 zeros

```
sph[θ1, φ1, θ2, φ2]  :=
{Cos[θ1]Cos[φ1], Cos[θ1]Sin[φ1], Sin[θ1],
 Cos[θ2]Cos[φ2], Cos[θ2]Sin[φ2], Sin[θ2]}
biq8pt[x,y,z,u,v,w]  :=
{u,v,w} · P_{a,μ,ν}[x,y,z]·{u,v,w} ≥ 0
RegionPlot3D[ Apply[And,
Map[biq8pt, Map[sph, RandomReal[2π, {9000, 4}]]]],
{μ, −2, 2}, {ν, −2, 2}, {a, 0, 1/2}  ]
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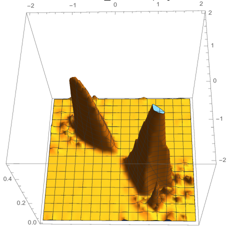


Nonnegative biquadratic form with 8 zeros

It is easy to check that for $\mu = -1/3$ and $\nu = 1/2$
the **extremal** PSD quadratic matrix is obtained at $a = 1/18$:

$$\begin{bmatrix} 3/2x^2+z^2 & -1/2xz & 1/2x(y-5z) \\ -1/2xz & x^2+1/2(z-2y)^2 & 1/2(y-3z)(2y-z) \\ 1/2x(y-5z) & 1/2(y-3z)(2y-z) & y^2+1/2(y-3z)^2 \end{bmatrix}.$$

The associated nonnegative biquadratic form is also extremal with zeros:

$$(1,1,1;1,1,1),\ (-1,1,1;-1,1,1),\ (1,-1,1;1,-1,1),\ (1,1,-1;1,1,-1),$$
$$(1,0,0;0,0,1),\quad (0,1,0;1,0,0),\quad (0,0,1;0,1,-1/3),\ (0,1/2,1;0,1,0)$$

Nonnegative biquadratic form with 9 zeros

Positive map $P =$

$$\begin{bmatrix} \big((3+2\sqrt{2})z_0+(3-2\sqrt{2})z_2+2z_5\big)/4 & -z_1 & -z_3 \\ -z_1 & (z_+z_2)/2 & 0 \\ -z_3 & 0 & \big((3-2\sqrt{2})z_0+(-1+2\sqrt{2})z_2+2z_5\big)/4 \end{bmatrix}$$

induces an extremal nonnegative biquadratic form $\mathbf{u}^T P(\mathbf{x}\,\mathbf{x}^T)\,\mathbf{u}$ with zeros:

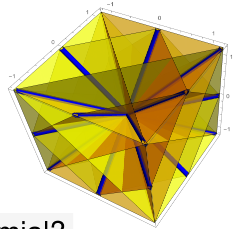
$$(1,1,1;1,1,1),\quad (-1,1,1;-1,1,1),\quad (1,-1,1;1,-1,1),(1,1,-1;1,1,-1),$$
$$(1,0,\tfrac{-1}{\sqrt{2}};1-\sqrt{2},0,1),\ (1,0,\tfrac{1}{\sqrt{2}};\sqrt{2}-1,0,1),$$
$$(1-\sqrt{2},1,0;1,\tfrac{-1}{\sqrt{2}},0),\ (\sqrt{2}-1,1,0;1,\tfrac{1}{\sqrt{2}},0),$$
$$(0,0,1;0,1,0).$$

“TO DO LIST”

- Find examples of non-negative polynomials that have **no** PSD quadratic determinantal representation.
  This would prove that det : $P^M_{3,6} \longrightarrow P_{3,6}$ is not surjective.

We believe that Robinson's polynomial is such, due to the particular configuration of its 10 zeros

$$\{[1,1,1], [-1,1,1], [1,-1,1], [1,1,-1], [1,\pm 1,0], [0,1,\pm 1], [\pm 1,0,1]\}.$$



What about Motzkin polynomial?

"TO DO LIST"

- Understand the map det : $P_{3,6}^{M} \longrightarrow P_{3,6}$.

Clearly, determinant of an SOS quadratic matrix is an SOS sextic polynomial. On the other hand, Quarez's example

$$\det \begin{bmatrix} x^2+z^2 & 0 & -xz \\ 0 & x^2+y^2 & -yz \\ -xz & -yx & y^2+z^2 \end{bmatrix} = x^4y^2 + y^4x^2 + z^4x^2 + y^4z^2$$

is a positive semidefinite quadratic matrix that is not SOS, but its determinant is an SOS sextic polynomial.

"TO DO LIST"

- Find geometric explanation for the containment $\Sigma_{3,6}^{M} \subset P_{3,6}^{M}$.

Follow Blekherman's explanation of the difference between the two cones in the polynomial case. The proof of Hilbert's 17th theorem for matrices is more constructive than for polynomials (because of the Cayley-Hamilton theorem).

What are the Cayley-Bacharach relations for matrix polynomials?

"TO DO LIST"

- Find algebraic boundaries $\partial P^M_{3,6}$ and $\partial \Sigma^M_{3,6}$.

We proved that $\partial P^M_{3,6}$ is the discriminant for biquadratic ternary forms. It is an ireducible hypersurface in $\mathbb{P}^{35}$ of degree 1328.

Recall that the non-discriminant boundary for $\Sigma_{3,6}$ consists of polynomials that are sums of three squares. Our "guess" is that the non-discriminant boundary $\partial \Sigma^M_{3,6} = \left\{ \sum_{j=1}^{5} A_j X A_j^T \right\}$: Take $P \in \Sigma^M_{3,6}$ that is a sum of 4 squares. This means that $P = M\,M^T$ for a linear 3×4 matrix M. By the Cauchy-Binet formula $\det P = \det M^2_{123} + \det M^2_{124} + \det M^2_{134} + \det M^2_{234}$. Therefore the set of real zeros equals to the determinantal variety $\text{rank} M \leq 2$ which consists of 6 points.

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