A three dimensional counterexample to conjecture of Cox and Katz about nef cones of toric hypersurfaces

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ABSTRACT

In this paper, a three–dimensional toric variety is constructed which provides a counterexample to [1, Conjecture 6.2.8] of Cox and Katz about nef cones of toric hypersurfaces.

Introduction

Let Δ be a reflexive polytope in lattice $M_{\mathbb{R}}$ and $\mathbb{P}_\Delta$ the corresponding projective variety. By Batyrev [2] there exists a maximal projective subdivision Σ of the normal fan of Δ obtained by inserting extra lattice points in $\Delta^* \subset N_{\mathbb{R}}$. Correspondingly, there is a toric partial resolution

$$\mathbb{X}_\Sigma \longrightarrow \mathbb{P}_\Delta.$$

$\mathbb{X}_\Sigma$ is nonsingular in codimension three. The proper transform of the general Calabi–Yau hypersurface $\overline{Z}$ of $\mathbb{P}_\Delta$ is the crepant resolution Z which is again Calabi–Yau hypersurface in $\mathbb{X}_\Sigma$.

$$\begin{array}{ccc} \mathbb{X}_\Sigma & \longrightarrow & \mathbb{P}_\Delta \\ \cup & & \cup \\ Z & & \overline{Z}. \end{array}$$

There also exists a restriction map $\mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma) = H^2(\mathbb{X}_\Sigma) \to \mathrm{Pic}_{\mathbb{R}}(Z)$. Its kernel is generated by the classes D_ρ of divisors associated to edges $\rho \in \Sigma(1)$ spanned by integral vectors which are in the interior of some facet of Δ^*. The image of the restriction map is by definition $\mathrm{Pic}_{\mathrm{toric}}(Z)$. The conjecture [1, Conjecture 6.2.8] is an attempt to describe the nef cone for the "toric part" of Z.

If X is a projective variety, recall that its *nef cone* is the closed cone $\mathcal{K}(X)$ in $\mathrm{Pic}_{\mathbb{R}}(X)$ generated by nef divisor classes, i.e. classes $D \in \mathrm{Pic}(X)$ satisfying $D \cdot C \geq 0$ for all effective curves $C \subset X$.

We will now slightly modify Batyrev construction in order to obtain injective restriction map $\mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma) \to \mathrm{Pic}_{\mathbb{R}}(Z)$. Given a maximal projective subdivision Σ, we remove all of the edges generated by interior points of facets of Δ^*. The resulting fan need not be simplicial, so we may need to further subdivide this fan without adding new edges to get a projective subdivision Σ'. We get a commutative diagram

$$\begin{array}{ccc} \mathbb{X}_\Sigma & \overset{\psi}{\dashrightarrow} & \mathbb{X}_{\Sigma'} \\ & \searrow \quad \swarrow & \\ & \mathbb{P}_\Delta & \end{array}$$

By construction ψ is an isomorphism except over the points of $\mathbb{P}_\Delta$ associated to the facets of Δ^* which contain interior points. Since a generic Calabi–Yau hypersurface

$\overline{Z} \in |-K_{\mathbb{P}_\Delta}|$ is disjoint from this finite set, its proper transforms Z and Z' which are Calabi–Yau hypersurfaces in $\mathbb{X}_\Sigma$ and $\mathbb{X}_{\Sigma'}$, are isomorphic. In particular Z is nonsingular iff Z' is nonsingular.

From now on we will fix the above described *simplified* projective subdivision Σ of the normal fan of Δ and consider $Z \subset \mathbb{X}_\Sigma$. The natural map $\mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma) \to \mathrm{Pic}_{\mathbb{R}}(Z)$ is injective and its image is $\mathrm{Pic}_{\mathrm{toric}}(Z)$. The closure of the nef cone of toric variety $\mathbb{X}_\Sigma$ is *Gelfand–Kapranov–Zelevinsky cone* $\mathrm{cpl}\,(\Sigma)$ defined in [1, Chapter 3] or in [4]. Hence we can regard $\mathrm{cpl}\,(\Sigma)$ as lying in $\mathrm{Pic}_{\mathrm{toric}}(Z)$. If we define the *toric part of the nef cone of* Z to be

$$\mathcal{K}(Z)_{\mathrm{toric}} = \mathcal{K}(Z) \cap \mathrm{Pic}_{\mathrm{toric}}(Z)$$

we have the obvious inclusion $\mathrm{cpl}\,(\Sigma) \subset \overline{\mathcal{K}(Z)}_{\mathrm{toric}}$, since ample classes on $\mathbb{X}_\Sigma$ restrict to ample classes on Z.

It is certainly possible that there are other subdivisions Σ' of Σ satisfying the property that the anticanonical hypersurface in $\mathbb{X}_{\Sigma'}$ is isomorphic to Z. To treat these fans, we recall some definitions from [1, 6.2].

Suppose Σ is a fan in $N_{\mathbb{R}}$, $\Sigma(1)$ the set of its one-dimensional cones. A *linear circuit* is a linearly dependent set $S \subset \Sigma(1)$, no subset of which is linearly dependent. One has a decomposition $S = S_+ \cup S_-$ where S_+, respectively S_- are the vectors appearing with positive, respectively negative coefficients in a linear relation. Correspondingly, there is a fan $\Sigma_+(S)$ given by cones spanned by $S \setminus n_i$ for $n_i \in S_+$ and their subcones, and a similar fan $\Sigma_-(S)$.

A linear circuit S is said to be *supported by* Σ, if the following two conditions are satisfied:

- $\Sigma_+(S)$ is a subfan of Σ.
- Let σ be a top-dimensional cone of $\Sigma_+(S)$. If there exists a subset $S' \subset \Sigma(1)$ such that $\sigma \cup S'$ generates a top-dimensional cone of Σ, then for all other top-dimensional cones σ' of $\Sigma_+(S)$, $\sigma' \cup S'$ also generates a top-dimensional cone of Σ.

Suppose that S is a linear circuit supported by Σ and both S_- and S_+ are non-empty. Then one can build a new fan $\mathrm{Flip}_S(\Sigma)$ from Σ by replacing the simplices of Σ spanned by $\sigma \cup S'$, where σ is a cone of $\Sigma_+(S)$ and $S' \subset \Sigma(1)$, by the simplices spanned by $\sigma' \cup S'$ where σ' is a cone of $\Sigma_-(S)$. Then the fan $\mathrm{Flip}_S(\Sigma)$ is simplicial and projective, and the cones $\mathrm{cpl}\,(\Sigma)$ and $\mathrm{cpl}\,(\mathrm{Flip}_S(\Sigma))$ touch along a common face in the GKZ decomposition (see [4]).

Corresponding to the two fans in $N_{\mathbb{R}}$, there is a birational map

$$\varphi : \mathbb{X}_\Sigma \dashrightarrow \mathbb{X}_{\mathrm{Flip}_S(\Sigma)}.$$

It is easy to check that in case S_- contains only one element, this map is in fact a morphism contracting a divisor. If however both S_- and S_+ contain more than one element, the birational map is a *generalized flop*, a small contraction followed by a small resolution. If the flop φ has exceptional locus disjoint from the anticanonical hypersurface of $\mathbb{X}_\Sigma$, it is called a *trivial flop* and in this case the flip attached to S is referred to as a *trivial flip*.

When Σ' is a trivial flip of Σ we get a map

$$\mathrm{Pic}\,(\mathbb{X}_{\Sigma'}) \to \mathrm{Pic}\,(Z') \simeq \mathrm{Pic}\,(Z)$$

which takes nef classes to nef classes. We therefore conclude $\operatorname{cpl}(\Sigma') \subset \overline{\mathcal{K}(Z)}_{\text{toric}}$. Thus we obtain

$$\bigcup_{\Sigma'} \operatorname{cpl}(\Sigma') \subset \overline{\mathcal{K}(Z)}_{\text{toric}} \qquad (\clubsuit)$$

where the union is over all fans Σ' which may be obtained from Σ by a sequence of trivial flips.

[1, Conjecture 6.2.8] states that this inclusion is in fact an equality. However we will construct a three–dimensional toric variety for which this inclusion will be strict.

Notation and conventions All schemes and varieties are defined over $\mathbb{C}$. A *Calabi–Yau* is a normal projective variety X with canonical Gorenstein singularities, satisfying $K_X \sim 0$ and $H^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq \dim(X) - 1$. A Calabi–Yau surface is called K3 *surface*. A birational morphism of projective Gorenstein varieties $f : Y \to X$ is *crepant* if $f^*K_X \sim K_Y$. Some statements and proofs use the language of *toric geometry*, in particular the ideas of *linear Gale transform* and *Gelfand–Kapranov–Zelevinsky cones*. The main references are Fulton [3], Oda–Park [4] and Cox–Katz [1, Chapters 3 and 6].

1 The construction

We will begin with

$$\mathcal{Q} = \left\{ \sum_{i=0}^{3} z_i^4 - 4t z_0 z_1 z_2 z_3 = 0 \right\} \subset \mathbb{P}^3.$$

$\mathcal{Q}$ is a K3 quartic in $\mathbb{P}^3$. Define a map $g : \mathbb{P}^3 \to \mathbb{P}^3$ by

$$g : [z_0 : z_1 : z_2 : z_3] \mapsto [z_0 : \xi z_1 : \xi z_2 : \xi^2 z_3]$$

where ξ is a fixed primitive fourth root of unity. This transformation acts on $\mathbb{P}^3$ and preserves $\mathcal{Q}$.

Proposition 1.1 *The group $\langle g \rangle$ acts on $\mathcal{Q}$. $\mathcal{Q}/\langle g \rangle$ is a canonical K3 surface with four $\frac{1}{4}(3,1)$ and two $\frac{1}{2}(1,1)$ isolated quotient singularities.*

Proof Since $g(0 : z_1 : z_2 : 0) = (0 : \xi z_1 : \xi z_2 : 0)$, $\langle g \rangle$ fixes a line $z_0 = z_3 = 0$ in $\mathbb{P}^3$. Its intersection with $\mathcal{Q}$

$$\{z_0 = z_3 = 0\} \cap \mathcal{Q} = \{z_1^4 + z_2^4 = 0\} = \{(0 : 1 : \sqrt{i} : 0),\ (0 : 1 : i\sqrt{i} : 0),\ (0 : 1 : -\sqrt{i} : 0),\ (0 : 1 : -i\sqrt{i} : 0)\}$$

are four points which do not identify under $\langle g \rangle$ action. On affine piece $z_1 = 1$, $\mathcal{Q}$ has local parameters z_0 and z_3. Our group acts on it by $(z_0, z_3) \mapsto (\xi^3 z_0, \xi z_3)$ which gives a $\frac{1}{4}(3,1)$ singularity for each of four points.

$\langle g \rangle$ acts freely on

$$\{z_2 = z_3 = 0\} \cap \mathcal{Q} = \{z_0^4 + z_1^4 = 0\} = \{(1 : \sqrt{i} : 0 : 0),\ (1 : i\sqrt{i} : 0 : 0),\ (1 : -\sqrt{i} : 0 : 0),\ (1 : -i\sqrt{i} : 0 : 0)\}$$

since $g^3(1 : \sqrt{i} : 0 : 0) = g^2(1 : i\sqrt{i} : 0 : 0) = g(1 : -\sqrt{i} : 0 : 0) = (1 : -i\sqrt{i} : 0 : 0)$. Similarly $\langle g \rangle$ acts freely on $\{z_1 = z_3 = 0\}$, $\{z_0 = z_1 = 0\}$, $\{z_0 = z_2 = 0\}$.

$\langle g \rangle$ also fixes a line $z_1 = z_2 = 0$ in $\mathbb{P}^3$. Its intersection with $\mathcal{Q}$

$$\{z_1 = z_2 = 0\} \cap \mathcal{Q} = \{(1:0:0:\sqrt{i}),\ (1:0:0:i\sqrt{i}),\ (1:0:0:-\sqrt{i}),\ (1:0:0:-i\sqrt{i})\}$$

are four points of which two identify under $\langle g \rangle$ action:
$g^2(1:0:0:\sqrt{i}) = (1:0:0:-\sqrt{i})$ and $g^2(1:0:0:i\sqrt{i}) = (1:0:0:-i\sqrt{i})$.
On affine piece $z_0 = 1$, $\mathcal{Q}$ has local parameters z_1 and z_2. Action $(z_1, z_2) \mapsto (\xi^2 z_1, \xi^2 z_2)$ gives a $\frac{1}{2}(1,1)$ singularity for both points.

□

2 The toric description

This Section gives a toric description of the construction.

Let $N = \mathbb{Z}^3$ be a lattice and $M = \mathrm{Hom}(N, \mathbb{Z})$. Consider the polyhedron

$$\tilde{\Delta} = \left\{ \sum m_i \leq 1, m_i \geq -1 \right\} \subset M_{\mathbb{R}}$$

together with its dual polyhedron

$$\tilde{\Delta}^* = \mathrm{span}\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\} \subset N_{\mathbb{R}}.$$

The data $(M, \tilde{\Delta})$ defines a toric variety $\mathbb{P}_{M,\tilde{\Delta}}$. It is well–known that $\mathbb{P}_{M,\tilde{\Delta}} \cong \mathbb{P}^3$.

Proposition 2.1 *$\mathbb{P}^3/\langle g \rangle \cong \mathbb{P}_{M,\Delta}$, where $M = \mathbb{Z}^3$ and Δ is the polyhedron*

$$\Delta = \mathrm{span}\{(0,-1,1), (1,0,-1), (1,0,3), (-2,1,-3)\} \subset M_{\mathbb{R}}.$$

Proof Let U_0 be the affine piece of $\mathbb{P}^3$ with the first coordinate nonzero. We know that $U_0 \cong \mathbb{C}^3 \cong \mathrm{Spec}\ \mathbb{C}[x_1, x_2, x_3]$. g acts on this piece by

$$(x_1, x_2, x_3) \longmapsto (\xi x_1, \xi x_2, \xi^2 x_3).$$

From here we can calculate the affine piece of $\mathbb{P}^3/\langle g \rangle$

$$\mathrm{Spec}\ \mathbb{C}[x_1, x_2, x_3]^{\langle g \rangle} = \mathrm{Spec}\ \mathbb{C}[x_1^4, x_2^4, x_3^2, x_1x_2x_3, x_1^2x_2^2, x_1^2x_3, x_2^2x_3].$$

Similarly g acts on $U_1 \cong \mathrm{Spec}\ \mathbb{C}[x_1^{-1}, x_1^{-1}x_2, x_1^{-1}x_3]$ by

$$(x_1^{-1}, x_1^{-1}x_2, x_1^{-1}x_3) \longmapsto (\xi^3 x_1^{-1}, x_1^{-1}x_2, \xi x_1^{-1}x_3),$$

on $U_2 \cong \mathrm{Spec}\ \mathbb{C}[x_2^{-1}, x_2^{-1}x_1, x_2^{-1}x_3]$ by

$$(x_2^{-1}, x_2^{-1}x_1, x_2^{-1}x_3) \longmapsto (\xi^3 x_2^{-1}, x_2^{-1}x_1, \xi x_2^{-1}x_3)$$

and on $U_3 \cong \mathrm{Spec}\ \mathbb{C}[x_3^{-1}, x_3^{-1}x_1, x_3^{-1}x_2]$ by

$$(x_3^{-1}, x_3^{-1}x_1, x_3^{-1}x_2) \longmapsto (\xi^2 x_3^{-1}, \xi^3 x_3^{-1}x_1, \xi^3 x_3^{-1}x_2).$$

The corresponding affine pieces of $\mathbb{P}^3/\langle g \rangle$ are

$$\begin{array}{rcl}
\mathrm{Spec}\ \mathbb{C}[x_1^{-1}, x_1^{-1}x_2, x_1^{-1}x_3]^{\langle g \rangle} & = & \mathrm{Spec}\ \mathbb{C}[(x_1^{-1})^4, x_1^{-1}x_2, (x_1^{-1}x_3)^4, x_1^{-1}(x_1^{-1}x_3)] \\
\mathrm{Spec}\ \mathbb{C}[x_2^{-1}, x_2^{-1}x_1, x_2^{-1}x_3]^{\langle g \rangle} & = & \mathrm{Spec}\ \mathbb{C}[(x_2^{-1})^4, x_2^{-1}x_1, (x_2^{-1}x_3)^4, x_2^{-1}(x_2^{-1}x_3)] \\
\mathrm{Spec}\ \mathbb{C}[x_3^{-1}, x_3^{-1}x_1, x_3^{-1}x_2]^{\langle g \rangle} & = & \mathrm{Spec}\ \mathbb{C}[(x_3^{-1})^2, (x_3^{-1}x_1)^4, (x_3^{-1}x_2)^4, x_3^{-1}(x_3^{-1}x_1)(x_3^{-1}x_2), \\
 & & \quad x_3^{-1}(x_3^{-1}x_1)^2, x_3^{-1}(x_3^{-1}x_2)^2, (x_3^{-1}x_1)^2(x_3^{-1}x_2)^2]
\end{array}$$

Our aim is to find lattices $M' \supset M = \mathbb{Z}^3$, $N' \subset N = \mathbb{Z}^3$ and polyhedron Δ in M such that

- $M' = \mathrm{Hom}(N', \mathbb{Z})$
- $\mathbb{P}^3/\langle g \rangle \cong \mathbb{P}_{M,\Delta}$
- $\mathbb{P}^3 \cong \mathbb{P}_{M',\Delta}$.

We have to find a fan in $N' \subset N$ spanned on vectors $a,\ b,\ c$ and $d = -a-b-c$ for which

$$\mathbb{C}[\check{\sigma}_0 \cap M] = \mathbb{C}[x_1^4, x_2^4, x_3^2, x_1x_2x_3, x_1^2x_2^2, x_1^2x_3, x_2^2x_3]$$

holds. In other words we need a basis $\vec{x}_1,\ \vec{x}_2,\ \vec{x}_3$ for $\check{\sigma}_0$ in M' such that

$$4\vec{x}_1,\ 4\vec{x}_2,\ 2\vec{x}_3,\ \vec{x}_1+\vec{x}_2+\vec{x}_3,\ 2\vec{x}_1+2\vec{x}_2,\ 2\vec{x}_1+\vec{x}_3,\ 2\vec{x}_2+\vec{x}_3 \in M.$$

Using duality of $M \subset M'$ and $N' \subset N$, this is equivalent to the problem of finding a sublattice $N' \subset N$ generated by

$$4e_1,\ 4e_2,\ 2e_3,\ e_1+e_2+e_3,\ 2e_1+2e_2,\ 2e_1+e_3,\ 2e_2+e_3$$

where $e_1,\ e_2,\ e_3$ are basis for N. Obviously

$$N' = 4\mathbb{Z}e_1 + 4\mathbb{Z}e_2 + 2\mathbb{Z}e_3 + \mathbb{Z}(e_1+e_2+e_3) + 2\mathbb{Z}(e_1+e_2) + \mathbb{Z}(2e_1+e_3) + \mathbb{Z}(2e_2+e_3).$$

Its dual is

$$M' = \frac{1}{4}\mathbb{Z}(e_1+e_2+2e_3) + \mathbb{Z}e_3 + \mathbb{Z}e_2.$$

Choose $\vec{x}_1 = \frac{1}{4}(-1,-1,2),\ \vec{x}_2 = \frac{1}{4}(-1,-1,-2),\ \vec{x}_3 = \frac{1}{2}(1,-1,2)$ for a basis for $\check{\sigma}_0$ in M'. This is the right choice since

$$\mathbb{C}[\check{\sigma}_0 \cap M] = \mathbb{C}[x_1^4, x_2^4, x_3^2, x_1x_2x_3, x_1^2x_2^2, x_1^2x_3, x_2^2x_3].$$

The same way we check that for $\check{\sigma}_1 = \mathrm{span}\{-\vec{x}_1,\ \vec{x}_2-\vec{x}_1,\ \vec{x}_3-\vec{x}_1\}$,

$$\mathbb{C}[\check{\sigma}_1 \cap M] = \mathbb{C}[(x_1^{-1})^4, x_1^{-1}x_2, (x_1^{-1}x_3)^4, x_1^{-1}(x_1^{-1}x_3)].$$

The expected equalities also hold for $\check{\sigma}_2$ and $\check{\sigma}_3$.

Finally $\sigma_0 = (\check{\sigma}_0)^\vee = \{a,b,c\} \subset N$ where

$$a = (1,-1,0),\ b = (0,-2,-1),\ c = (-2,0,1).$$

This gives us $\sigma_1 = \{a,b,d\},\ \sigma_2 = \{a,c,d\}$ and $\sigma_3 = \{b,c,d\}$ with $d = -a-b-c = (1,3,0)$.

We proved that $\mathbb{P}^3/\langle g \rangle \cong \mathbb{P}_{\Sigma,N}$, where fan $\Sigma \subset N$ is generated by vectors $a,\ b,\ c$ and d. So polyhedron Δ^* equals

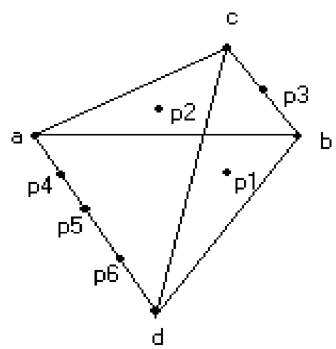
$$\Delta^* = \mathrm{span}\{a,\ b,\ c,\ d\}.$$

Easy calculation shows that Δ^* is reflexive and

$$\Delta = \mathrm{span}\{(0,-1,1),(1,0,-1),(1,0,3),(-2,1,-3)\}.$$

$\square$

From now on $\mathbb{P}^3/\langle g \rangle$ will be denoted by $\mathbb{P}_\Delta$. By Proposition 2.2.23 in [2] $\mathbb{P}_\Delta$ is Gornstein Fano variety.



- The only lattice point in the interior of Δ^* is the origin.
- $p_1 = (0, 1, 0)$ is a lattice point in the interior of two–dimensional face bcd and $p_2 = (0, -1, 0)$ is a lattice point in the interior of the face abc.
- $p_3 = (-1, -1, 0)$ is a lattice point in the interior of one–dimensional face bc and $p_4 = (1, 0, 0),\ p_5 = (1, 1, 0),\ p_6 = (1, 2, 0)$ lie in the interior of the face ad.

So the variety $\mathbb{P}_\Delta$ has two curves of canonical singularities corresponding to the one–dimensional faces of Δ^* and four singular points corresponding to two–dimensional faces of Δ^*.

Let $\bar{Z} \in |-K_{\mathbb{P}_\Delta}|$ be a generic K3 surface in $\mathbb{P}_\Delta$. Adding the lattice points $p_3,\ p_4,\ p_5$ and p_6 to the normal fan of Δ gives us simplified projective subdivision which we defined on page 2. We get the partial toric resolution

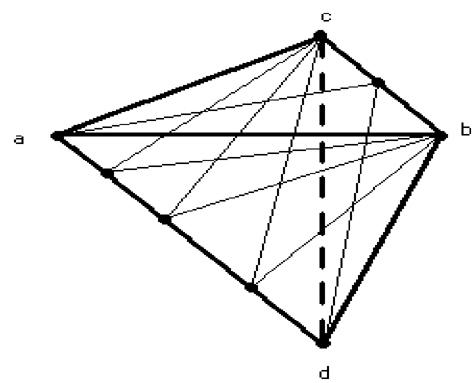
$$\mathbb{X}_\Sigma \to \mathbb{P}_\Delta$$

which induces crepant morphism $Z \to \bar{Z}$. We showed in the Introduction that proper transform Z is nonsingular.

3 Counterexample

Theorem 3.1 *The anticanonical surface Z in the toric variety $\mathbb{X}_\Sigma$ provides a counterexample to [1, Conjecture 6.2.8] of Cox and Katz. In the case constructed above the inclusion (♣) on page 3 is strict.*

PROOF Let Σ be a fan corresponding to simplified projective subdivision.



There exists a circuit $S = \{p_3, p_4, a\}$ supported on Σ with the linear relation $2p_4 + p_3 - a = 0$. Setting $S_+ = \{p_3, p_4\}$, it is easy to check that S is supported on Σ. The corresponding birational map π is a contraction of the divisor D_a given by the one–dimensional cone spanned by a. Easy calculation shows that the exceptional divisor $D_a = X(\mathrm{Star}\,(a))$ equals to $\mathbb{F}_4 = \mathbb{F}(4,0)$ and contracts to $X(\mathrm{Star}\,(bc)) = \mathbb{P}^1$

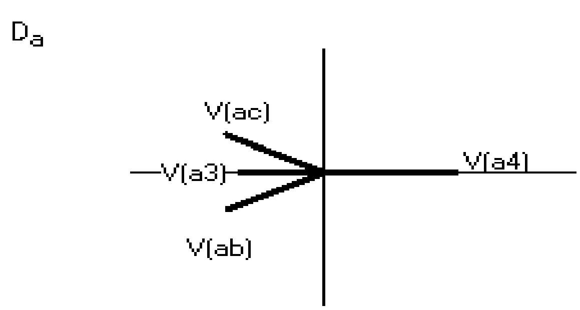
$$\begin{array}{ccc} \mathbb{X}_\Sigma & \xrightarrow{\pi} & \mathbb{X}_{\mathrm{Flip}\,\Sigma} \\ \cup & & \cup \\ \mathbb{F}_4 & \longmapsto & \mathbb{P}^1 \end{array}$$

This is not a trivial flip, so the corresponding face of $\mathrm{cpl}(\Sigma)$ is a face of $\bigcup_{\Sigma'} \mathrm{cpl}\,(\Sigma')$ from page 3.

Next we want to prove that

$$\pi|_Z \,:\, Z \longrightarrow \mathbb{X}_{\mathrm{Flip}\,\Sigma}$$

is an embedding. The only thing $\pi|_Z$ could contract is a curve $C = Z \cap \mathbb{F}_4$. All curves in $D_a = \mathbb{F}_4$ are numerically equivalent to linear combinations of $V(ab) = X(\mathrm{Star}\,(ab))$ and $V(a4) = X(\mathrm{Star}\,(a4))$. Their intersection numbers in D_a are $V(ab)^2 = 0,\ V(a4)^2 = 4,\ V(ab)V(a4) = 1$.



Writing $C \sim \alpha V(ab) + \beta V(a4)$,

$$-K_{\mathbb{X}_\Sigma} \sim D_a + D_b + D_c + D_d + D_3 + D_4 + D_5 + D_6 \overset{\square}{\sim} 4D_b + 2D_3$$

and using the calculations in Appendix we calculate

$$\begin{array}{rcl} (Z \cdot V(ab))_{\mathbb{X}_\Sigma} & = & (4D_b + 2D_3)V(ab) = 1 \\ \| & & \\ (C \cdot V(ab))_{D_a} & = & (\alpha V(ab) + \beta V(a4))V(ab) = \beta \\ & & \\ (Z \cdot V(a4))_{\mathbb{X}_\Sigma} & = & (4D_b + 2D_3)V(a4) = 4 \\ \| & & \\ (C \cdot V(a4))_{D_a} & = & (\alpha V(ab) + \beta V(a4))V(a4) = 4\beta + \alpha \end{array}$$

This proves that $C \sim V(a4)$. On the other hand we know that the only toric contraction from $\mathbb{F}_4$ to $\mathbb{P}^1$ is trivial projection. It contracts exactly all curves in the class of a fibre $V(ab)$. So $\pi|_Z$ maps C isomorphically to a curve in $\mathbb{X}_{\mathrm{Flip}\,\Sigma}$.

This however implies that the corresponding face is *not* in the boundary of $\mathcal{K}(Z)$. Thus the cone $\mathcal{K}(Z)$ is strictly larger than $\bigcup_{\Sigma'} \mathrm{cpl}\,(\Sigma')$ indeed, as claimed. $\square$

4 GKZ Decomposition

In the previous Section we discovered that in order to describe $\mathcal{K}(Z)$, it is not enough to consider all the cones $\mathrm{cpl}\,(\Sigma')$, where Σ' is obtained from Σ by a sequence of trivial flips. In this Section we will take another cone from GKZ decomposition which contains some divisors D such that $D|_Z$ are ample. We will check if then the whole cone is contained in $\mathcal{K}(Z)$.

The Chow group $A_{n-1}(\mathbb{X}_\Sigma)$ of Weil divisors modulo linear equivalence can be computed directly from the fan Σ. We have an exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(\mathbb{X}_\Sigma) \to 0.$$

Since $\mathbb{X}_\Sigma$ is simplicial $A_{n-1}(\mathbb{X}_\Sigma) \otimes \mathbb{R} = \mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma)$. In our case this is a five dimensional vector space with the following equivalence relations

$$\begin{array}{l} D_b \sim D_c, \\ 2D_a - 2D_d + D_4 - D_6 \sim 0, \\ -D_a - 2D_b + 3D_d - D_3 + D_5 + 2D_6 \sim 0. \end{array} \qquad (\square)$$

The theorem of Oda and Park [4] gives precise description of GKZ decomposition:

Theorem 4.1 *Let Ξ be a finite set of strongly convex rational 1–dimensional cones in $N_{\mathbb{R}}$. As Π ranges over all projective simplicial fans with $\Pi(1) \subset \Xi$, the cones* $\mathrm{cpl}(\Pi)$ *and their faces form a fan in $A_{n-1}(\mathbb{X}_\Sigma)$ supported on effective divisor classes.*

$\mathrm{cpl}(\Pi)$ can be computed by formula

$$\mathrm{cpl}(\Pi) = \{D \text{ effective divisor } : D \text{ is } \Pi\text{-convex}\}.$$

We say that $D = \sum a_i D_i$ is Π*–convex* if $\langle m_\sigma, v_i \rangle \geq -a_i$ for all σ in fan Π. Here v_i generate $\Pi(1)$ and $m_\sigma \in M_{\mathbb{R}}$ satisfies $\langle m_\sigma, v_i \rangle = -a_i$ whenever $v_i \in \sigma$.

In $\mathbb{X}_\Sigma$ we first take $\Xi = \{a, b, c, d, p_3, p_4, p_5, p_6\}$ and $\Pi = \Sigma$. Using the above formula we get

$$\begin{aligned}\mathrm{cpl}(\Sigma) = \{\alpha_b D_b + \alpha_d D_d + \alpha_3 D_3 + \quad & \alpha_5 D_5 + \alpha_6 D_6 \,:\, \alpha_5 \geq 0, \alpha_b - 2\alpha_3 \geq 0, \alpha_3 \geq 0, \\ & \alpha_6 - 2\alpha_5 \geq 0, \alpha_d + \alpha_5 - 2\alpha_6 \geq 0, -\alpha_d + \alpha_3 + 2\alpha_6 \geq 0\}.\end{aligned}$$

The vertices of cone $\mathrm{cpl}(\Sigma)$ in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma)$ are

$$3D_d + D_5 + 2D_6,\ 4D_d + D_5 + 2D_6,\ 2D_d + D_6,\ D_b,\ 2D_b + D_3,\ 2D_b + D_d + D_3.$$

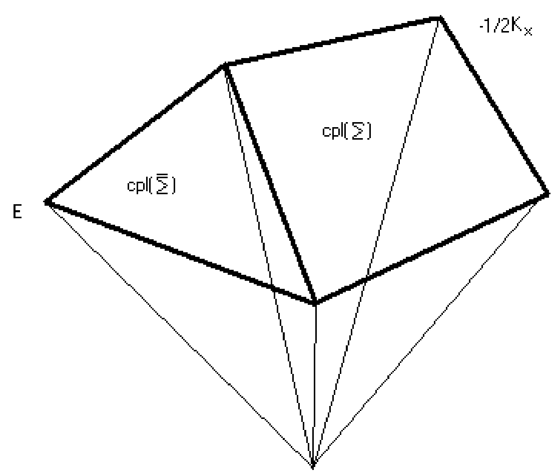
It is well known fact that $\mathrm{cpl}(\Sigma)$ is the closure of nef cone $\mathcal{K}(\mathbb{X}_\Sigma)$.

Next we take for Π a simplicial fan $\bar{\Sigma}$ generated by $\{b, c, d, p_3, p_4, p_5, p_6\}$ In this case

$$\begin{aligned}\mathrm{cpl}(\bar{\Sigma}) = \{\alpha_b D_b + \alpha_d D_d + \alpha_3 D_3 + \alpha_5 D_5 + \alpha_6 D_6 \,:\quad & \alpha_b - 2\alpha_3 \geq 0, \alpha_3 \leq 0, \alpha_6 - 2\alpha_5 \geq 0, \\ & \alpha_d + \alpha_5 - 2\alpha_6 \geq 0, -\alpha_d + \alpha_3 + 2\alpha_6 \geq 0\}\end{aligned}$$

and its vertices in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{X}_\Sigma)$ are

$$3D_d + D_5 + 2D_6,\ 4D_d + D_5 + 2D_6,\ 2D_d + D_6,\ D_b,\ -2D_b + 3D_d - D_3 + D_5 + 2D_6.$$



Note that $\mathrm{cpl}(\Sigma)$ and $\mathrm{cpl}(\bar{\Sigma})$ touch along the common face of codimension one in GKZ decomposition. This is exactly the face corresponding to contraction $\mathbb{X}_\Sigma \xrightarrow{\pi} \mathbb{X}_{\mathrm{Flip}\,\Sigma}$. Restrictions of divisors in this face to Z are ample since $\pi|_Z$ is embedding. One would expect that then the restrictions of the whole cone $\mathrm{cpl}(\bar{\Sigma})$ to Z are nef divisor classes. Choose $E = -2D_b + 3D_d - D_3 + D_5 + 2D_6 \in \mathrm{cpl}(\bar{\Sigma})$ and a curve $C \subset Z$ equivalent to $V(a4)$. As we proved in Section 3 we can take $C = Z \cap D_a \sim V(a4)$. Using calculations in Appendix we get

$$E|_Z \cdot C = (-2D_b + 3D_d - D_3 + D_5 + 2D_6 \cdot V(a4))_{\mathbb{X}_\Sigma} = -2 < 0$$

This however implies that $E|_Z$ is not nef.

Appendix: Intersection calculation

On a variety $\mathbb{X}_\Sigma$ the Chow group $A_1(\mathbb{X}_\Sigma)$ is defined to be the free abelian group on the 1-dimensional irreducible subvarieties of $\mathbb{X}_\Sigma$, modulo the subgroup generated by the cycles of the form $\mathrm{div}(f)$, where f is a rational function on a 2-dimensional subvariety of $\mathbb{X}_\Sigma$. By Proposition 5.1 in [3] $A_1(\mathbb{X}_\Sigma)$ is generated by the classes of the orbit closures $V(\sigma)$ of 2-dimensional cones in Σ. In our case we get

- $V(ab), V(ac), V(a3), V(a4) \subset D_a$ are related by $V(ac) \sim V(ab)$, $V(a3) \sim V(a4) - 4V(ab)$
- $V(b3), V(ab), V(b4), V(b5), V(b6), V(bd) \subset D_b$ are related by $V(ab) + V(b4) + V(b5) + V(b6) + V(bd) - V(b3) \sim 0$, $-V(ab) + V(b5) + 2V(b6) + 3V(bd) - V(b3) \sim 0$
- $V(c3), V(ac), V(c4), V(c5), V(c6), V(cd) \subset D_c$ are related by $V(ac) + V(c4) + V(c5) + V(c6) + V(cd) - V(c3) \sim 0$, $-V(ac) + V(c5) + 2V(c6) + 3V(cd) - V(c3) \sim 0$
- $V(bd), V(cd), V(d3), V(d6) \subset D_d$ are related by $V(bd) \sim V(cd)$, $V(d3) \sim V(d6) - 4V(bd)$
- $V(a3), V(b3), V(c3), V(d3) \subset D_3$ are related by $V(c3) \sim V(b3)$, $V(d3) \sim V(a3)$
- $V(a4), V(b4), V(c4), V(45) \subset D_4$ are related by $V(b4) \sim V(c4)$, $V(45) \sim V(a4) + 2V(b4)$
- $V(c5), V(b5), V(45), V(56) \subset D_5$ are related by $V(c5) \sim V(b5)$, $V(45) \sim V(56)$
- $V(b6), V(c6), V(d6), V(56) \subset D_6$ are related by $V(b6) \sim V(c6)$, $V(d6) \sim V(56) + 2V(b6)$

$A_1(\mathbb{X}_\Sigma) \otimes \mathbb{R}$ is so a five-dimensional vector space generated by $V(ab)$, $V(a4)$, $V(b4)$, $V(bd)$, $V(b3)$. There exists an intersection product between $A_{n-1}(\mathbb{X}_\Sigma)$ and $A_1(\mathbb{X}_\Sigma)$ defined by $D \cdot C = \deg_C D|_C$ and extended by linearity. In [3, Section 5.1] we find explicit description how to calculate this intersections, e.g.

$$D_a \cdot V(ab) = (-2D_c + 3D_d - D_3 + D_5 + 2D_6)V(ab) = 0 + 0 - \frac{\mathrm{mult}(ab)}{\mathrm{mult}(ab3)} V(ab3) + 0 + 0 = -\frac{1}{2}.$$

$$\begin{array}{llll}
D_a \cdot V(ab) = -\frac{1}{2} & D_b \cdot V(ab) = 0 & D_c \cdot V(ab) = 0 & D_d \cdot V(ab) = 0 \\
D_a \cdot V(b4) = 1 & D_b \cdot V(b4) = 0 & D_c \cdot V(b4) = 0 & D_d \cdot V(b4) = 0 \\
D_a \cdot V(bd) = 0 & D_b \cdot V(bd) = 0 & D_c \cdot V(bd) = 0 & D_d \cdot V(bd) = -\frac{1}{2} \\
D_a \cdot V(b3) = \frac{1}{2} & D_b \cdot V(b3) = 0 & D_c \cdot V(b3) = 0 & D_d \cdot V(b3) = \frac{1}{2} \\
D_a \cdot V(a4) = -2 & D_b \cdot V(a4) = 1 & D_c \cdot V(a4) = 1 & D_d \cdot V(a4) = 0
\end{array}$$

$$\begin{array}{llll}
D_3 \cdot V(ab) = \frac{1}{2} & D_4 \cdot V(ab) = 1 & D_5 \cdot V(ab) = 0 & D_6 \cdot V(ab) = 0 \\
D_3 \cdot V(b4) = 0 & D_4 \cdot V(b4) = -2 & D_5 \cdot V(b4) = 1 & D_6 \cdot V(b4) = 0 \\
D_3 \cdot V(bd) = \frac{1}{2} & D_4 \cdot V(bd) = 0 & D_5 \cdot V(bd) = 0 & D_6 \cdot V(bd) = 1 \\
D_3 \cdot V(b3) = 1 & D_4 \cdot V(b3) = 0 & D_5 \cdot V(b3) = 0 & D_6 \cdot V(b3) = 0 \\
D_3 \cdot V(a4) = 0 & D_4 \cdot V(a4) = 4 & D_5 \cdot V(a4) = 0 & D_6 \cdot V(a4) = 0
\end{array}$$

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