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Ice cream and orbifold Riemann–Roch

A. Buckley, M. Reid, and S. Zhou

Abstract. We give an orbifold Riemann–Roch formula in closed form for the Hilbert series of a quasismooth polarized n-fold (X,D), under the assumption that X is projectively Gorenstein with only isolated orbifold points. Our formula is a sum of parts each of which is integral and Gorenstein symmetric of the same canonical weight; the orbifold parts are called *ice cream functions*. This form of the Hilbert series is particularly useful for computer algebra, and we illustrate it on examples of K3 surfaces and Calabi–Yau 3-folds. These results apply also with higher dimensional orbifold strata (see [1] and [2]), although the precise statements are considerably trickier. We expect to return to this in future publications.

Keywords: orbifold, orbifold Riemann–Roch, Dedekind sum, Hilbert series, weighted projective varieties.

To Professor Igor Rostislavovich Shafarevich on his 90*th birthday*

§ 1. Introduction

Reid [3] introduced Riemann–Roch (RR) formulas for polarized orbifolds (X,D) with isolated orbifold locus, of the form

$$\chi(X,\mathcal{O}_X(D)) = \mathrm{RR}(X,D) + \sum_{P\in\mathcal{B}} c_P(D), \tag{1.1}$$

where $\mathrm{RR}(X,D)$ is a Riemann–Roch like expression and the $c_P(D)$ are certain fractional contributions from the orbifold points $\mathcal{B}$, depending only on the local type of (X,D). The orbifold RR formula of [3] has found numerous subsequent extensions and applications; see for example A. R. Iano-Fletcher [4], G. Brown, S. Altınok and M. Reid [5], A. Buckley and B. Szendrői [1], J. J. Chen, J. A. Chen and M. Chen [6] and M. Kawakita [7], and we expect these ideas to be equally applicable in the study of higher dimensional varieties.

A general RR formula for abstract orbifolds was first proved by T. Kawasaki [8] by analytic tools. B. Toën [9] gave another proof using the algebraic methods of Deligne–Mumford stacks. However, at present, it is not well understood how to use these abstract results in practice to compute the dimension of RR spaces. Toën's result was applied to weighted projective spaces by F. Nironi [10], to quasismooth varieties in weighted projective spaces by S. Zhou [2] and to twisted curves by D. Abramovich and A. Vistoli [11]. Edidin's recent treatment [12] clarifies orbifold

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RR considerably; our results and Zhou's thesis [2] provide many practical exercises. Our proof, like that of [3], is based on a reduction to Atiyah–Singer and Atiyah–Segal equivariant Riemann–Roch [13], [14].

Let D be an ample $\mathbb{Q}$-Cartier divisor on a normal projective n-fold X (we usually work over $\mathbb{C}$). The finite dimensional vector spaces $H^0(X, \mathcal{O}_X(mD))$ fit together as a finitely generated graded ring

$$R(X,D) = \bigoplus_{m\geqslant 0} H^0(X, \mathcal{O}_X(mD)), \tag{1.2}$$

with $X \cong \operatorname{Proj} R(X,D)$ and the divisorial sheaf $\mathcal{O}_X(mD)$ equal to the character sheaf $\mathcal{O}_X(m)$ of the Proj. A surjection from a graded polynomial ring

$$k[x_0,\dots,x_N] \twoheadrightarrow R(X,D) \quad \text{with variables } x_i \text{ of weight } a_i \tag{1.3}$$

corresponds to an embedding

$$i\colon X \cong \operatorname{Proj} R(X,D) \hookrightarrow \mathbb{P}(a_0,\dots,a_N) \tag{1.4}$$

of X into a weighted projective space as a projectively normal subscheme.

The Hilbert function $m \mapsto P_m(X,D) = h^0(X, \mathcal{O}_X(mD))$ and the *Hilbert series* $P_X(t) = \sum_{m\geqslant 0} P_m t^m$ encode the numerical data of $R(X,D)$. It is a standard result that $\prod(1-t^{a_i}) \cdot P_X(t)$ is a polynomial where, as above, the a_i are the weights of the generators. The multiplicative group $\mathbb{G}_{\mathrm{m}}$ ($= \mathbb{C}^\times$ if the ground field is $\mathbb{C}$) has a standard action on the graded ring $R(X,D) = \bigoplus_{m\geqslant 0} R_m$, with $\lambda \in \mathbb{C}^\times$ multiplying R_m by λ^m. Our aim is a *character formula* expressing the Hilbert series of R in closed form.

1.1. Plan of the paper. Section 1 recalls notation and background results from the literature, and states our Main Theorem 1.3. Section 2 defines the ice cream functions $P_{\mathrm{orb}}(\frac{1}{r}(a_1,\dots,a_n), k_X)$ as inverse polynomials modulo $1+t+\cdots+t^{r-1}$ that contain the same information as Dedekind sums (see especially 2.4). 3.1 deals with the existence of the RR formula for n-folds with isolated orbifold points and the precise nature of the term $\mathrm{RR}(D)$, as a preliminary to the proof of the main theorem in 3.2. 3.3 relates the new viewpoint of this paper to traditional formulas for the Hilbert series of K3 surfaces, Fano 3-folds and canonical 3-folds.

Although this paper mostly deals in isolated orbifold points, our ultimate aspiration is to find closed expressions for the Hilbert series of arbitrary orbifolds, having a stratification by orbifold loci of any dimension. Section 4 discusses briefly what we hope to do in this direction, and the difficulties associated with positive dimensional orbifold loci, especially their dissident strata (where the inertia group jumps); we exemplify this with Buckley's results on orbifold RR for polarized Calabi–Yau 3-folds [1]. The paper is backed up by a website,

`http://warwick.ac.uk/staff/Miles.Reid/Ice`,

containing additional material that does not fit in the paper, including implementations of our main algorithms in computer algebra and links to other papers.

A long term motivating question for us is the 'exact plurigenus formula' for Fano 4-folds and canonical 4-folds. We believe that terminal singularities are intractable in dimension $\geqslant 4$, so it is unreasonable to hope for a reduction in the style of [3] of general terminal singularities to orbifolds points. Nevertheless, the orbifold examples provide rich experimental material, and some modifications of the ideas of this paper should apply more generally.

Our term *dissident point* was originally coined in the 1970s as a reference to Professor Igor Rostislavovich Shafarevich, who created the Moscow school of algebraic geometry, and who has taught us so much. It is a pleasure to offer this paper to him as a birthday tribute.

1.2. Definitions and notation. A *Weil divisor* on a normal variety X is a formal linear combination of prime divisors with integer coefficients. A Weil divisor D is $\mathbb{Q}$-*Cartier* if mD is Cartier for some integer $m > 0$.

We write $\boldsymbol{\mu}_r \subset \mathbb{G}_\mathrm{m}$ for the multiplicative group of rth roots of unity, or the cyclic subgroup of $\mathbb{C}^\times$ generated by $\exp\frac{2\pi i}{r}$. A *cyclic orbifold point* or *cyclic quotient singularity* of type $\frac{1}{r}(a_1,\dots,a_n)$ is the quotient $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/\boldsymbol{\mu}_r$, where $\boldsymbol{\mu}_r$ acts on $\mathbb{A}^n$ by

$$\boldsymbol{\mu}_r \ni \varepsilon\colon (x_1,\dots,x_n) \mapsto (\varepsilon^{a_1}x_1,\dots,\varepsilon^{a_n}x_n). \tag{1.5}$$

We usually assume that no factor of r divides all the a_i, which is equivalent to the $\boldsymbol{\mu}_r$ action being effective; the orbifold point is isolated if and only if all the a_i are coprime to r. The sheaf $\pi_*\mathcal{O}_{\mathbb{A}^n}$ decomposes as a direct sum of divisorial eigensheaves

$$\mathcal{L}_i = \bigl\{f \mid \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \boldsymbol{\mu}_r\bigr\} \quad \text{for} \quad i \in \mathbb{Z}/r = \mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m}). \tag{1.6}$$

The notation $\frac{1}{r}(a_1,\dots,a_n)$ refers to *polarized* orbifold points. The orbinates x_j of degree a_j modulo r are local sections of $\mathcal{O}_X(a_j)$, which is locally isomorphic to $\mathcal{L}_{-a_j}$. In the terminology of [3], Definition 8.3, $\mathcal{O}_X(1) = \mathcal{O}_X(D)$ is of type ${}_{r-1}(\frac{1}{r}(a_1,\dots,a_n))$.

A polarized variety (X,D) is *quasismooth* if the corresponding affine cone $\mathcal{C}_X = \operatorname{Spec} R(X,D)$ is nonsingular outside the origin. In this case, the orbifold points of X arise from the orbits of the group action that are pointwise fixed by a nontrivial isotropy group, necessarily the cyclic subgroup $\boldsymbol{\mu}_r \subset \mathbb{G}_\mathrm{m}$ for some r. In terms of (X,D), quasismoothness holds if and only if X has locally cyclic quotient singularities $\frac{1}{r}(a_1,\dots,a_n)$ and the given Weil divisor $D = \mathcal{O}_X(1)$ generates the local class group $\mathbb{Z}/r = \mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{G}_\mathrm{m})$. Then the local index one cyclic cover defined by a local identification $\mathcal{O}_X(rD) \cong \mathcal{O}_X$ is nonsingular.

All our concrete examples are subvarieties in weighted projective spaces; see Iano-Fletcher [4] for definitions and properties. Our quasismoothness assumption implies that X has no orbifold behaviour in codimension 0 or 1, or is *well formed* in the terminology of [4]. This is right here because we work with n-folds for $n \geqslant 2$ with isolated orbifold locus; it means that the orbifold X as a scheme already knows its orbifold structure, the local universal cover of $X \setminus \operatorname{Sing} X$. This simplifies the treatment, allowing us to circumvent the language of stacks and the graded structure sheaf $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i)$ (see Canonaco [15]). Some of our examples involve fractional divisors on curves, and we leave the elementary treatment of the graded structure sheaf $\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_X(i)$ in this case to the conscientious reader.[1]

A polarized variety (X,D) is *projectively Gorenstein* if its affine cone or the corresponding graded ring $R(X,D)$ is Gorenstein. In this case $\omega_X \cong \mathcal{O}_X(k_X D)$ for

[1] See for example Exercise 2.14. Compare also Demazure [16] and Watanabe [17]; the latter also treats the graded dualizing sheaf for fractional divisors.

some $k_X \in \mathbb{Z}$, called the *canonical weight* of (X, D), and $H^j(X, \mathcal{O}_X(mD)) = 0$ for all j, $0 < j < \dim X$, and all m. Bruns and Herzog ([18], Corollary 4.3.8) give the following elementary result.

Lemma 1.1. *Let R be a graded Gorenstein ring of dimension $\dim R = n+1$ and canonical weight k_R, so that the canonical module of R is $\omega_R = R(k_R)$. Then the Hilbert series $P_R(t)$ of R satisfies the functional equation*

$$t^{k_R} P\left(\tfrac{1}{t}\right) = (-1)^{n+1} P(t). \tag{1.7}$$

We refer to property (1.7) of a rational function as *Gorenstein symmetry*. A palindromic polynomial or Laurent polynomial is Gorenstein symmetric. Examples: t and $t^{-1} + 1 + t^2 + t^3$ are both palindromic of degree 2.

Proof of Lemma 1.1. This follows from duality: R is a quotient of a weighted polynomial ring $A = k[x_0, \dots, x_N]$ with $\operatorname{wt} x_i = a_i$. A minimal free resolution

$$R \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{\text{cod}} \leftarrow 0 \tag{1.8}$$

has length equal to the codimension $\text{cod} = N - n$, and $F_{\text{cod}} = A(-\alpha)$ is the free module of rank one and degree $-\alpha$, where $\alpha = k_R + \sum a_i$ is the *adjunction number* for $X = \operatorname{Proj} R \subset \mathbb{P}(a_0, \dots, a_N)$. Duality gives $F_{\text{cod}-i} \cong \operatorname{Hom}_A(F_i, F_{\text{cod}})$ so that, over the denominator $\prod(1-t^{a_i})$ corresponding to $A = k[x_0, \dots, x_N]$, the numerator of the Hilbert series is a sum of terms of the form $t^d + (-1)^{\text{cod}} t^{\alpha-d}$. $\square$

For quasismooth X, the statement corresponds to Serre duality. However, the proof only uses the definition and basic properties of Gorenstein graded rings, without further assumptions on the singularities of $\operatorname{Spec} R$ or $\operatorname{Proj} R$.

Following Mukai [19], we write $c = k_X + n + 1$ for the *coindex* of (X, D). By the adjunction formula, the coindex is invariant under passing to a hyperplane section of degree 1. For nonsingular varieties, we have:

Example 1.2.

projective space $\mathbb{P}^n$	has coindex 0;
a quadric $Q \subset \mathbb{P}^{n+1}$	has coindex 1;
an elliptic curve, del Pezzo surface or Fano 3-fold of index 2	has coindex 2;
a canonical curve, K3 surface or anticanonical Fano 3-fold	has coindex 3;
a canonical surface, Calabi–Yau 3-fold or anticanonical Fano 4-fold	has coindex 4.

1.3. The main result. For a quasismooth projectively Gorenstein orbifold (X, D) with isolated orbifold points, Theorem 1.3 writes the Hilbert series of (X, D) as a sum of parts, each of which is integral and Gorenstein symmetric of the same degree k_X. We call the orbifold contributions $P_{\text{orb}}(Q, k_X)$ *ice cream functions*; see (2.29). The result expresses $P_X(t)$ in a closed form that can be calculated readily as a few lines of computer algebra.

Theorem 1.3. *Let (X, D) be a quasismooth orbifold of dimension $n \geqslant 2$. Suppose that (X, D) is projectively Gorenstein of canonical weight k_X, and has isolated orbifold points*

$$\mathcal{B} = \left\{ Q \text{ of type } \frac{1}{r}(a_1, \ldots, a_n) \right\}.$$

Then the Hilbert series of X is

$$P_X(t) = P_I(t) + \sum_{Q \in \mathcal{B}} P_{\text{orb}}(Q, k_X)(t), \tag{1.9}$$

where:

(i) *the **initial part** has the form $P_I = \frac{A(t)}{(1-t)^{n+1}}$, where $A(t)$ is the unique integral palindromic polynomial of degree $c = k_X + n + 1$ (the coindex) such that $P_I(t)$ equals the series $P_X(t)$ up to and including degree $\lfloor \frac{c}{2} \rfloor$; if $c < 0$ then $P_I = 0$;*

(ii) *each **orbifold part** for $Q \in \mathcal{B}$ of type $\frac{1}{r}(a_1, \ldots, a_n)$ is of the form $P_{\text{orb}}(Q, k_X) = \frac{B(t)}{(1-t)^n (1-t^r)}$, with*

$$B(t) = \text{InvMod}\left(\prod_{i=1}^{n} \frac{1 - t^{a_i}}{1 - t}, \frac{1 - t^r}{1 - t}, \left\lfloor \frac{c}{2} \right\rfloor + 1 \right), \tag{1.10}$$

the unique Laurent polynomial supported in the interval

$$\left[\left\lceil \frac{c-1}{2} \right\rceil + 1, \left\lfloor \frac{c-1}{2} \right\rfloor + r - 1 \right] \tag{1.11}$$

and equal to the inverse of $\prod_{i=1}^{n} \frac{1-t^{a_i}}{1-t}$ modulo $\frac{1-t^r}{1-t}$; the polynomial $B(t)$ has integral coefficients and is palindromic of degree $k_X + n + r$.

The quantities $\lceil \frac{c-1}{2} \rceil = \lfloor \frac{c}{2} \rfloor$ cause a few headaches of notation and computation. More conceptually, the numerator has Gorenstein symmetry of degree $k + n + r$, giving the part as a whole Gorenstein symmetry of degree k, and is a smallest residue modulo $\frac{1-t^r}{1-t}$. The interval (1.11) as written is manifestly symmetric, centred at $\frac{k+n+r}{2}$, and of length $\leqslant r - 2$; it contains $r - 1$ consecutive integers if c is odd, and $r - 2$ if c is even. The support of the numerator occupies the end points of the stated interval in about half the cases. Compare Exercise 2.1.

Addendum 1.4. *We suppose that (X, D) is as in Theorem 1.3, but relax the projectively Gorenstein assumption to assume only that K_X is $\mathbb{Q}$-Cartier and numerically equivalent to $k_X D$. (In other words, omit the projectively Cohen–Macaulay requirement.) Then the Hilbert series of X is*

$$P_X(t) = J(t) + P_I(t) + \sum_{Q \in \mathcal{B}} P_{\text{orb}}(Q, k_X)(t) \tag{1.12}$$

with P_I and P_{orb} as above, where $J(t) = \sum j_m t^m$ is a polynomial treating the irregularity of $\mathcal{O}_X(mD)$, with coefficients

$$\begin{aligned} j_m &= h^0(\mathcal{O}_X(mD)) + (-1)^n h^n(\mathcal{O}_X(mD)) - \chi(\mathcal{O}_X(mD)) \\ &= -\sum_{i=1}^{n-1} (-1)^i h^i(\mathcal{O}_X(mD)). \end{aligned} \tag{1.13}$$

In characteristic zero (*or if some form of Kodaira vanishing theorem holds*), $J(t)$ *has degree* $\leqslant k_X$.

Example 1.5. Consider the general hypersurface $X_{10} \subset \mathbb{P}^4(1,1,2,2,3)$ with coordinates x_1, x_2, y_1, y_2, z. Then X_{10} is a 3-fold with $5 \times \frac{1}{2}(1,1,1)$ orbifold points along $\mathbb{P}^1_{\langle y_1,y_2\rangle}$ and a $\frac{1}{3}(1,2,2)$ point at $P_z = (0,0,0,0,1)$. It has canonical weight $k_X = 1$ and coindex $c = k_X + n + 1 = 5$. The Hilbert series is as follows: the initial part

$$P_I = \frac{1-2t+3t^2+3t^3-2t^4+t^5}{(1-t)^4} = 1 + t + \frac{t+t^2}{(1-t)^2} + 2\frac{t^2+t^3}{(1-t)^4} \qquad (1.14)$$

takes care of $P_1 = 2$, $P_2 = 5$. The orbifold parts

$$P_{\mathrm{orb}}\bigl(\tfrac{1}{2}(1,1,1),1\bigr) = \frac{-t^3}{(1-t)^3(1-t^2)}, \qquad P_{\mathrm{orb}}\bigl(\tfrac{1}{3}(1,2,2),1\bigr) = \frac{-t^3-t^4}{(1-t)^3(1-t^3)} \qquad (1.15)$$

take care of the periodicity, giving

$$P_I + 5\cdot P_{\mathrm{orb}}\bigl(\tfrac{1}{2}(1,1,1),1\bigr) + P_{\mathrm{orb}}\bigl(\tfrac{1}{3}(1,2,2),1\bigr) = \frac{1-t^{10}}{(1-t)^2(1-t^2)^2(1-t^3)}.$$

Here the numerator of P_I is palindromic of degree $c = 5$, so that P_I is Gorenstein symmetric of degree 1. The two P_{orb} parts are also integral and Gorenstein symmetric of degree 1, and they start with t^3, so do not affect the first two plurigenera P_1 and P_2.

Caution 1.6. The initial part P_I handles the first plurigenera $P_1, \dots, P_{\lfloor \frac{c}{2} \rfloor}$, but is not the *leading term* of the Hilbert function controlling the order of growth of the plurigenera: in this example, $X_{10} \subset \mathbb{P}^4(1,1,2,2,3)$ is a canonical 3-fold with $K_X = \mathcal{O}_X(1)$, of degree $K_X^3 = \frac{10}{2\cdot 2\cdot 3} = \frac{5}{6}$, whereas P_I on its own would correspond to degree $K^3 = 4$ (for this, sum the coefficients in the numerator of P_I). In our formula, the orbifold parts contribute to the global order of growth of the plurigenera, in this case $5(-\frac{1}{2})$ and $-\frac{2}{3}$.

1.4. Appendix: Symmetric integral polynomials. The shape of our Hilbert series in the nonsingular case comes directly from the following result applied to Hilbert polynomials.

Proposition 1.7. (I) *Let* $\sum_{m\geqslant 0} \rho_m t^m$ *be a power series, and assume that* $\rho_m = F(m)$ *for all* $m \geqslant m_0$, *where* $F(x)$ *is a polynomial of degree* n *and* $m_0 \geqslant 0$ *an integer. Then* $(1-t)^{n+1}\bigl(\sum_{m\geqslant 0} \rho_m t^m\bigr)$ *is a polynomial in* t *of degree* $\leqslant m_0 + n + 1$.

(II) *Let* $F(x) \in \mathbb{Q}[x]$ *be a polynomial taking integer values* $F(m)$ *for all* $m \in \mathbb{Z}$. *Then* F *is an integral linear combination of the binomial coefficients*:

$$F(x) = \sum_{\nu=0}^{n} c_\nu \binom{x}{\nu} \quad \textit{with} \quad c_\nu \in \mathbb{Z}. \qquad (1.16)$$

Here $n = \deg F$, *and there are* $n+1$ *integral coefficients* c_ν *to specify.*

(III) *Let* $F(x) \in \mathbb{Q}[x]$ *be a polynomial taking integer values* $F(m)$ *for all* $m \in \mathbb{Z}$. *Assume that* F *satisfies* $(-1)^n F(k-x) \equiv F(x)$ *for an integer* k, *where* $\deg F = n$.

Then $F(X)$ and its associated power series $\sum_{m\geqslant 0} F(m)t^m$ are integral linear combinations of standard terms as follows:

1) *if* $n \equiv k+1 \mod 2$, *then*

$$F(x) = \sum_{\substack{-k\leqslant\nu\leqslant n\\ \nu\equiv n \mod 2}} b_\nu \binom{x+\frac{\nu-k-1}{2}}{\nu}, \qquad \sum_{m\geqslant 0} F(m)t^m = \sum_{\substack{-k\leqslant\nu\leqslant n\\ \nu\equiv n \mod 2}} b_\nu \frac{t^{\frac{\nu+k+1}{2}}}{(1-t)^{\nu+1}}; \tag{1.17}$$

2) *if* $n \equiv k \mod 2$, *then*

$$\begin{gathered} F(x) = \sum_{\substack{-k\leqslant\nu\leqslant n\\ \nu\equiv n \mod 2}} b_\nu \left(\binom{x+\frac{\nu-k}{2}}{\nu} + \binom{x+\frac{\nu-k-2}{2}}{\nu}\right), \\ \sum_{m\geqslant 0} F(m)t^m = \sum_{\substack{-k\leqslant\nu\leqslant n\\ \nu\equiv n \mod 2}} b_\nu \frac{(1+t)t^{\frac{\nu+k}{2}}}{(1-t)^{\nu+1}}. \end{gathered} \tag{1.18}$$

There are $\lfloor\frac{k+n+1}{2}\rfloor$ integral coefficients b_ν to specify.

In part (II) or (III), it is enough to assume that $F(m)\in\mathbb{Z}$ or $F(m)\in\mathbb{Z}$ and $(-1)^nF(k-m) = F(m)$ for all m in an interval of length $n+1$. The proof is a little exercise. Hint: Use induction based on $F(x)-F(x-1)$.

§ 2. Ice cream functions

2.1. Fun calculation. 'Income $\frac{3}{7}$ per day means ice cream on Wednesdays, Fridays and Sundays'. Consider the step function $i\mapsto\lfloor\frac{3i}{7}\rfloor$, where $\lfloor\,\rfloor$ denotes the rounddown or integral part. As a Hilbert series, it gives

$$P(t) = \sum_{i\geqslant 0}\left\lfloor\frac{3i}{7}\right\rfloor t^i = 0 + 0t + 0t^2 + t^3 + t^4 + 2t^5 + 2t^6 + 3t^7 + \cdots, \tag{2.1}$$

with closed form

$$P(t) = \frac{t^3+t^5+t^7}{(1-t)(1-t^7)}. \tag{2.2}$$

Indeed, $\lfloor\frac{3i}{7}\rfloor$ increments by 1 when $i = 0, 3, 5$ modulo 7, so that

$$(1-t)P(t) = t^3 + t^5 + t^7 + t^{10} + \cdots \tag{2.3}$$

is the sum over the jumps, that repeat weekly. Multiplying (2.3) by $1-t^7$ cuts the series down to the first week's ice cream ration:

$$(1-t)(1-t^7)P(t) = t^3 + t^5 + t^7. \tag{2.4}$$

The numerator $t^3+t^5+t^7$ can be seen as the inverse of $\frac{1-t^5}{1-t}=1+t+t^2+t^3+t^4$ mod $\frac{1-t^7}{1-t}=1+t+t^2+t^3+t^4+t^5+t^6$. Indeed, long multiplication gives

$$\begin{aligned}(1+t+t^2+t^3+t^4)&\times(t^3+t^5+t^7)\\ &= t^3+t^4+t^5+t^6+t^7\\ &\qquad +t^5+t^6+t^7+t^8+t^9\\ &\qquad\qquad +t^7+t^8+t^9+t^{10}+t^{11}\\ &= t^3+t^4+2t^5+2t^6+3t^7+2t^8+2t^9+t^{10}+t^{11}\\ &\equiv 3+2t+2t^2+2t^3+2t^4+2t^5+2t^6\equiv 1,\end{aligned} \tag{2.5}$$

where $\equiv$ denotes congruence modulo $\frac{1-t^7}{1-t}$. Here $5=\operatorname{InvMod}(3,7)$ is the inverse of 3 modulo 7. The product in (2.5) has $5\cdot 3=15\equiv 1 \mod 7$ terms that distribute themselves equitably among the 7 congruence classes, except that t^7 appears once for each of the 3 terms in the second factor.

There are several other meaningful expressions for $P(t)$. Under $\equiv$, the bounty $t^3+t^5+t^7$ can be viewed as famine $-t-t^2-t^4-t^6$ 'no ice cream on Mondays, Tuesdays, Thursdays or Saturdays'. In other words,

$$P(t)=\frac{t^3+t^5+t^7}{(1-t)(1-t^7)}=\frac{t}{(1-t)^2}+\frac{-t-t^2-t^4-t^6}{(1-t)(1-t^7)}. \tag{2.6}$$

Because $t^7\equiv 1$, we can shift the exponents of t up or down by 7:

$$\frac{t^{-4}+t^{-2}+1}{(1-t)(1-t^7)} \quad\text{or}\quad \frac{-t^{-1}-t-t^2-t^4}{(1-t)(1-t^7)} \tag{2.7}$$

so 'ice cream rations from Monday before the start of term' or 'famine from the previous Saturday'. More generally, working mod $\frac{1-t^r}{1-t}$, we can shift exponents $t^b\mapsto t^{b-ir} \mod r$ and subtract a multiple of $1+t+\cdots+t^{r-1}$ to *fold* any Laurent polynomial into any desired interval $[t^a,\dots,t^{a+r-2}]$ of length $r-2$. Of these possible shifts as Laurent polynomials with short support, $t^{7i}(t^3+t^5+t^7)$ is palindromic of degree $10+14i$, and $t^{7i}(-t^{-1}-t-t^2-t^4)$ is palindromic of degree $3+14i$, *and no other*.

In 'macroeconomic' terms, the order of growth is the linear function $\frac{3i}{7}$ with fractional seasonal corrections, that is,

$$P(t)=\frac{3}{7}\,\frac{t}{(1-t)^2}+\frac{-\frac{3}{7}t-\frac{6}{7}t^2-\frac{2}{7}t^3-\frac{5}{7}t^4-\frac{1}{7}t^5-\frac{4}{7}t^6}{1-t^7} \tag{2.8}$$

('on Mondays, we lose $\frac{3}{7}$ in small change', etc.). Notice the coefficient $\frac{1}{7}$ of t^5: the inverse of 3 modulo 7 is 5, so as we enjoy our second ice cream of the week on Fridays, we lose $\frac{1}{7}$, the unit of small change.

We can average out the seasonal corrections in (2.8) to sum to zero, giving

$$P(t)=\frac{3}{7}\,\frac{2t-1}{(1-t)^2}+\frac{\frac{3}{7}-\frac{3}{7}t^2+\frac{1}{7}t^3-\frac{2}{7}t^4+\frac{2}{7}t^5-\frac{1}{7}t^6}{1-t^7}, \tag{2.9}$$

where the coefficients $\frac{3}{7}$, 0, $-\frac{3}{7}$, $\frac{1}{7}$, $-\frac{2}{7}$, $\frac{2}{7}$, $-\frac{1}{7}$ are the Dedekind sums $\sigma_i\bigl(\frac{1}{7}(5)\bigr)$ (see Definition 2.5 and compare [3], Theorem 8.5).

The expressions (2.2)–(2.9) represent different views on numerical functions that grow with periodic corrections. Our main aim is to explain the orbifold contributions P_{orb} in Theorem 1.3 as minor variations on this simple-minded material.

Exercise 2.1 (One dimensional ice cream functions). Let $0 < a < r$ be coprime integers, $k \equiv -a \mod r$ and set $b = \mathrm{InvMod}(a, r)$. An integer is an *ice cream day* for $\frac{1}{r}(a)$ if its congruence class is one of the b distinct classes

$$\{0, a, 2a, \dots, a(b-1)\} \mod r, \tag{2.10}$$

and a *non ice cream day* if its congruence class is one of the $r - b$ classes

$$\{1, a+1, 2a+1, \dots, a(r-b-1)+1\} \mod r. \tag{2.11}$$

The two sets are complementary because $ai + 1 \equiv a(i+b) \mod r$.

Then the numerator $B(t)$ of (1.10) is one of

$$(\mathrm{I}) = \sum t^j \quad \text{summed over ice cream days in the interval (1.11)} \tag{2.12}$$

or

$$(\mathrm{II}) = \sum -t^j \quad \text{summed over non ice cream days in (1.11).} \tag{2.13}$$

Hint. Since $ab \equiv 1 \mod r$, the inverse of $\frac{1-t^a}{1-t} \mod 1 - t^r$ is $\frac{1-t^{ab}}{1-t^a} = \sum_{i=0}^{b-1} t^{ai}$.

2.2. The function Inverse Mod. We start with the following basic result.

Theorem 2.2. *Fix an integer γ and a monic polynomial $F \in \mathbb{Q}[t]$ of degree d with nonzero constant term.*

(I) *The quotient ring $\mathbb{Q}[t]/(F)$ is a d-dimensional vector space over $\mathbb{Q}$ and t is invertible in it, so that $\mathbb{Q}[t]/(F) = \mathbb{Q}[t, t^{-1}]/(F)$.*

(II) *Any range $[t^\gamma, \dots, t^{\gamma+d-1}]$ of d consecutive Laurent monomials maps to a $\mathbb{Q}$-basis of $\mathbb{Q}[t]/(F)$.*

(III) *If $A \in \mathbb{Q}[t]$ is coprime to F, we can write its inverse modulo F uniquely as a Laurent polynomial B with support in $[t^\gamma, \dots, t^{\gamma+\alpha-1}]$.*

Proof. This is all trivial. The leading term of F is nonzero, so $1, t, \dots, t^{d-1}$ base $\mathbb{Q}[t]/(F)$. The constant term of F is nonzero so t is coprime to F, and hence invertible modulo F. Multiplication by t is an invertible linear map, so multiplication by t^γ for any $\gamma \in \mathbb{Z}$ takes a basis to another basis. If A is coprime to F it is invertible in $\mathbb{Q}[t]/(F)$, and its inverse has a unique expression in any basis. □

Definition 2.3. For coprime polynomials $A, F \in \mathbb{Q}[t]$ we set

$$\mathrm{InvMod}(A, F, \gamma) = B \tag{2.14}$$

with B as in (III). That is, $B \in \mathbb{Q}[t, t^{-1}]$ is the uniquely determined Laurent polynomial supported in $[t^\gamma, \dots, t^{\gamma+d-1}]$ with $AB \equiv 1 \mod F$. For different $\gamma \in \mathbb{Z}$, these inverses are congruent modulo F, but of course different polynomials in general. We also write $\mathrm{InvMod}(A, F)$ with unspecified support for any inverse of A modulo F in $\mathbb{Q}[t]$.

Fix positive integers r and $a_1, \dots, a_n$ and set

$$A = \prod_{j=1}^{n} (1 - t^{a_j}), \quad h = \operatorname{hcf}(1 - t^r, A) \quad \text{and} \quad F = \frac{1 - t^r}{h}. \tag{2.15}$$

The polynomial F is the monic polynomial with simple roots only at the rth roots of unity with $A(\varepsilon) \neq 0$, or equivalently $\varepsilon^{a_j} \neq 1$ for all a_j. Since we take out the hcf, A and F are coprime. Theorem 2.2 applies to give $\operatorname{InvMod}(A, F, \gamma)$, the inverse of A modulo F with support in $[t^\gamma, \dots, t^{\gamma+d-1}]$, where $d = \deg F$ and $\gamma \in \mathbb{Z}$ is arbitrary.

We show how to compute InvMod:

Algorithm 2.4. If $\gamma \geqslant 0$ then $t^\gamma A$ and F are coprime polynomials. Set $d = \deg F$. The Euclidean algorithm in $\mathbb{Q}[t]$ provides a unique solution to

$$t^\gamma AB + FG = 1, \tag{2.16}$$

with $B \in \mathbb{Q}[t]$ a polynomial of degree $< d$. Then $\operatorname{InvMod}(A, F, \gamma) = t^\gamma B$.

If $\gamma < 0$, choose some m with $mr + \gamma \geqslant 0$, and solve

$$t^{mr+\gamma} AB + FG = 1 \tag{2.17}$$

by the Euclidean algorithm. Then $\operatorname{InvMod}(A, F, \gamma) = t^{mr+\gamma} B / t^{mr} = t^\gamma B$. This trick works because $t^{mr} \equiv 1 \mod F$. For more general polynomials F, one would need to calculate powers of the matrix M_t corresponding to multiplication by t in $\mathbb{Q}[t]/(F)$; in our case, $M_t^r = 1$.

The isolated case is when $a_1, \dots, a_n$ are coprime to r, so $h = 1 - t$ and $F = 1 + t + \cdots + t^{r-1}$ has degree $d = r - 1$ and roots $\varepsilon \in \boldsymbol{\mu}_r \setminus \{1\}$. If moreover r is prime then F is the cyclotomic polynomial, and working modulo F is essentially the same thing as setting $t = \varepsilon$ a primitive rth root of unity.

2.3. Dedekind sums as Inverse Mod. We now recall Dedekind sums, and relate them to the function InvMod.

Definition 2.5. We define the ith *Dedekind sum* σ_i by

$$\sigma_i\left(\frac{1}{r}(a_1, \dots, a_n)\right) = \frac{1}{r} \sum_{\substack{\varepsilon \in \boldsymbol{\mu}_r \\ \varepsilon^{a_j} \neq 1 \, \forall j = 1, \dots, n}} \frac{\varepsilon^i}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})}, \tag{2.18}$$

where ε runs over the rth roots of unity for which the denominator is nonzero. Proposition 2.6 characterizes the σ_i as solutions to a set of linear equations. We combine them into the *Dedekind sum polynomial*

$$\Delta\left(\frac{1}{r}(a_1, \dots, a_n), t\right) = \sum_{i=1}^{r} \sigma_{r-i} t^i \quad \text{with support in } [t, \dots, t^r]. \tag{2.19}$$

It is obvious that $\sigma_i = \sigma_{r+i}$. Therefore we only need to consider σ_i for $i = 0, 1, \dots, r-1$. In the coprime case, the sum in (2.18) runs over all nontrivial rth roots of unity. To stress that $a_1, \dots, a_n$ are not all coprime to r, we may call σ_i the ith *generalized* Dedekind sum.

Proposition 2.6. *Consider the $r \times r$ system of linear equations*

$$\sum_{i=0}^{r-1} \sigma_i \varepsilon^i = \begin{cases} 0 & \text{if } \varepsilon \in \boldsymbol{\mu}_{a_j} \text{ for some } j, \\ \dfrac{1}{(1-\varepsilon^{-a_1})\cdots(1-\varepsilon^{-a_n})} & \text{otherwise} \end{cases} \tag{2.20}$$

in unknowns σ_i indexed by $i \in \mathbb{Z}/r = \mathrm{Hom}(\boldsymbol{\mu}_r, \mathbb{C}^\times)$, with equations indexed by $\varepsilon \in \boldsymbol{\mu}_r$.

Then (2.20) is a nondegenerate system, with unique solution the Dedekind sums $\sigma_i = \sigma_i(\frac{1}{r}(a_1, \dots, a_n))$.

Proof. Fix a primitive root of unity $\varepsilon \in \boldsymbol{\mu}_r$. Then $(\varepsilon^{ij})_{i,j=0,\dots,r-1}$ is a Vandermonde matrix, with inverse $\frac{1}{r}(\varepsilon^{-ij})_{i,j=0,\dots,r-1}$. □

Lemma 2.7. *The polynomial Δ in (2.19) is divisible by $h = \mathrm{hcf}(A, 1-t^r)$.*

Proof. The roots of h are the $\varepsilon \in \boldsymbol{\mu}_r$ for which $\varepsilon^{a_j} = 1$ for some j. An equivalent statement is that if β is a common divisor of r and some a_j, then

$$\sum_{\substack{i=0,\dots,r-1 \\ i \equiv d \mod \beta}} \sigma_i = 0 \quad \text{for any integer } d. \tag{2.21}$$

In words, the average of the σ_i over any coset of $\beta\mathbb{Z}/r \subset \mathbb{Z}/r$ is zero. In particular, $\sum_{i=0}^{r-1} \sigma_i = 0$.

Note that $\varepsilon \in \boldsymbol{\mu}_r$ gives $\varepsilon^\beta \in \boldsymbol{\mu}_{r/\beta}$. Then by Definition 2.5,

$$\sigma_d + \sigma_{d+\beta} + \cdots + \sigma_{d+r-\beta} = \frac{1}{r} \sum_{\substack{\varepsilon \in \boldsymbol{\mu}_r \\ \varepsilon^{a_j} \neq 1 \, \forall j}} \frac{\varepsilon^d}{\prod_j (1-\varepsilon^{a_j})} (1 + \varepsilon^\beta + \varepsilon^{2\beta} + \cdots + \varepsilon^{\beta(\frac{r}{\beta}-1)}) = 0. \quad □$$

For example

$$\sigma_i\left(\frac{1}{14}(1,2,5,7)\right) = \frac{1}{14}\left\{-2,-2,-1,\frac{1}{2},0,-\frac{1}{2},1,2,2,1,-\frac{1}{2},0,\frac{1}{2},-1\right\}, \tag{2.22}$$

with $\sigma_i + \sigma_{7+i} = \sum_{l=0}^{6} \sigma_{2l+i} = \sum_{l=0}^{13} \sigma_{l+i} = 0$ for each i.

The next result was first stated and proved by Buckley [20], Theorem 2.2, following the ideas of [3].

Theorem 2.8. *Let A, h and F be as in (2.15) and Δ as in (2.19). Then*

$$\Delta = ht \cdot \mathrm{InvMod}(htA, F, 0). \tag{2.23}$$

Proof. Since F is coprime to t and h, multiplying by th before and after taking Inverse Mod does not change the result modulo F. The factor h makes the right-hand side of (2.23) divisible by h in accordance with Lemma 2.7. The factor t then folds it from a polynomial supported in $[0, \dots, r-1]$ to $[1, \dots, r]$, as in (2.16). It only remains to prove that

$$\Delta \equiv \mathrm{InvMod}(A, F, \gamma) \in \mathbb{Q}[t]/(F) \quad \text{for any } \gamma, \tag{2.24}$$

or equivalently, that $B(t) := A(t)\Delta$ is congruent to 1 modulo F. For this, substitute any root $t = \varepsilon$ of F in B and use (2.20) with the inverse value of ε. This gives

$$B(\varepsilon) = A(\varepsilon)\Delta = \frac{A(\varepsilon)}{\prod_j (1-\varepsilon^{a_j})} = 1. \tag{2.25}$$

This holds for every root ε of F, so $B(t) - 1$ is divisible by F, that is,

$$A(t)\Delta \equiv 1 \mod F. \qquad \square \tag{2.26}$$

Proposition 2.9. *Assume that all the* a_i *are coprime to* r*, so that* $h = 1-t$, $F = \frac{1-t^r}{1-t}$ *and* $d = \deg F = r-1$*. Then for any* γ,

$$(1-t)^n\Delta \equiv \operatorname{InvMod}\left(\prod_{j=1}^n \frac{1-t^{a_j}}{1-t}, F\right) \equiv \operatorname{InvMod}\left(\frac{A}{(1-t)^n}, F, \gamma+1\right) = \sum_{l=\gamma+1}^{\gamma+r-1} \theta_l t^l, \tag{2.27}$$

with integer coefficients $\theta_l = \sum_{s=0}^n (-1)^s \binom{n}{s}(\sigma_{s-l} - \sigma_{s-\gamma}) \in \mathbb{Z}$.

Proof. Replace the InvMod of a product by the product of InvMods. Each factor $\operatorname{InvMod}\big(\frac{1-t^{a_j}}{1-t}, F, 1\big)$ is a polynomial with integral coefficients; indeed, by the calculation in 2.1, or Exercise 2.1, it is the ice cream function for $\frac{b_j}{r}$ where $b_j = \operatorname{InvMod}(a_j, r)$. $\square$

Exercise 2.10 (Serre duality, Gorenstein symmetry)**.** If X is projectively Gorenstein of canonical weight k_X, prove the following:

(1) Each $Q = \frac{1}{r}(a_1, \dots, a_n)$ satisfies $k_X + \sum_{j=1}^n a_j \equiv 0 \mod r$.

(2) The σ_i are $(-1)^n$ symmetric under $i \mapsto \sum a_j - i$. (Hint: replace $\varepsilon \mapsto \varepsilon^{-1}$ in the characterization (2.20) of the σ_i, or in (2.18).)

(3) Now let θ_l be as in Proposition 2.9. Then $l_1 + l_2 \equiv k_X + n \mod r$ implies $\theta_{l_1} = \theta_{l_2}$. In particular, for c even and $\gamma = \frac{c}{2}$, we have $\theta_{\gamma+r-1} = 0$, since $\theta_\gamma = 0$ by definition.

2.4. Ice cream gives the correct periodicity. There are two expressions for the orbifold contributions to RR. The first, given in [3], is in terms of Dedekind sums:

$$\frac{\sum_{i=1}^{r-1}(\sigma_{r-i} - \sigma_0)t^i}{1-t^r}. \tag{2.28}$$

The alternative introduced here is the *ice cream function*

$$P_{\text{orb}}\left(\frac{1}{r}(a_1, \dots, a_n), k_X\right) = \frac{B(t)}{(1-t)^n(1-t^r)}, \tag{2.29}$$

with

$$B(t) = \operatorname{InvMod}\left(\prod_{i=1}^n \frac{1-t^{a_i}}{1-t}, \frac{1-t^r}{1-t}, \left\lfloor\frac{c}{2}\right\rfloor + 1\right) \tag{2.30}$$

as in (1.10). The first is strictly periodic (because of the denominator $1-t^r$), but fractional. The second is integral by Proposition 2.9, and Gorenstein symmetric of degree k, but has order of growth $O(m^n)$. They both give the same periodicity, as a simple consequence of Proposition 2.9. The point already appeared clearly in the different treatments of $P(t)$ in (2.1) and (2.8), (2.9).

Corollary 2.11.

$$P_{\text{orb}}\left(\frac{1}{r}(a_1,\dots,a_r),k_X\right)-\frac{\sum_{i=1}^{r-1}(\sigma_{r-i}-\sigma_0)t^i}{1-t^r}=\frac{C(t)}{(1-t)^{n+1}} \tag{2.31}$$

with numerator $C(t)\in\mathbb{Q}[t]$.

Indeed, put the left hand side over the common denominator $(1-t)^n(1-t^r)$ and use Proposition 2.9.

In the noncoprime case, the result is similar, and we leave the proof as an exercise.

Proposition 2.12. *Set* $s_i=\operatorname{hcf}(a_i,r)$, *so that* $\operatorname{hcf}(1-t^{a_i},1-t^r)=1-t^{s_i}$, *and write* $d=\deg F$. *Then for any* γ,

$$\begin{aligned}\prod(1-t^{s_i})\cdot\Delta &\equiv \operatorname{InvMod}\left(\prod_{j=1}^{n}\frac{1-t^{a_j}}{1-t^{h_j}},F,\gamma+1\right)\\ &=\operatorname{InvMod}\left(\frac{A}{\prod(1-t^{s_i})},F,\gamma+1\right)=\sum_{l=\gamma+1}^{\gamma+d}\theta_l t^l,\end{aligned} \tag{2.32}$$

with integer coefficients θ_l.

For an appropriate choice of $\gamma+1$, this gives a *generalized ice cream function* that is integral and Gorenstein symmetric, having the same r periodicity as $\frac{\Delta}{1-t^r}$. Compare (4.19).

Example 2.13. The ice cream function of 2.1 corresponds to $\sigma_i(\frac{1}{7}(5))$: the periodic rounding loss of (2.8), (2.9) is

$$\sum_{i=0}^{6}\sigma_{7-i}t^i=\frac{1}{7}(3-3t^2+t^3-2t^4+2t^5-t^6)\equiv\operatorname{InvMod}\left(1-t^5,\frac{1-t^7}{1-t},0\right).$$

Multiplication by $1-t$ gives a Gorenstein symmetric polynomial with integral coefficients θ_l:

$$\begin{aligned}&(1-t)\times\frac{1}{7}(3-3t^2+t^3-2t^4+2t^5-t^6)\\ &\qquad\equiv t^3+t^5+t^7=\operatorname{InvMod}\left(\frac{1-t^5}{1-t},\frac{1-t^7}{1-t},3\right).\end{aligned}$$

The fractional divisor $\frac{3}{7}P$ on a nonsingular curve is an orbifold point of type $\frac{1}{7}(5)$, with orbinate in $\mathcal{O}(5)$ having a genuine pole of order two, but a fractional zero of order $\frac{1}{7}$ in 'lost change'. As we saw in Exercise 2.1, the same considerations apply with $\frac{3}{7}$ replaced by a general reduced fraction $\frac{a}{r}$, corresponding to the orbifold point $\frac{1}{r}(b)$ with $b=\operatorname{InvMod}(a,r)$.

Consider for example the weighted projective line $X=\mathbb{P}(5,7)$. It has $k_X=-12$, and has two orbifold points of type $\frac{1}{7}(5)$ and $\frac{1}{5}(2)$. Its Hilbert series,

$$P_X(t)=\frac{1}{(1-t^5)(1-t^7)}, \tag{2.33}$$

satisfies Theorem 1.3: since $c = -10 < 0$, the initial part is $P_I = 0$. Then

$$P_X(t) = P_{\mathrm{orb}}\left(\frac{1}{7}(5), -12\right) + P_{\mathrm{orb}}\left(\frac{1}{5}(2), -12\right) = \frac{t^{-4}+t^{-2}+1}{(1-t)(1-t^7)} + \frac{-t^{-4}-t^{-2}}{(1-t)(1-t^5)},$$

where $-t^{-4} - t^{-2} = \mathrm{InvMod}(\frac{1-t^2}{1-t}, \frac{1-t^5}{1-t}, -4)$.

Exercise 2.14. Fun and games with the ice cream functions of 2.1.

(1) An elliptic curve polarized by $A = \frac{3}{7}P$ embeds as $C_{15} \subset \mathbb{P}(1,5,7)$ with canonical weight 2, that is, $K_{C,\mathrm{orb}} = 2A = \frac{6}{7}P$.

(2) A quasismooth complete intersection $C_{10,15} \subset \mathbb{P}(1,3,5,7)$ is a curve of genus 7 with $K_C = 3P + 9Q$ having P as an orbifold point of type $\frac{1}{7}(5)$, polarized by $A = \frac{3}{7}P + Q$ and having $K_{C,\mathrm{orb}} = 9A$. (Its initial part P_I is quite involved.)

(3) A curve of genus 2 polarized by $P + \frac{3}{7}Q$ with P a Weierstrass point embeds in $\mathbb{P}(1,2,3,5,7)$ as a Pfaffian with Hilbert numerator

$$1 - t^6 - t^7 - t^8 - t^9 - t^{10} + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} - t^{20}.$$

§ 3. Proof of Main Theorem

3.1. The existence of the Riemann–Roch formula. Let X be a normal projective n-fold; assume that the singularities of X are isolated, rational and $\mathbb{Q}$-factorial. We want to calculate $\chi(\mathcal{O}_X(D))$ for D a Weil divisor on X using the RR formula

$$\big(\mathrm{ch}(\mathcal{O}_X(D)) \cdot \mathrm{Td}(T_X)\big)[n], \tag{3.1}$$

that is, the component of top degree n of the product of

$$\mathrm{ch}(\mathcal{O}_X(D)) = \exp(D) = \sum \frac{D^i}{i!}$$

and

$$\mathrm{Td}(T_X) = \sum_{i=0}^{n} \mathrm{Td}_i(T_X) = 1 - \frac{1}{2}K_X + \frac{1}{12}(K_X^2 + c_2) - \frac{1}{24}K_X c_2 - \frac{1}{720}(K_X^4 - 4K_X^2 c_2 - 3c_2^2 + K_X c_3 + c_4) + \cdots .$$

We must get around the problem that the terms in (3.1) are not defined, because T_X is not a vector bundle on a singular X. For this, we use the following conventions. First, choose a resolution of singularities $f\colon Y \to X$ that is an isomorphism over the nonsingular locus of X.

(a) Replace the degree n term $\mathrm{Td}_n(T_X)$ in (3.1) by $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = \mathrm{Td}_n(T_Y)$.

(b) Replace the terms involving a product with D on X by the same expression on Y involving its pullback as a $\mathbb{Q}$-Cartier divisor. In more detail: the pullback of a $\mathbb{Q}$-Cartier divisor D is defined as usual by $f^*D = \frac{1}{m}f^*(mD)$ with mD Cartier. Except for $\mathrm{Td}_n(T_X)$, the terms in (3.1) are $D^i\,\mathrm{Td}_{n-i}(T_X)/i!$ with $i \geqslant 1$, and we replace

$$D^i\,\mathrm{Td}_{n-i}(T_X) \quad \text{by} \quad (f^*D)^i\,\mathrm{Td}_{n-i}(T_Y).$$

Remark 3.1. Our interpretation of (3.1) is independent of the choice of the resolution Y. Indeed, $\chi(\mathcal{O}_Y)$ is a birational invariant. Each of the other terms involves a product with the $\mathbb{Q}$-Cartier divisor D; now a multiple mD is linearly equivalent to a linear combination of nonsingular prime divisors disjoint from the singularities of X, so we can calculate $D^i \operatorname{Td}(T_X)$ for $i > 0$ on the nonsingular locus of X itself.

In (a), we use $\chi(\mathcal{O}_X)$ as a substitute for $\operatorname{Td}_n(T_X)$. In the 3-fold case, it is well known that the expression $\operatorname{Td}_3(T_X) = -\frac{1}{24} K_X \cdot c_2(T_X)$ can be defined using the same trick as in (b) (taking the pullback of the $\mathbb{Q}$-Cartier divisor K_X), but *is not equal* to $\chi(\mathcal{O}_X)$ in general. See [3], Corollary 10.3, and compare Kawamata [21], 2.2, and [22].

Theorem 3.2. *Let X be a normal projective n-fold with isolated, rational, $\mathbb{Q}$-factorial singularities and $f\colon Y \to X$ as above. Then the expression*

$$\operatorname{RR}(D) = \chi(\mathcal{O}_X) + \sum_{i=1}^{n} \frac{1}{i!} (f^*D)^i \operatorname{Td}_{n-i}(T_Y) = \text{“}\big(\operatorname{ch}(\mathcal{O}_X(D)) \cdot \operatorname{Td}(T_X)\big)[n]\text{”} \quad (3.2)$$

is a polynomial in the $\mathbb{Q}$-Cartier Weil divisor D such that for every D, the difference

$$\chi(X, \mathcal{O}_X(D)) - \operatorname{RR}(D) = \sum_{Q \in \operatorname{Sing} X} c_Q(D) \quad (3.3)$$

is a sum of fractional terms $c_Q(D) \in \mathbb{Q}$ depending only on the local analytic type of X and D at each singular point Q of X.

Plan of proof. We set $\mathcal{L} = f^*\mathcal{O}_X(D)/\text{torsion}$, which is a torsion free sheaf of rank 1 on Y, and write $\mathcal{O}_Y(D_Y) = \mathcal{L}^{\vee\vee}$ for its reflexive hull, which is an invertible sheaf. The proof has two parts: the first uses the Leray spectral sequence to compare $\chi(X, \mathcal{O}_X(D))$ with $\chi(Y, \mathcal{O}_Y(D_Y))$, given by RR on Y. After this, we compare the RR formula for D_Y on Y with our interpretation $\operatorname{RR}(D)$ of the RR formula for D on X. No sooner said than done.

The reflexive hull of $\mathcal{L}$ fits in a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_Y(D_Y) \to \mathcal{Q} \to 0, \quad (3.4)$$

where the cokernel $\mathcal{Q}$ has support of codimension $\geqslant 2$ in Y contained in the exceptional locus of f.

Now $f_*\mathcal{L} = f_*\mathcal{O}_Y(D_Y) = \mathcal{O}_X(D)$ because $\mathcal{O}_X(D)$ is saturated. Moreover, all the sheaves $R^if_*\mathcal{L}$ for $i \geqslant 1$ and $R^if_*\mathcal{Q}$ for $i \geqslant 0$ are finite dimensional vector spaces supported at the singular points of X.

Now the Leray spectral sequence together with the long exact sequence associated with (3.4) gives

$$\begin{aligned} \chi(Y, \mathcal{O}_Y(D_Y)) &= \chi(\mathcal{L}) + \chi(\mathcal{Q}) \\ &= \chi(X, \mathcal{O}_X(D)) + \sum_{i=1}^{n-1} (-1)^i h^0(X, R^if_*\mathcal{L}) + \sum_{i=0}^{n-1} (-1)^i h^0(X, R^if_*\mathcal{Q}). \end{aligned} \quad (3.5)$$

We deduce that $\chi(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(D_Y)) + \mathcal{P}$, where

$$\mathcal{P} = -\sum_i (-1)^i h^0(X, R^if_*\mathcal{L}) - \sum_i (-1)^i h^0(X, R^if_*\mathcal{Q}). \quad (3.6)$$

The second part of the proof depends on the exceptional locus of f. Write E_j for the exceptional divisors over the singular points, and set

$$f^*D = D_Y + F, \quad \text{where} \quad F = \sum_j m_j E_j \quad \text{with } m_j \in \mathbb{Q}. \tag{3.7}$$

The exceptional divisor F here is the fixed part of the birational transform of the linear system $|D + H|$ for any sufficiently ample Cartier divisor H on X.

Then

$$\chi(Y, \mathcal{O}_Y(D_Y)) - \mathrm{RR}(D) = \sum_{i>0} \frac{1}{i!}(-F)^i \operatorname{Td}_{n-i}(T_Y). \tag{3.8}$$

In fact, our interpretation $\mathrm{RR}(D)$ of $\mathrm{ch}(D) \cdot \mathrm{Td}(T_X)$ replaces $\mathrm{Td}_n(T_X)$ by $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = \mathrm{Td}_n(T_Y)$, and $D^i \operatorname{Td}_{n-i}(X)$ by $(f^*D)^i \operatorname{Td}_{n-i}(Y)$, whereas the terms in RR on Y are $D_Y^i \operatorname{Td}_{n-i}(Y)$. Therefore the difference in (3.8) is

$$\sum_{i>0} \frac{1}{i!}(D_Y^i - (f^*D)^i) \operatorname{Td}_{n-i}(T_Y). \tag{3.9}$$

However, f^*D is orthogonal to the exceptional divisors, and one checks using the binomial expansion that $D_Y^i - (f^*D)^i = (D_Y - f^*D)^i = (-F)^i$.

In conclusion, the difference required in Theorem 3.2 is

$$\chi(X, \mathcal{O}_X(D)) - \mathrm{RR}(D) = \mathcal{P} + \sum_{i>0} \frac{1}{i!}(-F)^i \operatorname{Td}_{n-i}(T_Y). \tag{3.10}$$

We can choose the resolution of singularities of X and D depending only on the local analytic type of X and D. The resolution determines the sheaves $\mathcal{L}$ and $\mathcal{Q}$ and their higher direct images, so the quantity $\mathcal{P}$, and it determines the fixed part F and its intersection numbers. This proves the theorem.

3.2. The main proof. The Main Theorem 1.3 follows formally from the above arguments together with Proposition 1.7. The plan of the proof: for an orbifold point Q, the local analytic type of $\big(X, \mathcal{O}_X(mD)\big)$ is periodic in m, so also the fractional contributions $c_Q(mD)$ of Theorem 3.2. The argument of [3], Theorem 8.5, calculates them by equivariant RR on a global quotient orbifold as Dedekind sums. 2.3 tells us how to replace the Dedekind sums by ice cream functions, which are integral and Gorenstein symmetric of the given canonical weight. After subtracting these off, we obtain an integer valued Hilbert polynomial for $m \gg 0$ that is Gorenstein symmetric, to which Proposition 1.7 applies.

Step 1. The local contributions of Theorem 3.2 making up the difference $\chi(mD) - \mathrm{RR}(mD)$ were calculated in [3], Theorem 8.5 for an isolated orbifold point of type $\frac{1}{r}(a_1, \dots, a_n)$.

Theorem 3.3. *Let X be a projective n-fold with a basket of isolated cyclic orbifold points $\mathcal{B} = \{Q = \frac{1}{r}(a_1, \dots, a_n)\}$, and D a $\mathbb{Q}$-Cartier Weil divisor. Then for $m \in \mathbb{Z}$,*

$$\chi(X, \mathcal{O}_X(mD)) = \mathrm{RR}(mD) + \sum_{Q \in \mathcal{B}} c_Q(mD), \tag{3.11}$$

where

$$c_Q(mD) = (\sigma_{r-m} - \sigma_0)\left(\frac{1}{r}(a_1, \dots, a_n)\right). \tag{3.12}$$

Recall the main idea of the proof: by Theorem 3.2, the contributions depend only on the analytic type of (X, mD). Thus we can reduce to the case of a global quotient $X = M/\boldsymbol{\mu}_r$ having all fixed points of the same type $\frac{1}{r}(a_1, \dots, a_n)$. The result then follows by equivariant RR (that is, the Lefschetz fixed point theorem).

Step 2. Ignoring for the moment finitely many initial terms, as is traditional in treating Hilbert polynomials, we replace the genuine Hilbert series $P_{X,D}(t) = \sum_{m\geqslant 0} h^0(X, mD)t^m$ by the series $P^{\chi}_{X,D}(t) = \sum_{m\geqslant 0} \chi(X, mD)t^m$. Since in (3.11) $\mathrm{RR}(mD)$ is a polynomial of degree n and the $c_Q(mD)$ are periodic, summing them gives a term of the form $A(t)/(1-t)^{n+1}$ with $A(t) \in \mathbb{Q}[t]$ plus periodic terms of the form $B(t)/(1-t^r)$ for each orbifold point.

Now Corollary 2.11 says that the mth term in $P_{\mathrm{orb}}(Q, k_X)$ matches the periodic correction $c_Q(mD)$, so that subtracting off our ice cream functions P_{orb} reduces us to a formal power series

$$P^0_I(t) = P^{\chi}_{X,D}(t) - \sum_{Q\in\mathcal{B}} P_{\mathrm{orb}}\left(\frac{1}{r}(a_1, \dots, a_n), k_X\right), \tag{3.13}$$

where $(1-t)^{n+1}P^0_I(t)$ is a polynomial. It follows as usual that the coefficient of t^m in $P^0_I(t)$ is a polynomial $H(m)$ of degree n for $m \gg 0$, a modified Hilbert polynomial.

Step 3. Now $H(x)$ satisfies the assumptions of Proposition 1.7. Indeed, it is integer valued because $\chi(\mathcal{O}_X(mD))$ and the coefficients of the power series P_{orb} are all integers by (2.29). Moreover, $H(k-x) = (-1)^n H(x)$ because $\chi(\mathcal{O}_X((k-m)D)) = (-1)^n\chi(\mathcal{O}_X(mD))$ by Serre duality, and we know by Exercise 2.10 that $\sigma_{k-m} = (-1)^n\sigma_m$.

Step 4. We define the initial part P_I in terms of the modified Hilbert polynomial:

$$P_I(t) = \sum_{m\geqslant 0} H(m)t^m. \tag{3.14}$$

By construction, the two formal power series $P_{X,D}(t)$ and $P_I(t) + \sum_Q P_{\mathrm{orb}}(t)$ coincide except for an initial segment (since the first $\lfloor\frac{c}{2}\rfloor$ coefficients of $P_{\mathrm{orb}}(t)$ are zero). This proves Addendum 1.4.

Step 5. By Appendix 1.4, $P_I(t)$ has denominator $(1-t)^{n+1}$ and numerator a palindromic polynomial of degree $n + k_X + 1$, and is therefore determined by its first $\lfloor\frac{c}{2}\rfloor$ coefficients. Finally, if $R(X, K_X)$ is Gorenstein, then these coefficients are equal to the first $\lfloor\frac{c}{2}\rfloor$ values of $h^0(X, mD)$. This completes the proof.

3.3. K3 surfaces and Fano 3-folds. Theorem 1.3 simplifies known results on K3s and Fano 3-folds (see Altınok, Brown and Reid [5]). Let (S, D) be a polarized K3 surface with a basket of orbifold points $\mathcal{B} = \{\frac{1}{r}(a, r-a)\}$. By [3], Appendix to Section 8,

$$\sigma_i = \frac{r^2-1}{12r} - \frac{\overline{bi}(r-\overline{bi})}{2r}, \tag{3.15}$$

where $ab = 1 \mod r$ and $\overline{}$ denotes the smallest nonnegative residue $\mathrm{mod}\, r$. By Theorem 2.8,

$$\mathrm{InvMod}\left((1-t^a)(1-t^{r-a}), \frac{1-t^r}{1-t}\right) \equiv -\frac{1}{2r}\sum_{i=1}^{r-1} \overline{bi}(r-\overline{bi})t^i$$

and

$$\mathrm{InvMod}\left(\frac{(1-t^a)(1-t^{r-a})}{(1-t)^2}, \frac{1-t^r}{1-t}\right) \equiv -\frac{(1-t)^2}{2r}\sum_{i=1}^{r-1}\overline{bi}(r-\overline{bi})t^i.$$

Applying RR for surfaces ([5], Theorem 4.6) gives the Hilbert series

$$P_S(t) = \frac{1+t}{1-t} + \frac{t+t^2}{(1-t)^3}\,\frac{D^2}{2} - \sum_{\mathcal{B}}\frac{1}{1-t^r}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})}{2r}t^i. \tag{3.16}$$

We can parse $P_S(t)$ into the ice cream functions of Theorem 1.3 as follows. Comparing the coefficients of t in (3.16) yields

$$D^2 = 2g-2+\sum_{\mathcal{B}}\frac{b(r-b)}{r}, \tag{3.17}$$

where the genus g is defined by $P_1 = h^0(S,\mathcal{O}_S(D)) = g+1$. Then $P_S(t) = P_I + \sum_{\mathcal{B}} P_{\mathrm{orb}}$, where

$$P_I = \frac{1+(g-2)t+(g-2)t^2+t^3}{(1-t)^3} = \frac{1+t}{1-t} + (g-1)\frac{t+t^2}{(1-t)^3}, \tag{3.18}$$

and one checks as above that

$$P_{\mathrm{orb}} = \frac{\mathrm{InvMod}\left(\frac{(1-t^a)(1-t^{r-a})}{(1-t)^2}, \frac{1-t^r}{1-t}, 2\right)}{(1-t)^2(1-t^r)} \tag{3.19}$$

$$= \frac{t+t^2}{(1-t)^3}\,\frac{b(r-b)}{2r} - \frac{1}{1-t^r}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})}{2r}t^i. \tag{3.20}$$

Indeed, the coindex is $c=3$ and the numerator of (3.20) is supported in $[2,\dots,r]$.

Corollary 3.4. *Let V be a $\mathbb{Q}$-Fano 3-fold with basket $\mathcal{B} = \left\{\frac{1}{r}(1,a,r-a)\right\}$ of terminal quotient singularities. The Hilbert series of its anticanonical ring is $P_V(t) = P_I + \sum_{\mathcal{B}} P_{\mathrm{orb}}$, with*

$$P_I = \frac{1+(g-2)t+(g-2)t^2+t^3}{(1-t)^4}, \tag{3.21}$$

where $h^0(-K_X) = g+2$ and $-K^3 = 2g-2+\sum\frac{b(r-b)}{r}$, and

$$P_{\mathrm{orb}} = \frac{\mathrm{InvMod}\left(\frac{(1-t)(1-t^a)(1-t^{r-a})}{(1-t)^3}, \frac{1-t^r}{1-t}, 2\right)}{(1-t)^3(1-t^r)}. \tag{3.22}$$

Proof. By [5], Theorem 4.6, the Hilbert series of $(V,-K_V)$ equals

$$P_V(t) = \frac{1+t}{(1-t)^2} - \frac{t+t^2}{(1-t)^4}\,\frac{K_V^3}{2} - \sum_{\mathcal{B}}\frac{1}{(1-t)(1-t^r)}\sum_{i=1}^{r-1}\frac{\overline{bi}(r-\overline{bi})}{2r}t^i. \tag{3.23}$$

The coefficient of t gives the stated value of $-K_V^3$. Clearly, Fano 3-folds and K3 surfaces have coindex 3 and the same Inverse Mod polynomials:

$$\operatorname{InvMod}\left(\frac{(1-t)(1-t^b)(1-t^{r-b})}{(1-t)^3}, \frac{1-t^r}{1-t}\right) \equiv \operatorname{InvMod}\left(\frac{(1-t^b)(1-t^{r-b})}{(1-t)^2}, \frac{1-t^r}{1-t}\right). \qquad \square$$

Exercise 3.5. Consider the following general weighted projective hypersurfaces:
1) $S_5 \subset \mathbb{P}(1,1,1,2)$ with an orbifold point of type $\frac{1}{2}(1,1)$ at $Q = (0,0,0,1)$;
2) $S_7 \subset \mathbb{P}(1,1,2,3)$ with basket $\{\frac{1}{2}(1,1), \frac{1}{3}(1,2)\}$;
3) $S_{11} \subset \mathbb{P}(1,2,3,5)$ with basket $\{\frac{1}{2}(1,1), \frac{1}{3}(1,2), \frac{1}{5}(2,3)\}$.

All three are K3 surfaces and have $k_{S_i} = 0$ and $c = 3$. Their Hilbert series parsed as $P_{S_i}(t) = P_I + \sum_{\mathcal{B}_i} P_{\text{orb}}$ are as follows:

$$P_{S_5}(t) = \frac{1-t^5}{(1-t)^3(1-t^2)} = \frac{1+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)},$$

$$P_{S_7}(t) = \frac{1-t^7}{(1-t)^2(1-t^2)(1-t^3)} = \frac{1-t-t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)},$$

$$P_{S_{11}}(t) = \frac{1-t^{11}}{(1-t)(1-t^2)(1-t^3)(1-t^5)} = \frac{1-2t-2t^2+t^3}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^2)} + \frac{t^2+t^3}{(1-t)^2(1-t^3)} + \frac{2t^2+t^3+t^4+2t^5}{(1-t)^2(1-t^5)}.$$

§ 4. Towards the nonisolated case

This final section speculates on the shape of the Hilbert series of quasismooth orbifolds with higher dimensional orbifold loci, and discusses some partial results. We now have abundant experience of working with these, and definitive results in special cases, such as the Calabi–Yau 3-fold orbifolds of Buckley's thesis [1]. Our conjectures in the case of 1-dimensional orbifold locus are fairly specific, and in principle within reach of our methods, although we do not yet venture formal proofs.

It is traditional to discuss Hilbert functions in terms of Zariski's notion of *cyclic polynomial*, an integral function $H(n)$ represented for $n \gg 0$ by r polynomials $f_0, \dots, f_{r-1}$ according to n modulo r. In the isolated orbifold case discussed so far, the f_i differ only in their constant terms, so that $H(n)$ behaves periodically with period r, giving the Hilbert series $P(t) = \sum H(n)t^n$ a simple pole at $\boldsymbol{\mu}_r$. In the more general case, the f_i differ by terms that grow, giving $P(t)$ higher order poles. The order of poles corresponds to one plus the dimension of the strata: for if X has a $\frac{1}{s}$ orbifold stratum of dimension ν, its graded ring $R(X,D)$ has at least $\nu+1$ generators x_i of degree b_i divisible by s, and then $P(t)$ usually has poles of order $\nu+1$ at the primitive sth roots of 1.

Dissident points are nonisolated orbifold points Q where the inertia group jumps. Experiments and the results of [1] and [2] suggest that for these, the fractional strictly periodic contribution $\frac{\Delta}{1-t^r}$ given by generalized Dedekind sums (Definition 2.5) can be replaced by an integral term, at the expense of modifying the contributions corresponding to adjacent strata.

4.1. Calabi–Yau 3-folds. The following results illustrate these points. A quasismooth Calabi–Yau 3-fold (X, D) has orbifold locus consisting of:

(a) curves C of generic transverse type $\frac{1}{s}(a, s-a)$;

(b) points Q of type $\frac{1}{r}(a_1, a_2, a_3)$ with $a_1 + a_2 + a_3 \equiv 0 \mod r$; if $\operatorname{hcf}(r, a_i) = s_i > 1$, then Q is a dissident point on a $\frac{1}{s_i}$ curve C.

To handle the periodicity of the Hilbert series of X, we have a choice between two alternative strategies. First, a partial fraction decomposition over $\mathbb{Q}$ with parts having small denominators, directly linked to the strata of the orbifold locus.

Theorem 4.1. *Let (X, D) be as above. Then its Hilbert series is*

$$P_{X,D}(t) = \mathrm{I}_{X,D} + \sum_Q \mathrm{II}_Q + \sum_C (\mathrm{III}_C + \mathrm{IV}_C). \tag{4.1}$$

Here the parts are:

$$\mathrm{I}_{X,D} = 1 + \frac{t}{(1-t)^2}\,\frac{Dc_2}{12} + \frac{(t+4t^2+t^3)}{(1-t)^4}\,\frac{D^3}{6} \tag{4.2}$$

with Dc_2 and D^3 as in RR (*they are*, *however*, *rational numbers*; *the same applies to the degree DC below*);

$$\mathrm{II}_Q = \frac{\sum(\sigma_i - \sigma_0)t^i}{1-t^r} = \frac{\Delta\big(\frac{1}{r}(a_1, a_2, a_3)\big)}{1-t^r}, \tag{4.3}$$

with $\sigma_i = \sigma_i\big(\frac{1}{r}(a_1, a_2, a_3)\big)$ the Dedekind sum for Q (we note that $\sigma_0 = 0$ and $\sigma_i = -\sigma_{r-i}$ by Exercise 2.10);

$$\mathrm{III}_C = \left(\frac{st^s\Delta}{(1-t^s)^2} + \frac{t\Delta'}{1-t^s} - \frac{\sigma_0 t}{(1-t)^2}\right)DC, \tag{4.4}$$

where σ_0 and Δ are now the Dedekind sums for the transverse section $\frac{1}{s}(a, s-a)$ of Γ, $\Delta' = d\Delta/dt$, and $\sigma_0\big(\frac{1}{s}(a, s-a)\big) = \frac{s^2-1}{12s}$;

$$\mathrm{IV}_C = \frac{N_C}{72 s \tau_C} \times \frac{B}{1-t^s}, \tag{4.5}$$

where the second term has numerator

$$B = t \cdot \operatorname{InvMod}\left(t(1-t^a)^2(1-t^{s-a}), \frac{1-t^s}{1-t}\right) - t \cdot \operatorname{InvMod}\left(t(1-t^a)(1-t^{s-a})^2, \frac{1-t^s}{1-t}\right) \in \mathbb{Q}[t] \tag{4.6}$$

with support $[t, \dots, t^{s-1}]$, *determined by*

$$(1 - t^a)^2 (1 - t^{s-a})^2 B \equiv t^a - t^{s-a} \mod \frac{1 - t^s}{1 - t}, \tag{4.7}$$

and the coefficient $\frac{N_C}{12s^2\tau_C} \in \mathbb{Q}$ *depends on the topology of a tubular neighbourhood of* C *in* X (*as described in* [1], *Theorem* 2.1). *The dissident points give rise to the extra denominator* τ_C *and, in spirit,* $N_C \in \mathbb{Z}$ *is the difference between the degrees of the isotypical components of the normal bundle to* C; *interchanging* $a \leftrightarrow s - a$ *sends* $N_C \mapsto -N_C$.

This result follows on replacing the individual terms P_m in the formulas of [1], Theorem 2.1, by their corresponding Hilbert series $\sum P_m t^m$ in closed form (with further calculations that are somewhat involved); it also follows from the results of [2], Section 5.1. We omit the details since our main aim is to motivate the philosophy of higher dimensional ice cream, and the detailed statements are not really the issue.

The second partial fraction decomposition has bigger denominators: each part has denominator $\prod(1 - t^a)$, a product of $n + 1 = 4$ factors. The parts are one further step removed from the topological characters of (X, D) appearing in RR. The advantage, however, is that each part is integral and Gorenstein symmetric of the same degree $k_X = 0$.

Theorem 4.2. *Let* X, D *be as above. Then its Hilbert series is*

$$P_X(t) = P_I + \sum_Q P_{\mathrm{orb}}(Q, k_X) + \sum_C A_C + \sum_C B_C, \tag{4.8}$$

where, as in Theorem 1.3, *the initial part* P_I *depends on the first* $\lfloor \frac{c}{2} \rfloor = 2$ *plurigenera and* $P_{\mathrm{orb}}(Q, k_X)$ *are ice cream functions at the point strata*;

$$A_C = \frac{P_{\mathrm{orb}}(\frac{1}{s}(a, s-a), s)}{1 - t^s}\, \delta C \tag{4.9}$$

with δC *an integer corresponding to the degree of* C *appropriately modified by the dissident points* (*see* 4.3), *and*

$$B_C = \frac{\mathrm{Num}\, B_C}{(1 - t)^3 (1 - t^s)}, \tag{4.10}$$

with numerator an integral palindromic polynomial of symmetric degree $s + 3$ *and short support* $[3, \dots, s]$.

The quantities δC and $\mathrm{Num}\, B_C$ are deduced by recombining the result of Theorem 4.1, and can be calculated in any particular example without difficulty, but the formulas are cumbersome to state. The part B_C depends on the degree DC and the isotypical components of its normal bundles, as modified by the dissident points (see 4.3). It introduces a periodicity mod s, whereas A_C grows linearly in s. We shall return to the general significance of these two expressions and the relation between them in 4.3.

Example 4.3. The Calabi–Yau 3-fold

$$X_{40} \subset \mathbb{P}(2,5,8,10,15)_{\langle x,y,z,t,u\rangle} \tag{4.11}$$

has the $\frac{1}{2}(1,1)$ curve $\Gamma_2 = X_{40} \cap \mathbb{P}^2(2,8,10)_{\langle x,z,t\rangle}$ of degree $\frac{1}{2}$, and the $\frac{1}{5}(2,3)$ curve $C_5 = X_{40} \cap \mathbb{P}^2(5,10,15)_{\langle y,t,u\rangle}$ of degree $\frac{4}{15}$ passing through the $\frac{1}{15}(2,5,8)$ dissident point P_u. One computes the two alternative expressions for its Hilbert series:

$$\begin{aligned} P(t) &= \frac{1-t^{40}}{\prod_{a\in\{2,5,8,10,15\}}(1-t^a)} = \mathrm{I} + \mathrm{II} + \mathrm{III}_{\Gamma_2} + \mathrm{IV}_{\Gamma_2} + \mathrm{III}_{C_5} + \mathrm{IV}_{C_5} \\ &= P_I + P_{\mathrm{orb}}\left(\frac{1}{15}(2,5,8),0\right) + A_{\Gamma_2} + B_{\Gamma_2} + A_{C_5} + B_{C_5}. \end{aligned} \tag{4.12}$$

Here the parts of the first expression are

$$\begin{gathered} \mathrm{I} = 1 + \frac{113}{240}\,\frac{t}{(1-t)^2} + \frac{1}{1800}\,\frac{t+4t^2+t^3}{(1-t)^4}, \qquad \mathrm{II} = \frac{\Delta(\frac{1}{15}(2,5,8))}{1-t^{15}}, \\ \mathrm{III}_2 = -\frac{1}{8}\,\frac{t+t^3}{(1-t^2)^2}, \qquad \mathrm{IV}_2 = 0, \\ \mathrm{III}_5 = \frac{4}{15}\,\frac{1+t^2+t^3}{(1-t^5)^2}, \qquad \mathrm{IV}_5 = \frac{4}{25}\,\frac{t^4-\frac{8}{3}t^3+t^2-t-5}{1-t^5}, \end{gathered} \tag{4.13}$$

where $\Delta\big(\frac{1}{15}(2,5,8)\big)$ is the Dedekind sum polynomial of Theorem 2.8:

$$\begin{aligned} \Delta &= t(1-t^5)\,\mathrm{InvMod}\big(t(1-t^5)\cdot(1-t^2)(1-t^5)(1-t^8), 1+t^5+t^{10}\big) \\ &= \frac{1}{9}(t+2t^2+t^4-t^5-2t^7+2t^8+t^{10}-t^{11}-2t^{13}-t^{14}). \end{aligned}$$

The parts of the second expression are

$$\begin{gathered} P_I = 1 + \frac{t^2}{(1-t)^4} = \frac{1-4t+7t^2-4t^3+t^4}{(1-t)^4}, \\ P_{\mathrm{orb}}\left(\frac{1}{15}(2,5,8),0\right) = \frac{t^8-t^9+t^{10}-t^{11}+t^{12}-t^{13}+t^{14}}{(1-t)^2(1-t^5)(1-t^{15})}, \\ A_{\Gamma_2} = \frac{P_{\mathrm{orb}}(\frac{1}{2}(1,1),2)}{1-t^2}, \qquad B_{\Gamma_2} = 0, \\ A_{C_5} = \frac{P_{\mathrm{orb}}(\frac{1}{5}(2,3),5)}{1-t^5}, \qquad B_{C_5} = \frac{-3t^3+2t^4-3t^5}{(1-t)^3(1-t^5)}. \end{gathered}$$

As a little exercise, we propose the analogous calculations for the Calabi–Yau 3-fold $X_{80} \subset \mathbb{P}(3,4,15,20,38)$ (or other cases from the vast lists of Kreuzer and Skarke).

4.2. A general conjecture. Let $P(t) = \frac{H(t)}{\prod_{i=1}^N(1-t^{b_i})}$ be a rational function with integral numerator $H(t) \in \mathbb{Z}[t]$ satisfying Gorenstein symmetry (1.7). For example, P might be the Hilbert series of a Gorenstein graded ring R of dimension $n+1$ and canonical weight k_R (more generally, a finite Gorenstein graded module M over a polynomial ring with $(n+1)$-dimensional support).

Conjecture 4.4. *Under the above assumptions, $P(t)$ has a unique partial fraction decomposition of the form*

$$P(t) = \sum_A \frac{N_A}{\prod_{a\in A}(1-t^a)}\,. \tag{4.14}$$

The sum runs over sequences $A = \{a_1, \dots, a_{n+1}\}$ consisting of a main period $r = a_{n+1}$ and some divisors $a_i \mid r$ (some or all of the a_i may be 1 or r); each a_i divides one of the original b_j, so that a priori only finitely many A occur. The numerator N_A of each part is an integral polynomial that is symmetric of degree $k_A = k + \sum_{a\in A} a$, so that the part as a whole has the same Gorenstein symmetry; moreover $N_A(t)$ is 'of shortest support', a minimal residue modulo

$$F_A = \frac{1-t^r}{\operatorname{hcf}\bigl(1-t^r, \prod_{a\in A, a<r}(1-t^a)\bigr)} \tag{4.15}$$

(as in (2.15)) supported in an interval of length $< \deg F_A$ centred at $k_A/2$.

To be on the safe side, we could restrict to quasismooth orbifolds.

If all the $b_i = 1$, there is only one part, and the result follows from Proposition 1.7. We expect the proof to be formal. The idea is to take account of the poles of $P(t)$ at roots of unity in terms of its principal parts. The A part should deal with the highest order principal part of $P(t)$ at primitive rth roots of unity, while possibly modifying the principal parts of higher order poles at nonprimitive rth roots. The parts document the periodicity of an integral cyclic polynomial, so should have coefficients in $\mathbb{Z}$.

Remark 4.5. Conjecture 4.4 delinks Hilbert series and the topological terms in RR. The conventional narrative is that RR expresses the coherent cohomology invariants of a projective variety in terms of topological data. However, in dimension $\geqslant 4$ one does not necessarily aspire to a detailed understanding of all the Chern numbers in the RR formula. For example, no-one claims intimate acquaintance with $c_1^2c_2$, c_1c_3 and c_2^2 in the Todd genus

$$\mathrm{Td}_4 = \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4). \tag{4.16}$$

In fact, in work with 3-folds, we commonly treat the quantity $\frac{Dc_2}{12}$ as a basic invariant, deducing its numerical value from the plurigenera, rather than the other way around. For a canonical 4-fold (say), the plurigenera P_1, P_2, P_3 are just integers, and in our treatment, the initial part

$$P_I = \frac{1 + a_1t + a_2t^2 + a_3t^3 + a_2t^4 + a_1t^5 + t^6}{(1-t)^5} \tag{4.17}$$

with $a_1 = P_1 - 5$, $a_2 = P_2 - 5P_1 + 10$, $a_3 = P_3 - 5P_2 + 10P_1 - 10$ holds comparatively few terrors for us, and is arguably a better starting point than the Chern numbers; for example, they are integers with no implicit congruences of the type $12 \mid c_1^2 + c_2$.

In the same way, even without tying the orbifold parts P_A very closely to topological invariants (such as the degree of curve orbifold strata and the isotypical components of their normal bundle), we have formulas that depend in principle only on a small basket of integers.

4.3. The case of curve orbifold locus. Let (X, D) be a quasismooth projectively Gorenstein n-fold orbifold of dimension $n \geqslant 2$ with orbifold locus of dimension $\leqslant 1$. As before, its orbifold strata are:

(a) curves Γ of transverse type $\frac{1}{s}(a_1, \dots, a_{n-1})$;

(b) points Q of type $\frac{1}{r}(a_1, \dots, a_n)$.

Write $s_i = \operatorname{hcf}(a_i, r)$. Dissident points are characterized by having some s_i a nontrivial factor of r, with $1 < s_i < r$. The x_i-axis is then pointwise fixed by $\boldsymbol{\mu}_{s_i}$, so its image is contained in a $\frac{1}{s_i}$ orbifold stratum of X. Our assumption on the dimension of the orbifold locus implies that the s_i are pairwise coprime, and Q is in the closure of orbifold curve strata Γ_i of transverse type $\frac{1}{s_i}(a_1, \dots, \widehat{a}_i, \dots, a_n)$.

We summarize the logic of Conjecture 4.4 in this case. We treat the $\frac{1}{s}$ orbifold curves (a) by adding contributions of the form

$$c_\Gamma(t) = \frac{\mathrm{Num}_{D_C}}{(1-t^s)^2(1-t)^{n-1}} + \frac{\mathrm{Num}_{N_C}}{(1-t^s)(1-t)^n}, \tag{4.18}$$

where the numerators are integral, Gorenstein symmetric of the appropriate degree, and with short support. We expect to see the $(1-t^s)^2$ in the denominator for the reason outlined at the start of Section 4. In the numerators, D_C and N_C refer to quantities involving the degree of C, respectively of the isotypical components of its normal bundle. Multiplying (4.18) by $1-t^{ms}$, corresponding to taking a transverse section by a general hypersurface in $|msD|$ for some m, leaves Num_{D_C} distinguished as the numerator of an isolated orbifold point (times $m \times \deg C$). The N_C term is destroyed by taking a hyperplane section, and cannot be recovered after so doing.

We deal with points (b) by putting in ice cream of the form

$$P_{\mathrm{orb}}(Q, k_X) = \frac{\mathrm{InvMod}(A, F, \gamma)}{\prod_{a \in [s_1, \dots, s_n, r]}(1-t^a)}, \quad \text{with} \quad A = \prod \frac{1-t^{a_i}}{1-t^{s_i}}, \tag{4.19}$$

with F as in (2.15), and γ chosen to make the numerator Gorenstein symmetric of degree $k_X + r + \sum s_i$.

The contribution $P_{\mathrm{orb}}(Q, k_X)$ is well defined, integral and Gorenstein symmetric of degree k (see Proposition 2.12). It has the right periodicity modulo r by an argument similar to Corollary 2.11. The curious point, however, is that when $s_i > 1$, it usually contains contributions with denominator $(1-t^{s_i})^2(1-t)^{n-1}$ and $(1-t^{s_i})(1-t)^n$ that might at first sight appear to be native to the curves C_i of type $\frac{1}{s_i}(a_1, \dots, \widehat{a}_i, \dots, a_n)$ through P.

A dissident point Q of type $\frac{1}{r}$ on an orbifold $\frac{1}{s}$ curve Γ commonly forces the degree of Γ and of the isotypical components of its normal bundle to become fractional with denominator r, thus adding fractional terms into (4.18). Attributing fractional terms with denominator $(1-t^{s_i})^2(1-t)^{n-1}$ and $(1-t^{s_i})(1-t)^n$ to the dissident point is the same idea as adding a global fractional term with denominator $(1-t)^{n+1}$ into the local contribution from an isolated orbifold point, as discussed in Caution 1.6.

The meaning of Proposition 2.12 is that $P_{\mathrm{orb}}(Q, k_X)$ can be viewed as obtained from the Dedekind sum term $\frac{\Delta}{1-t^r}$ by multiplying top and bottom of the fraction by $\prod(1-t^{s_i})$, then folding the numerator back into the required interval. Since the denominator of $P_{\mathrm{orb}}(Q, k_X)$ is $\prod_{a\in[s_1,\dots,s_n,r]}(1-t^a)$, the difference

$$\frac{\Delta}{1-t^r} - P_{\mathrm{orb}}(Q, k_X) \tag{4.20}$$

between the Dedekind polynomial and the ice cream function has $\prod(1-t^{s_i})^2$ in its denominator. This difference has a unique partial fraction decomposition defined over $\mathbb{Q}$ with parts having respective denominators

$$(1-t)^{n-2}(1-t^{s_i})^2, \quad (1-t)^{n-1}(1-t^{s_i}) \quad \text{and} \quad (1-t)^n, \tag{4.21}$$

and with numerators of short support. Our assertion is that if we parse the Hilbert series allocating these local parts to the adjacent curves and to the initial part, we achieve a decomposition with every part integral and Gorenstein symmetric. Thus using ice cream $P_{\mathrm{orb}}(Q, k_X)$ in place of the more obvious r periodicity contribution $\frac{\Delta}{1-t^r}$ effectively cuts the dissident points out of the curve strata C, modifying their degrees and those of the isotypical components of their normal bundle to be integral.

We believe that for 1-dimensional orbifold locus, the formal statement and proof of Conjecture 4.4 should be within reach of the strategies of Buckley's thesis [1]. Her proof in the Calabi–Yau case involves two ingredients: taking cyclic covers in the style of [3] introduces the Dedekind sums at the dissident points. She deals with the 1-dimensional orbifold locus by resolving singularities by standard toric resolutions, then calculating exceptional divisors in the style of our 3.1. Localizing around the orbifold strata more generally is a stacky business, and when we are forced to wear that hair shirt, we can also hope for progress by combining the stacky methods of [12] and [2].

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Anita Buckley
Department of Mathematics,
University of Ljubljana, Slovenia
E-mail: Anita.Buckley@fmf.uni-lj.si

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Miles Reid
Mathematics Institute, University of Warwick,
England
E-mail: Miles.Reid@warwick.ac.uk

Shengtian Zhou
Høgskolen i Telemark, Notodden,
Norway
E-mail: Shengtian.Zhou@hit.no