CONSTRUCTION OF SELF-ADJOINT DETERMINANTAL REPRESENTATIONS OF SMOOTH CUBIC SURFACES

Anita Buckley and Tomaž Košir

Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

`tomaz.kosir@fmf.uni-lj.si(T. Košir)`

Abstract

We consider a smooth cubic surface S and its determinantal representations. The equivalence classes of determinantal representations correspond to sixes of skew lines on S. There are 72 such sixes of lines on S and thus there are 72 nonequivalent determinantal representations of S. The aim of our paper is to provide two procedures for computation of determinantal representations of cubic surfaces. For smooth real cubic surface we also construct self-adjoint and definite determinantal representations when they exist. For the first procedure we assume that S is given as a blow-up of six points in a projective plane and for the second that we are given equations of a line on S or an equation a tritangent plane of S. The key step in the constructions is computation of explicit equations of all the 27 lines on S. Exact computations are possible if S is given as a blow-up of six points or if we are given an equation of a tritangent plane. It is known that if we are given a defining polynomial for S then, in general, computation of a line or a tritangent plane requires transcendental methods since the Galois group of the corresponding equation is not solvable. One can then use transcendental methods introduced by Klein and Coble or use numerical methods and approximate computations instead of exact ones.

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Presenting Author's Biography

Tomaž Košir is a professor of mathematics at the Faculty of Mathematics and Physics, University of Ljubljana. He did his Ph.D. in 1993 at the University of Calgary in Canada. After that he spent two years at Dalhousie University, Halifax, Canada as a postdoctoral fellow. Later, he was a visiting professor at the University of Southern California in Los Angeles and at Dalhousie University. His main research interest are in algebra, geometry, and their applications in partial differential equations.

1 Introduction

We consider a smooth cubic surface S in $\mathbb{P}^3 = \mathbb{P}^3(\mathbb{C})$ given by equation

$$F(z_0, z_1, z_2, z_3) = 0,$$

where F is a homogeneous cubic polynomial over $\mathbb{R}$ or $\mathbb{C}$. If F is real then we say that S is a real cubic surface. It is well known that a smooth cubic surface S contains 27 lines. A plane intersecting S in three lines is called a *tritangent plane.* Every line on S lies exactly on 5 tritangent planes and there are 45 tritagent planes for S. We refer to Henderson [9] or Reid [13] for the geometry of the 27 lines. A great source for the geometry of real cubic surfaces is Segre [16]. One can find chapters on cubic surfaces also in Shafarevich [17] and Dolgachev [6].

The most elegant way to study curves on S (our particular interest will be in lines) is by defining S as a blow up of 6 points in the plane, no three collinear and not on a conic. Every nonsingular cubic surface in $\mathbb{P}^3(\mathbb{C})$ can be obtained this way [8].

A *determinantal representation* of a smooth cubic surface is a 3×3 matrix of linear forms

$$M = M(z_0, z_1, z_2, z_3) = z_0M_0+z_1M_1+z_2M_2+z_3M_3$$

satisfying $\det M = cF$, where $M_0, M_1, M_2, M_3 \in \mathrm{M}_3(\mathbb{C})$ and $c \in \mathbb{C}$, $c \neq 0$. Two determinantal representations M and M' are *equivalent* if there exist $X, Y \in \mathrm{GL}_3(\mathbb{C})$ such that $M' = XMY$.

It is known that a smooth cubic surface S allows exactly 72 nonequivalent determinantal representations. In fact, there is a one-to-one correspondence between:

- equivalence classes of determinantal representations of S,
- linear systems of twisted cubic curves on S,
- sets of six lines on S that do not intersect each other.

This was most likely known in the 19th century (see [3, 5, 14]). Since it is hard to find a modern reference we provided a proof in [2]. There we also studied self-adjoint and definite determinantal representations of smooth real cubic surfaces.

A determinantal representation $M = z_0M_0 + z_1M_1 + z_2M_2 + z_3M_3$ is *self-adjoint* if $M_j^* = M_j$ for all j. Two self-adjoint determinantal representations M and M' are *equivalent* if there exist $X \in \mathrm{GL}_3(\mathbb{C})$ such that $M' = XMX^*$. A self-adjoint determinantal representation is *definite* if there exist $c_0,\ c_1,\ c_2,\ c_3 \in \mathbb{R}$ such that the matrix $c_0M_0+c_1M_1+c_2M_2+c_3M_3$ is positive-definite. All smooth cubic surfaces are divided into 5 types according to the geometry of the corresponding 27 lines (see Segre [16]). The number of nonequivalent self-adjoint determinantal representations depends on the Segre type F_i, $i = 1, \ldots, 5$, of S. A surface of type F_i, $i = 1, \ldots, 4$ has exactly $2(i-1)$ nonequivalent self-adjoint determinantal representations none of which is definite, while a surface of type F_5 has 24 nonequivalent self-adjoint determinantal representations, 16 of which are definite [2].

The main topic of our current presentation is explicit construction of determinantal representations of a smooth cubic surface and in particular of all self-adjoint and definite representations when they exist. We discuss two procedures. In the first, we assume that we are given six points in the plane such that S is blow-up at these six points. In the second, we assume that we are given equations of a line on S or an equation of a tritangent plane to S. It is known that if we are given a defining polynomial F then in general it is not possible to express a line on S by radicals since the corresponding Galois group is not solvable. We refer to Hunt [10] for a nice review of the classical construction of a line on S using transcendental methods of Klein and Coble. One could use Coble's hexahedral form of S (see Coble's original papers [4] or Hunt [10]) to find explicit equations of a line.

Our motivation to study explicit constructions of determinantal representations comes from possible application to multiparameter spectral theory [11]. Important motivation to study self-adjoint and definite determinantal representations is provided by Vinnikov [19, 20]. Such representations appear as determinantal representations of discriminant varieties in the theory of commuting nonselfadjoint operators in a Hilbert space [12]. Cubic surfaces are used also in modeling with algebraic surfaces [1, 15, 18].

2 Algorithms if given six points of a blow-up

A possible construction of a cubic surface is by blowing-up of six points in general position in a projective plane $\mathbb{P}^2$ (see e.g. [7, 8, 17]). Suppose that $\mathbb{X} = \{P_1, P_2, \ldots, P_6\}$ is a set of six points in $\mathbb{P}^2$, no three collinear and not on a conic. Denote by x_0, x_1, x_2 the homogeneous coordinates of $\mathbb{P}^2$. Then it is easy to see that the vector space of all cubic forms f in x_0, x_1, x_2 such that $f(P_i) = 0$ for $i = 1, 2, \ldots, 6$, is four dimensional. Assume that f_0, f_1, f_2, f_3 form a basis for this vector space. Then we consider the rational mapping

$$\Phi : \mathbb{P}^2 \to \mathbb{P}^3$$

given by

$$\Phi(Q) = [f_0(Q), f_1(Q), f_2(Q), f_3(Q)].$$

It is well defined at every point away from $\mathbb{X}$. It gives a blow-up constructions at points in $\mathbb{X}$. The closure of its image is a smooth cubic surface S. We refer to Geramita [7] for elementary proofs of the above statements. The 27 lines on S are then the following (see e.g. [8, Theorem V.4.8.]):

- $a_1, \ldots, a_6$ are the exceptional lines of the blow-up Φ,

- $c_{ij} = c_{ji}$ is the strict transform of the line through P_i and P_j in $\mathbb{P}^2$, where $1 \leq i < j \leq 6$.
- $b_1, \ldots, b_6$, with b_j being the strict transform of the plane conic through the five $P_i,\ i \neq j$.

Observe that $a_1, \ldots, a_6$ are mutually skew, $b_1, \ldots, b_6$, are mutually skew and a_i intersects b_j if and only if $i \neq j$. Every configuration of 12 lines on S with this property is called a *Schläfli's double-six.* Every smooth cubic surface S contains 36 double-sixes of lines. The 27 lines have a high degree of symmetry: for any set $l_1, \ldots, l_6$ of mutually skew lines on S there exist 6 points in $\mathbb{P}^2$ and a blow-up for which $l_1, \ldots, l_6$ are the exceptional lines. Proof of this can be found in [8, Proposition V.4.10.]. These lines then uniquely determine another set of 6 mutually skew lines to form together a double-six. Using the above notation the double-sixes on S are:

$$\begin{pmatrix} a_1 & \ldots & a_6 \\ b_1 & \ldots & b_6 \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} a_i & b_i & c_{kl} & c_{km} & c_{kn} & c_{kp} \\ a_k & b_k & c_{il} & c_{im} & c_{in} & c_{ip} \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} a_i & a_k & a_l & c_{mn} & c_{mp} & c_{np} \\ c_{kl} & c_{il} & c_{ik} & b_p & b_n & b_m \end{pmatrix}. \tag{3}$$

Here i, k, l, m, n, p are all distinct.

The 27 lines lie in triples on 45 tritangent planes. The coplanar triples of lines are

$$a_i \quad b_j \quad c_{ij} \quad \text{and} \quad c_{ik} \quad c_{lm} \quad c_{np} \tag{4}$$

for distinct i, j and i, k, l, m, n, p, respectively.

Next we give an explicit procedure to find equations for the 27 lines given a set of six points $\mathbb{X}$ in general position.

Suppose that

$$L_{ij} = L_{ij}(s,t) = [l_{0ij}(s,t), l_{1ij}(s,t), l_{2ij}(s,t)],$$

where l_{kij} are linear forms in s and t, is a parametrization of line through P_i and P_j, $i \neq j$, in $\mathbb{P}^2$. Assume that

$$L_{ij}(1,0) = P_i \text{ and } L_{ij}(0,1) = P_j. \tag{5}$$

Then $[f_0(L_{ij}), \ldots, f_3(L_{ij})]$ is a parametrization of c_{ij} in $\mathbb{P}^3$. From this it is easy to find explicit equations of the c_{ij}. The assumption (5) and the choice of f_k imply that each $f_k(L_{ij}(s,t))$ is divisible by both s and t. Denote by

$$m_{kij}(s,t) \tag{6}$$

the remaining linear factor of $f_k(L_{ij}(s,t))$. Observe that

$$m_{kij}(1,0) \text{ and } m_{kij}(0,1) \tag{7}$$

are points on lines a_i and a_j, respectively. Using these points for various k we can determine the equations for the lines a_i. Since b_j is the intersection of the planes $\langle a_i, c_{ij}\rangle \cap \langle a_k, c_{kj}\rangle$ for two distinct i and k we can obtain equations for the lines b_j as well. Here we denote by $\langle a, b\rangle$ the plane spanned by two intersecting lines a and b in $\mathbb{P}^3$.

The procedure to find all the nonequivalent determinantal representations of a surface given by the blow-up of points in $\mathbb{X}$ is then the following:

Algorithm 2.1 *Given* $\mathbb{X} = \{P_1, P_2, \ldots .P_6\}$.

Find a basis f_0, f_1, f_2, f_3.

For all pairs of distinct i, j *parametrise the line through* P_i *and* P_j. *Compute equations for the line* c_{ij}.

Find linear factors m_{kij} *and compute equations for the lines* a_i.

Find equations for 45 tritangent planes. The lines that span them are given in (4).

Among the equations of the tritagent planes find equations for the lines b_j.

For each of the double-sixes in the list (1)-(3) do: If

$$\begin{pmatrix} a_1 & \ldots & a_6 \\ b_1 & \ldots & b_6 \end{pmatrix} \tag{8}$$

is a double-six, then consider the tritangent planes

$$\pi_{12}, \pi_{23}, \pi_{31}, \pi_{13}, \pi_{21}, \pi_{32}, \tag{9}$$

where $\pi_{ij} = \langle b_i, a_j\rangle$. *Use a point outside the lines* a_i *and* b_j *to determine* λ *such that*

$$F = \pi_{12}\pi_{23}\pi_{31} + \lambda\pi_{13}\pi_{21}\pi_{32}. \tag{10}$$

Modify one of the equations π_{ij} *so that* $\lambda = 1$. *Then*

$$\Re = \begin{pmatrix} 0 & \pi_{12} & \pi_{13} \\ \pi_{21} & 0 & \pi_{23} \\ \pi_{31} & \pi_{32} & 0 \end{pmatrix} \tag{11}$$

and

$$\Re^T = \begin{pmatrix} 0 & \pi_{21} & \pi_{31} \\ \pi_{12} & 0 & \pi_{32} \\ \pi_{13} & \pi_{23} & 0 \end{pmatrix} \tag{12}$$

are two nonequivalent determinantal representations corresponding to (8).

Example 2.2 We used `Mathematica 5.0` (software for symbolic computations) to compute the steps of the above procedure. Since a complete list of equations of lines, tritangent planes and determinantal representations is large we include only a small sample.

Suppose that $\mathbb{X} = \{P_1, P_2, \ldots .P_6\}$ is the set of points

$$\{[1,0,0], [0,1,0], [0,0,1], [1,1,1], [1,2,3], [1,3,7]\}.$$

It is easy to verify that these points are in general position. We choose the cubic forms

$$\begin{aligned} f_0 &= 4x_0^2x_1 - 6x_0^2x_2 + x_0x_1^2 + x_0x_1x_2 \\ f_1 &= 9x_0x_1^2 + 15x_0^2x_1 - 25x_0^2x_2 + x_0x_2^2 \\ f_2 &= 10x_0x_1^2 + 19x_0^2x_1 - 30x_0^2x_2 + x_1^2x_2 \\ f_3 &= 40x_0x_1^2 + 55x_0^2x_1 - 96x_0^2x_2 + x_1x_2^2 \end{aligned}$$

determined by

$$f_i(P_j) = 0, \ \text{ for } \ i = 0, 1, 2, 3, \ j = 1, 2, \ldots, 6,$$

which define a blow up $\mathbb{P}^2 \to \mathbb{P}^3$. The line through P_1 and P_2 in $\mathbb{P}^2$ is parametrized by $[s, t, 0]$. Its direct transform is the line $c_{12} = c_{21}$ parametrized by

$$st[4s + t, 15s + 9t, 19s + 10t, 55s + 40t].$$

Let z_0, z_1, z_2, z_3 be the coordinates of $\mathbb{P}^3$. It is now easy to calculate the equations of c_{12}:

$$\begin{array}{rcl} z_1 + z_2 - z_3 & = & 0 \\ 5z_1 - 5z_2 + z_4 & = & 0. \end{array}$$

Similarly,

the line P_1P_3 is parametrized by $[s, 0, t]$,

the line P_2P_3 is parametrized by $[0, s, t]$,

and the line P_4P_5 is parametrized by $[s + t, s + 2t, s + 3t]$. Their direct transforms c_{13}, c_{23}, c_{45} are parametrized by

$$st[-6s, -25s + t, -30s, -96s],$$
$$st[0, 0, s, t],$$
$$st[-3(s + t), -13(s + t), -17s - 19t, -52s - 56t],$$

respectively. The corresponding equations in $\mathbb{P}^3$ are

$$\begin{array}{lrcl} c_{13}: & 5z_1 - z_3 & = & 0 \\ & 16z_1 - z_4 & = & 0, \\ & & & \\ c_{23}: & z_1 & = & 0 \\ & z_2 & = & 0, \\ & & & \\ c_{45}: & 13z_1 - 3z_2 & = & 0 \\ & 6z_1 + 2z_3 - z_4 & = & 0. \end{array}$$

From the parametrisations of c_{ij} we can read the linear forms m_{kij} defined in (6) and calculate points given by (7) that lie on various a_j. Therefore

$[4, 15, 19, 55]$ and $[6, 25, 30, 96]$ are points on a_1,

$[1, 9, 10, 40]$ and $[0, 0, 1, 0]$ are points on a_2,

$[0, 1, 0, 0]$ and $[0, 0, 0, 1]$ are points on a_3.

Then a_1, a_2, a_3 are given by the equations

$$\begin{array}{lrcl} a_1: & 25z_1 + 6z_2 - 10z_3 & = & 0 \\ & 65z_1 - 54z_2 + 10z_4 & = & 0, \\ & & & \\ a_2: & 9z_1 - z_2 & = & 0 \\ & 40z_1 - z_4 & = & 0, \\ & & & \\ a_3: & z_1 & = & 0 \\ & z_3 & = & 0. \end{array}$$

Recall the tritangent planes listed in (9). From the lines constructed above we find that their equations are

$$\begin{array}{lcl} \pi_{21} =< a_1, c_{12} > & \equiv & 4z_1 - 6z_2 + z_3 + z_4 = 0 \\ \pi_{31} =< a_1, c_{13} > & \equiv & 29z_1 - 9z_3 + z_4 = 0 \\ \pi_{32} =< a_2, c_{23} > & \equiv & 9z_1 - z_2 = 0 \\ \pi_{12} =< a_2, c_{12} > & \equiv & 5z_1 - 5z_2 + z_4 = 0 \\ \pi_{13} =< a_3, c_{13} > & \equiv & 5z_1 - z_3 = 0 \\ \pi_{23} =< a_3, c_{23} > & \equiv & z_1 = 0. \end{array}$$

It is easy to check that the point $[3, 13, -6, 6]$ lies on c_{45} and on no other line of S. This determines λ in (10). Indeed, $\lambda = -1$ is the solution of $\pi_{12}\pi_{23}\pi_{31} + \lambda\pi_{13}\pi_{21}\pi_{32} = 0$ evaluated at $[3, 13, -6, 6]$.

Then the two nonequivalent determinantal representations corresponding to

$$\begin{pmatrix} a_1 & \ldots & a_6 \\ b_1 & \ldots & b_6 \end{pmatrix}$$

are

$$\begin{pmatrix} 0 & 5z_1 - 5z_2 + z_4 & -5z_1 + z_3 \\ 4z_1 - 6z_2 + z_3 + z_4 & 0 & -z_1 \\ 29z_1 - 9z_3 + z_4 & -9z_1 + z_2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 4z_1 - 6z_2 + z_3 + z_4 & 29z_1 - 9z_3 + z_4 \\ 5z_1 - 5z_2 + z_4 & 0 & 9z_1 - z_2 \\ 5z_1 - z_3 & -z_1 & 0 \end{pmatrix}$$

both defining

$$\begin{array}{rl} F = & -35z_1^3 + 145z_1^2z_2 - 30z_1z_2^2 - 54z_1^2z_3 \\ & -8z_1z_2z_3 + 6z_2^2z_3 + 9z_1z_3^2 - z_2z_3^2 \\ & -11z_1^2z_4 - z_2z_3z_4 + z_1z_4^2 \end{array}$$

as their determinant.

In the same way all 45 tritangent planes, 27 lines, 36 double-sixes and 72 determinantal representations on the surface given by F can be computed. $\square$

Next we recall from [2] the results on existence and number of self-adjoint and definite determinantal representations of real cubic surfaces.

A double-six $\begin{pmatrix} a_1 & \ldots & a_6 \\ b_1 & \ldots & b_6 \end{pmatrix}$ is called *mutually self-conjugate* if $\{b_1, \ldots, b_6\} = \{\overline{a_1}, \ldots, \overline{a_6}\}$ as sets. Here $\overline{a}$ is the line obtained from line a by conjugation of all points of a. After a suitable permutation of indexes, a mutually self-conjugate double-six is one of the following 4 kinds: a double-six *of the I-st kind* is of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix},$$

a double-six *of the II-nd kind* is of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_3} & \overline{a_4} & \overline{a_5} & \overline{a_6} \end{pmatrix},$$

a double-six *of the III-rd kind* is of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_5} & \overline{a_6} \end{pmatrix}$$

and a double-six *of the IV-th kind* is of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} & \overline{a_6} & \overline{a_5} \end{pmatrix}.$$

All mutually self-conjugate double-sixes are specified by

Type	F_1	F_2	F_3	F_4	F_5
Number_{Kind}	0	1_I	2_{II}	3_{III}	12_{IV}

The four kinds of mutually self-conjugate double-sixes were introduced by Cremona [5]. See [16] for further details on double-sixes and types of real cubic surfaces.

Every mutually self-conjugate double-six induces two nonequivalent self-adjoint determinantal representations. A real cubic surface has the following number of nonequivalent self-adjoint determinantal representations:

Type of the surface	F_1	F_2	F_3	F_4	F_5
Number of s.a. reps	0	2	4	6	24

Only real cubic surfaces of type F_5 have definite determinantal representations. Each surface of type F_5 has up to equivalence 16 definite determinantal representations (among the 24 nonequivalent self-adjoint determinantal representations).

The procedure to determine self-adjoint and definite determinantal representations (when they exits) of a real cubic surface is an extension of the procedure to compute all nonequivalent determinantal representations. For each of mutually self-conjugate double-six we find two nonequivalent self-adjoint determinantal representations. Given a determinantal representation M corresponding to a mutually self-conjugate double-six as in Algorithm 2.1 one has to additionally find a matrix $X \in \mathrm{GL}_4(\mathbb{C})$ such that $XM = M^*X^*$. Such X always exists.

A cubic surface given by a blow-up of a set $\mathbb{X}$ in $\mathbb{P}^2$ is real if $\mathbb{X}$ is invariant under complex conjugation. The type of a real cubic surface then depends on the number of complex conjugate pairs in $\mathbb{X}$. It is F_i, $i = 1, 2, 3$, if there are $i-1$ pairs of complex conjugate points in $\mathbb{X}$, and either F_4 or F_5 if $\mathbb{X}$ consists of three complex conjugate pairs of points. This follows from the number of real lines on real cubic surfaces of various types [16].

Since the set $\mathbb{X}$ in Example 2.2 consists of real points the corresponding surface is of type F_1 and it has no self-adjoint representations. Next we give an example of a surface of type F_2 to illustrate the procedure of finding self-adjoint determinantal representations.

Example 2.3 Suppose that $\mathbb{X}$ contains the following six points:

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], [1, i, 1-i]$$

and

$$[1, -i, 1+i].$$

It is obvious that it is invariant under complex conjugation. One can easily check that these points are in general position. We choose the cubic forms

$$\begin{aligned}
f_0 &= -4x_0^2x_1 - x_0^2x_2 + 2x_0x_1^2 + 3x_0x_1x_2,\\
f_1 &= x_0^2x_1 - x_0^2x_2 - x_0x_1^2 + x_0x_2^2\\
f_2 &= -2x_0^2x_1 - 2x_0x_1^2 + x_0^2x_2 + 3x_1^2x_2\\
f_3 &= -x_0^2x_1 + x_0x_1^2 - x_0^2x_2 + x_1x_2^2
\end{aligned}$$

for a basis of cubic forms defining the blow-up and such that

$$f_i(P_j) = 0, \text{ for } i = 0, 1, 2, 3, \ j = 1, 2, \ldots, 6.$$

The corresponding cubic surface contains one self-conjugate double-six which is of the I-st kind. It is

$$\begin{pmatrix} a_5 & b_5 & c_{16} & c_{26} & c_{36} & c_{46} \\ a_6 & b_6 & c_{15} & c_{25} & c_{35} & c_{45} \end{pmatrix},$$

where $a_6 = \overline{a_5}, b_6 = \overline{b_5}$ and $c_{i6} = \overline{c_{i5}}$ for $i = 1, 2, 3$.

The line through P_1 and P_5 in $\mathbb{P}^2$ is parametrized by $[s+t, it, (1-i)t]$ and its direct transform c_{15} is parametrized by

$$\begin{aligned}
st[&(-1-3i)(s+t), (-1+2i)(s+t),\\
&(1-3i)s + (4-6i)t, -s-3t].
\end{aligned}$$

Its equations are

$$\begin{aligned}
-z_0 + 2z_2 + (3-3i)z_3 &= 0\\
z_0 + (1-i)z_1 &= 0.
\end{aligned}$$

Similarly, we see that lines P_2P_5 and P_3P_5 are parametrized as follows:

$$\begin{aligned}
&P_2P_5 \text{ by } [t, s+it, (1-i)t],\\
&P_3P_5 \text{ by } [t, it, s+(1-i)t].
\end{aligned}$$

Their direct transforms c_{25}, c_{35} are parametrised by

$$st[2s-(1-i)t, -s+(1-2i)t, (1-3i)s+(4+2i)t, s-t]$$

and

$$st[(-1+3i)t, s+(1-2i)t, (-2t, is+(1+2i)t],$$

respectively. The corresponding equations in $\mathbb{P}^3$ are

$$\begin{aligned}
c_{25}: \quad -5z_0 + (3-9i)z_1 + (4+3i)z_2 &= 0\\
(3-i)z_0 + (1-2i)z_1 - 5z_3 &= 0,\\
\\
c_{35}: \quad z_0 + (1+2i)z_1 + (-2+i)z_3 &= 0\\
2z_1 - (1+i)z_2 + 2iz_3 &= 0.
\end{aligned}$$

We will also use a point on a line c_{12}. It is a direct transform of the line P_1P_2, which is parametrized by $[s, t, 0]$. Then c_{12} is parametrized by

$$st[-4s + 2t, s-t, -2(s+t), -s+t].$$

Its equations are

$$\begin{aligned}
2z_0 + 6z_1 - z_2 &= 0\\
z_1 + z_3 &= 0
\end{aligned}$$

and $[3,-1,0,1]$ is a point on it.

The tritangent planes are listed in (4). We need the tritangent planes $\sigma_{ij} = \langle c_{j5}, \overline{c_{i5}}\rangle = \langle c_{j5}, c_{i6}\rangle$ for $i,j \in \{1,2,3\}$. From the lines constructed above we find that their equations are

$$\begin{aligned}
\sigma_{12}: &\quad (2-2i)z_0 + 3z_1 + (-1+i)z_2 - 3z_3 = 0,\\
\sigma_{13}: &\quad (1-2i)z_0 + 3z_1 + (-1+i)z_2 + 3iz_3 = 0,\\
\sigma_{23}: &\quad z_0 + 3iz_1 + z_2 - 3z_3 = 0.
\end{aligned}$$

An equation of the corresponding cubic surface S is then of the form

$$\sigma_{12}\sigma_{23}\overline{\sigma_{13}} + \lambda\overline{\sigma_{12}\sigma_{23}}\sigma_{13} = 0.$$

We use a point on S to determine λ. Note that we always have $|\lambda| = 1$. If θ is such that $\theta^2 = \lambda$ then $\theta^{-1} = \overline{\theta}$ and

$$\sigma_{12}\sigma_{23}(\overline{\theta}\overline{\sigma_{13}}) + \overline{\sigma_{12}\sigma_{23}}(\theta\sigma_{13}) = 0$$

is an equation of our surface that gives two self-adjoint determinantal representations

$$\begin{pmatrix} 0 & \sigma_{12} & \theta\sigma_{13} \\ \overline{\sigma_{12}} & 0 & \sigma_{23} \\ \overline{\theta}\overline{\sigma_{13}} & \overline{\sigma_{23}} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \overline{\sigma_{12}} & \overline{\theta}\overline{\sigma_{13}} \\ \sigma_{12} & 0 & \overline{\sigma_{23}} \\ \theta\sigma_{13} & \sigma_{23} & 0 \end{pmatrix}.$$

Using the point $[3,-1,0,1]$ we find that $\lambda = 1$ and we choose $\theta = 1$. We compute the determinant and find that the polynomial

$$\begin{aligned}
F = \quad & 2z_0^3 + 9z_0^2z_1 + 3z_0^2z_2 - 9z_0^2z3 \\
& +27z_0z_1^2 - 72z_0z_1z_2 - 9z_0z_2^2 + 9z_0z_3^2 \\
& -81z_1^2z_2 + 108z_1z_2^2 + 27z_1z_2z_3
\end{aligned}$$

defines our surface. □

3 Algorithms if given a line or a tritangent plane on S

The problem of computing equations of a line on a cubic surface if given a defining polynomial F is, in general, not solvable by radicals. Transcendental methods for finding equations of a line were described by Klein and Coble [4, 10]. Numerical methods to compute equations of a line are given by Szilágy in her Ph.D. thesis [18].

The procedure to find all nonequivalent determinantal representations from an equation $F = 0$ for S is therefore not as direct as it was for their construction from the six points of the blow-up. However, once we obtain an equation of a tritangent plane to S we can explicitly compute equations of all tritangent planes. Then we proceed as in Algorithm 2.1.

If we are given an equation of a line then we still need to solve an equation of degree 5 to find an equation of a tritangent plane. Once we have an equation of a tritangent plane it is possible to compute exactly equations for all 27 lines and 45 tritangent planes.

Suppose we know the tritangent plane π_{21} and the lines a_1, b_2 and c_{12}. Since all the tritangent planes that contain a given line on S can be computed explicitly by solving a quintic equation [13, pp. 106-107] we can explicitly find the remaining four tritangent planes that contain either of a_1, b_2 or c_{12} since we know π_{21}. Continuing in this manner we can explicitly compute equations for all 45 tritangent planes and 27 lines on S.

For real cubic surfaces we can compute self-adjoint and definite determinantal representations once we know equations of lines and tritangent planes of S.

To conclude, we briefly discuss two examples taken from [2].

Example 3.1 Consider Fermat surface S given by the equation

$$F = z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0.$$

We represent a line

$$\begin{aligned}
\alpha_0 z_0 + \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 &= 0\\
\beta_0 z_0 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 &= 0
\end{aligned}$$

by a 2×4 matrix

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

Because of the symmetry of F it is easy to obtain the 27 lines on S:

$$\begin{pmatrix}1&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&1&\omega\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&0&\omega&1\end{pmatrix},$$
$$\begin{pmatrix}1&\omega&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&\omega&0&0\\0&0&1&\omega\end{pmatrix}, \begin{pmatrix}1&\omega&0&0\\0&0&\omega&1\end{pmatrix},$$
$$\begin{pmatrix}\omega&1&0&0\\0&0&1&1\end{pmatrix}, \begin{pmatrix}\omega&1&0&0\\0&0&1&\omega\end{pmatrix}, \begin{pmatrix}\omega&1&0&0\\0&0&\omega&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&1&0&\omega\end{pmatrix}, \begin{pmatrix}1&0&1&0\\0&\omega&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&\omega&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}1&0&\omega&0\\0&1&0&\omega\end{pmatrix}, \begin{pmatrix}1&0&\omega&0\\0&\omega&0&1\end{pmatrix},$$
$$\begin{pmatrix}\omega&0&1&0\\0&1&0&1\end{pmatrix}, \begin{pmatrix}\omega&0&1&0\\0&1&0&\omega\end{pmatrix}, \begin{pmatrix}\omega&0&1&0\\0&\omega&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&\omega&0\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&\omega&1&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&\omega\\0&1&1&0\end{pmatrix}, \begin{pmatrix}1&0&0&\omega\\0&1&\omega&0\end{pmatrix}, \begin{pmatrix}1&0&0&\omega\\0&\omega&1&0\end{pmatrix},$$
$$\begin{pmatrix}\omega&0&0&1\\0&1&1&0\end{pmatrix}, \begin{pmatrix}\omega&0&0&1\\0&1&\omega&0\end{pmatrix}, \begin{pmatrix}\omega&0&0&1\\0&\omega&1&0\end{pmatrix},$$

where ω is a primitive third root of unity.

Consider determinantal representation

$$M = \begin{pmatrix} 0 & z_0 + z_1 & z_2 + z_3 \\ \omega z_2 + z_3 & 0 & z_0 + \omega z_1 \\ \omega z_0 + z_1 & z_2 + \omega z_3 & 0 \end{pmatrix}.$$

Together M and M^t correspond to the double six

$$\left(\begin{smallmatrix} \binom{\omega\,1\,0\,0}{0\,0\,\omega\,1}, & \binom{1\,1\,0\,0}{0\,0\,1\,\omega}, & \binom{1\,\omega\,0\,0}{0\,0\,1\,1}, & \binom{1\,0\,0\,1}{0\,1\,1\,0}, & \binom{1\,0\,0\,\omega}{0\,1\,\omega\,0}, & \binom{\omega\,0\,0\,1}{0\,\omega\,1\,0} \\ \binom{1\,1\,0\,0}{0\,0\,1\,1}, & \binom{1\,\omega\,0\,0}{0\,0\,\omega\,1}, & \binom{\omega\,1\,0\,0}{0\,0\,1\,\omega}, & \binom{\omega\,0\,0\,1}{0\,1\,\omega\,0}, & \binom{1\,0\,0\,1}{0\,\omega\,1\,0}, & \binom{1\,0\,0\,\omega}{0\,1\,1\,0} \end{smallmatrix}\right)$$

Observe that determinantal representations M is not equivalent to a self-adjoint one, since the double-six contains real lines.

Consider next

$$M' = \begin{pmatrix} 0 & z_2 + \omega z_3 & z_0 + \omega z_1 \\ \omega z_0 + z_1 & 0 & z_2 + z_3 \\ \omega z_2 + z_3 & z_0 + z_1 & 0 \end{pmatrix}.$$

The determinantal representations M' and $(M')^t$ correspond to the double-six

$$\left(\begin{smallmatrix} \binom{\omega\,1\,0\,0}{0\,0\,\omega\,1}, & \binom{1\,1\,0\,0}{0\,0\,1\,\omega}, & \binom{1\,\omega\,0\,0}{0\,0\,1\,1}, & \binom{1\,0\,0\,1}{0\,1\,\omega\,0}, & \binom{\omega\,0\,0\,1}{0\,1\,1\,0}, & \binom{1\,0\,0\,\omega}{0\,\omega\,1\,0} \\ \binom{1\,\omega\,0\,0}{0\,0\,1\,\omega}, & \binom{\omega\,1\,0\,0}{0\,0\,1\,1}, & \binom{1\,1\,0\,0}{0\,0\,\omega\,1}, & \binom{1\,0\,0\,\omega}{0\,1\,1\,0}, & \binom{1\,0\,0\,1}{0\,\omega\,1\,0}, & \binom{\omega\,0\,0\,1}{0\,1\,\omega\,0} \end{smallmatrix}\right)$$

which is mutually self-conjugate of the III-rd kind. Therefore M' is equivalent to a self-adjoint determinantal representation. Indeed,

$$\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot M' =$$

$$\begin{pmatrix} 0 & \omega^2 z_2 + z_3 & \omega^2 z_0 + z_1 \\ \omega z_2 + z_3 & z_0 + z_1 & 0 \\ \omega z_0 + z_1 & 0 & z_2 + z_3 \end{pmatrix}.$$

We conclude that Fermat surface is of the Segre type F_4 and thus it has 6 nonequivalent self-adjoint and no definite determinantal representations. $\square$

Example 3.2 Let S be a surface defined by equation

$$\begin{aligned} \left(\frac{100}{24} z_0^2 + z_1^2\right)(z_0 + z_2) & \\ -z_3 \left(z_3 - \frac{1}{2} z_2\right)\left(z_3 - \frac{2}{3} z_2\right) &= 0. \end{aligned}$$

It is easy to check that S is of type F_5. It has 3 real lines on the plane $z_0 + z_2 = 0$. Through the line

$$z_0 + z_2 = 3z_3 - 2z_2 = 0$$

there are 4 real tritangent planes, each containing two intersecting complex conjugate lines:

$$\begin{aligned} z_0 + 0.98987 z_2 + 0.01519 z_3 &= 0, \\ z_0 + 0.01345 z_2 + 1.47982 z_3 &= 0, \\ z_0 - 3.00333 z_2 + 6.00499 z_3 &= 0, \\ 3z_3 - 2z_2 &= 0. \end{aligned}$$

Determinantal representation

$$\begin{pmatrix} -z_0 - 0.98987 z_2 - 0.01519 z_3 & 0 & \frac{2.04124 z_0 - i z_1 + 8.14425 z_3}{28.68441(1-i)} \\ 0 & 3z_3 - 2z_2 & (1+i)(\frac{1}{2} z_0 - i \frac{\sqrt{6}}{10} z_1) \\ \frac{2.04124 z_0 + i z_1 + 8.14425 z_3}{28.68441(1+i)} & (1-i)(\frac{1}{2} z_0 + i \frac{\sqrt{6}}{10} z_1) & -0.02020 z_3 \end{pmatrix}$$

is definite. Indeed, for example evaluate the representation at

$$z_0 = 0.02, \ z_1 = 0, \ z_2 = -1.2, \ z_3 = -0.3$$

and see that its eigenvalues $1.50013, \ 1.17540, \ 0.00293$ are all positive.

On the other hand, determinantal representation

$$\begin{pmatrix} \frac{z_0 - 3.00333 z_2 + 6.00499 z_3}{-0.12000} & 0 & \frac{2.04124 z_0 - i z_1 - 0.00679 z_3}{0.49979(1-i)} \\ 0 & 3z_3 - 2z_2 & (1+i)(\frac{1}{2} z_0 - i \frac{\sqrt{6}}{10} z_1) \\ \frac{2.04124 z_0 + i z_1 - 0.00679 z_3}{0.49979(1+i)} & (1-i)(\frac{1}{2} z_0 + i \frac{\sqrt{6}}{10} z_1) & 0.00666 z_3 \end{pmatrix}$$

is non-definite. The proof of non-definiteness is longer and can be found in [2]. $\square$

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