Rend. Istit. Mat. Univ. Trieste
Volume 47 (2015), 1–25 (electronic preview)

Simultaneously self-adjoint sets of 3×3 matrices

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"Dedicated to Emilia Mezzetti."

Abstract. *For a generic set $\mathcal{M}$ of 3×3 matrices over $\mathbb{C}$ we find necessary and sufficient conditions when $\mathcal{M}$ is simultaneously self-adjoint. Moreover, for a set of complex hermitean matrices we can tell if there exists a linear combination of matrices which is positive definite. Every $\mathcal{M}$ can be identified with a determinantal representation of a cubic hypersurface. This allows us to use the tools of algebraic geometry. The question of definiteness can be solved by using semidefinite programming.*

Keywords: simultaneously self-adjoint sets of matrices, matrix definiteness, determinantal representation, linear matrix inequalities
MS Classification 2010: 15B48, 15A15, 14Q10

1. Introduction

The article addresses the following two natural questions:

(1) Consider a set of matrices $\mathcal{M} \subset \mathbb{C}^{d \times d}$. When are all the elements of $\mathcal{M}$ simultaneously equivalent to hermitian matrices under the natural action of $\mathrm{GL}_d(\mathbb{C}) \times \mathrm{GL}_d(\mathbb{C})$? In other words, when do there exist $A, B \in \mathrm{GL}_d(\mathbb{C})$ such that AMB is hermitian for all $M \in \mathcal{M}$?

(2) Assume that the answer to (1) is positive. Is there an element in $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$ that is equivalent (under this simultaneous equivalence) to a positive definite matrix? In other words, given a set of hermitian $d \times d$ matrices, when do these matrices admit a positive definite linear combination?

Computationally both questions are straightforward. Question (1) reduces to a system of linear equations over $\mathbb{R}$,

$$CM_i^* = M_i C^*, \quad i = 0, 1, \ldots, n,$$

where $C = A^{-1}B^*$ and $\{M_0, M_1, \ldots, M_n\}$ is a basis of the $\mathbb{R}-$linear span of the set $\mathcal{M}$. Question (2) is solved by semidefinite programming at least for moderate d and n.

For sets of 3×3 matrices we interlace different approaches to obtain the answers: linear algebra (simultaneous reduction of a set of matrices to hermitian (or symmetric form), algebraic geometry (cubic curves, surfaces and hypersurfaces as zero loci of determinantal representations) and semidefinite programming (linear matrix inequality representations).

Let $\mathcal{M} \subset \mathbb{C}^{d\times d}$ be a set of square matrices of order d over $\mathbb{C}$. We call $\mathcal{M}$ *simultaneously self-adjoint* if there exist invertible $A,\ B \in \mathrm{GL}_d(\mathbb{C})$ such that AMB are complex hermitean matrices for all $M \in \mathcal{M}$.

We can think of $\mathbb{C}^{d\times d}$ as an $2d^2$ dimensional vector space over $\mathbb{R}$ and thus restrict ourselves to finite sets. The following statements are clearly equivalent:

- $\mathcal{M}$ is simultaneously self-adjoint;
- $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$ is simultaneously self-adjoint;
- Any basis of $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$ is simultaneously self-adjoint.

We call a subset $\{M_0, M_1, \ldots, M_n\}$ *a basis* of the set $\mathcal{M}$ if it is a basis of $\mathcal{L}in_{\mathbb{R}}\mathcal{M}$.

A set $\mathcal{M}$ is *regular* if it contains an invertible matrix, i.e. $\mathcal{M} \cap \mathrm{GL}_d(\mathbb{C}) \neq \emptyset$. If $\mathcal{M}$ is not regular we say that it is *singular.*

A set $\mathcal{M}$ of complex hermitean matrices is *definite* if there exist $k_0, \ldots, k_n \in \mathbb{R}$ and a basis $\{M_0, \ldots, M_n\}$ of $\mathcal{M}$ such that

$$k_0M_0 + k_1M_1 + \cdots + k_nM_n > 0$$

and is *indefinite* otherwise. When $\mathcal{M}$ is indefinite, it is sometimes possible to find a self-orthogonal vector. Vector $v \in \mathbb{C}^d$ is *self-orthogonal* for $\mathcal{M}$ if

$$vMv^* = 0 \quad \text{for all} \quad M \in \mathcal{M}.$$

The study of simultaneous classification of n-tuples of matrices can be related to the geometric problem of determinantal representations in the following way:

A set $\mathcal{M}$ is *regular* if it contains an invertible matrix, i.e. $\mathcal{M} \cap \mathrm{GL}_d(\mathbb{C}) \neq \emptyset$. If $\mathcal{M}$ is not regular we say that it is *singular.*

To $\mathcal{M}$ with a basis $\{M_0, \ldots, M_n\}$ we assign matrix

$$M(x) = M(x_0, \ldots, x_n) = x_0M_0 + x_1M_1 + \ldots + x_nM_n$$

whose entries are linear in $x_0, \ldots, x_n$. When $\mathcal{M}$ is regular, we call the matrix $M(x)$ a *determinantal representation* of the hypersurface

$$\{(x_0, \ldots, x_n) \subset \mathbb{P}^n\ ;\ \det M(x_0, \ldots, x_n) = 0\},$$

or of the polynomial F

$$\det M(x_0, \dots, x_n) = c\, F(x_0, \dots, x_n), \quad 0 \neq c \in \mathbb{C}.$$

We say that the set $\mathcal{M}$ *has a determinantal representation.* Furthermore, we say that $\mathcal{M}$ is *regular and irreducible*, resp. *regular and reducible*, if the corresponding polynomial F is irreducible, resp. reducible.

Note that F is a homogeneous polynomial of degree d. We consider singular sets with determinant constantly 0 in Section 6. On the other hand, a generic set $\mathcal{M}$ defines a smooth hypersurface of degree d in $\mathbb{P}^n$.

Choose another basis $\{N_0, N_1, \dots, N_n\}$ of $\mathcal{M}$. The corresponding determinantal representation $x_0N_0 + \ldots + x_nN_n$ is related to $x_0M_0 + \ldots + x_nM_n$ via a real projective change of the coordinates $x_0, \dots, x_n$. Thus for different choices of bases of $\mathcal{M}$ we obtain different representations whose determinants are projectively equivalent polynomials. We see that $\mathcal{M}$ being simultaneously self-adjoint or definite or having a self-orthogonal vector does not depend on the choice of a basis. Therefore, from now on we will describe $\mathcal{M}$ by a finite number of matrices $\{M_0, \dots, M_n\}$ or equivalently by $M(x) = x_0M_0 + \ldots + x_nM_n$. By a slight abuse of terminology we call $M(x)$ a determinantal representation of $\mathcal{M}$.

Determinantal representations M and M' (necessarily of the same polynomial) are *equivalent* if

$$M' = AMB \quad \text{for some} \quad A,\, B \in \mathrm{GL}_d(\mathbb{C}).$$

Naturally, M is called a *self-adjoint representation* if all its coefficient matrices are complex hermitean. From the above definitions it is obvious that

Lemma 1.1. *Suppose that $\mathcal{M}$ is regular. Then it is simultaneously self-adjoint if and only if any (and therefore every) corresponding determinantal representation $M(x)$ is equivalent to some self-adjoint determinantal representation.*

After multiplying a given self-adjoint determinantal representation from left and right by an invertible matrix and its adjoint, we get another self-adjoint determinantal representation of the same hypersurface. We say that two self-adjoint determinantal representations M, M' are *hermitean equivalent* if

$$M' = AMA^* \quad \text{or} \quad M' = -AMA^* \quad \text{for some } A \in \mathrm{GL}_d(\mathbb{C}).$$

Note that hermitean equivalence preserves definiteness.

Question (2) about definiteness arises and is partly answered by semidefinite programming (SDP). According to Vinnikov [27], SDP is probably the most important new development in optimization in the last 20 years. The semidefinite programme minimizes an affine linear functional l on $\mathbb{R}^n$ subject to a linear matrix inequality (LMI) constraint

$$U_0 + x_1U_1 + \cdots + x_nU_n \geq 0, \text{ where all } U_i \in \mathbb{S}^d,$$

where $\mathbb{S}^d$ is the set of all $d \times d$ self-adjoint (i.e. complex hermitean) matrices. This can be solved either by finding an approximate solution (the running time of the algorithm increases only polynomially with the input size of the problem and $\log(\frac{1}{\varepsilon})$, where the parameter ε controls the accuracy of the result), or in many concrete situations by using interior point methods.

Our aim is to establish the link between Question (2) and SDP. Assume that the set of matrices $\mathcal{M}$ is simultaneously self-adjoint. Therefore each corresponding determinantal representation is equivalent to some self-adjoint determinantal representation

$$x_0U_0 + x_1U_1 + \cdots + x_nU_n, \text{ where all } U_i \in \mathbb{S}^d.$$

Matrices admit a positive definite linear combination if and only if

$$\{(x_0, x_1, ..., x_n) \in \mathbb{P}^n(\mathbb{R}) \ ; \ x_0U_0 + x_1U_1 + \cdots + x_nU_n \geq 0\} \neq \emptyset.$$

Next consider the reverse problem: given a convex set $\mathcal{C} \subset \mathbb{R}^n$, do there exist complex hermitian matrices such that

$$\mathcal{C} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n \ ; \ U_0 + x_1U_1 + \cdots + x_nU_n \geq 0\}?$$

We refer to the above as a *linear matrix inequality (LMI) representation* of $\mathcal{C}$. Sets having a LMI representation are called *spectrahedra*. Thus we can rephrase our Question (2): given a determinantal representation of a self-adjoint set of matrices $\mathcal{M}$, is it also a LMI representation? By the abuse of notation we will also call LMI representations *definite representations*.

In order to describe feasible sets for SDP, we examine the determinant of a LMI representation. Let $q(x) = \det(U_0 + x_1U_1 + \cdots + x_nU_n)$. Take $x^0 = (x_1^0, \ldots, x_n^0) \in \operatorname{Int}\mathcal{C}$ and normalize the LMI representation by $U_0 + x_1^0U_1 + \cdots + x_n^0U_n = \mathrm{I}$ (after conjugation with a unitary matrix). Here I is the identity matrix. We restrict the polynomial q to a straight line through x^0: for any $x \in \mathbb{R}^n$ consider

$$q(x^0 + tx) = \det(\mathrm{I} + t(x_1U_1 + \cdots + x_nU_n)).$$

Since all the eigenvalues of $x_1U_1 + \cdots + x_nU_n$ are real, we conclude that $q(x^0 + tx) \in \mathbb{R}[t]$ has only real zeroes. We say that it satisfies the *real zero (RZ) condition* with respect to $x^0 \in \mathbb{R}^n$. An algebraic interior $\mathcal{C}$ whose minimal defining polynomial satisfies the RZ condition with respect to one (and therefore every [18]) point of $\operatorname{Int}\mathcal{C}$ is *rigidly convex*.

Remark 1.2. Note that a LMI representation is a definite self-adjoint determinantal representation of some multiple of the minimal defining polynomial of $\mathcal{C}$. We defined RZ polynomials and rigidly convex algebraic interiors in the affine setting. In the homogeneous coordinates they correspond to hyperbolic polynomials and hyperbolicity sets, respectively.

The above considerations show that, for a set of matrices $\mathcal{M}$ to admit a positive definite linear combination, it is necessary that any determinantal representation of $\mathcal{M}$ induces a hyperbolic polynomial.

We conclude Introduction by a brief summary of classical results and conjectures. For $n = 2$, the famous Helton-Vinnikov Theorem [18] asserts that every RZ polynomial of degree d has a definite determinantal representation (with matrices of size d).

THEOREM 1.3. *A necessary and sufficient condition for $\mathcal{C} \subset \mathbb{R}^2$ to admit a LMI representation is that $\mathcal{C}$ is a rigidly convex algebraic interior. Moreover, the size of the matrices in a LMI representation is equal to the degree a minimal defining polynomial of $\mathcal{C}$.*

For $n \geq 3$ and d sufficiently large, by a simple parameter count, most polynomials do not admit a determinantal representation of size d (see [12]). If we allow matrices of arbitrary size, every real polynomial has a self-adjoint determinantal representation [17], though not necessarily a definite one (in this case it is not possible to normalize the representation by setting the constant matrix to be the identity). The generalized Lax conjecture, whether every real-zero polynomial has a definite determinantal representation of any size, has been disproved by Brändén [3]. However, the "new" form of the Lax conjecture is still open: for every RZ polynomial p there exists another RZ polynomial q such pq has a definite determinantal representation and q is non-negative on the rigidly convex set of p.

At TULS 2006 (a regional meeting in algebraic geometry) Emilia Mezetti suggested to consider sets of matrices being simultaneously self-adjoint. The authors have been introduced to the subject through GEOLMI (Geometry and Algebra of Linear Matrix Inequalities with Systems Control Applications) and in particular wishes to thank Didier Henrion and Daniele Faenzi for pointing out the connections between real algebraic geometry and semidefinite programming.

In this paper we present a complete set of conditions when a set of 3×3 matrices is simultaneously self-adjoint or definite. These conditions follow from results of Vinnikov [24], [25] and of our paper [9] (see also [6]). Sets of 4×4 matrices correspond to quartic hypersurfaces. LMI representations of quartic curves with respect to their 28 bitangents were constructed in [21]. We are not aware of any similar results for quartic surfaces. General self-adjoint representations of real curves are presented in [26].

2. Examples

Geometricaly the most interesting cases occur for $n = 2$ and 3 that correspont to curves and surfaces.

Example 2.1. The "flat TV screen" $\{(x_1,x_2)\in\mathbb{R}^2\ ;\ x_1^4+x_2^4\le 1\}$ is not a rigidly convex algebraic interior. Therefore any set of matrices whose determinantal representation induces $-x_0^4+x_1^4+x_2^4$ does not have a definite linear combination. For example,

$$\mathcal{M}=\left\{\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\begin{pmatrix}0&0&i&0\\0&1&0&0\\-i&0&0&0\\0&0&0&-1\end{pmatrix}\right\}.$$

Example 2.2. Let M_0,M_1,M_2 be three 3×3 matrices over $\mathbb{C}$. Then

$$\{(x_0,x_1,x_2)\ ;\ \det(x_0M_0+x_1M_1+x_2M_2)=0\}$$

defines a cubic curve in $\mathbb{P}^2$. Determinantal representations of smooth cubic curves were extensively studied in [24] and [25]. It is a classic result [11] that, given a smooth cubic curve F, there exists a 1-1 correspondence between nonequivalent determinantal representations of F and affine points on F. The same result holds for singular irreducible cubics.

Example 2.3. A general $\mathcal{M}$ generated by 4 matrices of size 3 defines

$$\det(x_0M_0+x_1M_1+x_2M_2+x_3M_3)=c\,F(x_0,x_1,x_2,x_3),\ 0\ne c\in\mathbb{C},$$

a smooth cubic surface in $\mathbb{P}^3$. It is well known that there are exactly 72 equivalence classes of determinantal representations defining the same smooth F.

Remark 2.4. Another interesting question is when a set of matrices is simultaneously symmetric. We remark that, for matrices of fixed size, this is a stronger condition than the condition of being simultaneously self-adjoint. Indeed, it is well known [13, Example 4.2.18] that an irreducible smooth, nodal or cuspidal cubic curve has respectively 3, 2 or 1 symmetric determinantal representations of size 3×3. Also in the case of surfaces it was proved [9, Corollary 3.6] or [10] that four 3×3 matrices over $\mathbb{C}$ defining a smooth cubic surface can not be simultaneously symmetric. See also [20].

It would be interesting to consider analogous questions for sets of skew-symmetric matrices. Given a hypersurface of degree d in $\mathbb{P}^n$, the moduli space of Pfaffian representations (described by $n+1$ skew-symmetric matrices of size $2d\times 2d$) is much bigger than the moduli space of determinantal representations (described by $n+1$ matrices of size $d\times d$). Pfaffian representations of cubic hypersurfaces have been intensively studied in [8] and [23], following the Beauville's survey [2].

3. Quadrics

We start with the simple case $d = 2$. Sets of 2×2 matrices already induce some interesting geometry, so we will use them to describe our methods. Pick a basis for a regular set $\mathcal{M}$ and assign to it the determinantal representation:

$$\sum_{i=0}^{n} x_i \begin{pmatrix} m_{11}^i & m_{12}^i \\ m_{21}^i & m_{22}^i \end{pmatrix}.$$

Its determinant is a quadric in $\mathbb{P}^n$ with equation

$$(x_0, \ldots, x_n)\, Q \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0,$$

where the $ij-$th element in Q equals $m_{11}^i m_{22}^j + m_{11}^j m_{22}^i - m_{12}^i m_{21}^j - m_{12}^j m_{21}^i$. If $\mathcal{M}$ is simultaneously self-adjoint, its $\mathbb{R}-$basis contains at most 4 matrices. Indeed, a basis for $\mathbb{S}^2$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}.$$

Therefore, the obtained nontrivial hypersurfaces are either two points or a double point ($n = 1$), quadric curves ($n = 2$) or quadric surfaces ($n = 3$).

Over $\mathbb{R}$, the corresponding quadric is projectively equivalent to one of the following:

$$\begin{aligned}
n = 0: &\quad x_0^2, \\
n = 1: &\quad x_0^2,\ x_0^2 + x_1^2,\ -x_0^2 + x_1^2, \\
n = 2: &\quad x_0^2 + x_1^2 + x_2^2,\ -x_0^2 + x_1^2 + x_2^2, \\
n = 3: &\quad x_0^2 + x_1^2 + x_2^2 + x_3^2,\ -x_0^2 + x_1^2 + x_2^2 + x_3^2,\ -x_0^2 + x_1^2 + x_2^2 - x_3^2.
\end{aligned}$$

Suppose first that $\det M = x_0^2$. Since $\det M_0 \neq 0$, we can multiply M by M_0^{-1} and from now on assume that $M_0 = \mathrm{I}$. Then any other nonzero M_i, $i \neq 1$ is nilpotent and similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\det M = x_0^2$ it follows that $n \leq 2$. Then it is easy to verify that either $M = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & x_0 \\ x_0 & x_1 \end{pmatrix}$.

Next we prove that each of other possible polynomials has exactly one determinantal representation (up to equivalence). Consider for example $M = x_0 M_0 + x_1 M_1 + x_2 M_2$ with determinant $x_0^2 + x_1^2 + x_2^2$. Since $\det M_0 \neq 0$, we can multiply M by M_0^{-1} and from now on assume that $M_0 = \mathrm{I}$. Then the eigenvalues of M_1 are $\pm i$. Indeed, $\det(-\lambda\,\mathrm{I}+M_1) = \lambda^2+1$. Thus, there exists such a matrix A that the map $M_i \mapsto AM_iA^{-1}$ preserves $M_0 = \mathrm{I}$ and brings M_1 into the diagonal form. Then M_2 has to be antidiagonal. Finally, the action BMB^{-1} by an antidiagonal B preserves the diagonal form of I, $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and reduces M_2 to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If we multiply M by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we get a self-adjoint representation.

By analogous reasoning we obtain determinantal representations for all of the above hypersurfaces. If it exists, we give a self-adjoint one:

$$
\begin{aligned}
n=0: &\quad \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix}, \\
n=1: &\quad \begin{pmatrix} 0 & x_0 \\ x_0 & x_1 \end{pmatrix}, \begin{pmatrix} 0 & x_0+ix_1 \\ x_0-ix_1 & 0 \end{pmatrix}, \begin{pmatrix} x_0+x_1 & 0 \\ 0 & x_0-x_1 \end{pmatrix}, \\
n=2: &\quad \begin{pmatrix} x_2 & x_0+ix_1 \\ x_0-ix_1 & -x_2 \end{pmatrix}, \begin{pmatrix} x_0+x_1 & x_2 \\ x_2 & x_0-x_1 \end{pmatrix}, \\
n=3: &\quad \begin{pmatrix} x_2+ix_3 & x_0+ix_1 \\ x_0-ix_1 & -x_2+ix_3 \end{pmatrix}, \begin{pmatrix} x_0+x_1 & x_2+ix_3 \\ x_2-ix_3 & x_0-x_1 \end{pmatrix}, \begin{pmatrix} x_0+x_1 & x_2+x_3 \\ x_2-x_3 & x_0-x_1 \end{pmatrix}.
\end{aligned}
$$

The eigenvalues of $\begin{pmatrix} x_2 & x_0+ix_1 \\ x_0-ix_1 & -x_2 \end{pmatrix}$ are $\pm\sqrt{x_0^2+x_1^2+x_2^2}$, so this representation can not be definite. However, it has no self-orthogonal vector. On the other hand, $-x_0^2+x_1^2+x_2^2$ and $-x_0^2+x_1^2+x_2^2+x_3^2$ define a rigidly convex algebraic interior. Their determinantal representations are indeed LMI representations (the coefficient matrix at x_0 is definite). Note that sphere is the only surface with self-adjoint representation. We summarize the above:

$$
\begin{aligned}
n=0: &\quad \binom{\text{self-adjoint}}{\text{definite}}, \\
n=1: &\quad \binom{\text{self-adjoint}}{\text{not definite}}\text{with self-orthogonal}\binom{1}{0}, \ \binom{\text{self-adjoint}}{\text{definite}}, \\
n=2: &\quad \binom{\text{self-adjoint}}{\text{not definite}} \text{ no self-orthogonal vector, } \binom{\text{self-adjoint}}{\text{definite}}, \\
n=3: &\quad \binom{\text{not self-adjoint}}{}, \ \binom{\text{self-adjoint}}{\text{definite}}, \ \binom{\text{not self-adjoint}}{}.
\end{aligned}
$$

Every real reducible quadric is projectively equivalent to $(x_0+x_1)(x_0-x_1)$ and thus definite. If $\mathcal{M}$ is singular then it is equivalent to one of the following spaces:

$$
\begin{pmatrix} x_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_0 & x_1 \\ 0 & 0 \end{pmatrix}.
$$

The first case is self-adjoint while it is easy to see that the second is not self-adjoint.

4. Regular sets of 3×3 matrices

In this section we address/answer Question 1, when a regular set $\mathcal{M}$ is a simultaneously self-adjoint. For $n \geq 2$ we also assume that $\mathcal{M}$ is irreducible. The reducible case is studied in section 5.

We equate $\mathcal{M}$ with $M = x_0M_0 + \ldots + x_nM_n$, where $n+1 = \dim \mathcal{L}in_{\mathbb{R}}\mathcal{M}$. Then $\det M = c\,F(x_0,\ldots,x_n),\quad 0 \neq c \in \mathbb{C}$ is a nonzero polynomial in $\mathbb{P}^n$. If M is equivalent to a self-adjoint representation, the corresponding F has real coefficients (after factoring out c) and $n < 9$.

n=0

Since F is nonzero, $M_0 \in \mathrm{GL}_3(\mathbb{C})$. Therefore we can always multiply M_0 by its inverse to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ which is self-adjoint and definite.}$$

n=1

First check whether $\det(x_0M_0 + x_1M_1) = c\,F$ is nonzero and F has real coefficients. If this holds, a real projective change of coordinates transforms F to

$$F = x_0^3 + x_1\, f(x_0, x_1)$$

for some real quadric f. This implies that $\det M_0 \neq 0$. The group action

$$x_0M_0 + x_1M_1 \;\longrightarrow\; AM_0^{-1}(x_0M_0 + x_1M_1)A^{-1}, \quad A \in \mathrm{GL}_3(\mathbb{C})$$

is the same as the group acting on the pair

$$(M_0, M_1) \;\longrightarrow\; (I, AM_0^{-1}M_1A^{-1}), \quad A \in \mathrm{GL}_3(\mathbb{C}),$$

which reduces M_0 to the identity I and M_1 to one of the canonical forms

$$\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \text{ or } \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Since F is real, either $a, b, d \in \mathbb{R}$ or $a \in \mathbb{R},\ d = \overline{b} \in \mathbb{C}$. These canonical forms can be made self-adjoint by suitable premultiplication:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \left(x_0 I + x_1 \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \right) = x_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 0 & a \\ 0 & a & 1 \\ a & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(x_0 I + x_1 \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right) = x_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 0 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & b \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \left(x_0 I + x_1 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \overline{b} \end{pmatrix} \right) = x_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_1 \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & \overline{b} \\ 0 & b & 0 \end{pmatrix},$$

$$x_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Thus we proved

LEMMA 4.1. *Every pair of* 3×3 *matrices whose determinant induces a real polynomial is simultaneously self-adjoint.*

n=2 Cubic curve
First check if F is a real irreducible cubic curve. Then by a real projective change of coordinates F can be brought into the Weierstrass form

$$x_1^2 x_2 = x_0^3 + p\, x_0^2 x_2 + q\, x_2^3$$

(check [16] or [7]), where $p, q \in \mathbb{R}$. Recall that the coordinate change only changes the basis of $\mathcal{L}in_{\mathbb{R}}\{M_0, M_1, M_2\}$.

Following Vinnikov's methods [24], the group action

$$x_0 M_0 + x_1 M_1 + x_2 M_2 \longrightarrow A(x_0 M_0 + x_1 M_1 + x_2 M_2)B, \quad A, B \in \mathrm{GL}_3(\mathbb{C})$$

in a unique way reduces the representation to

$$x_0 I + x_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} \frac{t}{2} & l & p + \frac{3}{4}t^2 \\ 0 & -t & -l \\ -1 & 0 & \frac{t}{2} \end{pmatrix},$$

where $t, l \in \mathbb{C}$ satisfy $l^2 = t^3 + p\,t + q$. Act on the above from the right by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

to get

$$x_0\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} + x_1\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + x_2\begin{pmatrix} p+\frac{3}{4}t^2 & l & \frac{t}{2}\\ -l & -t & 0\\ \frac{t}{2} & 0 & -1\end{pmatrix}.$$

This representation is self-adjoint if and only if t is real and l purely imaginary. Vinnikov [24] also proved that all self-adjoint representations of a given curve are of this form.

Therefore we obtain

Proposition 4.2. *Let $M = x_0M_0 + x_1M_1 + x_2M_2$ define a cubic curve $x_1^2x_2 = x_0^3 + p\,x_0^2x_2 + q\,x_2^3$ with $p, q \in \mathbb{R}$. Then M can be in unique way transformed to an equivalent representation*

$$\begin{pmatrix} x_2(p+\frac{3}{4}t^2) & x_1 + x_2 l & x_0 + x_2\frac{t}{2}\\ x_1 - x_2 l & x_0 - x_2 t & 0\\ x_0 + x_2\frac{t}{2} & 0 & -x_2\end{pmatrix},$$

where $l^2 = t^3 + p\,t + q$.

The set $\{M_0, M_1, M_2\}$ is simultaneously self-adjoint if and only if

$$t \in \mathbb{R} \quad \textit{and} \quad l \in i\mathbb{R}.$$

We conclude the curve case by another characterization that can be easily used for verification by a computer:

Let $M(x_0, x_1, x_2)$ be a determinantal representation of a cubic curve F. Define the corresponding kernel sheaf (or vector bundle if F smooth) $\epsilon(x_0, x_1, x_2)$ along F by

$$\epsilon(x_0, x_1, x_2) = \ker M(x_0, x_1, x_2).$$

Equivalent determinantal representations clearly induce equivalent vector bundles.

The best way to compute a section of ϵ is as a column of the adjoint matrix

$$\operatorname{adj} M(x_0, x_1, x_2),$$

whose entries are the signed $(n-1)\times(n-1)$ minors of M. Since the adjoint matrix adj$M$ has rank 1, all its columns are proportional along F [24, Proposition 2]. Then

Corollary 4.3. *Determinantal representation $M(x_0, x_1, x_2)$ is equivalent to a self-adjoint determinantal representation if and only if*

$$\overline{ker M(x_0, x_1, x_2)} \;\equiv\; ker M^*(x_0, x_1, x_2)$$

as sheaves (or vector bundles if F is smooth).

n=3 Cubic surface
A generic fourtuple of matrices $\mathcal{M}$ induces a determinantal representation $M(x_0, x_1, x_2, x_3)$ and a smooth irreducible cubic surface with the equation

$$F(x_0, x_1, x_2, x_3) = \frac{1}{c} \det M = 0,\ \ 0 \neq c \in \mathbb{C}.$$

Singular and reducible sets are considered in Sections 6 and 5, respectively.

Every smooth cubic surface can be obtained as a blow-up of $\mathbb{P}^2$ in 6 generic points. We will use the relation between the determinantal representation M and the six points of the blow-up, which can be found in [14]:

Define a 3×4 matrix L of linear forms in z_1, z_2, z_3 by

$$M \cdot \left(\begin{array}{c} z_0 \\ z_1 \\ z_2 \end{array} \right) \quad = \quad L \cdot \left(\begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right). \tag{1}$$

The minors of L form a basis of the 4-dimensiona linear system of plane cubic curves, which defines the blow-up. At the base points $P_i = (\zeta_i, \eta_i, \xi_i) \in \mathbb{P}^2,\ i = 1, \ldots, 6$ the rank of L equals 2 and equals 3 elsewhere. In other words, the rank of L in $P = (\zeta, \eta, \xi) \in \mathbb{P}^2$ equals 2 if and only if the three planes in $\mathbb{P}^3$ with equations

$$M \cdot \left(\begin{array}{c} \zeta \\ \eta \\ \xi \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$$

intersect in a line. Note that the lines obtained this way are exactly the exceptional lines of the blow-up [14]. They are mutually skew and we call them the *six skew lines corresponding to determinantal representation M.*

In the same way M^t determines another set of six skew lines.

A configuration of 12 lines with the property that $a_1, \ldots, a_6$ are mutually skew, $b_1, \ldots, b_6$, are mutually skew and a_i intersects b_j if and only if $i \neq j$, is called a *Schläfli double-six* and is denoted by

$$\left(\begin{array}{ccc} a_1 & \ldots & a_6 \\ b_1 & \ldots & b_6 \end{array} \right).$$

In [9, Corollary 3.5] we proved that the lines corresponding to M and M^t form a double-six. More precisely, for a given surface F there is a 1-1 correspondence between pairs M, M^t and double-sixes on F. From [9, Proposition 5.1, Theorem 5.3] it follows

PROPOSITION 4.4. *Let $M(x_0, x_1, x_2, x_3)$ be a determinantal representation of a real smooth cubic surface F. Then M is equivalent to a self-adjoint representation if and only if the double-six corresponding to M, M^t is mutually* *self-conjugate, i.e.*

$$\begin{pmatrix} a_1 & \dots & a_6 \\ b_1 & \dots & b_6 \end{pmatrix}$$

equals to one of the following:

$$I-st\ kind: \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix},$$

$$II-nd\ kind: \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix},$$

$$III-rd\ kind: \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_5} & \overline{a_6} \end{pmatrix},$$

$$IV-th\ kind: \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_6} & \overline{a_5} \end{pmatrix}.$$

It is easy to collect the above considerations in an answer to our Question 1:

input $\mathcal{M}$;

check $cF = \det M$ smooth, F real;

find the corresponding double-six and check its type;

result $\mathcal{M}$ is simultaneously self-adjoint if and only if the corresponding double-six is mutually self-conjugate.

It is well known that every smooth surface has exactly 72 nonequivalent determinantal representations. The number of self-adjoint representations depends on the geometric type of the surface (see [9] and [22]).

The geometry of singular cubic surfaces is also regulated by their configurations of lines. Every singular surface is a limit of nonsingular ones [22, page 40]. More on these and their determinantal representations can be found in [5, 4].

$\mathbf{n \geq 4}$

For a set $\mathcal{M}$ with $5 \leq m \leq 9$ independent matrices it is enough to check if $m-3$ of its 4 dimensional subsets are simultaneously self-adjoint. We will prove this claim only for 5-dimensional $\mathcal{M}$. The generalization to sets of higher dimension is straightforward.

THEOREM 4.5. *To a 5-dimensional $\mathcal{M}$ we assign a determinantal representation $M = x_0M_0 + \cdots + x_4M_4$ which defines a cubic threefold $F(x_0, \dots, x_4)$ in $\mathbb{P}^4$.*

Let π_1 and π_2 be hyperplanes in $\mathbb{P}^4$ such that $F \cap \pi_2$ and $F \cap \pi_2$ are smooth cubic surfaces. Then $\mathcal{M}$ is simultaneously self-adjoint if and only if $M|_{\pi_1}$ and $M|_{\pi_2}$ are equivalent to some self-adjoint representations.

Thus our answer to Question 1 can be extended to $n = 4$:

input $\mathcal{M}$;

check F real for some $0 \neq c \in \mathbb{C}$ such that $cF = \det M$;

find two hyperplanes π_1, π_2 such that $M|_{\pi_1}$ and $M|_{\pi_2}$ are determinantal representations of smooth cubic surfaces;

find the double-sixes corresponding to $M|_{\pi_1}$, $M|_{\pi_2}$ and check their type;

result $\mathcal{M}$ is simultaneously self-adjoint if and only if both double-sixes are mutually self-conjugate.

Proof. Both equations $\pi_i = 0$ can be seen as linear combinations of matrices in $\mathcal{M}$. Then $\mathcal{M}|_{\pi_i=0}$ is a 4 dimensional set. It is obvious that $\mathcal{M}$ being simultaneously self-adjoint implies that $\mathcal{M}|_{\pi_1=0}$ and $\mathcal{M}|_{\pi_2=0}$ are simultaneously self-adjoint.

Conversely, assume that $M|_{\pi_1=0}$ and $M|_{\pi_2=0}$ are equivalent to some self-adjoint representations. We can change the coordinates so that

$$\pi_1 = \{x_3 = 0\},\ \pi_2 = \{x_4 = 0\}$$

and so that the representation $M|_{\{x_3=x_4=0\}}$ defines a Weierstrass cubic curve

$$x_1^2 x_2 = x_0^3 + p\,x_0 x_2^2 + q\,x_2^3.$$

Moreover, as in the case of $n = 2$, the $\mathrm{GL}_3(\mathbb{C})$ action from left and right reduces M to the form

$$x_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} p + \frac{3}{4}t^2 & l & \frac{t}{2} \\ -l & -t & 0 \\ \frac{t}{2} & 0 & -1 \end{pmatrix} + x_3 M_3 + x_4 M_4$$

for a pair $t, l \in \mathbb{C}$ satisfying $l^2 = t^3 + p\,t + q$.

By our assumption $\{M_0, M_1, M_2, M_3\}$ are simultaneously self-adjoint. Observe that

$$A(M_0,\, M_1,\, M_2,\, M_3)B, \quad A, B \in \mathrm{GL}_3(\mathbb{C})$$

are self-adjoint if and only if

$$A^{-1}A(M_0,\, M_1,\, M_2,\, M_3)BA^{*-1}$$

are self-adjoint. Therefore it is enough to check for which

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

the matrices

$$M_0C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} C = \begin{pmatrix} c_{31} & c_{32} & c_{33} \\ c_{21} & c_{22} & c_{23} \\ c_{11} & c_{12} & c_{13} \end{pmatrix},$$

$$M_1C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} C = \begin{pmatrix} c_{21} & c_{22} & c_{23} \\ c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_2C = \begin{pmatrix} \star & \star & (p+\frac{3}{4}t^2)c_{13} + lc_{23} + \frac{t}{2}c_{33} \\ \star & \star & -lc_{13} - tc_{23} \\ \frac{t}{2}c_{11} - c_{31} & \frac{t}{2}c_{12} - c_{32} & \star \end{pmatrix},$$

$$M_3C$$

are complex hermitean. From the first two equalities it follows that

$$c_{13} = c_{23} = c_{12} = 0,\ c_{21} = c_{32} \in \mathbb{R},\ c_{22} = c_{11} = c_{33} \in \mathbb{R},\ c_{31} \in \mathbb{R}$$

and the third equality implies $c_{32} = c_{31} = 0$. Thus C is a multiple of the identity. This proves that if $\{M_0, M_1, M_2, M_3\}$ are simultaneously self-adjoint, then M_3 is complex hermitean.

The same way we prove that if $\{M_0, M_1, M_2, M_4\}$ are simultaneously self-adjoint, then M_4 is complex hermitean.

This concludes the proof since the reduced $x_0M_0 + x_1M_1 + x_2M_2 + x_3M_3 + x_4M_4$ is already a self-adjoint representation. □

Remark 4.6. Recall that not every cubic threefold has a determinantal representation with 3×3 matrices. Determinantal cubic threefolds are a closed $(5-2)3^2 + 2$ dimensional subvariety in the $\binom{3+5-1}{3}$ dimensional variety of all cubic threefolds.

For $n > 4$ the same argument works. Without loss of generality we only need to test the sets

$$\{M_0, M_1, M_2, M_k\}, \quad k = 3, \ldots, n.$$

5. Reducible sets

Now, we assume that a subset $\mathcal{M}$ is regular and reducible. The corresponding polynomial $F = \det M$ is a reducible polynomial. It can be a product of an irreducible quadratic and a linear polynomial or a product of three linear factors (counting multiplicities). We apply a result of Kerner and Vinnikov [19, Thm. 3.1], which tells us that the corresponding kernel sheaf $\ker M(x_1,\ldots,x_n)$ is globally a direct sum of kernel sheaves over distinct irreducible components of F. This can be viewed as a matrix version of a generalized M. Noether's $AF+BG$ Theorem [1, p. 139].

LEMMA 5.1. *If $\mathcal{M}$ is a regular and reducible subspace of 3×3 matrices that is self-adjoint then* $\dim_{\mathbb{R}}\mathcal{M}\le 5$.

Proof. First suppose that $F=\det M$ has two distinct irreducible factors. One has to be linear of multiplicity 1. So $F=lq$, where l is linear, q quadratic and l does not divide q. Then the kernel sheaf of $\mathcal{M}$ is globally decomposable by [19, Thm. 3.1], i. e.,

$$M=\begin{pmatrix} M_{(1)} & 0\\ 0 & M_{(2)}\end{pmatrix},$$

where $\det M_{(1)}=cl$ and $\det M_{(2)}=c^{-1}q$ for a nonzero scalar c. We saw in Section 3 that the dimension over $\mathbb{R}$ of a selfadjoint $\mathcal{M}\subset\mathbb{C}^{2\times 2}$ is at most 4. Hence $\dim\mathcal{M}\le 5$.

Assume next that F is of the form l^3 for some linear form l. Without loss we can take $F=x_0^3$. Further we can assume that $M_0=\mathrm{I}$. Then any other matrix M_i in the basis of $\mathcal{M}$ is nilpotent. The maximal possible dimension over $\mathbb{C}$ of a subspace of 3×3 nilpotent matrices is 3 (see [15]). Thus it is at most 6 over $\mathbb{R}$. A straightforward analysis then shows that the selfadjointness condition implies that also $\dim_{\mathbb{R}}\mathcal{M}=3$. If the dimension over $\mathbb{C}$ is at most 2 then over $\mathbb{R}$ it is at most 4. Thus it follows that $\dim\mathcal{M}\le 5$. $\square$

The case $n=0$ and $n=1$ were studied in Section 4.

Then an elementary analysis of all possible cases using the results of Section 3 yields a complete list of all possible cases. It is straightforward but cumbersome to write down, so we omit it.

6. Singular Sets of 3×3 matrices

It remains to consider sets $\mathcal{M}$ with determinant constantly 0. In other words, $\operatorname{rank}M=x_1M_1+\cdots+x_nM_n\le 2$ for all $x_1,\ldots,x_n\in\mathbb{R}$. In this case the $\mathrm{GL}_3(\mathbb{C})$ action transforms each M_i separately into either

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

We say that $\mathcal{M}$ is of rank i, $i=1,2$, if $\mathcal{M}$ is singular, $\operatorname{rank}N\le i$ for all $N\in\mathcal{M}$ and $\operatorname{rank}N=i$ for at least one $N\in\mathcal{M}$.

Proposition 6.1. *Suppose that $M_1, M_2, \dots, M_n$ is a basis of a subspace $\mathcal{M}$ which is of rank 1. Then $\mathcal{M}$ is simultaneously self-adjoint if and only if $x_1 M_1 + \cdots + x_n M_n$ is equivalent to*

$$\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where m_{11} is a real linear form in $x_1, \dots, x_n$.

Proposition 6.2. *Suppose that $M_1, M_2, \dots, M_n$ is a basis of a subspace $\mathcal{M}$ which is of rank 2 and that $\operatorname{rank} M_1 = 2$. Then $\mathcal{M}$ is simultaneously self-adjoint iff $x_1 M_1 + \cdots + x_n M_n$ is equivalent to one of the following:*

$$\begin{pmatrix} x_1 + m_{11} & m_{12} & 0 \\ \overline{m_{12}} & x_1 + m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_{11} & x_1 + m_{12} & 0 \\ x_1 + \overline{m_{12}} & m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} m_{11} & x_1 + m_{12} & m_{13} \\ x_1 + \overline{m_{12}} & 0 & 0 \\ \overline{m_{13}} & 0 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} -\gamma m_{11} & x_1 + i\delta m_{11} - m_{12} & i m_{11} \\ x_1 + i\gamma \overline{m_{22}} - m_{12} & \delta m_{22} & m_{22} \\ -i m_{11} & m_{22} & 0 \end{pmatrix}.$$

Here m_{ij} are linear forms in $x_2, \dots, x_n$, m_{11} and m_{22} are real. Moreover, in the last matrix above we have $\gamma, \delta \in \mathbb{R}$ and $m_{12} - \overline{m_{12}} = i(\gamma m_{22} + \delta m_{11})$.

7. Definite linear combinations of matrices

In this section we examine Question 2, when a set of 3×3 matrices is definite. We only need to consider the case of $\mathcal{M}$ regular. In order to stress that the elements of $\mathcal{M}$ are complex hermitean, we denote them by U_i. As before, equate $\mathcal{M}$ with $U = x_0 U_0 + \ldots + x_n U_n$, where $n + 1 = \dim \mathcal{L}in_{\mathbb{R}} \mathcal{M}$.

Our question is whether U is a LMI representation. In other words, do there exist $k_0, \dots, k_n \in \mathbb{R}$ such that

$$k_0 U_0 + k_1 U_1 + \cdots + k_n U_n > 0.$$

The property of being definite is an open condition. More precisely, all $n+1-$ tuples $(k_0, \dots, k_n)$ inside the spectrahedron (hyperbolicity set) satisfy $U(k_0, \cdots, k_n) > 0$. Small perturbations of k_i or of the entries in U_i preserve definiteness, therefore we can afford small errors occuring by numeric computations.

Throughout this section we will need the following elementary result.

LEMMA 7.1. *Let* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.
Then $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0$ *if and only if*

$$\lambda_1 + \lambda_2 + \lambda_3 > 0,\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0,\ \lambda_1\lambda_2\lambda_3 > 0.$$

The same way $\lambda_1 < 0,\ \lambda_2 < 0,\ \lambda_3 < 0$ *if and only if*

$$\lambda_1 + \lambda_2 + \lambda_3 < 0,\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0,\ \lambda_1\lambda_2\lambda_3 < 0.$$

Proof. We will prove the first statement (the second proof is the same). Implication $\Rightarrow$ is obvious. Conversely, assume that $\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 < 0$. We will prove that this implies either $\lambda_1 + \lambda_2 + \lambda_3 < 0$ or $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 < 0$, which finishes the proof.

Indeed, if $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$, then

$$\begin{array}{rcl} \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 & = & \\ \lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 & \leq & \\ -(\lambda_2 + \lambda_3)^2 + \lambda_2\lambda_3 & = & \\ -\lambda_2^2 - \lambda_3^2 - \lambda_2\lambda_3 & < & 0, \end{array}$$

since $\lambda_1 \geq -(\lambda_2 + \lambda_3) > 0$. □

Consider now $U = (u_{ij})_{1 \leq i,j \leq 3}$, a complex hermitean matrix with eigenvalues

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

By Lemma 7.1 the signs of λ_i can be read from the characteristic polynomial

$$\begin{array}{rcl} \det(\Lambda I - U) & = & (\Lambda - \lambda_1)(\Lambda - \lambda_2)(\Lambda - \lambda_3) \\ & = & \Lambda^3 - \Lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \Lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3. \end{array}$$

On the other hand

$$\begin{array}{l} \det(\Lambda I - U) = \\ \quad \Lambda^3 - \Lambda^2(u_{11} + u_{22} + u_{33}) + \\ \quad \Lambda(u_{11}u_{22} - u_{12}\overline{u_{12}} + u_{11}u_{33} - u_{13}\overline{u_{13}} + u_{22}u_{33} - u_{23}\overline{u_{23}}) - \det U. \end{array} \tag{2}$$

Like in Section 4 we consider different n separately.

$\mathbf{n = 0}$

Calculate the eigenvalues of U_0. If they are all of the same sign, then U_0 is definite.

$\mathbf{n = 1}$

Recall that the coordinates in $x_0U_0 + x_1U_1$ can be chosen so that

$$\det(x_0U_0 + x_1U_1) = x_0^3 + x_0x_1(\cdots).$$

In particular $\det U_0 \neq 0$ and $\det U_1 = 0$.

First check if U_0 is already definite. If not, we need to check whether

$$U_0 + tU_1$$

is definite for some $t \in \mathbb{R}$. The characteristic polynomial of $U_0 + tU_1$ by (2) equals

$$\Lambda^3 - \Lambda^2 \operatorname{trace}(U_0 + tU_1) + \Lambda\, q(t) - \det(U_0 + tU_1),$$

where $\operatorname{trace}(U_0 + tU_1)$ is a linear, $q(t)$ is a quadratic and $\det(U_0 + tU_1)$ is a cubic polynomial in t. It is easy to check from their graphs if there exists $t \in \mathbb{R}$ for which either

$$\operatorname{trace}(U_0 + tU_1) > 0, \quad q(t) > 0, \quad \det(U_0 + tU_1) > 0$$

or

$$\operatorname{trace}(U_0 + tU_1) < 0, \quad q(t) > 0, \quad \det(U_0 + tU_1) < 0.$$

If such t exists, then U_0+tU_1 is definite by Lemma 7.1. Otherwise it is indefinite.

$\mathbf{n = 2}$

For cubic curves we use the following beautiful result

Theorem 7.2. *[24, Theorems 8 & 9] Let*

$$U = x_0U_0 + x_1U_1 + x_2U_2$$

be a self-adjoint determinantal representation of a smooth cubic curve with equation $x_1^2x_2 = x_0^3 + px_0^2x_2 + qx_2^3$. There exists a unique $P \in \mathrm{GL}_3(\mathbb{C})$ for which either PUP^* or $-PUP^*$ equals

$$x_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} p + \frac{3}{4}t^2 & l & \frac{t}{2} \\ -l & -t & 0 \\ \frac{t}{2} & 0 & -1 \end{pmatrix}.$$

Here $t \in \mathbb{R},\ l \in i\mathbb{R}$ and $l^2 = t^3 + pt + q$.

Observe that

$$l^2 = t^3 + pt + q$$

defines an affine curve C in $\mathbb{R}^2 \equiv \mathbb{C}$. When equation $E :\ t^3 + pt + q = 0$ has 3 real solutions, C consists of two components, one compact and the other non-compact. When E has a pair of complex conjugate solutions, C consists of a single non-compact component.

The representation U is definite if and only if the corresponding point (t, l) lies on the compact component of C. Moreover, U is either definite or the coefficient matrices U_0, U_1, U_2 have a common self-orthogonal vector.

The same result holds for representations of singular irreducible curves [7].

Corollary 7.3. *A pair of complex hermitean matrices U_0, U_1 is either definite or U_0, U_1 have a common self-orthogonal vector.*

Proof. Let U_0, U_1 have a common self-orthogonal vector $v \in \mathbb{C}^3$. Then $x_0U_0 + x_1U_1$ is indefinite, because by definition $vU_0v^* = vU_1v^* = 0$.

Next assume that $x_0U_0 + x_1U_1$ is indefinite. Find a matrix U_2 such that (after a real projective change of coordinates)

$$\det(x_0U_0 + x_1U_1 + x_2U_2)$$

is a Weierstrass curve and the graph in $\mathbb{R}^2$ only has one non-compact component. Then U_0, U_1, U_2 have a common self-orthogonal vector by Theorem 7.2. $\square$

n = 3

Let $U(x_0, x_1, x_2, x_3)$ be a self-adjoint determinantal representation of a smooth cubic surface. Every self-adjoint representation induces one of the 4 kinds of double-sixes specified in Proposition 4.4. In [9, Theorem 6.2.] we proved that the representations corresponding to the $I-$st, $II-$nd or $III-$rd kind always contain a self-orthogonal vector and are therefore indefinite.

The $IV-$th kind needs to be considered separately. Before we state the result, recall some facts ([9] & [22]) about the geometry of a real cubic surface F which contains a double-six

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_6} & \overline{a_5} \end{pmatrix}$$

of the $IV-$th kind.

Let

$$\pi_{11} = \langle a_1, \overline{a_1} \rangle\ ,\ \ \pi_{22} = \langle a_2, \overline{a_2} \rangle$$

be tritangent planes spanned by the lines of F. A *tritangent plane* in $\mathbb{P}^3$ by definition intersect F in three lines. Observe that the equations π_{11}, π_{22} are real. The planes π_{11} and π_{22} divide $\mathbb{P}^3(\mathbb{R})$ into two wedges where π_{11}, π_{22} either have the same or different signs.

The real part of $F(\mathbb{R})$ consists of two disconnected components, one of which is ovoidal. There are two possibilities: either the ovoidal and non-ovoidal component of $F(\mathbb{R})$ both lie in the same wedge, or each component lies in a different wedge. From [9, Theorem 6.4.] we conclude:

Theorem 7.4. *Representation $U(x_0, x_1, x_2, x_3)$ is definite if and only if the ovoidal and non-ovoidal piece of F lie in different wedges cut out by π_{11} and π_{22}.*

The best way to calculate which wedge contains the ovoidal piece is to view our surface affinely. Then consider an orientation on the pencil of real planes with the axis $\pi_{11} \cap \pi_{22}$. The axis is a real line, since π_{11} and π_{22} intersect in a real line on F .

It follows from the above, that we can answer Question 2 using the following algorithm:

input $\{U_0, \ldots, U_3\}$;

check all U_i complex hermitean, $\det U$ smooth;

find the corresponding double-six and check its type;

if of kind I, II or III then $\{U_i\}_{i=0,\ldots,3}$ is indefinite;

else construct the corresponding tritangent planes π_{11}, π_{22};

check which wedge contains the ovoidal and nonovoidal parts of the surface by rotating real planes around the axis $\pi_{11} \cap \pi_{22}$.

In the case of surfaces of the IV–th kind indefiniteness does not imply the existence of a self-orthogonal vector. It is easy to construct a self-adjoint representation which is not definite and has no self-orthogonal vector [9, Example 6.5].

$\mathbf{n \geq 4}$

To a $n+1$ dimensional $\mathcal{M}$ we assign a self-adjoint determinantal representation

$$U = x_0 U_0 + \cdots + x_n U_n$$

which defines a real cubic hypersurface $F(x_0, \ldots, x_n)$ in $\mathbb{P}^n$. With growing n it is more likely that the representation becomes definite. On the other hand, the geometry of higher dimensional cubic hypersurfaces gets much more complicated. Note that for U to be a LMI representation, $F = 0$ needs to have a compact "ovoidal" piece. This is exactly the hyperbolicity set (spectrahedron or rigidly convex algebraic interior in the affine setting).

Consider the eigenvalues $\lambda_i(x_0, \ldots, x_n)$ of U. They are the solutions of the characteristic polynomial $\det(\Lambda I - U)$ which we computed in (2). By Lemma 7.1, U is definite if and only if there exist $k_0, \ldots, k_n \in \mathbb{R}$ such that

$$\begin{array}{rc} L: & u_{11} + u_{22} + u_{33}, \\ Q: & u_{11}u_{22} - u_{12}\overline{u_{12}} + u_{11}u_{33} - u_{13}\overline{u_{13}} + u_{22}u_{33} - u_{23}\overline{u_{23}}, \\ F: & \det U \end{array}$$

evaluated in $(k_0, \ldots, k_n)$ are all strictly positive.

Note that u_{ij} are linear functions of $x_0, \ldots, x_n$. Then $L = 0$ defines a real hyperplane, $Q = 0$ a quadratic form and $F = 0$ our cubic in $\mathbb{P}^n$. Write

$$Q = (x_0, \ldots, x_n)\, S\, (x_0, \ldots, x_n)^t,$$

where S is a real symmetric $n+1 \times n+1$ matrix. Observe that S negative definite implies $Q \leq 0$ for all values of $x_0, \ldots, x_n$. In this case U can not be definite.

On the other hand, S positive definite implies $Q > 0$ for all $x_0, \ldots, x_n \neq 0^{n+1}$. Since $L = 0$ and $F = 0$ always intersect in $\mathbb{R}^{n+1}$ (a real cubic equation has a real solution), there exist $k_0, \ldots, k_n \in \mathbb{R}$ in which $L > 0$, $F > 0$. In this case U is definite.

The last option we need to consider is the case when S is indefinite. Then $Q = 0$ is a nonempty conic in $\mathbb{P}(\mathbb{R}^{n+1})$. Recall that $\mathbb{P}(\mathbb{R}^{n+1})$ can be divided into two parts by the equations of L and F: points in which L, F are both of the same sign and points in which L, F have different sign. Denote the first part $L \cdot F > 0$ and the second part $L \cdot F < 0$. Under these assumptions we get

Proposition 7.5. *The representation U is indefinite if and only if the conic $Q = 0$ and its interior $Q > 0$ are entirely included in the $L \cdot F < 0$ part.*

In particular, $Q \cap L$ must be empty, which implies that $Q|_{L=0}$ is a definite quadratic form.

Proof. The statement follows easily from Figure 1.

The interior of the sphere represents $Q > 0$. Then U is indefinite if and only if L and F have different signs along the whole area defined by $Q > 0$. In other words, U is definite if either

- $Q = 0$ intersects $L = 0$,
- $Q = 0$ intersects $F = 0$,
- $Q > 0$ intersects the part $L \cdot F > 0$.

□

We finish the section by summarizing the above observations:

input $\{U_i\}_{i=0,\ldots,n}$ complex hermitean;

find $L = \operatorname{trace} U,\ Q = (x_0, \ldots, x_n) S (x_0, \ldots, x_n)^t,\ F = \det U$;

if S negative definite, then U indefinite;

if S positive definite, then U definite;

else check the position and sign of Q with respect to the parts $L \cdot F > 0$ and $L \cdot F < 0$. Then use Proposition 7.5.

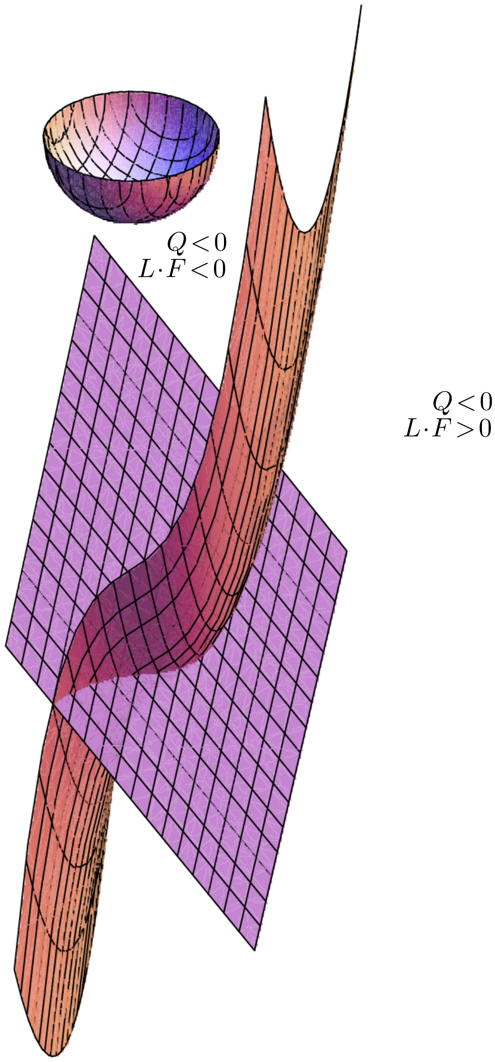

Figure 1: L and F always intersect.

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Received December 12, 2014
Revised March 5, 2015