RANK 2 ACM BUNDLES ON COMPLETE INTERSECTION CALABI-YAU THREEFOLDS

MATEJ FILIP

Abstract. The aim of this paper is to classify indecomposable rank 2 arithmetically Cohen-Macaulay (ACM) bundles on general complete intersection Calabi-Yau (CICY) threefolds and prove the existence of some of them. New geometric properties of the curves corresponding to rank 2 ACM bundles (by Serre correspondence) are obtained. These follow from minimal free resolutions of curves in suitably chosen fourfolds (containing Calabi-Yau threefolds as hypersurfaces). A strong indication leading to existence of bundles with $c_1 = 2$, $c_2 = 13$ on a quintic conjectured in [8] and [21] is found.

1. Introduction

Curves and vector bundles on a general threefold $X \subset \mathbb{P}^n$ have been considered as an important tool for the description of the geometry of X.

The existence of ACM bundles is linked to the existence of some arithmetically Cohen-Macaulay curves, via Serre correspondence between rank 2 bundles on threefolds and locally complete intersection subcanonical curves (see e.g. [13] or [18]).

We will consider smooth Calabi-Yau threefolds, i.e. smooth 3-dimensional projective varieties with trivial canonical class. In particular, if $X = X_{d_1 \cdots d_k} \subset \mathbb{P}^{3+k}$ is a complete intersection of hypersurfaces of degrees $d_1, ..., d_k$ then X is called a CICY threefold. By adjunction formula (see e.g. [11, p. 59]), we obtain the following CICY threefolds:

- quintic threefold X_5 in $\mathbb{P}^4$,
- complete intersection X_8 of type (2,4) in $\mathbb{P}^5$,
- complete intersection X_9 of type (3,3) in $\mathbb{P}^5$,
- complete intersection X_{12} of type (2,2,3) in $\mathbb{P}^6$,
- complete intersection X_{16} of type (2,2,2,2) in $\mathbb{P}^7$.

Chiantini and Madonna classify ACM rank 2 bundles on a general quintic threefold [8, p. 247] and prove the existence of some of them. They use the tools of deformation theory from Kley [15]. Rao and Kumar [21] prove the existence of all ACM bundles from [8, p. 247], where some of the proofs involve computer calculations (for bunldes with $c_1 = 2$, $c_2 = 13$ and $c_1 = 3$, $c_2 = 20$ there are no quality proofs). They proved the existence of bundles using the Pfaffian matrix representation of the quintic threefold, so their result cannot be generalized to other CICY threefolds. A list of indecomposable ACM bundles on CICY threefolds is given in Madonna's paper [17], however it contains some miscalculations (see Theorem 1.1).

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Recently Knutsen [16] and Yu [25] explored the existence of smooth isolated curves on general CICY threefolds. Their results and the theory of elliptic and canonical curves (see [2], [5], [10], [23] and [24]) will help us to prove the existence of indecomposable rank 2 bundles on general CICY threefolds.

For the existence of rank 4 indecomposable vector bundles on a quintic threefold see Madonna [19].

The main result of this paper is

Theorem 1.1. *Let $X_r \subset \mathbb{P}^{3+k}$ (where $k = \lfloor \frac{r}{4} \rfloor$) be a general CICY threefold and let $\mathcal{E}$ be an indecomposable ACM rank 2 vector bundle on it. Then the normalization of $\mathcal{E}$ has one of the following Chern classes:*

- $c_1 = -2$, $c_2 = 1$,
- $c_1 = -1$, $c_2 = 2$,
- $c_1 = 0$, $3 \leq c_2 \leq 4 + k$,
- $c_1 = 1$, $4 \leq c_2 \leq 6 + 2k$ *and* c_2 *is even,*
- $c_1 = 2$, $c_2 \leq 7 + 2k + r$
- $c_1 = 3$, $c_2 = 8 + 2k + 2r$
- $c_1 = 4$,
    - $c_2 = 30$ *if* $r = 5$,
    - $c_2 = 44$ *if* $r = 8$,
    - $c_2 = 48$ *if* $r = 9$,
    - $c_2 = 62$ *if* $r = 12$,
    - $c_2 = 80$ *if* $r = 16$.

We prove the existence of $\mathcal{E}$ for $c_1 = -2, -1$ and 0, for all possible r and c_2 listed above, except $c_1 = 0$, $c_2 = 3$ on X_{16}. There also exist bundles for $r = 8$ with $c_1 = 1$, $c_2 = 6$ and $c_1 = 1$, $c_2 = 10$; for $r = 9$ with $c_1 = 1$, $c_2 = 6$ and $c_1 = 1$, $c_2 = 10$; for $r = 12$ with $c_1 = 1$, $c_2 = 8$ and $c_1 = 1, c_2 = 12$.

Section 2 includes definitions, notations, the Grothendieck-Riemann-Roch formula for rank 2 bundle on smooth CICY threefolds and states the Serre correspondence. Our Theorem 3.1 in Section 3 explicitely relates minimal resolutions of arithmetically Gorenstein curves and the corresponding rank 2 bundles. This is a generalization of Faenzi and Chiantini result [7] for rank 2 bundles on surfaces. In Section 4 we classify bundles using case by case analysis. We obtain some interesting properties of the corresponding curves and some minimal free resolutions which lead to the proof of Theorem 1.1 in Section 5. We also discuss why the existence of indecomposable rank 2 bundles with $c_1 = 1$ surprisingly indicates the existence of rank 2 bundle with $c_1 = 2$ and $c_2 = 13$ on a quintic threefold, conjectured in [8] and [21]. We also analyse the existence of an indecomposable bundle of higher rank on X_8.

2. Generalities

We work over the field of complex numbers. A vector bundle on a projective scheme X is a locally free coherent sheaf on X. We denote by $\mathcal{O}_X$ the structure sheaf of X and for any vector bundle $\mathcal{E}$ we write $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{O}_X(n)$, where $\mathcal{O}_X(1)$ is the twisting sheaf of Serre (see [12, p. 117]).

Since the Picard group of X_r is isomorphic to $\mathbb{Z}$ and second Chern class $c_2(\mathcal{E})$ is a multiple of the class of a line, we identify Chern classes $c_1(\mathcal{E})$, $c_2(\mathcal{E})$ and line bundles with integers. We write c_i for $c_i(\mathcal{E})$. If $\mathcal{E}$ is a rank 2 vector bundle on X_r, we have

$$c_1(\mathcal{E}(n)) = c_1(\mathcal{E}) + 2n,$$

$$c_2(\mathcal{E}(n)) = c_2(\mathcal{E}) + rnc_1(\mathcal{E}) + rn^2.$$

Lemma 2.1. *The Grothendieck-Riemann-Roch formula (GRR) for a rank 2 bundle $\mathcal{E}$ on a smooth CICY threefold X_r is*

$$\chi(\mathcal{E}) = \frac{r}{6}c_1^3 - \frac{c_1c_2}{2} + \frac{c_1}{12}(12(k+4) - 2r), \text{where} \quad k = \left\lfloor \frac{r}{4} \right\rfloor. \tag{1}$$

Proof. Let h denote the generator of the Picard group and let l be the class of a line. From the Grothendieck-Riemann-Roch formula (see e.g. p. 431 in [12]) we get

$$\deg(\mathrm{ch}(\mathcal{E})\mathrm{td}(\mathcal{T}_X))_3 = \frac{1}{6}(c_1^3 - 3c_1c_2) + \frac{1}{4}d_1(c_1^2 - 2c_2) + \frac{1}{12}(d_1^2 + d_2)c_1 + \frac{1}{12}d_1d_2,$$

where $c_i = c_i(\mathcal{E})$, $d_i = c_i(\mathcal{T}_X)$, and $\mathcal{T}_X$ is the tangent sheaf of X. By the adjunction formula ([11, p. 59]) we have $d_1 = 0$ and in the case $r = 5$ we obtain $d_2 = 10h^2$, in the case $r = 8$ we obtain $d_2 = 7h^2$, in the case $r = 9$ we obtain $d_2 = 6h^2$, in the case $r = 12$ we obtain $d_2 = 5h^2$ and in the case $r = 16$ we obtain $d_2 = 4h^2$. We have $h^2 = r \cdot l$ and after identifying the Chern classes with integers we get (1). □

Definition 2.2. Let $\mathcal{I}_V$ be the saturated ideal of a closed subscheme V of $\mathbb{P}^n$. Then V is *arithmetically Cohen-Macaulay* (ACM) if

$$\dim S/\mathcal{I}_V = \operatorname{depth} S/\mathcal{I}_V,$$

where $S = \mathbb{C}[x_0, ..., x_n]$.

Another equivalent definition is that V is ACM if the *projective dimension* (i.e. the length of a minimal free resolution of $S/\mathcal{I}_V$) is equal to the codimension of V (see e.g. [20, p. 10]). It holds (see [20, Lemma 1.2.3]) that if $\dim V = r \geq 1$, then V is ACM if and only if $(M^i)(V) = 0$, for $1 \leq i \leq r$. Here $(M^i)(V)$ is the deficiency module of V, defined as the i-th cohomology module of the ideal sheaf of V:

$$(M^i)(V) = H^i_*(\mathcal{I}_V) = \bigoplus_{k \in \mathbb{Z}} H^i(X, \mathcal{I}_V(k)).$$

A locally complete intersection projective variety $V \subset \mathbb{P}^n$ is *subcanonical* if the canonical sheaf ω_V is isomorphic to $\mathcal{O}_V(k)$, for some integer k.

If $V \subset \mathbb{P}^n$ is ACM and the last bundle in the minimal free resolution of $\mathcal{I}_V$ is a line bundle, then we call V *arithmetically Gorenstein* (AG). Note that variety V is AG if and only if it is ACM and subcanonical (see e.g. [20, Proposition 4.1.1]). If there exists a variety X ($\mathbb{P}^n \supseteq X \supseteq V$) such that the last bundle in the minimal free resolution of $\mathcal{I}_V$ in X is a line bundle then we say that V is *AG in X*.

Theorem 2.3. *Smooth elliptic curves in $\mathbb{P}^n$ are AG.*

Proof. By [10, Chapter 6D] smooth elliptic curves are ACM and since they have trivial canonical sheaf (see [12, Example IV.1.3.6]) they are AG. □

Theorem 2.4. *Smooth canonical curves in $\mathbb{P}^n$ are AG.*

Proof. Following Noether [23], Schreyer [24] wrote the minimal resolutions of smooth canonical curve C in $\mathbb{P}^{g-1}$

$$0 \to \mathcal{O}(-g-1) \to \mathcal{P}_{g-3} \cdots \to \mathcal{P}_1 \to \mathcal{I}_C \to 0, \tag{2}$$

where g is the genus of C. It follows that C is ACM since the projective dimension of C is equal to the codimension. Curve C is also AG since the last bundle in resolution (2) is line bundle $\mathcal{O}(-g-1)$. □

A sheaf $\mathcal{E}$ on a k-dimensional projective variety X is *arithmetically Cohen-Macaulay* (ACM) if it is locally Cohen-Macaulay (i.e. depth $\mathcal{E}_x = \dim\ \mathcal{E}_x$ for every $x \in X$) and it holds

$$H^i(X, \mathcal{E}(n)) = 0 \ \text{ for } i = 1, ..., k-1 \ \text{ and all } n \in \mathbb{Z}. \tag{3}$$

If X is smooth, then [1, Lemma 3.2] implies that a sheaf $\mathcal{E}$ is ACM if and only if (3) holds.

We say that a sheaf $\mathcal{E}$ on X is *normalized* if the number

$$b(\mathcal{E}) := \max\{n \mid H^0(X, \mathcal{E}(-n)) \neq 0\}$$

is equal to zero. Clearly, the normalization of $\mathcal{E}$ is $\mathcal{E}(-b(\mathcal{E}))$.

The Serre correspondence between bundles and curves (see e.g. [13] or [18]) is the following:

Theorem 2.5. *Let X be a smooth 3-dimensional variety. If a curve $C \subset X$ is local complete intersection and subcanonical, then C occurs as the zero-locus of a section of a rank 2 vector bundle $\mathcal{E}$ on X. More precisely, for any fixed invertible sheaf $\mathcal{L}$ on X with $h^1(\mathcal{L}^\vee) = h^2(\mathcal{L}^\vee) = 0$, there exists a bijection between the following set of data:*

1. *the set of triples $(\mathcal{E}, s, \phi)$, where $s \in H^0(X, \mathcal{E})$ and $\phi : \wedge^2\mathcal{E} \to \mathcal{L}$ is an isomorphism, modulo the equivalence relation $(\mathcal{E}, s, \phi) \sim (\mathcal{E}', s', \phi')$ if there is an isomorphism $\psi : \mathcal{E} \to \mathcal{E}'$ and an element $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that $s' = \lambda\psi(s)$ and $\phi' = \lambda^2\phi \circ (\wedge^2\psi)^{-1}$.*
2. *the set of pairs $(C, \mathcal{E})$, where C is a locally complete intersection curve in X and $\mathcal{L} \otimes \omega_X \otimes \mathcal{O}_C$ and ω_C are isomorphic.*

A normalized bundle $\mathcal{E}$ has a section whose zero locus C is a curve and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(c_1(\mathcal{E})), \tag{4}$$

where $\mathcal{I}_C$ is an ideal sheaf of C on X. The curve C is ACM if and only if $\mathcal{E}$ is ACM and C has degree $c_2(\mathcal{E})$. Moreover, if X is a Calabi-Yau threefold, the genus of C is $\frac{c_1(\mathcal{E})c_2(\mathcal{E})}{2} + 1$.

3. Minimal resolutions of ACM bundles on complete intersection threefolds

Let $\mathcal{E}$ be a vector bundle of rank 2 on a general complete intersection threefold $X \subset \mathbb{P}^{3+k}$ of type $(d_1, ..., d_k)$. Denote by Y_i the complete intersection fourfold of type $(d_1, .., d_{i-1}, d_{i+1}, ..., d_k)$ which contains X and let a_j be the degrees of the minimal generators for $\mathcal{E}$ in Y_i. By abuse of notation write $\mathcal{E}$ for the sheaf $i_*\mathcal{E}$, where $i : X \to Y_i$ is the natural inclusion. Clearly, for $k = 1$ take $Y_1 = \mathbb{P}^4$. From the Auslander-Buchsbaum formula (see [10, Chapter 19]) we get a resolution in Y_i for any i

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{5}$$

where $\mathcal{G} = \bigoplus_{j=1}^{k} \mathcal{O}_{Y_i}(-a_j)$ and $\mathcal{F}$ is an ACM bundle on Y_i.

From now on we will assume that $\mathcal{F}$ splits. If Y_i is $\mathbb{P}^4$ (which is the case when X is a quintic threefold), then by Horrock's criterion $\mathcal{F}$ splits. X and Y have Picard group isomorphic to $\mathbb{Z}$ by the Grothendieck-Lefschetz theorem.

After dualising the exact sequence (5), we get

$$0 \longrightarrow \mathcal{H}om(\mathcal{G}, \mathcal{O}_{Y_i}) \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{O}_{Y_i}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) \longrightarrow 0. \tag{6}$$

Indeed, $\mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) = 0$ since the support of $\mathcal{E}$ is in X and $\mathcal{E}xt^i_{\mathcal{O}_{Y_i}}(\mathcal{F}, \mathcal{O}_{Y_i}) = \mathcal{E}xt^i_{\mathcal{O}_{Y_i}}(\mathcal{G}, \mathcal{O}_{Y_i}) = 0$, for $i > 0$ by [12, Proposition III.6.3] since $\mathcal{F}$ splits.

The same argument as in [22] for a quintic yields that even when Y_i is not equal to $\mathbb{P}^4$, we have an isomorphism $\mathcal{E}xt^1_{Y_i}(\mathcal{E}, Y_i) \cong \mathcal{E}^\vee(d_i)$. Indeed, applying the functor $\mathcal{H}om(\mathcal{E}, \cdot)$ on the exact sequence

$$0 \to \mathcal{O}_{Y_i} \to \mathcal{O}_{Y_i}(d_i) \to \mathcal{O}_X(d_i) \to 0,$$

we obtain

$$\begin{aligned} 0 \longrightarrow\ & \mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}(d_i)) \longrightarrow \mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_X(d_i)) \longrightarrow \\ \longrightarrow\ & \mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) \xrightarrow{f} \mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}(d_i)) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_X(d_i)) \longrightarrow \cdots \end{aligned}$$

where f is multiplication by the defining polynomial of X in Y_i. As above we see that $\mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) = \mathcal{H}om_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}(d_i)) = 0$. Since $\mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i})$ and $\mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}(d_i))$ are both supported on X, we obtain an isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) \cong \mathcal{E}^\vee(d_i).$$

Since $\mathcal{E}^\vee \cong \mathcal{E}(-c_1)$ by [12, Ex. II.5.16], we have

$$\mathcal{E}xt^1_{\mathcal{O}_{Y_i}}(\mathcal{E}, \mathcal{O}_{Y_i}) \cong \mathcal{E}(d_i - c_1).$$

Thus we see that resolutions (5) and (6) twisted by $c_1 - d_i$ are equivalent resolutions for $\mathcal{E}$ and uniqueness of the minimal resolution implies $\mathcal{F} \cong \mathcal{G}^\vee(c_1 - d_i)$. Thus we have a minimal resolution

$$0 \to \mathcal{G}^\vee(c_1 - d) \to \mathcal{G} \to \mathcal{E} \to 0, \tag{7}$$

which will also appear in the following theorem.

Theorem 3.1. *Assume that a smooth complete intersection threefold X contains a subcanonical local complete intersection curve C, which is AG in at least one of the Y_i. Then the ideal sheaf of C has a minimal resolution*

$$0 \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{I}_{C,Y_i} \longrightarrow 0, \tag{8}$$

where

$$\mathcal{P}_0 = \bigoplus_{j=1}^{2b+1} \mathcal{O}_{Y_i}(-r_j), \quad \mathcal{P}_1 = \bigoplus_{j=1}^{2b+1} \mathcal{O}_{Y_i}(r_j - c), \quad \mathcal{P}_2 = \mathcal{O}_{Y_i}(-c).$$

Here r_j are the degrees of minimal generators of $\mathcal{I}_{C,Y_i}$ and $2b+1$ is the number of these generators. Assume also that $c_1(\mathcal{E}) = c - d_i$, where $\mathcal{E}$ is a normalized rank 2 ACM bundle on X, corresponding to the curve C. Then $\mathcal{E}$ has a minimal resolution

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{E} \longrightarrow 0, \tag{9}$$

where

$$\mathcal{L}_1 = \mathcal{O}_{Y_i}(c-2d_i) \oplus \left(\bigoplus_{j=1}^{2b+1} \mathcal{O}_{Y_i}(r_j - d_i) \right),$$

$$\mathcal{L}_0 = \mathcal{O}_{Y_i} \oplus \left(\bigoplus_{j=1}^{2b+1} \mathcal{O}_{Y_i}(-r_j + c - d_i) \right).$$

From now on, we fix the fourfold Y_i satisfying the assumptions of Theorem 3.1 and denote it by Y. We also denote the degree of the defining polynomial of X in Y by d instead of d_i. Observe that (9) is of type (7).

Proof of Theorem 3.1. By Eisenbud and Buchsbaum [6] every ideal sheaf of an AG curve in Y has a resolution of type (8). From (8) we obtain two short exact sequences:

$$0 \longrightarrow \mathcal{O}_Y(-c) \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{K} \longrightarrow 0, \tag{10}$$

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{I}_{C,Y} \longrightarrow 0. \tag{11}$$

We also have the following exact sequences:

$$0 \longrightarrow \mathcal{O}_Y(-d) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{12}$$

$$0 \longrightarrow \mathcal{O}_Y(-d) \longrightarrow \mathcal{I}_{C,Y} \longrightarrow \mathcal{I}_{C,X} \longrightarrow 0, \tag{13}$$

$$0 \longrightarrow \mathcal{O}_X(d-c) \longrightarrow \mathcal{E}(d-c) \longrightarrow \mathcal{I}_{C,X} \longrightarrow 0, \tag{14}$$

where (14) is given by Serre correspondence.

Let $\mathcal{Q}$ be the kernel of the surjective map $\mathcal{P}_0 \to \mathcal{I}_{C,X}$ induced by (11) and (13). Thus we have

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{I}_{C,X} \longrightarrow 0. \tag{15}$$

By the snake lemma applied to (11) and (15), $\mathcal{Q}$ fits into

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}(-d) \longrightarrow 0. \tag{16}$$

Next, apply $\mathrm{Hom}(\mathcal{P}_0, \cdot)$ to (14). We have $\mathrm{Ext}^1(\mathcal{P}_0, \mathcal{O}_X(d-c)) = 0$ because $\mathcal{P}_0$ is a direct sum of line bundles. Therefore the map $\mathcal{P}_0 \to \mathcal{I}_{C,X}$ lifts to the map $\mathcal{P}_0 \to \mathcal{E}(d-c)$ and thus we can connect (14) and (15). The mapping cone determines the surjectivity of $\mathcal{P}_0 \oplus \mathcal{O}_X(d-c) \to \mathcal{E}(d-c)$. Thus $\mathcal{P}_0 \oplus \mathcal{O}_Y(d-c) \to \mathcal{E}(d-c)$ is also surjective and we have

$$0 \to \mathcal{R} \to \mathcal{P}_0 \oplus \mathcal{O}_Y(d-c) \to \mathcal{E}(d-c) \to 0, \tag{17}$$

where $\mathcal{R}$ denotes the kernel of the surjective map.

From the snake lemma applied to (17) and (12) twisted by $d-c$ and using (11) and (14), we obtain

$$0 \to \mathcal{O}_Y(-c) \to \mathcal{R} \to \mathcal{Q} \to 0. \tag{18}$$

Again apply the snake lemma to (10) and (18) and using (16) to get

$$0 \to \mathcal{P}_1 \to \mathcal{R} \to \mathcal{O}_Y(-d) \to 0,$$

which splits since $\mathrm{Ext}^1(\mathcal{O}_Y(-d), \mathcal{P}_1) = 0$. Thus we proved

$$0 \to \mathcal{P}_1 \oplus \mathcal{O}_Y(-d) \to \mathcal{P}_0 \oplus \mathcal{O}_Y(d-c) \to \mathcal{E}(d-c) \to 0. \tag{19}$$

Observe, that twisting (19) by $c - d$ gives (9). It can be easily verified that the obtained resolution is minimal. □

Remark 3.2. In [4, Lemma 2, Lemma 5, Proposition 2] some minimal resolution of ACM rank 2 bundles on cubic and quartic threefolds are determined. If we apply Theorem 3.1 on these cases we obtain the same resolutions except in the case of [4, Lemma 5], where X is a quartic threefold and the curve C corresponding to the bundle $\mathcal{E}$ with $c_1 = 2$, $c_2 = 8$ is of type (2,2,2). We have a minimal resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-6) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-4)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0$$

and by Theorem 3.1 ($c = 6$, $d_i = 4$, $c_1 = c - d_i$, $r_j = 2$, for $j = 1, ..., 3$) we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^4 \longrightarrow \mathcal{O}_{\mathbb{P}^4}^4 \longrightarrow \mathcal{E} \longrightarrow 0.$$

However, in [4] the obtained resolution of $\mathcal{E}$ is of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-2)^4 \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^k \longrightarrow \mathcal{O}_{\mathbb{P}^4}^4 \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^k \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $k \in \{0, 2, 4\}$. It is then deduced that $k = 0$ implies $\mathcal{E}$ is 0-regular (see definition of regularity in [20, p. 8]), so $k \neq 0$. We believe this is incorrect, since $\mathcal{E}$ in this case is 1-regular. Theorem 3.1 immediately implies that the case $k = 0$ is the correct choice.

4. Classification and existence of ACM bundles on CICY threefolds

By [18, Theorem 3.9] a normalized rank 2 ACM bundle on a smooth CICY threefold splits unless $-5 < -c_1(\mathcal{E}) < 3$. Recall from the Serre correspondence that if $\mathcal{E}$ is an indecomposable ACM rank 2 bundle, then the corresponding curve is ACM of degree $c_2(\mathcal{E})$ and genus $c_1(\mathcal{E})c_2(\mathcal{E})/2 + 1$. In this section we will classify the indecomposable rank 2 ACM bundles on general CICY threefolds.

4.1. **Quintic threefold.** In this subsection we write $\mathcal{O}$ for $\mathcal{O}_{\mathbb{P}^4}$. For a quintic threefold X in $\mathbb{P}^4$ the minimal resolution of a curve can be determined from the minimal resolution of the corresponding bundle:

Theorem 4.1. *Let $\mathcal{E}$ be a normalized indecomposable rank 2 vector bundle on X with a minimal resolution*

(20)

$$0 \longrightarrow \mathcal{O}(c_1(\mathcal{E})-5) \oplus \left(\bigoplus_{i=1}^{2b+1} \mathcal{O}(r_i + c_1(\mathcal{E}) - 5) \right) \longrightarrow \mathcal{O} \oplus \left(\bigoplus_{i=1}^{2b+1} \mathcal{O}(-r_i) \right) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then the corresponding curve has a minimal resolution

$$0 \longrightarrow \mathcal{O}(-c_1(\mathcal{E}) + 5) \longrightarrow \bigoplus_{i=1}^{2b+1} \mathcal{O}(r_i - 5) \longrightarrow \bigoplus_{i=1}^{2b+1} \mathcal{O}(-r_i - c_1(\mathcal{E})) \longrightarrow \mathcal{I}_C \longrightarrow 0.$$

Proof. The proof is similar to the proof of [7, Theorem 2.1] so we omit it (note that $c_1(\mathcal{E}) = c - 5$, where c is the integer from Theorem 1.1). □

Remark 4.2. Every normalized bundle on X has resolution of type (20) (see Section 3 or Beauville [3])

The classification of indecomposable ACM rank 2 bundles on a quintic can be found in [8]. In particular, the lower bound for c_2 is also given: $11 \leq c_2 \leq 14$. Theorem 4.1 allows us to determine geometric properties of the curves that correspond to these bundles. Many of these properties are already established in [8] and [21]. Here we only describe the new ones.

Let $\mathcal{E}$ be a normalized bundle with $c_1 = 2$, $c_2 = 11$. From GRR we have $h^0(\mathcal{E}) = 4$, $h^0(\mathcal{E}(1)) = 18$, $h^0(\mathcal{E}(2)) = 52$ and because a minimal resolution of $\mathcal{E}$ is of type (20), we have a minimal resolution

$$0 \longrightarrow \mathcal{O}(-1)^2 \oplus \mathcal{O}(-3)^4 \longrightarrow \mathcal{O}(-2)^2 \oplus \mathcal{O}^4 \longrightarrow \mathcal{E} \longrightarrow 0.$$

From Theorem 4.1 the minimal resolution of the corresponding curve C is

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{O}(-5)^3 \oplus \mathcal{O}(-3)^2 \longrightarrow \mathcal{O}(-2)^3 \oplus \mathcal{O}(-4)^2 \longrightarrow \mathcal{I}_C \longrightarrow 0$$

where the degree matrix (see [14]) of C is

$$\begin{bmatrix} 3 & 3 & 3 & 1 & 1 \\ 3 & 3 & 3 & 1 & 1 \\ 3 & 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

By [14, Theorem 1.2] C is singular.

In the case of c_1 even, the minimal resolution of $\mathcal{E}$ can be uniquely determined as above. When c_1 is odd, the rank of the direct summands in a minimal resolution of $\mathcal{E}$ cannot be determined, similarly to the problem in Remark 3.2. For example, without taking into account the geometry of the corresponding curve, a normalized ACM bundle $\mathcal{E}$ with $c_1 = -1$ and $c_2 = 2$ has a minimal resolution

$$0 \longrightarrow \mathcal{O}(-6) \oplus \mathcal{O}^2(-4) \oplus \mathcal{O}(-3)^j \longrightarrow \mathcal{O} \oplus \mathcal{O}(-2)^2 \oplus \mathcal{O}(-3)^j \longrightarrow \mathcal{E} \longrightarrow 0.$$

Since the corresponding curve is a conic, Theorem 3.1 implies that $j = 1$.

In the sequel we collect minimal resolutions of all possible indecomposable normalized ACM bundles $\mathcal{E}$ of rank 2. By the above methods all these resolutions are uniquely determined, except for $c_1 = 3$. We have:

- $c_1 = -2$, $c_2 = 1$

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-2)^3 \longrightarrow \mathcal{O}(-1)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-7) \oplus \mathcal{O}(-4)^3 \longrightarrow \mathcal{O} \oplus \mathcal{O}(-3)^3 \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = -1$, $c_2 = 2$

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3)^2 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1)^2 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-6) \oplus \mathcal{O}(-4)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-2)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = 0$, $c_2 = 3$

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-4)^2 \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-4)^2 \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 0$, $c_2 = 4$

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3)^2 \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2)^2 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-3)^2 \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-4) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)^2 \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 0$, $c_2 = 5$

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-3)^5 \longrightarrow \mathcal{O}(-2)^5 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-3)^5 \longrightarrow \mathcal{O} \oplus \mathcal{O}(-2)^5 \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 1$, $c_2 = 4$

$$0 \longrightarrow \mathcal{O}(-6) \longrightarrow \mathcal{O}(-5)^2 \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1)^2 \oplus \mathcal{O}(-4) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-4)^3 \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}^3 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = 1$, $c_2 = 6$

$$0 \longrightarrow \mathcal{O}(-6) \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-3) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-4)^2 \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}^2 \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = 1$, $c_2 = 8$

$$0 \longrightarrow \mathcal{O}(-6) \longrightarrow \mathcal{O}(-4)^3 \longrightarrow \mathcal{O}(-2)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-3)^3 \oplus \mathcal{O}(-4) \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1)^3 \longrightarrow \mathcal{E} \longrightarrow 0$$

- $c_1 = 2$, $c_2 = 11$

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{O}(-5)^3 \oplus \mathcal{O}(-3)^2 \longrightarrow \mathcal{O}(-2)^3 \oplus \mathcal{O}(-4)^2 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-1)^2 \oplus \mathcal{O}(-3)^4 \longrightarrow \mathcal{O}(-2)^2 \oplus \mathcal{O}^4 \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = 2$, $c_2 = 12$

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{O}(-5)^2 \oplus \mathcal{O}(-4) \longrightarrow \mathcal{O}(-2)^2 \oplus \mathcal{O}(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-3)^3 \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}^3 \oplus \mathcal{O}(-1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

- $c_1 = 2$, $c_2 = 13$

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-4)^4 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3)^4 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-3)^2 \oplus \mathcal{O}(-2)^4 \longrightarrow \mathcal{O}^2 \oplus \mathcal{O}(-1)^4 \longrightarrow \mathcal{E} \longrightarrow 0, \tag{21}$$

- $c_1 = 2$, $c_2 = 14$

$$0 \longrightarrow \mathcal{O}(-7) \longrightarrow \mathcal{O}(-4)^7 \longrightarrow \mathcal{O}(-3)^7 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2)^7 \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1)^7 \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 3$, $c_2 = 20$

$$0 \longrightarrow \mathcal{O}(-8) \longrightarrow \mathcal{O}(-5)^4 \oplus \mathcal{O}(-4)^k \longrightarrow \mathcal{O}(-3)^4 \oplus \mathcal{O}(-4)^k \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-2)^5 \oplus \mathcal{O}(-1)^k \longrightarrow \mathcal{O}^5 \oplus \mathcal{O}(-1)^k \longrightarrow \mathcal{E} \longrightarrow 0,$$

  where k is odd.
- $c_1 = 4$, $c_2 = 30$

$$0 \longrightarrow \mathcal{O}(-9) \longrightarrow \mathcal{O}(-5)^9 \longrightarrow \mathcal{O}(-4)^9 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-1)^{10} \longrightarrow \mathcal{O}^{10} \longrightarrow \mathcal{E} \longrightarrow 0.$$

4.2. **CICY of type (2,4).** Let X_8 be a general CICY threefold of type (2,4) and Y a fourfold of degree 2 in $\mathbb{P}^5$, containing X_8. The only attempt of classification of indecomposable rank 2 on X_8 bundles known to the autor can be found in [17].

- $c_1 = -2$

  The bundle $\mathcal{E}$ has a section whose 0-locus C is a curve. From the exact sequence
  $$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(-2) \longrightarrow 0,$$
  we obtain by Serre duality $h^3(\mathcal{E}) = h^0(\mathcal{E}(2)) = 20$, so $\chi(\mathcal{E}) = -19$. From GRR we have $\chi(\mathcal{E}) = -20 + c_2$, so $c_2 = 1$ and the corresponding curve is a line.

  Since a line exists on X_8 (see e.g. [16]), then by Serre correspondence a bundle with $c_1 = -2$ and $c_2 = 1$ exists on X_8. We easily see that the minimal resolution of $\mathcal{E}$ is not of type (7) on either of the two fourfolds containing X_8.
- $c_1 = -1$

  As above we have an exact sequence
  $$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(-1) \longrightarrow 0$$
  and thus $h^3(\mathcal{E}) = h^0(\mathcal{E}(1)) = 6$. From GRR we have $-5 = \chi(\mathcal{E}) = -6 + \frac{c_2}{2}$, so $c_2 = 2$. From the above sequence we compute $h^0(\mathcal{I}(1)) = 23 - 20 = 3$, so C is a plane conic.

  We have
  $$0 \longrightarrow \mathcal{O}_Y(-3) \longrightarrow \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0$$
  and Theorem 3.1 implies
  $$0 \longrightarrow \mathcal{O}_Y(-5) \oplus \mathcal{O}_Y(-3)^3 \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{E} \longrightarrow 0. \tag{22}$$
- $c_1 = 0$

  We start with an exact sequence
  $$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C \longrightarrow 0.$$
  From GRR we obtain $h^0(\mathcal{E}(1)) = 12 - c_2$, thus we have $c_2 = 6 - h^0(\mathcal{I}_C(1))$, which gives four possibilities for c_2: 3, 4, 5, 6. If $c_2 = 4$ the curve C is a space curve of type (2,2) and we have
  $$0 \longrightarrow \mathcal{O}_Y(-4) \longrightarrow \mathcal{O}_Y(-3)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{O}_Y(-1)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$
  $$0 \longrightarrow \mathcal{O}_Y(-4) \oplus \mathcal{O}_Y(-3)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{E} \longrightarrow 0.$$
- $c_1 = 1$

  We have
  $$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(1) \longrightarrow 0.$$
  Because $h^3(\mathcal{E}) = h^0(\mathcal{E}(-1)) = 0$, then $1 + h^0(\mathcal{I}_C(1)) = h^0(\mathcal{E}) = \chi(\mathcal{E})$. From GRR follows $c_2 = 10 - 2h^0(\mathcal{I}_C(1))$. So we have four choices for c_2, which are 4,6,8,10.

  If $c_2 = 4$ the corresponding curve is a plane quartic with a resolution
  $$0 \longrightarrow \mathcal{O}_Y(-3) \longrightarrow \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0$$
  and Theorem 3.1 yields
  $$0 \longrightarrow \mathcal{O}_Y(-1)^4 \longrightarrow \mathcal{O}_Y^4 \longrightarrow \mathcal{E} \longrightarrow 0.$$

If $c_2 = 6$ the corresponding curve is a complete intersection of type (2,3). We have

$$0 \longrightarrow \mathcal{O}_Y(-5) \longrightarrow \mathcal{O}_Y(-4)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{O}_Y(-1)^2 \oplus \mathcal{O}_Y(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y(-3)^3 \oplus \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y^3 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{E} \longrightarrow 0.$$

If $c_2 = 8$ the corresponding curve is a complete intersection of type (2,2,2) and has a resolution

$$0 \longrightarrow \mathcal{O}_Y(-5) \longrightarrow \mathcal{O}_Y(-4) \oplus \mathcal{O}_Y(-3)^2 \longrightarrow \mathcal{O}_Y(-2)^2 \oplus \mathcal{O}_Y(-1) \longrightarrow \mathcal{I}_C \longrightarrow 0$$

and as before

$$0 \longrightarrow \mathcal{O}_Y(-3)^2 \oplus \mathcal{O}_Y(-2)^2 \longrightarrow \mathcal{O}_Y^2 \oplus \mathcal{O}_Y(-1)^2 \longrightarrow \mathcal{E} \longrightarrow 0.$$

If $c_2 = 10$ the corresponding curve is canonical of genus 6. We will show the existence of this bundle in the next section, which will also give us an indecomposable bundle of higher rank.

- $c_1 = 2$

  As before we have

$$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(2) \longrightarrow 0.$$

Because $h^3(\mathcal{E}) = h^0(\mathcal{E}(-2)) = 0$, then $1 + h^0(\mathcal{I}_C(2)) = h^0(\mathcal{E}) = \chi(\mathcal{E})$. With GRR we see $c_2 = 19 - h^0(\mathcal{I}_C(2))$. So we have $c_2 \leq 19$.

If $c_2 = 16$, then the corresponding ACM curve is a complete intersection of type (2,2,2,2). We have

$$0 \longrightarrow \mathcal{O}_Y(-6) \longrightarrow \mathcal{O}_Y(-4)^3 \longrightarrow \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y(-2)^4 \longrightarrow \mathcal{O}_Y^4 \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 3$

  As above we have

$$0 \longrightarrow \mathcal{O}_{X_8} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_C(3) \longrightarrow 0$$

and thus $h^3(\mathcal{E}(-1)) = h^0(\mathcal{E}(-2)) = 0$. GRR implies $c_2 = 28$.

- $c_1 = 4$ As above we have $h^3(\mathcal{E}(-1)) = 0$ and therefore $c_2 = 44$.

Using similar methods as in the case of CICY of type (2,4), we will obtain the remaining cases CICY threefolds of types (3,3), (2,2,3) and (2,2,2,2).

4.3. **CICY of type (3,3).** Let X_9 be a general CICY of type (3,3) and Y a fourfold of degree 3 in $\mathbb{P}^5$, containing X_9.

- $c_1 = -2$

  We have $c_2 = 1$ and the corresponding curve is a line. The existence of a line on X_9 was showed in [16].
- $c_1 = -1$

  We have $c_2 = 2$ and the corresponding curve is a conic. The existence of a conic on X_9 can also be found in [16].
- $c_1 = 0$

  We have $6 + h^0(\mathcal{I}_C(1)) = 12 - c_2$. This gives four possible choices for c_2, which are 3,4,5,6. If $c_2 = 3$ the curve C is a plane cubic and we have

$$0 \longrightarrow \mathcal{O}_Y(-3) \longrightarrow \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y(-3) \oplus \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{E} \longrightarrow 0. \tag{23}$$

- $c_1 = 1$
  We have $6 - \frac{c_2}{2} = 1 + h^0(I_C(1))$ thus by GRR $c_2 = 10 - 2h^0(I(1))$. We get four options for c_2: 4,6,8,10. If $c_2 = 6$ the curve C is a complete intersection of type (2,3) and suitable resolutions are

$$0 \longrightarrow \mathcal{O}_Y(-4) \longrightarrow \mathcal{O}_Y(-3)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{O}_Y(-1)^2 \oplus \mathcal{O}_Y(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y(-2)^3 \oplus \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y^3 \oplus \mathcal{O}_Y(-1) \longrightarrow \mathcal{E} \longrightarrow 0.$$

- $c_1 = 2$
  Because $h^3(\mathcal{E}) = h^0(\mathcal{E}(-2)) = 0$, we have $1 + h^0(\mathcal{I}_C(2)) = h^0(\mathcal{E}) = \chi(\mathcal{E})$. From GRR we get $c_2 = 20 - h^0(\mathcal{I}_C(2))$ and therefore $c_2 \leq 20$.
- $c_1 = 3$ We have $h^3(\mathcal{E}(-1)) = h^0(\mathcal{E}(-2)) = 0$ and by GRR we obtain $c_2 = 30$.
- $c_1 = 4$ We have $h^3(\mathcal{E}(-1)) = 0$ and by GRR we obtain $c_2 = 48$.

4.4. **CICY of type (2,2,3) and (2,2,2,2).** Let X_{12} and X_{16} be general CICY threefolds of type (2,2,3) and (2,2,2,2), respectively. Using the above methods concludes the classification of indecomposable rank 2 bundles listed in Theorem 1.1. For $c_1 = -2$ and $c_1 = -1$, Knutsen [16] proved the existence of a line and a conic on X_{12} and X_{16}.

In the next section we will also need the following interesting example, when X is CICY of type (2,2,3) and $\mathcal{E}$ is a bundle with $c_1 = 0$ and $c_2 = 4$. In this case we have

$$0 \longrightarrow \mathcal{O}_Y(-3) \to \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_Y(-3) \oplus \mathcal{O}_Y(-2)^3 \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)^3 \longrightarrow \mathcal{E} \longrightarrow 0, \tag{24}$$

where Y is a complete intersection of type (2,2) in $\mathbb{P}^6$, containing X_{12}.

5. Proof of Theorem 1.1

In the previous section we proved part of the Theorem 1.1 regarding classification. We proved the existence of the bundles with $c_1 = -2$ and $c_1 = -1$. Bundles with $c_1 = 0$ correspond to elliptic curves. Knutsen [16] showed the existence of smooth elliptic curves on all X_r of degree $d \geq 3$, except for $d = 3$ on X_{16}. By Theorem 2.3 smooth elliptic curves are AG and thus the Serre correspondence gives the existence of the ACM bundles with $c_1 = 0$ for all c_2 listed in Theorem 1.1.

Bundles with $c_1 = 1$ correspond to canonical curves. Knutsen [16] showed the existence of smooth curves of degree 10 and genus 6 on X_8 and X_9 and smooth curves of degree 12 and genus 7 on X_{12}. These curves are canonical and thus by Theorem 2.4 they are AG. The Serre correspondence then implies the existence of ACM bundles with $c_1 = 1$, $c_2 = 12$ on X_{12} and $c_1 = 1$, $c_2 = 10$ on X_8 and X_9.

In the sequel we will prove the existence of the remaining bundles from Theorem 1.1.

In order to obtain the last three cases of bundles with $c_1 = 1$ we tensor the exact sequence (7) with $\mathcal{O}_X$ to obtain

$$\cdots \to \mathcal{L}_0^\vee(c_1 - d) \otimes \mathcal{O}_X \xrightarrow{A} \mathcal{L}_0 \otimes \mathcal{O}_X \to \mathcal{E} \to 0.$$

After tensoring (7) with $\mathcal{H}om(\cdot, \mathcal{O}_X(c_1 - d))$ we see that $\ker A = \mathcal{E}(-d)$. Thus we have

$$0 \to \mathcal{E}(-d) \to \mathcal{L}_0^\vee(c_1 - d) \otimes \mathcal{O}_X \xrightarrow{A} \mathcal{L}_0 \otimes \mathcal{O}_X \to \mathcal{E} \to 0. \tag{25}$$

First we prove the existence of a bundle with $c_1 = 1$ and $c_2 = 6$ on X_8. Recall the minimal resolution (22) of a bundle with $c_1 = -1$ and $c_2 = 2$ on X_8. As in (25) we get an exact sequence

$$0 \longrightarrow \mathcal{E}(-4) \longrightarrow \mathcal{O}_{X_8}(-5) \oplus \mathcal{O}_{X_8}(-3)^3 \longrightarrow \mathcal{O}_{X_8} \oplus \mathcal{O}_{X_8}(-2)^3 \longrightarrow \mathcal{E} \longrightarrow 0,$$

and from this we get a short exact sequence

$$0 \longrightarrow \mathcal{E}(-4) \longrightarrow \mathcal{O}_{X_8}(-5) \oplus \mathcal{O}_{X_8}(-3)^3 \longrightarrow \mathcal{F}(-3) \longrightarrow 0, \tag{26}$$

where $\mathcal{F}$ is a rank 2 bundle on X_8 by the Auslander-Buchsbaum formula. From (26) also follows that $\mathcal{F}$ is indecomposable and normalized with $c_1(\mathcal{F}) = 1$ and $c_2(\mathcal{F}) = 6$. The bundle $\mathcal{F}$ exists on X_8 since $\mathcal{E}$ exists.

Next we prove the existence of a bundle with $c_1 = 1$ and $c_2 = 6$ on X_9. Consider the minimal resolution (23) of a bundle with $c_1 = 0$ and $c_2 = 3$ on X_9. As above we get a short exact sequence

$$0 \longrightarrow \mathcal{E}(-3) \longrightarrow \mathcal{O}_{X_9}(-3) \oplus \mathcal{O}_{X_9}(-2)^3 \longrightarrow \mathcal{F}(-2) \longrightarrow 0,$$

where $\mathcal{F}$ is a normalized indecomposable rank 2 bundle with $c_1 = 1$ and $c_2 = 6$ on X_9.

Finally the existence of a bundle with $c_1 = 1$ and $c_2 = 8$ on X_{12} will be obtained from the minimal resolution (24) of a bundle with $c_1 = 0$ and $c_2 = 4$ on X_{12}. Again, we get a short exact sequence

$$0 \longrightarrow \mathcal{E}(-3) \longrightarrow \mathcal{O}_{X_{12}}(-3) \oplus \mathcal{O}_{X_{12}}(-2)^3 \longrightarrow \mathcal{F}(-2) \longrightarrow 0,$$

with $\mathcal{F}$ a normalized indecomposable rank 2 bundle with $c_1 = 1$ and $c_2 = 8$ on X_{12}. This finishes the proof of Theorem 1.1.

5.1. **Bundles of higher rank.** Using similar methods as in Section 5 from exact sequence (21) we obtain

$$0 \to \mathcal{E}(-3) \to \mathcal{O}_{X_5}(-1)^2 \oplus \mathcal{O}_{X_5}^4 \to \mathcal{F} \to 0,$$

where $\mathcal{E}$ is rank 2 bundle with $c_1 = 2$, $c_2 = 13$ on X_5 and $\mathcal{F}$ is rank 4 bundle on X_5. From exact sequence we compute chern classes of $\mathcal{F}$: $c_1(\mathcal{F}) = 2$, $c_2(\mathcal{F}) = 17$, $c_3(\mathcal{F}) = 12$. Notice that the same chern classes have bundles $\mathcal{G} \oplus \mathcal{G}$ and $\mathcal{G}_1 \oplus \mathcal{G}_2$, where $\mathcal{G}$ is rank 2 bundle with $c_1 = 1$, $c_2 = 6$, $\mathcal{G}_1$ is rank 2 bundle with $c_1 = 1$, $c_2 = 4$ and $\mathcal{G}_2$ is rank 2 bundle with $c_1 = 1$ and $c_2 = 8$. Moreover, we have

$$h^0(\mathcal{F}(k)) = h^0(\mathcal{G}(k) \oplus \mathcal{G}(k)) = h^0(\mathcal{G}_1(k) \oplus \mathcal{G}_2(k)) = \frac{1}{3}(1+2k)(12+5k(1+k)).$$

Thus we have $h^i(X_5, \mathcal{F}(k)) = h^i(X_5, (\mathcal{G} \oplus \mathcal{G})(k)) = h^i(X_5, (\mathcal{G}_1 \oplus \mathcal{G}_2)(k))$ for all $i \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. It seems natural to us the following

Conjecture 5.1. $\mathcal{F} \cong \mathcal{G} \oplus \mathcal{G} \cong \mathcal{G}_1 \oplus \mathcal{G}_2$.

Since all rank 2 bundles with $c_1 = 1$ exist on X_5 then follows that bundle with $c_1 = 2$ and $c_2 = 13$ exists on X_5.

We conclude paper by analysing interesting rank 4 bundle on X_8. A canonical curve which correspond to bundle with $c_1 = 1$, $c_2 = 10$ on X_8 is the intersection of general quadric and Del Pezzo surface S of degree 5 in $\mathbb{P}^5$ (see [2]). By [5, Theorem 2.2] S is the Pfaffian variety defined by the five Pfaffians of a skew symmetric 5×5 matrix of linear forms. The minimal free resolution of S is given by the Buchsbaum-Eisenbud complex [6]. Thus we have

$$0 \to \mathcal{O}_Y(-5) \longrightarrow \mathcal{O}_Y(-3)^5 \longrightarrow \mathcal{O}_Y(-2)^5 \longrightarrow \mathcal{I}_C \longrightarrow 0,$$

where Y is fourfold of degree 2 in $\mathbb{P}^5$, containing X_8. Theorem 3.1 gives a minimal resolution of a bundle $\mathcal{E}$ with $c_1 = 1$ and $c_2 = 10$:

$$0 \to \mathcal{O}_Y(-3) \oplus \mathcal{O}_Y(-2)^5 \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)^5 \longrightarrow \mathcal{E} \longrightarrow 0.$$

As above, we construct a short exact sequence

$$0 \to \mathcal{E}(-4) \longrightarrow \mathcal{O}_{X_8}(-3) \oplus \mathcal{O}_{X_8}(-2)^5 \longrightarrow \mathcal{F}(-2) \longrightarrow 0.$$

The chern classes are $c_1(\mathcal{F}) = 2$, $c_2(\mathcal{F}) = 22$ and $c_3(\mathcal{F}) = 14$. The same chern classes have bundles $\mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_3 \oplus \mathcal{G}_4$, where $\mathcal{G}_1$ is rank 2 bundle with $c_1 = 1$, $c_2 = 4$, $\mathcal{G}_2$ is rank 2 bundle with $c_1 = 1$, $c_2 = 10$, $\mathcal{G}_3$ is rank 2 bundle with $c_1 = 1$, $c_2 = 6$ and $\mathcal{G}_4$ is rank 2 bundle with $c_1 = 1$, $c_2 = 8$. Similarly as above we can verify that

$$\begin{aligned} h^0(X_8, \mathcal{F}(k)) &= h^0(X_8, (\mathcal{G}_1 \oplus \mathcal{G}_2)(k)) = h^0(X_8, (\mathcal{G}_3 \oplus \mathcal{G}_4)(k)) = \\ &= \tfrac{1}{3}(1+2k)(15+8k(1+k)). \end{aligned}$$

We see that $\mathcal{F}$ is either indecomposable rank 4 bundle or direct sum of indecomposable rank 3 bundle and line bundle or most likely isomorphic to $\mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_3 \oplus \mathcal{G}_4$ (in this case the existence of $\mathcal{E}$ implies also the existence of all indecomposable rank 2 bundles with $c_1 = 1$, $c_2 = 4$ and $c_1 = 1$, $c_2 = 8$ on X_8).

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University of Ljubljana, Faculty of Math. and Phys., Dept. of Math.
E-mail address: matej.filip@student.fmf.uni-lj.si