DETERMINANTAL REPRESENTATIONS OF CUBIC CURVES

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Abstract. For every (irreducible) cubic curve we explicitly construct all nonequivalent determinantal representations. The results of Vinnikov [4], [5] for smooth curves are extended to all cubics. We give a new proof of the criteria when a self-adjoint determinantal representation is definite.

1. Introduction

In this paper we study determinantal representations of plane cubic curves defined over an algebraically closed field k.

Every irreducible curve can be brought by a projective change of coordinates into the Weierstrass form

$$yz^2 = x^3 + pxy^2 + qy^3, \quad p, q \in k$$

or equivalently

$$yz^2 = x(x + \theta_1 y)(x + \theta_2 y), \quad \theta_1, \theta_2 \in k$$

and every reduced curve is projectively equivalent to one of the

$$x^3,\ x^2y,\ xy(x+y),\ xyz \ \text{ or } \ (\alpha x + \beta y + \gamma z)(x^2 - yz) \ \text{ for some } \ \alpha, \beta, \gamma \in k. \tag{1}$$

Determinantal representations of a smooth irreducible plane curve F can be parametrised by all but finitely many points of the Jacobian variety of F. The Jacobian of a smooth cubic is the curve itself. It follows that nonequivalent determinantal representations of a curve of degree 3 correspond to the affine points on the curve [4].

For singular curves the notion of the Jacobian can not be used. It turns out that we can still parametrise all determinantal representations of a cubic (up to equivalence via $\mathrm{GL}(3, k)$ action by left and right multiplycation) by affine points of the curve.

Rank of the determinantal representation is 2 in the smooth points of F and changes in singular points. In other words, the line bundle defined as the kernel of the determinantal representation needs to be replaced by Cohen-Macaulay sheaves.

For $k = \mathbb{C}$, all self-adjoint determinantal representations of real cubics are found. We give necessary and sufficient condition when there exist points in $\mathbb{R}^3$ in which a self-adjoint representation is positive definite. This condition can be expressed as the sign of some parameter t that can be easily verified in practise.

All necessary definitions will be introduced at the beginning of each section.

2. Cubic curves up to projective equivalence

Let $\mathcal{C}$ be a plane curve defined by a cubic polynomial $F(x, y, z)$ over k, an algebraically closed field. It is well known that every curve in $\mathbb{P}^2$ of degree ≥ 3 has an inflection point. By a projective change of coordinates we can assume that the inflection point is $(0,0,1)$ and the inflection tangent is $y=0$. This is equivalent to saying that $F(x,y,z)$ has no term in z^3, xz^2 or x^2z [3, Chapter I.1.5.]. This means

$$F(x,y,z) = ayz^2 + (bx+cy)yz - g(x,y),$$

where g is a polynomial in x,y and $a,b,c \in k$.

If $a \neq 0$ we can set new coordinates $z \mapsto z - \frac{1}{2a}(bx+cy)$ and $y \mapsto \frac{y}{a}$. This reduces the equation to the form

$$yz^2 - g(x,y),$$

where $g(x,y) = \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3$, $\alpha,\beta,\gamma,\delta \in k$. If $\alpha \neq 0$, we can assume $\alpha = 1$. Transformation $x \mapsto x - \frac{1}{3}\beta y$ brings it to the Weierstrass normal form

$$yz^2 = x^3 + pxy^2 + qy^3, \quad p,q \in k.$$

If $\alpha = 0$ the curve is reducible and projectively equivalent to one of the following:

$$yz^2 = x^2 y, \quad yz^2 = x(x+y)y, \quad yz^2 = xy^2, \quad yz^2 = y^3.$$

If $a = 0$ the inflection point is singular. Then we can set new coordinates

$$\begin{array}{lll} x_1 = bx + cy & \text{if } b \neq 0 & \text{or} \\ y_1 = cy & \text{if } b = 0, & \end{array}$$

which bring the equation into the form

$$xyz - g(x,y) \ \text{ or } \ y^2 z - g(x,y).$$

By $z \mapsto z + \beta x + \gamma y$ we transform

$$xyz = \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 \ \text{ into } \ xyz = \alpha x^3 + \delta y^3.$$

Thus g reduces to one of the forms:

$$xyz = \begin{cases} x^3 + y^3, \\ x^3, \\ y^3, \\ 0. \end{cases}$$

The same way $z \mapsto z + \gamma x + \delta y$ transforms

$$y^2 z = \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 \ \text{ into } \ y^2 z = \alpha x^3 + \beta x^2 y.$$

If $\alpha \neq 0$ we can assume $\alpha = 1$. Next make the right hand side into the cube $(x + \frac{\beta}{3}y)^3$ by $z \mapsto z - \frac{\beta^2}{3}x - \frac{\beta^3}{27}y$. Thus g reduces to either:

$$y^2 z = \begin{cases} x^3, \\ x^2 y, \\ 0. \end{cases}$$

It is easy to chech that the change of coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -\frac{1}{2}x + \frac{1}{6}y - \frac{1}{2}z \\ -\frac{1}{2}x + \frac{1}{6}y + \frac{1}{2}z \\ 3x + 3y \end{pmatrix}$$

makes $xyz = x^3 + y^3$ projectively equivalent to

$$yz^2 = x^3 - \frac{1}{3}xy^2 + \frac{2}{27}y^3.$$

The same way interchanging y and z makes $y^2z = x^3$ projectively equivalent to

$$yz^2 = x^3.$$

The above curves are both in the Weierstrass form.

Thus we proved:

Proposition 2.1. *Every irreducible cubic curve $F(x, y, z)$ can be brought by a projective change of coordinates into the form*

$$yz^2 = x^3 + pxy^2 + qy^3,$$

where $p, q \in k$.

Any of the reduced curves considered in this section is projectively equivalent to one in (1).

3. Determinantal representations

In this section we determine all determinantal representations of the curves from Proposition 2.1.

Let $\mathcal{C}$ be a projective cubic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 3 over k. A *determinantal representation* of $\mathcal{C}$ is a 3×3 linear matrix

$$M(x, y, z) = xA + yB + zC$$

satisfying

$$\det M = cF,$$

where $A, B, C \in \mathrm{GL}(3, k)$ and $0 \neq c \in k$.

Two determinantal representations M and M' are *equivalent* if there exist $X, Y \in \mathrm{GL}(3, k)$ such that

$$M' = XMY.$$

Proposition 3.1. *The nonequivalent determinantal representations of*

$$F(x, y, z) = -z^2y + x^3 + pxy^2 + qy^3$$

can be parametrised by the affine points $(x, 1, z)$ on F.

Proof. Consider triples $(A,\ B,\ C)$ of constant matrices, such that

$$\det(xA + yB + zC) = cF(x, y, z).$$

The group action

$$(A,\ B,\ C) \sim (XAY,\ XBY,\ XCY) \qquad X, Y \in \mathrm{GL}(3, k)$$

preserves the equivalence of determinantal representations. Since $\det A = 1$ we can assume that $A = I$ and $Y = X^{-1}$. Inserting $x = -\lambda,\ y = 0,\ z = 1$ in the above equation shows that the characteristic polynomial of C is λ^3. Thus $C \sim XCX^{-1}$ can be brought to the Jordan form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since F contains the monomial z^2y the nilpotent must has Jordan cell. Using the computer program **Mathematica 5.0** we get

$\mathbf{Expand}[\mathbf{Det}[\{\{\mathbf{x,0,0}\},\{\mathbf{0,x,0}\},\{\mathbf{0,0,x}\}\}+\{\{\mathbf{0,z,0}\},\{\mathbf{0,0,z}\},\{\mathbf{0,0,0}\}\}+$
$\mathbf{y}\{\{\mathbf{b11,b12,b13}\},\{\mathbf{b21,b22,b23}\},\{\mathbf{b31,b32,b33}\}\}]]$

$\mathtt{x^3+b11x^2z+b22x^2z+b33x^2z-b21xyz-b32xyz+b31y^2z-}$
$\mathtt{b12b21xz^2+b11b22xz^2-b13b31xz^2-b23b32xz^2+b11b33xz^2+b22b33xz^2+}$
$\mathtt{b12b31yz^2+b23b31yz^2-b11b32yz^2-b21b33yz^2-}$
$\mathtt{b13b22b31z^3+b12b23b31z^3+b13b21b32z^3-b11b23b32z^3-b12b21b33z^3+b11b22b33z^3}$

This implies $b31=-1,\ b21+b32=0,\ b11+b22+b33=0$. The identity and the nilpotent are preserved by the action of

$$X=\begin{pmatrix}1&\xi&\zeta\\0&1&\xi\\0&0&1\end{pmatrix}\quad\text{and}\quad Y=X^{-1}.$$

We will show that a for suitable choice of ξ,ζ matrix B can be brought into the form

$$XBX^{-1}=\begin{pmatrix}\frac{t}{2}&l&p+\frac{3}{4}t^2\\0&-t&-l\\-1&0&\frac{t}{2}\end{pmatrix}$$

where $l^2=t^3+pt+q$. Indeed, in **Mathematica 5.0** we compute

$\mathbf{Simplify}[\mathbf{ReplaceAll}[\{\{\mathbf{1},\xi,\zeta\},\{\mathbf{0,1},\xi\},\{\mathbf{0,0,1}\}\}\cdot$
$\{\{\mathbf{b11,b12,b13}\},\{\mathbf{b21,b22,b23}\},\{\mathbf{b31,b32,b33}\}\}\cdot\mathbf{Inverse}[\{\{\mathbf{1},\xi,\zeta\},\{\mathbf{0,1},\xi\},\{\mathbf{0,0,1}\}\}],$
$\{\mathbf{b31}\to-\mathbf{1},\mathbf{b21}\to-\mathbf{b32},\mathbf{b11}\to-\mathbf{b22}-\mathbf{b33},\xi\to-\mathbf{b32},\zeta\to(-\mathbf{b22}-\mathbf{2b33}+\mathbf{b32^2})/\mathbf{2}\}]]$

$\mathtt{\{\{1/2(-b22+b32^2),b12-2b22b32+b32^3-b32b33,b13+}$
$\mathtt{1/4(-b22^2+4b12b32-4b23b32-6b22b32^2+3b32^4-4b22b33-4b33^2)\},}$
$\mathtt{\{0,b22-b32^2,b23-b32(-b22+b32^2+b33)\},\{-1,0,1/2(-b22+b32^2)\}\}}$

Moreover, two different matrices of the above form can not be equivalent.

Observe that (l,t) is exactly an affine point on F. Thus we found a 1-1 correspondence between determinantal representations of F and the affine points $(x,1,z)$ of F. □

Note that F is singular if and only if

$$4p^3+27q^2=0.$$

For $p=q=0$ the singularity $(0,1,0)$ is a cusp:

$$F(x,1,z)=-z^2+x^3.$$

For $p\neq 0$ and $q\neq 0$ the singularity $(-\frac{3q}{2p},1,0)$ is a node:

$$F(x,1,z)=-z^2-\frac{9q}{2p}\left(x+\frac{3q}{2p}\right)^2+\left(x+\frac{3q}{2p}\right)^3.$$

Lemma 3.2 ([4], Proposition 1)**.** *If (a_x,a_y,a_z) is a regular point of F, then the determinantal representation $M(a_x,a_y,a_z)$ has rank 2 at (a_x,a_y,a_z).*

Alternatively, ker$M(x,y,z)$ is a vector bundle of rank 1 along the smooth points of F.

Observe that the affine curve $y\neq 0$ contains the singularity of F. Determinantal representation corresponding to the singular point changes rank in the singularity:

For $F(x,1,z) = -z^2 + x^3$ with the cusp $(0,1,0)$ we get

$$M(0,1,0) = \begin{pmatrix} \frac{t}{2} & l & \frac{3}{4}t^2 \\ 0 & -t & -l \\ -1 & 0 & \frac{t}{2} \end{pmatrix}.$$

Thus $M(x,y,z)$ evaluated in $(0,1,0)$ has rank 1 if $t = 0$ and rank 2 if $t \neq 0$.

For $F(x,1,z)$ with $p, q \neq 0$ and the node $(-\frac{3q}{2p}, 1, 0)$ we get

$$M(-\frac{3q}{2p}, 1, 0) = \begin{pmatrix} \frac{t}{2} - \frac{3q}{2p} & l & p + \frac{3}{4}t^2 \\ 0 & -t - \frac{3q}{2p} & -l \\ -1 & 0 & \frac{t}{2} - \frac{3q}{2p} \end{pmatrix}.$$

In this case $M(x,y,z)$ evaluated in $(-\frac{3q}{2p}, 1, 0)$ has rank 1 if $t = -\frac{3q}{2p}$ and rank 2 otherwise.

Alternatively, consider the kernel of the determinantal representation corresponding to the singular point on F. It defines a Cohen-Macaulay sheaf of rank 1 along the smooth points and of rank 2 along the singular points of F. On the other hand, the kernel of the determinantal representation corresponding to a regular point on F has rank 1 along the whole F.

Example 3.3. Determinantal representations of reduced curves can also be calculated by the above method. Consider for example

$$x(x-y)(x+y) = 0.$$

The zero locus are three lines intersecting in a point. Up to equivalence their determinantal representations are

$$\begin{pmatrix} x & 0 & 0 \\ 0 & x+y & 0 \\ 0 & 0 & x-y \end{pmatrix} + z \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix},$$

where

$$c_{12}c_{21} - c_{13}c_{31} = 0, \ c_{12}c_{21} + c_{13}c_{31} + c_{23}c_{32} = 0, \ c_{12}c_{23}c_{31} + c_{21}c_{32}c_{13} = 0.$$

Next multiply the representation by a diagonal matrix $P = \operatorname{diag}(p_1, p_2, p_3)$ from the left and by P^{-1} from the right. This preserves the matrix of x and y and acts on

$$c_{ij} \ \mapsto \ \frac{p_i}{p_j} c_{ij}.$$

We can this way reduce at most two of the nonzero c_{ij} to 1. We are left with a finite number of one parameter families of determinantal representations.

4. Self-adjoint determinantal representations

From now on let the ground field be $\mathbb{C}$. A determinantal representation

$$M(x,y,z) = xA + yB + zC$$

is *self-adjoint* if A, B, C are complex hermitean matrices. Naturally M and M' are *hermitean equivalent* if there exists $P \in \mathrm{GL}(3,\mathbb{C})$ such that $M' = PMP^*$.

Note that the determinant of a self-adjoint representation is a real polynomial and thus defines a real curve.

In this section we show that every real irreducible cubic curve has a self-adjoint determinantal representation. In [5] we find the following

Theorem 4.1. *The complete set of nonequivalent self-adjoint determinantal representations of a real Weierstrass cubic*

$$z^2y = x^3 + pxy^2 + qy^3, \quad p, q \in \mathbb{R}$$

is

$$x\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y\begin{pmatrix} p+\frac{3}{4}t^2 & l & \frac{t}{2} \\ -l & -t & 0 \\ \frac{t}{2} & 0 & -1 \end{pmatrix},$$

where t is real, l purely imaginary and

$$l^2 = t^3 + pt + q.$$

Let $-\theta$ be the biggest real zero of $x^3+pxy^2+qy^3 = 0$. The real projective change of coordinates $x \mapsto x - \theta y$ brings our cubic to

$$z^2y = x(x+\theta_1 y)(x+\theta_2 y)$$

where $\theta_1,\ \theta_2$ are either real nonpositive numbers or complex conjugates.

It is easy to check that the curve is singular if and only if $\theta_1\theta_2(\theta_1 - \theta_2) = 0$ in which case it is projectively equivalent to one of the following:

$$z^2y = x^3, \qquad z^2y = x^2(x-y), \qquad z^2y = x(x-y)^2.$$

The first curve has a cusp in $(0,1,0)$ and the other two have a node in $(0,1,0)$ and $(1,1,0)$ respectively.

Using the above notation, we can rewrite Theorem 4.1 into

Theorem 4.2. *The complete set of (nonequivalent) self-adjoint determinantal representations of a real Weierstrass cubic*

$$z^2y = x(x+\theta_1 y)(x+\theta_2 y)$$

is

$$\begin{aligned} M_{s,t}(x,y,z) = \ & x\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \\ & y\begin{pmatrix} \frac{3}{4}t^2 + \frac{1}{2}t(\theta_1+\theta_2) - \frac{1}{4}(\theta_1-\theta_2)^2 & is & \frac{1}{2}(\theta_1+\theta_2+t) \\ -is & -t & 0 \\ \frac{1}{2}(\theta_1+\theta_2+t) & 0 & -1 \end{pmatrix} \end{aligned}$$

where t, s are real and

$$-s^2 = t(t+\theta_1)(t+\theta_2).$$

The proof of the above theorems is similar to Proof 3.1 and works for all irreducible cubics.

The set of nonequivalent self-adjoint determinantal representations of F can thus be parametrised by affine points on F whose first coordinate is real and the second coordinate is purely imaginary. Figures 1 and 2 show all different types of irreducible cubic curves.

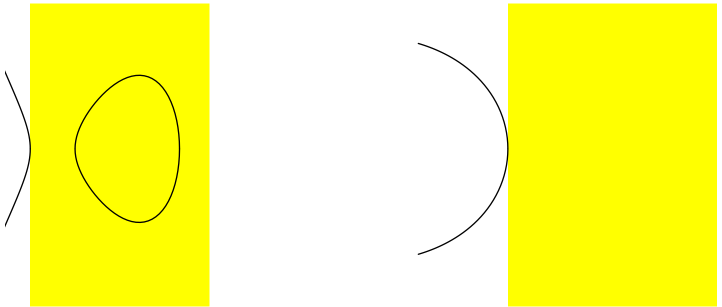

FIGURE 1. Smooth cubics $-s^2 = t(t+\theta_1)(t+\theta_2)$ with $\theta_1, \theta_1 \in \mathbb{R}$ or $\theta_1 = \bar{\theta_1}$.

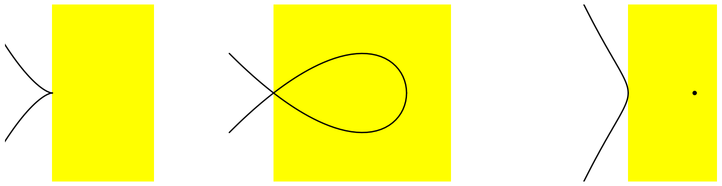

FIGURE 2. Singular cubics $-s^2 = t^3,\ -s^2 = t^2(t-1),\ -s^2 = t(t-1)^2$.

5. Definite detrminantal representations

In this section we investigate when do three complex hermitean matrices admit a real linear combination that is positive definite.

Definition 5.1. A self-adjoint determinantal representation $M(x, y, z) = xA+yB+zC$ is called *definite* if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha A + \beta B + \gamma C > 0$$

and is *indefinite* otherwise.

A vector (a, b, c) is *self-orthogonal* for M if

$$(a, b, c) \cdot M(x, y, z) \cdot \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} = 0 \quad \text{for all } x, y, z.$$

Note that the above definitions are projectively invariant. Definitness is also preserved under equivalent representations. When M has a self-orthogonal vector, it is clearly indefinite.

In Section 4 we proved that every irreducible real cubic is projectively equivalent to

$$z^2 y = x(x + \theta_1 y)(x + \theta_2 y)$$

with either $\theta_2 \leq \theta_1 \leq 0$ real or complex conjugates. Moreover, the real part of the complex conjugates is 0, or it can be made 1 or -1 by

$$y \mapsto \frac{1}{|\mathrm{Re}\theta_1|} y, \qquad z \mapsto \sqrt{|\mathrm{Re}\theta_1|} z.$$

From now on we restrict to cubics satisfying the above assumptions.

Theorem 5.2. *Let $\mathcal{C}$ be a real cubic*

$$z^2y = x(x+\theta_1 y)(x+\theta_2 y).$$

Determinantal representations $M_{s,t}$ from Theorem 4.2 are definite for $t>0$.

Proof. Observe from Figure 1 and 2 that t parametrises the affine points of the real part of $\mathcal{C}$. The shaded areas correspond to $t>0$. Then $\mathcal{C}$ has definite representations if and only if $0 \neq \theta_2 \in \mathbb{R}$. More precisely, for a definite $M_{s,t}$ the parameter t takes values on the interval $[-\theta_1, -\theta_2]$.

Choose $t>0$. We will find real α, β, γ such that

$$(a,b,c)\cdot M_{s,t}(\alpha,\beta,\gamma)\cdot \begin{pmatrix} \bar a \\ \bar b \\ \bar c \end{pmatrix} > 0 \tag{2}$$

for all nontrivial vectors (a,b,c). Assume that $\beta=-1$. Then (2) equals

$$a\bar a(-t^2+(\theta_1+\theta_2-\alpha)(\alpha-t)-\theta_1\theta_2)+a\bar b(\gamma-is)+\bar a b(\gamma+is)+b\bar b(\alpha+t)$$

$$+(c-a\tfrac{1}{2}(\theta_1+\theta_2+t-2\alpha))(\bar c-\bar a\tfrac{1}{2}(\theta_1+\theta_2+t-2\alpha))$$

If we can find $\alpha,\gamma\in\mathbb{R}$ such that the system

$$\begin{array}{rcl} v\bar v &=& -t^2+(\theta_1+\theta_2-\alpha)(\alpha-t)-\theta_1\theta_2,\\ w\bar w &=& \alpha+t,\\ v\bar w &=& \gamma-is,\\ \bar v w &=& \gamma+is \end{array} \tag{3}$$

has a solution for some v and w in $\mathbb{C}$, then (2) equals

$$(av+bw)(\overline{av+bw})+(c-a\frac{1}{2}(\theta_1+\theta_2+t-2\alpha))(\bar c-\bar a\frac{1}{2}(\theta_1+\theta_2+t-2\alpha)).$$

In this case (2) is positive which concludes the proof.

It remains to prove the existence of v,w in (3). Write

$$v=r_v(\cos\varphi_v+i\sin\varphi_v) \quad \text{and} \quad w=r_w(\cos\varphi_w+i\sin\varphi_w)$$

and substitute $\alpha=r_w^2-t$ in the first equation. This transforms (3) into

$$\begin{aligned} r_v^2 = -&\left(r_w^2-\tfrac{1}{2}\left(\theta_1+\theta_2+3t-\sqrt{(\theta_1-\theta_2)^2-2t(\theta_1+\theta_2)-3t^2}\right)\right)\\ &\left(r_w^2-\tfrac{1}{2}\left(\theta_1+\theta_2+3t+\sqrt{(\theta_1-\theta_2)^2-2t(\theta_1+\theta_2)-3t^2}\right)\right), \end{aligned}$$

$$r_v r_w(\cos(\varphi_v-\varphi_w)+i\sin(\varphi_v-\varphi_w))=\gamma-is.$$

This will have a solution if and only if for every $t>0$ on the interval $[-\theta_1,-\theta_2]$ exist positive $r_v,\ r_w$ such that

$$\begin{aligned} r_v^2 = -&\left(r_w^2-\tfrac{1}{2}\left(\theta_1+\theta_2+3t-\sqrt{(\theta_1-\theta_2)^2-2t(\theta_1+\theta_2)-3t^2}\right)\right)\\ &\left(r_w^2-\tfrac{1}{2}\left(\theta_1+\theta_2+3t+\sqrt{(\theta_1-\theta_2)^2-2t(\theta_1+\theta_2)-3t^2}\right)\right) \end{aligned}$$

and $r_v r_w \ \geq\ |s|$.

The first equation is a quadric in r_w^2. It is easy to see that its graph runs through the first quadrant. Indeed, for $t \in [-\theta_1, -\theta_2]$ the value of

$$\frac{1}{2}\left(\theta_1 + \theta_2 + 3t + \sqrt{(\theta_1 - \theta_2)^2 - 2t(\theta_1 + \theta_2) - 3t^2}\right)$$

strictly increases from $-\theta_1$ to $-\theta_2$. Thus for the first equality can be always fulfilled.

Consider now $r_v^2 r_w^2$ which can be seen as a cubic in r_w^2. It is straightforward to compute that its local maximum in the first quadrant is

$$\frac{1}{27}(\theta_1+\theta_2)(2\theta_1^2+2\theta_2^2-5\theta_1\theta_2)+\frac{2}{27}(\theta_1^2+\theta_2^2-\theta_1\theta_2)^{\frac{3}{2}}-t(t+\theta_1)(t+\theta_2).$$

This means $r_v^2 r_w^2 \;\geq\; s^2 = -t(t+\theta_1)(t+\theta_2)$ if and only if

$$\frac{1}{27}(\theta_1+\theta_2)(2\theta_1^2+2\theta_2^2-5\theta_1\theta_2)+\frac{2}{27}(\theta_1^2+\theta_2^2-\theta_1\theta_2)^{\frac{3}{2}} \geq 0.$$

For $\theta_2 \leq \theta_1 \leq 0$ the last inequality always holds which finishes the proof. □

Theorem 5.3. *Let $\mathcal{C}$ be a real cubic*

$$z^2 y = x(x+\theta_1 y)(x+\theta_2 y).$$

Determinantal representations $M_{s,t}$ from Theorem 4.2 are indefinite for $t \leq 0$. In particular, for every $M_{s,t}$ with $t \leq 0$ there exists a self-orthogonal vector.

Proof. For every representation $M_{s,t}$ with $t \leq 0$ we will find a self-orthogonal vector (a,b,c). In particular (a,b,c) is self-orthogonal for all three matrices in the representation. This yields

$$\begin{gathered} a\bar{c}+\bar{a}c+b\bar{b}=0, \\ a\bar{b}+\bar{a}b=0, \\ a\bar{a}\left(\tfrac{3}{4}t^2+\tfrac{1}{2}t(\theta_1+\theta_2)-\tfrac{1}{4}(\theta_1-\theta_2)^2\right)+\tfrac{1}{2}(a\bar{c}+\bar{a}c)(\theta_1+\theta_2+t)+is(a\bar{b}-\bar{a}b)-b\bar{b}t-c\bar{c}=0. \end{gathered}$$

We can always take $a=1$, which implies

$$b = i\beta \quad\text{and}\quad c = -\frac{1}{2}\beta^2 + i\gamma \quad\text{for some}\quad \beta,\ \gamma \in \mathbb{R}.$$

The last equation then becomes

$$-\beta^4 - 2\beta^2(\theta_1+\theta_2+3t) + 8\beta s - 4\gamma^2 - (\theta_1-\theta_2)^2 + t(2\theta_1+2\theta_2+3t).$$

This is a polynomial function $h(\beta)$, which is negative for big β.

First consider the case $\theta_1+\theta_2 \leq 0$. Then for $\gamma=0$ and β such that

$$\beta^2 = -\theta_1 - \theta_2 \quad\text{and}\quad \beta s \geq 0$$

the value of $h(\beta)$ is positive since all its summands are positive:

$$4\theta_1\theta_2 + 8\beta s + 8t(\theta_1+\theta_2) + 3t^2.$$

For all $t \leq 0$ we found values of β in which $h(\beta)$ is positive and negative. Therefore there exist β and $\gamma=0$ for which $h(\beta)=0$.

Recall that the coordinaes of $\mathcal{C}$ were chosen so that

$$\theta_2 \leq \theta_1 \leq 0 \quad \text{if} \quad \theta_1, \theta_2 \in \mathbb{R}$$

or there exists $\vartheta > 0$ such that either

$$\theta_1 = \begin{cases} i\vartheta, \\ 1+i\vartheta, \\ -1+i\vartheta \end{cases} \quad \text{if} \quad \theta_1 = \overline{\theta_2} \notin \mathbb{R}.$$

The only remaining case that needs to be considered is $\theta_1 = 1+i\vartheta$. Again set $\gamma = 0$. Now $h(\beta)$ equals

$$-\beta^4 - 2\beta^2(2+3t) + 8\beta s + 4\vartheta^2 + t(4+3t).$$

In **Mathematica 5.0** we can draw the implicit function $h(\beta, t) = 0$ for various $\vartheta > 0$. From the graph we observe that the function is defined for all $t < 0$. ☐

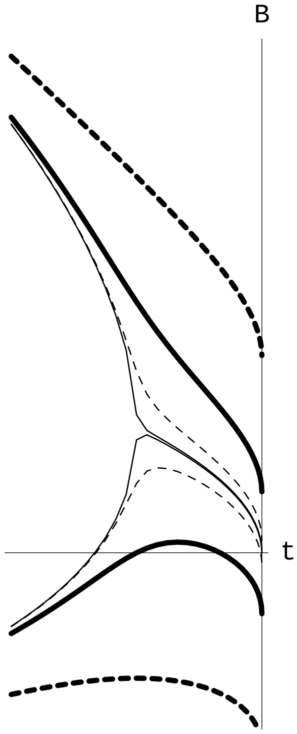

FIGURE 3. — $\vartheta = 0.005$, - - - $\vartheta = 0.1$, — $\vartheta = 0.5$, - - - $\vartheta = 2$.

Example 5.4. Consider a nodal curve

$$z^2y = x(x-y)^2.$$

The complete set of self-adjoint determinantal representations is

$$M_{s,t} = x\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y\begin{pmatrix} \frac{3}{4}t^2 - t & is & \frac{t}{2} - 1 \\ -is & -t & 0 \\ \frac{t}{2} - 1 & 0 & -1 \end{pmatrix}$$

where t, s are real and

$$-s^2 = t(t-1)^2.$$

From Figure 2 we see that $M_{s,t}$ is definite for $t = 1,\ s = 0$ and indefinite for $t \leq 0$. Indeed, check that

$$M_{0,1} = \begin{pmatrix} -\frac{y}{4} & z & x - \frac{y}{2} \\ z & x - y & 0 \\ x - \frac{y}{2} & 0 & -y \end{pmatrix}$$

is definite for $x = -1,\ y = -4,\ z = 1$.

On the other hand, for every $t \leq 0$ find a self-orthogonal vector for $M_{s,t}$ of the form

$$\left(1,\, i\beta,\, -\frac{\beta}{2}\right), \quad \beta \in \mathbb{R}.$$

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