Orbifold Riemann–Roch for Threefolds and Applications to Calabi–Yaus

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Declaration

This thesis is the result of my own work except when explicitly stated otherwise. Chapter 1 (except 1.4) and the Appendix recall material from literature. To the best of my knowledge, the formula in 1.4 and the proofs in Chapter 2 are original. Chapters 3 and 4 apply the formula to the construction and study of new examples.

Introduction

The aim of this thesis is to state and prove the Riemann–Roch formula for threefolds with quotient singularities. Let X be a normal projective threefold and D a Weil divisor which is $\mathbb{Q}$–Cartier and Cartier outside $\operatorname{Sing} X$. We prove

$$\chi(X, \mathcal{O}_X(D)) = \text{ RR type expression in } D + \sum_P c_P(D) + \sum_C s_C(D),$$

where $\sum_P c_P(D)$ are the contributions from the singular points of $\mathcal{O}_X(D)$ and $\sum_C s_C(D)$ are the contributions from the curves of singularities. The contributions depend on the type of quotient singularities and on the embedding of $\operatorname{Sing} X$ in X.

Chapter 1 recalls notations and results from literature. It also states Theorem 1.4.2, which is the main result of the thesis.

In Chapter 2 we prove the stated formula. The contributions from the A_{r-1} curves of singularities are computed in two ways. First we resolve the singularities

$$f \colon Y \longrightarrow X,$$

so that $f_* \mathcal{O}_Y(D') = \mathcal{O}_X(D)$, and then use the Riemann–Roch formula for D' on the smooth Y. The resolution also determines the shape of the formula. In the second proof we use the Atiyah–Singer equivariant Riemann–Roch formula on the auxiliary cover

$$\pi \colon Z \longrightarrow X,$$

where π is a (local) quotient morphism for $\mu_r \oplus \mu_r$ group action. The formula follows from comparing the singularities of $\mathcal{O}_X(D)$ and $\mathcal{L}_{i,j}$, which is the (i,j)th eigensheaf of the $\mu_r \oplus \mu_r$ action on $\pi_* \mathcal{O}_Z$.
The contributions from the singular points

$$\frac{1}{s}(a_1, a_2, a_3)$$

are local analytic invariants and depend only on the type of singularity. Thus, they can be computed on a suitable model by equating $\mathcal{O}_X(D)$ with $\mathcal{L}_i$, the eigensheaf of the μ_s action with weights a_1, a_2, a_3.

The Riemann–Roch formula is proved with the purpose to construct new examples of Calabi–Yau threefolds. In Chapter 3 we state the formulae for

$h^0(X, \mathcal{O}_X(D))$ and the Hilbert series. We describe the algorithm for the computer search of Calabi–Yaus in weighted projective spaces, by using the properties of the following graded ring

$$R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n)).$$

In Chapter 4 we find examples of Calabi–Yaus in codimensions $1, \ldots, 5$. We give a thorough analysis of Calabi–Yau 3–folds that are complete intersections in the weighted Grassmannian $\mathrm{wGr}(2,5)$ or orthogonal Grassmannian $\mathrm{wOGr}(5,10)$ of Corti and Reid [8]. A method for computing their Hodge numbers by using the Type I unprojection is described in Section 4.3.
The technique of unprojections is also used to construct varieties in codimension 4.
In Section 4.1 we classify the quotient singularities of complete intersections in weighted projective spaces.

Chapter 5 explores interesting questions, new ideas and possibilities that arise throughout the thesis. A number of these should serve as suggestions for further research. We suggest using our computer algorithm to generate new examples; and then applying the technique of unprojection to analyse these examples, their Hodge numbers and mirror pairs.

The Appendix recalls some additional material from literature. We prove the Riemann–Hurwitz formula for smooth 3–folds and define the mirror pairs for smooth Calabi–Yaus. We include the definition of orbifolds and the dualising sheaf ω_X.

1 Preliminaries

1.1 Basic definitions and notation

Let X be a normal projective threefold. We define:

$\mathrm{Div}\,(X) =$ the group of Weil divisors

$\mathrm{PDiv}\,(X) =$ the group of principal divisors

$\mathrm{Cl}\,(X) = \mathrm{Div}\,(X)\,/\,\mathrm{PDiv}\,(X)$

$\mathrm{Pic}\,(X) =$ Cartier divisors $/\,\mathrm{PDiv}\,(X) =$ the group of line bundles.

A $\mathbb{Q}$–*divisor* is a formal linear combination of prime divisors with coefficients in $\mathbb{Q}$. A divisor D is $\mathbb{Q}$–*Cartier* if mD is Cartier for some positive integer $m \neq 0$. The *canonical divisor* K_X is defined so that its restriction to the nonsingular part of X is a divisor of a rational 3–form. The corresponding sheaf $\omega_X = \mathcal{O}_X(K_X)$ is called the *canonical sheaf.*

Definition 1.1.1 Let $f\colon Y \to X$ be a resolution of singularities of a normal variety X. We say that X has *canonical singularities* if

(i) mK_X is Cartier for some positive integer m,

(ii) $mK_Y = f^*(mK_X) + \sum_i \alpha_i E_i$ with $\alpha_i \geq 0$,

where E_i are prime exceptional divisors.

If all $\alpha_i > 0$, then X has *terminal singularities.*

It is known [31] that canonical 3–fold singularities are Cohen–Macaulay, which means that the local rings are Cohen–Macaulay. A variety with canonical singularities is *Gorenstein* if K_X is Cartier. X is $\mathbb{Q}$–*factorial* if every Weil divisor is $\mathbb{Q}$–Cartier.

We sometimes add the condition that $\mathrm{rank}\,\mathrm{Pic}_{\mathbb{Q}}\,X = \mathrm{Pic}\,X \otimes \mathbb{Q}$ is 1, or even stronger $\mathrm{Cl}\,X = \mathbb{Z}\cdot D$ for some Weil divisor D. For example, if X is embedded in a weighted projective space as a complete intersection, then $\mathrm{Pic}_{\mathbb{Q}}\,X$ has rank 1 by [7, Lemma 3.5].

1.2 Embedding in a weighted projective space

A *weighted projective space* $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)$ is defined by

$$\mathbb{P} = \mathrm{Proj}\ \mathbb{C}[x_0, \ldots, x_n],$$

where $\deg x_i = a_i$. Note that $\mathbb{P}$ is a quotient of $\mathbb{C}^{n+1} - \{0\}$ by the $\mathbb{C}^*$ action

$$x_0, \ldots, x_n \mapsto \lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n.$$

Every weighted projective space is a normal projective variety with quotient singularities.

Let X be a normal projective threefold over $\mathbb{C}$ with canonical singularities. On X choose a Weil divisor D which is $\mathbb{Q}$–Cartier and Cartier outside $\mathrm{Sing}\, X$. By [27, Appendix to §1], there exists a bijective correspondence

$$\left\{ \begin{array}{c} \text{Weil divisors} \\ \text{on } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{divisorial subsheaves} \\ \mathcal{L} \subset k(X) \end{array} \right\} / \Gamma(X, \mathcal{O}_X)^*$$

$$D \quad \leftrightarrow \quad \mathcal{O}_X(D).$$

Definition 1.2.1 Let r be the index of a $\mathbb{Q}-$Cartier divisor D. Locally identify $\mathcal{O}_X(rD)$ with $\mathcal{O}_X$. Define an $\mathcal{O}_X$ algebra structure on

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \cdots \oplus \mathcal{O}_X((r-1)D),$$

by the multiplication of sections $\mathcal{O}_X(aD) \otimes \mathcal{O}_X(bD) \to \mathcal{O}_X((a+b)D)$.

The pair (X, D) is *well formed* if the above multiplication is an isomorphism in codimension zero and codimension one, and is *quasismooth* if the local $\mathbb{Z}_r$ covers $\mathrm{Spec}_X \mathcal{A} \to X$ are nonsingular at each point.

From here on we will always assume that (X, D) is quasismooth and well formed.

On X there exists an ample Cartier divisor H such that $D' = D + mH$ is also ample for some positive integer m. Note that the sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ have the same type of singularities. The same way as in Definition 1.2.1 define locally

$$\mathcal{B} = \mathcal{O}_X \oplus \mathcal{O}_X(D') \oplus \cdots \oplus \mathcal{O}_X((r-1)D').$$

Since $\mathrm{Spec}_X\mathcal{A}$ and $\mathrm{Spec}_X\mathcal{B}$ are isomorphic, (X, D') is well defined and quasismooth.

The finite dimensional vector spaces

$$H^0(X, nD') = H^0(X, \mathcal{O}_X(nD')) = \{f \in \mathbb{C}(X) \mid \mathrm{div}\ f + nD' \geq 0\}$$

define the graded ring

$$R(X, D') = \bigoplus_{n \geq 0} H^0(X, nD').$$

Since D' is ample and $\mathbb{Q}$–Cartier, $R(X, D')$ is finitely generated.

In all the cases we consider in this thesis, $R(X, D')$ is a Gorenstein ring. This happens if and only if K_X is $\mathbb{Q}$–Cartier and K_X is linearly equivalent to tD' for some $t \in \mathbb{Z}$. We say that such (X, D') is *projectively Gorenstein.*

In other words, Proj $R(X, D')$ is an embedding $i\colon X \hookrightarrow \mathbb{P}$ into a weighted projective space, where $\mathcal{O}_X(D')$ corresponds to $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}}(1)$. Here $\mathcal{O}_X(1)$ denotes the *twisting sheaf* of Serre.

Remark 1.2.2 Our assumptions imply that the embedded variety $X \subset \mathbb{P}(a_1, \ldots, a_n)$ is quasismooth and well formed (check [11] for definitions).

When X is quasismooth, its singularities are cyclic quotient singularities induced by the weights $a_1, \ldots, a_n$. Another important fact is that a quasismooth variety is an orbifold, that is, locally a complex manifold divided by a finite group of automorphisms. This will be used in Section 2.3.

When X is also well formed, the adjunction formula holds [11, Formula 6.14].

1.3 Quotient singularities

Definition 1.3.1 The quotient $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/\mu_r$, where μ_r acts on $\mathbb{A}^n$ by

$$\mu_r \ni \varepsilon\colon\ (x_1, \ldots, x_n) \mapsto (\varepsilon^{b_1}x_1, \ldots, \varepsilon^{b_n}x_n)$$

is a *cyclic quotient singularity of type* $\frac{1}{r}(b_1, \ldots, b_n)$. For the sake of clearer notation we always assume that all the b_i and r have no common divisor, which is equivalent to the action μ_r being effective. The sheaf $\pi_*\mathcal{O}_{\mathbb{A}^n}$ decomposes into r eigensheaves

$$\mathcal{L}_i = \{f \mid \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r\},$$

for $i = 1, \dots, r-1$. A singularity $Q \in X$ polarised by a Weil divisor D is a *cyclic quotient singularity of type* ${}_i\big(\frac{1}{r}(b_1, \dots, b_n)\big)$, if Q is locally isomorphic to a point of type $\frac{1}{r}(b_1, \dots, b_n)$ and $\mathcal{O}_X(D) \cong \mathcal{L}_i$ near the singular point.

The cyclic quotient singularities are special cases of quotient singularities. We say that $x \in X$ has a *quotient singularity* if locally

$$(x \in X) \cong (0 \in \mathbb{A}^n)/G$$

for a finite group G.

In Section 1.2 we restricted our study to a quasismooth, well formed threefold X embedded in a weighted projective space $\mathbb{P}$. The singularities of X are induced by the singularities of $\mathbb{P}$, which are cyclic quotient singularities. Since X is normal its singular locus has codimension 2. Thus $\operatorname{Sing} X$ consists of a union of intersecting curves and isolated points.

Examples 1.3.2
(1) $\mathbb{P}(1,4,5,15)$ with variables x, y, z_1, z_2 has an isolated singular point P_Y of type $\frac{1}{4}(1,1,3)$ and a curve of singularities C, which is the (z_1, z_2) line. A general point on C is of transverse type $\frac{1}{5}(1,4)$, but there is also a dissident point $P_{z_2} \in C$ which is of type $\frac{1}{15}(1,4,5)$.

(2) $X_{13} \subset \mathbb{P}(1,1,2,3,6)$ with variables x_1, x_2, y, z, t is a Calabi–Yau threefold with two curves of singularities. The first $C_1 = \{x_1 = x_2 = y = 0\}$ is of type $\frac{1}{3}(1,2)$ and the second $C_2 = \{x_1 = x_2 = z = 0\}$ is of type $\frac{1}{2}(1,1)$. They intersect in $(0,0,0,0,1)$ of type $\frac{1}{6}(1,2,3)$.

Through the entire thesis we will assume that all curves of singularities are compound Du Val singularities. It is well known that the only cyclic Du Val singularities are of type

$$A_{r-1} = {}_k\Big(\frac{1}{r}(1,-1)\Big).$$

The precise definition is the following:

Definition 1.3.3 Let X be a normal projective threefold with a curve C of singularities and D a divisor which is $\mathbb{Q}$–Cartier and Cartier away from C.

Take a generic hyperplane S which intersects C transversely in a finite number of points. Assume that every point in the intersection is a singular point of type $_k\big(\frac{1}{r}(1,-1)\big)$ on the polarised surface $(S, D|_S)$. A curve with this property will be named a *curve of singularities of transverse type* $_k\big(\frac{1}{r}(1,-1)\big)$, or shorter a $_k\big(\frac{1}{r}(1,-1)\big)$ *curve.* When the index k is not important we will name it an A_{r-1} *curve.*

Every A_{r-1} curve can contain a finite number of points of different type. We call them *dissident points.* In order to emphasise that C contains dissident points, we call it a *dissident* A_{r-1} *curve.*

Note that r and k in the above definition are not necessarily coprime.

Remark 1.3.4 Let P be a point of type $\frac{1}{r}(a,b,c)$. The numbers a,b,c,r determine whether P is an isolated singular point, or it lies on a number of dissident curves. There are 4 possibilities:

- If a,b,c are all coprime to r, then P is an isolated point singularity. Consider for example $(0,0,0,1) \in \mathbb{P}(1,1,1,3)$.

- If a,b are coprime to r, and $\mathrm{hcf}(c,r) = \gamma$, then P is a dissident point on a curve of generically transverse type $\frac{1}{\gamma}(a,b)$. A model for such singularity is $\mathbb{P}(1,4,20,25)$ with coordinates x,y,z,t. In this example, P_t is of type $\frac{1}{25}(1,4,20)$ and lies on a dissident $A_4 = \frac{1}{5}(1,4)$ curve (z,t).

- The same example demonstrates the case where a is coprime to r and $\mathrm{hcf}(b,r) = \beta$ and $\mathrm{hcf}(c,r) = \gamma$. Note that if b,c,r had a common divisor, this would contradict the well formedness. Thus P is an intersection of two curves of different types $\frac{1}{\beta}(a,c)$ and $\frac{1}{\gamma}(a,b)$. In our example $P_z \in \mathbb{P}(1,4,20,25)$ is of type $\frac{1}{20}(1,4,5)$ and lies on the intersection of the line (y,z) of type $\frac{1}{4}(1,1)$ with the line (z,t) of type $\frac{1}{5}(1,4)$.

- If $\mathrm{hcf}(a,r) = \alpha$, $\mathrm{hcf}(b,r) = \beta$ and $\mathrm{hcf}(c,r) = \gamma$, then P is an intersection of three curves. As in the previous case, all the curves are of different types $\frac{1}{\alpha}(b,c)$, $\frac{1}{\beta}(a,c)$ and $\frac{1}{\gamma}(a,b)$. Consider for example $P_t \in \mathbb{P}(3,4,35,42)$ which is the intersection of a $\frac{1}{3}(1,2)$ line (x,t), a $\frac{1}{2}(1,1)$ line (y,t) and a $\frac{1}{7}(3,4)$ line (z,t).

Therefore, the positive integers a,b,c,r corresponding to P do not only determine the type of a point singularity, but also determine the transverse type of curve singularities for all curves that pass through P.

Often there is some confusion with the notion of the types of singularities in a weighted projective space, which we will now try to clarify.

Lemma 1.3.5 *Consider X in a weighted projective space $\mathbb{P}(b_1, \ldots, b_m)$. Let $P \in X$ be a singular point of type $\frac{1}{r}(a_1, \ldots, a_n)$, which is induced by the singularities of $\mathbb{P}$. This means that $a_i = b_{p_i} \bmod r$, for $i = 1, \ldots, n$ and some weights b_{p_i} in $\mathbb{P}$. Then locally at P*

$$\mathcal{O}_X(i) \text{ is of type } {}_{\overline{-i}}\Big(\frac{1}{r}(a_1, \ldots, a_n)\Big) \text{ for all } i \in \mathbb{N},$$

where $\overline{}$ denotes the smallest residue mod r.

Proof A divisor D on a projective variety by definition contains a singular point Q of type ${}_j\big(\frac{1}{r}(a_1, \ldots, a_n)\big)$ if Q is locally isomorphic to a point of type $\frac{1}{r}(a_1, \ldots, a_n)$ and

$$\mathcal{O}_X(D) \cong \mathcal{L}_j = \{f \mid \varepsilon(f) = \varepsilon^j \cdot f,\ \forall \varepsilon \in \mu_r\}$$

near the singular point.

Now, consider X in a weighted projective space $\mathbb{P}(b_1, \ldots, b_m)$ which has a singular point P of type $\frac{1}{r}(a_1, \ldots, a_n)$. The singularities of X are induced by the weights of $\mathbb{P}$, thus we can write the following relations (after permuting the weights in $\mathbb{P}$)

$$\begin{array}{rcl} b_1 & = & a_1 \bmod r \\ & \vdots & \\ b_n & = & a_n \bmod r \\ b_{n+1} & = & 0 \bmod r. \end{array}$$

Without loss of generality we can assume that $P = (\underbrace{0, \ldots, 0}_{n}, 1, \ldots)$ are the coordinates of P in $\mathbb{P}$ and that $b_{n+1} = r$. Moreover, if $x_1, \ldots, x_m$ denote the variables of $\mathbb{P}$, then $x_1, \ldots, x_n$ define local parameters of P. On the affine piece $x_{n+1} \neq 0$ the group $\varepsilon \in \mu_r$ acts by

$$x_1, \ldots, x_n \mapsto \varepsilon^{a_1} x_1, \ldots, \varepsilon^{a_n} x_n.$$

Recall that

$$\mathcal{O}_X(k) = \left\{ \frac{p}{q} \mid p, q \text{ homogeneous of } \deg p = \deg q + k \right\}.$$

Observe that $\mathcal{O}_X(b_1)$ has a section $x_1 = 0$. It defines a divisor $D_1 = \{x_1 = 0\}$ and $\mathcal{O}_X(b_1) \cong \mathcal{O}_X(D_1)$. By definition $\mathcal{O}_X(D_1) = \{$ functions with poles at most at $x_1\}$. It is generated by

$$f = \frac{\sqrt[r]{x_{n+1}}^{b_1}}{x_1}$$

in the neighborhood of P. It is obvious that $f \in \mathcal{L}_{-a_1}$, since $\varepsilon(f) = \varepsilon^{-a_1} \cdot f$. Thus we proved that locally $\mathcal{O}_X(b_1) \cong \mathcal{O}_X(D_1) \cong \mathcal{L}_{-a_1}$.

For the same reason

$$\mathcal{O}_X(i) \cong \mathcal{L}_{\overline{-i}} \text{ for all } i \in \mathbb{Z},$$

where $\overline{}$ denotes the smallest residue mod r. $\square$

For varieties with quotient singularities the following holds:

Proposition 1.3.6 ([21], Prop. 5.15) *Let X be a variety over $\mathbb{C}$ with quotient singularities only. Then X is $\mathbb{Q}$–factorial and has rational singularities.*

1.4 The Riemann–Roch formula

Definition 1.4.1 Let an A_{r-1} curve C contain a finite number of dissident points $\{P_\lambda \ : \ \lambda \in \Lambda\}$ of types $\frac{1}{r\tau_\lambda}(a_{1\lambda}, a_{2\lambda}, a_{3\lambda})$. We define the *index of C* to be $\tau = \operatorname{lcm}\{\tau_\lambda\}$.

Note that for A_{r-1} curves with no dissident points $\tau = 1$.

Consider a threefold X and a $\mathbb{Q}$–Cartier divisor D. Then tD is Cartier for some $t \in \mathbb{N}$. For a curve C on X we define the *degree* of D on C by:

$$\deg D|_C = \frac{1}{t} \cdot \deg_C \mathcal{O}_X(tD) = \frac{1}{t} C \cdot tD.$$

We also define the intersection $D \cdot c_2(X)$ either as $\frac{1}{t} f^*(tD) \cdot c_2(Y)$ computed on a resolution $f\colon Y \to X$, or we compute $\frac{1}{t} tD \cdot c_2(X)$ on the nonsingular part of X.

At this stage we can state the main theorem of this thesis:

Theorem 1.4.2 *Let X be a normal projective threefold with canonical singularities and let D be a $\mathbb{Q}$–Cartier divisor. Assume that (X, D) is quasismooth and well formed. Let the singularities of $\mathcal{O}_X(D)$ consist of*

- *points P of type ${}_n\big(\frac{1}{s}(a_1, a_2, a_3)\big)$,*
- *curves C of generic type ${}_k\big(\frac{1}{r}(1, -1)\big)$ with index τ_C.*

Then for all positive integers m,

$$\begin{aligned} \chi(X, \mathcal{O}_X(mD)) &= \chi(\mathcal{O}_X) + \tfrac{1}{12} mD(mD - K_X)(2mD - K_X) + m\tfrac{D \cdot c_2(X)}{12} \\ &\qquad + \textstyle\sum_P c_P(mD) + \sum_C s_C(mD), \end{aligned}$$

where

$$c_P(mD) = \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s, \\ \varepsilon^{a_i} \neq 1 \ \forall i = 1,2,3}} (\varepsilon^{-nm} - 1) \frac{1}{(1 - \varepsilon^{-a_1})(1 - \varepsilon^{-a_2})(1 - \varepsilon^{-a_3})}$$

and

$$\begin{aligned} s_C(mD) &= -m\tfrac{\overline{mk}(r - \overline{mk})}{2r} \deg D|_C + \tfrac{\overline{mk}(r - \overline{mk})}{4r} \deg K_X|_C \\ &\qquad + \tfrac{\overline{mk}(r - \overline{mk})(r - 2 \cdot \overline{mk})}{12 r^2 \tau_C} N_C. \end{aligned}$$

Here $\overline{}$ denotes the smallest residue mod r and the integer N_C is an invariant of X in the neighbourhood of C.

Remark 1.4.3 Note that P is either
(i.) an isolated point singularity if $\mathrm{hcf}(a_i, s) = 1$ for all $i = 1, 2, 3$,
or
(ii.) a dissident point on some curve if $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$ for some i.

In the first case $c_P(mD)$ equals to the so called basket contribution to the Riemann–Roch formula.

In the second case we always assume that $a_j + a_k = 0 \bmod \alpha_i$ for different indexes i, j, k. Thus, the singularities along the curve are of $A_{\alpha_i - 1}$ transverse type. Since X is well formed a_i, a_j, s have no common divisor.

This formula is a version of the orbifold Riemann–Roch (RR) theorem for the singular $\mathcal{O}_X(mD)$. It is of the type

$$\begin{aligned}\chi(X, \mathcal{O}_X(mD)) = \ & \text{RR–type expression in } mD + \\ & \text{contributions from singular points} + \\ & \text{contributions from curves of singularities.}\end{aligned}$$

It only deals with the $\mathbb{Q}$–divisor class $D \in \operatorname{Pic} X \otimes \mathbb{Q}$.

These types of formulae were first studied by Miles Reid in [30]. Reid considers the plurigenus formula of 3–folds with canonical singularities for which K_X generates $\operatorname{Cl} X$. The orbifold RR for isolated cyclic singularities can be found in [23, Chapter 4]. In Chapters 3 and 4 of this thesis we apply Theorem 1.4.2 to Calabi–Yau threefolds. The Riemann–Roch formula for Calabi–Yaus with only A_1 curves is particularly nice and was proved in [32].

2 Proof of the Riemann–Roch formula

In this chapter we prove Theorem 1.4.2. The contributions from curves will be computed in two ways: first by resolving the singularities and second by the method of cyclic group coverings. The contributions from singular points will be computed on suitable models.

2.1 Existence of the Riemann–Roch formula

We will use resolution of singularities on many occasions. We need the following version of Hironaka's resolution of singularities.

Theorem 2.1.1 ([21], Thm 0.2) *Let X be an irreducible and reduced projective threefold over $\mathbb{C}$ with canonical singularities. On X pick a $\mathbb{Q}$–Cartier Weil divisor D. There exists a smooth threefold Y with a Cartier divisor D' and a projective morphism $f\colon Y \to X$ such that $f^*\mathcal{O}_X(D) \to \mathcal{O}_Y(D')$ is surjective and $f_*\mathcal{O}_Y(D') \cong \mathcal{O}_X(D)$.*

Consider the surjection $f^*\mathcal{O}_X(D) \to \mathcal{O}_Y(D')$. Since X is normal $f_*\mathcal{O}_Y = \mathcal{O}_X$. Hence the corresponding short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow f^*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_Y(D') \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{array}{cccccccc} 0 \longrightarrow & f_*\mathcal{K} & \longrightarrow & \mathcal{O}_X(D) & \xrightarrow{\cong} & f_*\mathcal{O}_Y(D') & \longrightarrow \\ & R^1f_*\mathcal{K} & \longrightarrow & R^1f_*f^*\mathcal{O}_X(D) & \longrightarrow & R^1f_*\mathcal{O}_Y(D') & \longrightarrow \\ & R^2f_*\mathcal{K} & \longrightarrow & R^2f_*f^*\mathcal{O}_X(D) & \longrightarrow & R^2f_*\mathcal{O}_Y(D') & \longrightarrow \dots \end{array}$$

By Theorem (3.8) in [30] the canonical singularities in characteristic zero are rational. Thus $R^if_*\mathcal{O}_Y = 0$ for $i > 0$. Using the projection formula we get

$$R^if_*f^*\mathcal{O}_X(D) = R^if_*(\mathcal{O}_Y \otimes f^*\mathcal{O}_X(D)) = R^if_*\mathcal{O}_Y \otimes \mathcal{O}_X(D) = 0$$

for every $i > 0$. Hence the long exact sequence becomes

$$\begin{array}{cccccccc} 0 \longrightarrow & f_*\mathcal{K} & \longrightarrow & \mathcal{O}_X(D) & \xrightarrow{\cong} & f_*\mathcal{O}_Y(D') & \longrightarrow \\ & R^1f_*\mathcal{K} & \longrightarrow & 0 & \longrightarrow & R^1f_*\mathcal{O}_Y(D') & \xrightarrow{\cong} \\ & R^2f_*\mathcal{K} & \longrightarrow & 0 & \longrightarrow & R^2f_*\mathcal{O}_Y(D') & \xrightarrow{\cong} \\ & R^3f_*\mathcal{K} & \dots & & & & \end{array}$$

Note that $\mathcal{K}$ is supported on the fibres of f. The fibre dimension of f is ≤ 2, thus

$$R^i f_* \mathcal{O}_Y(D') \cong R^{i+1} f_* \mathcal{K} = 0$$

for $i \geq 2$. Finally, the Leray spectral sequence yields

$$\begin{aligned} \chi(Y, \mathcal{O}_Y(D')) &= \sum_i (-1)^i \chi(X, R^i f_* \mathcal{O}_Y(D')) \qquad \text{(E. i)} \\ &= \chi(X, \mathcal{O}_X(D)) - \chi(X, R^2 f_* \mathcal{K}). \end{aligned}$$

Note that $R^2 f_* \mathcal{K}$ is supported only over the isolated and dissident singularities.

On the smooth Y we can use the Riemann–Roch formula

$$\chi(Y, \mathcal{O}_Y(D')) = \mathrm{ch}(D') \cdot \mathrm{td}(\mathcal{T}_Y).$$

In the next section we will express D' and K_Y with D, K_X and the exceptional divisors of the resolution. This will prove

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) \quad = \quad & \text{Riemann–Roch type expression in } D + \\ & \text{contributions from singularities.} \end{aligned}$$

2.2 Resolution of singularities

We start by a technical lemma.

Lemma 2.2.1 *Let X be a quasismooth and well formed threefold embedded in a weighted projective space. Assume that $(X, \mathcal{O}_X(1))$ contains a ${}_k\big(\frac{1}{r}(1,-1)\big)$ curve C. Then r and k are coprime.*

Proof A generic point P on the curve is of transverse type A_{r-1}. For P on C choose local transverse parameters x and y. Since X is quasismooth, all singularities of X come from the singularities of the weighted projective space $\mathbb{P}$. In other words, $\mathbb{P}$ has at least two variables of degrees $r*$ (since it contains a singular curve), where $*$ is an arbitrary integer. Moreover, there exists an $0 < a < r$ such that variables x, y are of degrees $a + r*, -a + r*$. The cyclic group $\varepsilon \in \mu_r$ then acts by

$$(x, y) \sim (\varepsilon^a x, \varepsilon^{r-a} y)$$

.

We will first prove that in the above set-up r and a have to be coprime. Assume that there exists $s > 1$ such that $r = sr'$ and $a = sa'$ and argue by contradiction. The quasismooth X is locally a complete intersection in the neighbourhood of P. It is given by independent equations

$$f_1, f_2, \ldots, f_{n+m+l-4} \subset \mathbb{P}(b_1, \ldots, b_n, rc_1, \ldots, rc_m, sd_1, \ldots, sd_l),$$

where f_i have degrees a_i, entries sd_i are not divisible by r, and b_i are not divisible by s for all i. Since X contains a curve of transverse type A_{r-1}, $m-2$ of degrees a_i must be divisible by r, say the last $m-2$ of them $a_{n+l-1}, \ldots, a_{n+m+l-4}$. Without loss of generality we can assume that the type of singularity is induced by the last two variables of degrees sd_{l-1}, sd_l. More precisely, $sd_{l-1} = a + r*$ and $sd_l = -a + r*$.

Threefold X is quasismooth along the curve, so we can use the implicit function theorem in the neighbourhood of P to expand out the variables in the remaining degrees $b_1, \ldots, b_n, sd_1, \ldots, sd_{l-2}$. This gives us

$$\begin{array}{rcl} a_1 & = & b_1 + r* \\ & \vdots & \\ a_n & = & b_n + r* \\ a_{n+1} & = & sd_1 + r* \\ & \vdots & \\ a_{n+l-2} & = & sd_{l-2} + r* \end{array}$$

Thus, $(l-2) + (m-2) = m + l - 4$ of equations have degrees divisible by s. They define a threefold

$$f_{n+1}, \ldots, f_{n+l-2}, f_{n+l-1}, \ldots, f_{n+m+l-4} \subset \mathbb{P}(0, \ldots, 0, rc_1, \ldots, rc_m, sd_1, \ldots, sd_l),$$

which means that X contains a threefold fixed by μ_s action. This contradicts the fact that X is well formed.

Using the notation in Lemma 1.3.5, we proved that the singularities of $\mathcal{O}_X(1)$ are of type

$${}_{r-1}\Big(\frac{1}{r}(a, -a)\Big) \,=\, {}_1\Big(\frac{1}{r}(a, -a)\Big),$$

where a and r are coprime. This is the same as ${}_k\big(\frac{1}{r}(1,-1)\big)$ where $ak = 1 \mod r$. Clearly such k and r are coprime. $\square$

We will prove Theorem 1.4.2 by resolving the singularities. By [29, Main Theorem] we can choose a resolution $f\colon Y \to X$ which factors as $Y \xrightarrow{h} \bar{X} \xrightarrow{g} X$, where g is a partial crepant resolution, $\bar{X}$ only has terminal singularities and h is an isomorphism outside terminal points.

Before we proceed, we describe the precise configuration of exceptional surfaces: Let ${}^CE_i$ and ${}^PG_j$ be the exceptional surfaces of the chosen f.

Every singular point P (dissident or isolated) is locally analytically isomorphic to $\mathbb{A}^3/\mu_s$, thus toric. Using the terminology of [12], the resolution of toric singularities depends only on the quotient type of the singularity. It corresponds to subdividing the defining fan in a lattice. Every divisor corresponds to a ray in the fan and is determined by its Star. Therefore ${}^PG_j$ depend only on the singularity type of P. They are contracted to P by f.

A generic point of a ${}_k\big(\frac{1}{r}(1,-1)\big)$ curve C allows a crepant resolution, which is a composite of $\lfloor\frac{r}{2}\rfloor$ blow–ups. Thus Y has no discrepant divisors over the generic point of C. A dissident point on C has a toric resolution.

In more detail we can write

$$K_Y = f^*K_X + S, \text{ where } S = \sum_P \sum_j {}^P\gamma_j\, {}^PG_j,$$

and

$$f^*D = D' + \sum_C R_C + M, \text{ where } R_C = \sum_i {}^C\alpha_i\, {}^CE_i \text{ and } M = \sum_P \sum_j {}^P\beta_j\, {}^PG_j,$$

with γ_j and $\alpha_i, \beta_j \in \mathbb{Q}$. Here f^*D is by definition $\frac{1}{t}f^*(tD)$ for a Cartier divisor tD.

Lemma 2.2.2 *$R_C = \frac{k}{r}\,{}^CE_1 + \frac{\overline{2k}}{r}\,{}^CE_2 + \ldots + \frac{\overline{(r-1)k}}{r}\,{}^CE_{r-1}$, where $\overline{}$ denotes the smallest residue mod r.*

PROOF The type of singularities determines the geometry of ${}^CE_i$ and the coefficients ${}^C\alpha_i$ in

$$R_C = \sum {}^C\alpha_i\, {}^CE_i.$$

Choose a generic point Q on C and a transverse hypersurface S that contains P. On the polarised $(S, D|_S)$ the type of Q is ${}_k\big(\frac{1}{r}(1,-1)\big)$. Such a point has unique crepant resolution $f|_{\tilde{S}}\colon \tilde{S} \to S$ and coefficients in

$$f^*D|_S = D'|_{\tilde{S}} + \frac{k}{r}F_1 + \frac{\overline{2k}}{r}F_2 + \ldots + \frac{\overline{(r-1)k}}{r}F_{r-1},$$

where F_i are exceptional lines that form the configuration on Figure I.

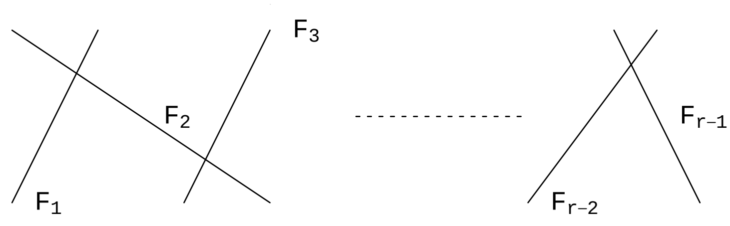

Figure I: Resolution of an A_{r-1} point

Each F_i is a fibre in one of the surfaces ${}^CE_j$. By Lemma 2.2.1 k and r are coprime, thus all numbers $k, \overline{2k}, \ldots, \overline{(r-1)k}$ are different. Hence $F_1, \ldots, F_{r-1}$ must be fibres in different surfaces ${}^CE_1, \ldots, {}^CE_{r-1}$ and ${}^C\alpha_i = \frac{\overline{ik}}{r}$. $\square$

In particular, every ${}^CE_i$ is isomorphic to a blow–up of a few points (over the dissident points) of a smooth ruled surface. The base of the ruling is isomorphic to C. Moreover, ${}^CE_i$ and ${}^CE_j$ intersect along a curve isomorphic to C if $j = i + 1$ and are disjoint otherwise. Define ${}^C\gamma_i = {}^CE_i \cap {}^CE_{i+1}$. The situation is shown on Figure II. For two different curves C and $\bar{C}$, the surfaces ${}^CE_i$ and ${}^{\bar{C}}E_j$ intersect only over the dissident points.

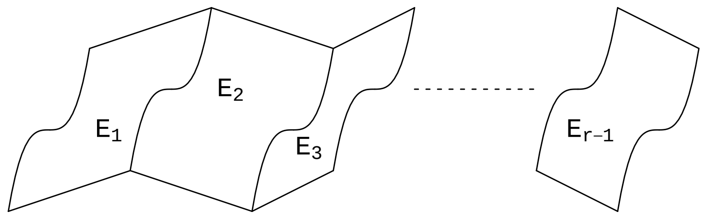

Figure II: Resolution of an A_{r-1} curve

The Riemann-Roch formula for the smooth threefold Y gives

$\chi(Y, \mathcal{O}_Y(D')) =$

$\chi(\mathcal{O}_Y) + \frac{1}{12}D'(D' - K_Y)(2D' - K_Y) + \frac{1}{12}D'c_2(Y) =$

$$\chi(\mathcal{O}_Y) + \tfrac{1}{12}\left(f^*D - \textstyle\sum_C R_C - M\right)\left(f^*D - \textstyle\sum_C R_C - M - f^*K_X - S\right)$$
$$\left(2f^*D - 2\textstyle\sum_C R_C - 2M - f^*K_X - S\right) +$$

$\frac{1}{12}\left(f^*D - \sum_C R_C - M\right) \cdot c_2(Y).$

Take t big, so that the zero locus of $\bar{D} = tD$ defines a smooth surface. The restriction $tD|_C$ is a finite number of nondissident points on C. Then $f^*\bar{D} \cdot {}^C E_i$ is a finite union of fibres in ${}^C E_i$, since $f^*\bar{D}|_{{}^C E_i}$ is a pull back of $tD|_C$. Now, using the projection formula

$$f^*\bar{D} \cdot f^*\bar{D} \cdot {}^C E_i = f^*\bar{D} \cdot (\text{fibres}) = tD \cdot f_*(\text{fibres}) = 0.$$

This holds for all i, thus $f^*D \cdot f^*D \cdot R_C = 0$ and the same way

$$f^*D \cdot f^*K_X \cdot R_C = 0.$$

The projection formula also gives

$$(f^*D)^3 = D^3,\ (f^*D)^2 f^*K_X = D^2 K_X,\ f^*D(f^*K_X)^2 = DK_X^2.$$

The intersection numbers involving M or S depend only on the isolated or dissident singular points of X. Moreover, for different curves of singularities, the intersections

$$f^*D \cdot R_C R_{\bar{C}}, \ f^*K_X \cdot R_C R_{\bar{C}} \ \text{ and } \ (R_C)^2 R_{\bar{C}}$$

can only be nonzero because of intersections over dissident points.

Recall Formula (E. i) from Section 2.1

$$\chi(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(D')) + \chi(X, R^2 f_* \mathcal{K}).$$

The fibre dimension of f over a generic point on a curve of singularities is 1. Thus $R^1 f_* \mathcal{O}_Y(D') \cong R^2 f_* \mathcal{K}$ is supported on the isolated or dissident singular points of X.

The above considerations and the fact that $\chi(\mathcal{O}_X)$ is a birational invariant simplify the RR formula to

$$\begin{aligned} \chi(X, \mathcal{O}_X(D)) \quad = \quad & \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} f^*D \cdot c_2(Y) + \\ & \frac{1}{2} f^*D \cdot \sum_C R_C^2 - \frac{1}{4} f^*K_X \cdot \sum_C R_C^2 - \\ & \frac{1}{6} \sum_C R_C^3 - \frac{1}{12} \sum_C R_C \cdot c_2(Y) + \\ & \text{contributions from singular points of } X. \end{aligned} \tag{E. ii}$$

The contributions from the singular points of X only depend on the analytic neighbourhood of these points. We will compute them on suitable models in Section 2.4. In this section we focus our attention to the contributions from the curves of singularities. For this reason we will denote any expression which depends only on the singular points by $\mathcal{P}$ and will not worry about its exact value.

The next two lemmas will analyse the intersection numbers in (E. ii).

Lemma 2.2.3 *For a* ${}_k\big(\frac{1}{r}(1, -1)\big)$ *curve* C *in* (X, D) *the following holds*

$$\frac{1}{2} f^*D \cdot R_C^2 = -\deg D|_C \cdot \frac{\overline{k}(r - \overline{k})}{2r}$$

and

$$\frac{1}{4}f^*K_X\cdot R_C^2 = -\deg K_X|_C\cdot\frac{\overline{k}(r-\overline{k})}{4r}.$$

Proof Choose a surface $S\subset X$ which intersects C transversely. By Bertini's theorem the only singularities on the polarised $(S,D|_S)$ are points $C\cap S$ which are all of type ${}_k\big(\frac{1}{r}(1,-1)\big)$. Write down Riemann–Roch for a resolution $f|_{\tilde{S}}\colon \tilde{S}\to S$ of these points

$$\chi(S,\mathcal{O}_S(D)) = \chi(\mathcal{O}_S)+\frac{1}{2}D|_S(D|_S-K_S)+\frac{1}{2}B_{\overline{k}}^2,$$

as in the proof of Theorem 9.1 in [30]. The term $\frac{1}{2}B_{\overline{k}}^2$ equals to the so called basket contribution $\sum_P c_P(D|_S)$, where each point $P\in C\cap S$ contributes

$$c_P(D|_S) = -\frac{\overline{k}(r-\overline{k})}{2r}.$$

In our notation $R_C|_{\tilde{S}} = B_{\overline{k}}$.

Note that S can be defined by a section of tD in X for t big enough. Finally we can compute

$$\tfrac{1}{2}f^*D\cdot R_C^2 = \tfrac{1}{2t}f^*S\cdot R_C^2 = \tfrac{1}{2t}\cdot B_{\overline{k}}^2 =$$

$$\tfrac{1}{t}\textstyle\sum_{P\in C\cap S}c_P(D|_S) = -\deg D|_C\cdot\frac{\overline{k}(r-\overline{k})}{2r}.$$

Similarly, if we define S by a section of a multiple of K_X, we get

$$\frac{1}{4}f^*K_X\cdot R_C^2 = -\deg K_X|_C\cdot\frac{\overline{k}(r-\overline{k})}{4r}$$

which ends the proof of Lemma 2.2.3.

$\square$

Lemma 2.2.4 *Let C be a ${}_k\big(\frac{1}{r}(1,-1)\big)$ curve in (X,D). Then*

$$R_C^3+\tfrac{1}{2}R_C\cdot c_2(Y) =$$

$$-\tfrac{1}{r^2}\overline{k}(r-\overline{k})(r-2\overline{k})\left[(r-2)\left(1-g+\tfrac{1}{2}K_Y\,{}^C\!E_1\,{}^C\!E_2\right)+{}^C\!E_{r-2}^2\,{}^C\!E_{r-1}\right]+\mathcal{P},$$

where g is the genus of C and $\mathcal{P}$ depends only on the singular points of X.

Proof By Lemma 2.2.2

$$R_C = \frac{\overline{k}}{r}E_1 + \frac{\overline{2k}}{r}E_2 + \ldots + \frac{\overline{(r-1)k}}{r}E_{r-1}.$$

We drop the upper index in ${}^{C}E_i$ for the sake of clearer notation. Then

$$\begin{aligned} R_C^3 \;&=\; \textstyle\sum_{i=1}^{r-1} \left(\frac{\overline{ik}}{r}\right)^3 E_i^3 + \\ & 3\textstyle\sum_{i=1}^{r-2} \left(\frac{\overline{ik}}{r}\right)^2 \frac{\overline{(i+1)k}}{r} E_i^2 E_{i+1} + \frac{\overline{ik}}{r}\left(\frac{\overline{(i+1)k}}{r}\right)^2 E_i E_{i+1}^2 \end{aligned}$$

and

$$R_C \cdot c_2(Y) = \left(\frac{\overline{k}}{r}E_1 + \frac{\overline{2k}}{r}E_2 + \ldots + \frac{\overline{(r-1)k}}{r}E_{r-1}\right) \cdot c_2(Y).$$

These expressions will simplify, using the properties of ruled surfaces. Recall that E_i equals to a smooth ruled surface $\hat{E}_i$ blown–up in a number of points for $i = 1, \ldots, r-1$. The blow–ups $E_i \xrightarrow{\pi_i} \hat{E}_i$ happen only over the dissident points. In other words, the resolution $f\colon Y \to X$ contracts $\mathrm{Exc}(\pi_i) \subset E_i$ to the dissident points on C.

Denote $\gamma_i = E_i \cap E_{i+1}$ which is isomorphic to C of genus g. Note that $\pi_i(\gamma_i) = \pi_{i+1}(\gamma_i)$ is a section in both ruled surfaces $\hat{E}_i$ and $\hat{E}_{i+1}$. Then

$$K_{E_i} = \pi_i^* K_{\hat{E}_i} + \mathcal{P}$$

is numerically equivalent to

$$-2\gamma_i + (2g - 2 + \gamma_i^2) f_i + \mathcal{P},$$

where f_i is a generic fibre on E_i. In particular $K_{E_i}^2 = 8(1-g) + \mathcal{P}$. Similarly

$$K_{E_{i+1}} \sim -2\gamma_i + (2g - 2 + \gamma_i^2) f_{i+1} + \mathcal{P}.$$

By the adjunction formula we can compute

$$\begin{aligned} &\gamma_i^2 \text{ in } E_{i+1} = E_i|_{E_{i+1}} E_i|_{E_{i+1}} = E_i^2 E_{i+1} = \\ &E_i|_{E_i} E_{i+1}|_{E_i} = (K_{E_i} - K_Y|_{E_i}) E_{i+1}|_{E_i} = K_{E_i}\gamma_i - K_Y E_i E_{i+1} = \\ &-\gamma_i^2 \text{ in } E_i + 2(g-1) + \mathcal{P} - K_Y E_i E_{i+1}. \end{aligned}$$

Similarly

$$\gamma_i^2 \text{ in } E_i = E_i E_{i+1}^2 = -\gamma_i^2 \text{ in } E_{i+1} + 2(g-1) + \mathcal{P} - K_Y E_i E_{i+1}.$$

Both equalities together give

$$E_i^2 E_{i+1} + E_i E_{i+1}^2 = 2(g-1) - K_Y E_i E_{i+1} + \mathcal{P} \quad \text{for } i = 1, \ldots, r-2.$$

Next,

$$E_i^2 E_{i+1} + E_{i+1} E_{i+2}^2 = \gamma_i^2 \text{ in } E_{i+1} + \gamma_{i+1}^2 \text{ in } E_{i+1} = \mathcal{P} \quad \text{for} \quad i = 1, \ldots, r-3$$

since $\pi_{i+1}(\gamma_i)$ and $\pi_{i+1}(\gamma_{i+1})$ are disjoint sections in $\hat{E}_{i+1}$. And the sum of the self–intersections of any two disjoint sections in a ruled surface is 0.

From expanding the equalities

$$\begin{aligned} 0 &= f^* K_X \cdot f^* D \cdot E_i = \\ & (K_Y + \mathcal{P}) \left(D' + \overline{\tfrac{k}{r}} E_1 + \overline{\tfrac{2k}{r}} E_2 + \ldots + \overline{\tfrac{(r-1)k}{r}} E_{r-1} + \mathcal{P} \right) E_i \end{aligned}$$

for various i it follows

$$K_Y E_{i-1} E_i - K_Y E_i E_{i+1} = \mathcal{P} \text{ and } K_Y E_i^2 + 2 K_Y E_i E_{i+1} = \mathcal{P}.$$

Finally, putting together all of the above yields

$$\begin{aligned} E_i^2 E_{i+1} &= -(r-2-i) 2(g-1) + (r-2-i) K_Y E_1 E_2 + E_{r-2}^2 E_{r-1} + \mathcal{P}, \\ E_i E_{i+1}^2 &= (r-1-i) 2(g-1) - (r-1-i) K_Y E_1 E_2 - E_{r-2}^2 E_{r-1} + \mathcal{P}, \end{aligned}$$

for all $i = 1, \ldots, r-2$.

The second Chern class $c_2(Y)$ can also be computed without difficulty. Let $\mathcal{N}_i$ be the normal bundle of E_i in the smooth threefold Y, and $\mathcal{T}_{E_i}$, $\mathcal{T}_Y$ the tangent sheaves. There is an exact sequence

$$0 \longrightarrow \mathcal{T}_{E_i} \longrightarrow j_i^* \mathcal{T}_Y \longrightarrow \mathcal{N}_i \longrightarrow 0,$$

where $j_i \colon E_i \to Y$ is the inclusion. The Chern polynomial for E_i is

$$c_t(\mathcal{T}_{E_i}) = 1 - K_{E_i} t + c_2(E_i) t^2.$$

By the self–intersection formula

$$c_t(\mathcal{N}_i) = 1 + E_i|_{E_i} t.$$

By comparing the coefficients of t^2 and then using the adjunction formula, we find

$$\begin{aligned} c_2(Y) \cdot E_i = c_2(E_i) - K_{E_i} E_i|_{E_i} = \\ c_2(E_i) - (K_Y + E_i)|_{E_i} E_i|_{E_i} = c_2(E_i) - (K_Y + E_i) E_i^2. \end{aligned} \tag{E. iii}$$

Recall that $f\colon Y \to X$ is crepant over a generic point on C, which implies $K_Y^2 E_i = (f^* K_X + \mathcal{P})^2 E_i = \mathcal{P}$. This yields another relation

$$8(1-g) + \mathcal{P} = K_{E_i}^2 = (K_Y + E_i)^2 E_i = E_i^3 + 2K_Y E_i^2 + \mathcal{P}.$$

Also $c_2(E_i)$ can be expressed with other invariants

$$\frac{1}{12}\left(K_{E_i}^2 + c_2(E_i)\right) = 1 + p_a(E_i) = 1 - g,$$

where the last equality holds since $\hat{E}_i$ is a ruled surface and p_a is a birational invariant. This in particular implies $c_2(E_i) = 4(1-g) + \mathcal{P}$.

Pluging all this in (E. iii) we get,

$$c_2(Y) \cdot E_i = -\frac{1}{2} E_i^3 + \mathcal{P},$$

where $E_i^3 = 8(1-g) + 4K_Y E_1 E_2 + \mathcal{P}$.

This reduces the expression for R_C^3 to

$$\begin{aligned}
&\tfrac{8(1-g)+4K_Y E_1 E_2}{r^3} \textstyle\sum_{i=1}^{r-1} && \left(\overline{ki}\right)^3 + \\
&\tfrac{3}{r^3} \textstyle\sum_{i=1}^{r-2} (\overline{ik})^2 \overline{(i+1)k} && [-(r-2-i)2(g-1) + \\
& && (r-2-i) K_Y E_1 E_2 + E_{r-2}^2 E_{r-1}] + \\
&\tfrac{3}{r^3} \textstyle\sum_{i=1}^{r-2} \overline{ik} (\overline{(i+1)k})^2 && [(r-1-i)2(g-1) - \\
& && (r-1-i) K_Y E_1 E_2 - E_{r-2}^2 E_{r-1}] + \\
&\mathcal{P}
\end{aligned}$$

and $R_C \cdot c_2(Y)$ reduces to

$$-\frac{4(1-g)+2K_Y E_1 E_2}{r}\sum_{i=1}^{r-1}\overline{ki}+\mathcal{P}.$$

Together $R_C^3 + \frac{1}{2}c_2(Y)R_C$ equals

$$(1-g+\tfrac{1}{2}K_Y E_1E_2)\Big[\tfrac{8}{r^3}\textstyle\sum_{i=1}^{r-1}\overline{ki}^3 -$$

$$\tfrac{2}{r}\textstyle\sum_{i=1}^{r-1}\overline{ki}+\tfrac{6}{r^3}\sum_{i=1}^{r-2}\overline{ki}^2\overline{k(i+1)}(r-2-i)-\overline{ki}\,\overline{k(i+1)}^2(r-1-i)\Big]+$$

$$\tfrac{3}{r^3}E_{r-2}^2E_{r-1}\textstyle\sum_{i=1}^{r-2}\overline{ki}^2\overline{k(i+1)}-\overline{ki}\,\overline{k(i+1)}^2+\mathcal{P}.$$

We will rewrite this into a more compact form. First observe that $-\overline{ki}+\overline{k(i+1)}$ is either k or $k-r$. More precisely, it has value k exactly $r-k-1$ times and value $k-r$ exactly $k-1$ times. This can be easily seen, since the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & r-1 \\ k & \overline{2k} & \overline{3k} & \dots & r-k \end{pmatrix}$$

changes the sign exactly $k-1$ times. This and

$$\sum_{i=1}^{r-2}\left(-\overline{ki}+\overline{k(i+1)}\right)^3=\sum_{i=1}^{r-2}-\overline{ki}^3+3\overline{ki}^2\overline{k(i+1)}-3\overline{ki}\,\overline{k(i+1)}^2+\overline{k(i+1)}^3$$

yields

$$3\sum_{i=1}^{r-2}\overline{ki}^2\overline{k(i+1)}-\overline{ki}\,\overline{k(i+1)}^2=-rk(r-k)(r-2k). \tag{E. iv}$$

In the next step we prove that

$$\frac{8}{r^3}\sum_{i=1}^{r-1}\overline{ki}^3-\frac{2}{r}\sum_{i=1}^{r-1}\overline{ki}+\frac{3}{r^3}\sum_{i=1}^{r-2}\overline{ki}^2\overline{k(i+1)}(r-2-2i)- \overline{ki}\,\overline{k(i+1)}^2(r-2i)=0. \tag{E. v}$$

By Formula 8.30. in [19] define

$$a_j=\frac{\overline{kj}}{r}-\frac{\overline{k(j-1)}}{r}\quad\text{for}\quad j=1,\dots,r,$$

which in particular means $a_1 = \frac{k}{r}$ and $a_r = \frac{k-r}{r}$. Then

$$\begin{aligned}\textstyle\sum_{1\le i<j\le r}(a_i-a_j)^3 &= \textstyle\frac{8}{r^3}\sum_{i=1}^{r-1}\overline{ki}^3 - \frac{3}{r^3}\sum_{i=1}^{r-1}\overline{ki}^2\overline{k(i+1)} + \overline{ki}\,\overline{k(i+1)}^2 + \\ &\quad \textstyle\frac{3}{r^3}\sum_{i=1}^{r-1}(r-2i-1)(\overline{ki}^2\overline{k(i+1)} + \overline{ki}\,\overline{k(i+1)}^2).\end{aligned}$$

This rewrites (E. v) into

$$-\frac{2}{r}\sum_{i=1}^{r-1}\overline{ki} + \sum_{1\le i<j\le r}(a_i-a_j)^3.$$

By Lemma 2.2.1 r and k are coprime, thus $\overline{ki}\neq 0$ for all i. The sum

$$\sum_{i=1}^{r-1}\overline{ki} = \frac{r(r-1)}{2}$$

does not depend on k. We can write the terms $a_i - a_j$ into an upper–triangular matrix

$$\begin{array}{ccccc} a_1-a_2 & a_1-a_3 & \dots & a_1-a_{r-1} & a_1-a_r \\ & a_2-a_3 & \dots & a_2-a_{r-1} & a_2-a_r \\ & & \ddots & & \vdots \\ & & & a_{r-2}-a_{r-1} & a_{r-2}-a_r \\ & & & & a_{r-1}-a_r \end{array}$$

Observe that

$$\begin{array}{ll} a_1-a_r = 1 & \text{and} \\ a_1-a_j = -(a_{r-j+1}-a_r)+1, & \\ a_i-a_j = -(a_{r-j+1}-a_{r-i+1}), & \\ a_i-a_{r-i+1} = 0 & \text{for}\ \ i,j=2,\dots,r-1. \end{array}$$

This means that the $(1r)$th minor is always antisymmetric along the anti–diagonal and has zeros along the anti–diagonal. Thus,

$$\sum_{1\le i<j\le r}(a_i-a_j)^3 = \sum_{i=2}^{r-1}(a_1-a_i)^3 + \sum_{i=2}^{r-1}(a_i-a_r)^3 + (a_1-a_r)^3.$$

The same as in the proof of (E. iv), a_i has value $\frac{k-r}{r}$ exactly $k-1$ times and value $\frac{k}{r}$ exactly $r-k-1$ times. We get

$$\sum_{1\le i<j\le r}(a_i-a_j)^3 = (k-1)+(r-k-1)+1 = r-1,$$

which finishes the proof of (E. v).

Equations (E. iv) and (E. v) simplify $R_C^3 + \frac{1}{2} R_C \cdot c_2(Y)$ into the beautiful form

$$-\frac{1}{r^2}\overline{k}(r-\overline{k})(r-2\overline{k})\left[(r-2)\left(1-g+\frac{1}{2}K_Y E_1 E_2\right)+E_{r-2}^2 E_{r-1}\right]+\mathcal{P}.$$

This ends the proof of Lemma 2.2.4. $\square$

Corollary 2.2.5 *Let (X, D) satisfy the conditions of Theorem 1.4.2. Then for all positive integers m*

$$\begin{aligned}\chi(X, \mathcal{O}_X(mD)) &= \chi(\mathcal{O}_X) + \tfrac{1}{12} mD(mD-K_X)(2mD-K_X) + m\tfrac{D\cdot c_2(X)}{12} \\ &\quad + \textstyle\sum_C s_C(mD) + \mathcal{P},\end{aligned}$$

where

$$\begin{aligned}s_C(mD) &= -m\tfrac{\overline{mk}(r-\overline{mk})}{2r}\deg D|_C + \tfrac{\overline{mk}(r-\overline{mk})}{4r}\deg K_X|_C \\ &\quad + \tfrac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r^2}\left((r-2)(1-g)+\tfrac{r-2}{2}K_Y\,{}^C\!E_1\,{}^C\!E_2 + {}^C\!E_{r-2}^2\,{}^C\!E_{r-1}\right).\end{aligned}$$

and $\mathcal{P}$ only depends on the singular points of X.

Proof For $m=1$ we only need to put the results of Lemma 2.2.3 and Lemma 2.2.4 into Formula (E. ii) on page 16. We define $D \cdot c_2(X)$ to be $f^*D \cdot c_2(Y)$. The existence of the formula proves that $K_Y\,{}^C\!E_1\,{}^C\!E_2$, ${}^C\!E_{r-2}^2\,{}^C\!E_{r-1}$ and $f^*D \cdot c_2(Y)$ are independent of the resolution and are thus invariants of the threefold X.

For $m>1$ consider $\mathcal{O}_X(mD)$ which is of transverse type $_{\overline{mk}}\big(\frac{1}{r}(1,-1)\big)$ on C. We can write

$$f^*(mD) = D^{(m)} + \sum_C R_C^{(m)} + \mathcal{P},$$

where $D^{(m)}$ is a Cartier divisor on Y and

$$R_C^{(m)} = \frac{\overline{mk}}{r}\,{}^C\!E_1 + \frac{\overline{2mk}}{r}\,{}^C\!E_2 + \ldots + \frac{\overline{(r-1)mk}}{r}\,{}^C\!E_{r-1}$$

by Lemma 2.2.2. Repeat all arguments in Lemma 2.2.3 and Lemma 2.2.4 for $R_C^{(m)}$. The only assumption that might fail is that $\overline{mk}$ and r are coprime. This fact was only used in the proof of (E. v).

We will show that (E. v) also holds when k and r are not coprime. Write $h = \mathrm{hcf}(k, r)$ and $k' = \frac{k}{h}$, $r' = \frac{r}{h}$. Then

$$\begin{array}{ccccccccc} k, & \overline{2k}, & \ldots, & \overline{(r'-1)k}, & \overline{r'k}, & \overline{(r'+1)k}, & \ldots, & \overline{(r-1)k}, & \overline{rk} \\ k, & \overline{2k}, & \ldots, & \overline{(r'-1)k}, & 0, & k, & \ldots, & \overline{(r'-1)k}, & 0, \end{array} \quad \text{equal to}$$

where $\overline{}$ denotes mod r.
The sum $\frac{2}{r}\sum_{i=1}^{r-1} \overline{ki}$ consists of h equal parts, each of them being $r' - 1$.

The matrix of $(a_i - a_j)_{1 \le i < j \le r}$ has the form

$$\begin{array}{ccccccccccc}
a_1-a_2 & a_1-a_3 & \cdots & a_1-a_{r'-1} & a_1-a_{r'} & a_1-a_1 & a_1-a_2 & \cdots & a_1-a_{r'} & a_1-a_1 & \cdots \\
 & a_2-a_3 & \cdots & a_2-a_{r'-1} & a_2-a_{r'} & a_2-a_1 & \cdots & \cdots & a_2-a_{r'} & a_2-a_1 & \cdots \\
 & & \ddots & & & \vdots & & & & \vdots & \\
 & & & & a_{r'-1}-a_{r'} & a_{r'-1}-a_1 & \cdots & \cdots & a_{r'-1}-a_{r'} & a_{r'-1}-a_1 & \cdots \\
 & & & & & a_{r'}-a_1 & a_{r'}-a_2 & \cdots & a_{r'}-a_{r'} & a_{r'}-a_1 & \cdots \\
 & & & & & & a_1-a_2 & \cdots & a_1-a_{r'} & a_1-a_1 & \cdots \\
 & & & & & & & \ddots & & \vdots &
\end{array}$$

Observe that by symmetry most of the terms cancel out. Only the triangular blocks contribute to the sum $\sum_{1 \le i < j \le r}(a_i - a_j)^3$. More precisely, the same way as in the coprime case each triangle contributes $r' - 1$. There is h of the triangles. Thus we proved that also when r and k are not coprime

$$-\frac{2}{r}\sum_{i=1}^{r-1} \overline{ki} + \sum_{1 \le i < j \le r} (a_i - a_j)^3 = 0.$$

□

2.3 Contributions from A_{r-1} curves

Until now we studied contributions to the Riemann–Roch formula via the resolution $f\colon Y \to X$. The aim of this section is to describe another approach, using cyclic quotients. We will construct a smooth threefold Z such that $X = Z/G$ for a finite group G. We will give a new proof of Corollary 2.2.5 for curves of singularities with no dissident points.

2.3.1 Quotients by finite groups

Theorem 2.3.1 ([16], Ex. 10.25) *The quotient of a projective variety $Z \subset \mathbb{P}^n$ by a finite group G always exists and is a projective variety.*

Not every projective threefold with an A_{r-1} curve can be realised as a global quotient by μ_r. We begin with an example:

Example 2.3.2 Consider $\mathbb{P}(1,2,3,3)$ with the variables x, y, t_1, t_2. It contains an A_2 curve $C = \{x = y = 0\}$ and an isolated singular point $(0,1,0,0)$. In this section we are only interested in the contributions to RR induced by C.

Lemma 2.3.3 $\mathbb{P}(1,2,3,3)$ *can not be realised as a global quotient of a smooth threefold under μ_3 action.*

Proof We prove this by contradiction. Assume that there exists Z, which is smooth in the neighbourhood of the curve, such that

$$Z/\mu_3 = \mathbb{P}(1,2,3,3).$$

On the affine piece $U_1 = \{t_1 \neq 0\}$ of $\mathbb{P}(1,2,3,3)$ choose coordinates a_1, b_1, c_1, which are identified by $a_1, b_1, c_1 \sim \varepsilon a_1, \varepsilon^2 b_1, c_1$, for all $\varepsilon \in \mu_3$. This defines a cyclic μ_3 cover

$$\begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & U_1 \\ a_1, b_1, c_1 & \longmapsto & a_1^3, a_1 b_1, b_1^3, c_1. \end{array}$$

The same way denote by a_2, b_2, c_2 the coordinates of $U_2 = \{t_2 \neq 0\}$, which are identified by $a_2, b_2, c_2 \sim \varepsilon a_2, \varepsilon^2 b_2, c_2$. This defines another cyclic cover $\mathbb{C}^3 \to U_2$.

Since $\mathbb{P}(1,2,3,3) = Z/\mu_3$, we can obtain Z by gluing together the affine pieces of the covers. The transition function is induced by identification between U_1 and U_2 :

$$a_1, b_1, c_1 \leftrightarrow a_1 c_1^{-\frac{1}{3}}, b_1 c_1^{-\frac{2}{3}}, c_1^{-1}.$$

But these are not rational functions, which gives the contradiction. $\square$

On the other hand, $\mathbb{P}(1,2,3,3)$ is isomorphic to

$$\mathbb{P}(1,2,1,1)/\mu_3 \oplus \mu_3.$$

The elements $(\varepsilon, \varepsilon') \in \mu_3 \oplus \mu_3$ act on $\mathbb{P}(1,2,1,1)$ via $1, 1, \varepsilon, \varepsilon'$ and fix either points, curve or surfaces. Note that $\mathbb{P}(1,2,1,1)$ is smooth in the neighbourhood of C.

The above example demonstrates what happens in a general case. In Subsection 2.3.4 we will prove that a projective threefold with an A_{r-1} curve can be realised as a global quotient by the group $\mu_r \oplus \mu_r$. Various elements of $\mu_r \oplus \mu_r$ fix different loci in the cover Z, i.e. points, curves or surfaces.

2.3.2 The Atiyah–Singer equivariant Riemann–Roch formula

Through the entire section we will use the **Atiyah-Singer equivariant RR formula.** The version that we need can be found on page 565 in [3]:
Let Z be a compact complex manifold, $\mathcal{V}$ a holomorphic vector bundle over Z and $\mathcal{O}(\mathcal{V})$ the sheaf of germs of holomorphic sections of $\mathcal{V}$. Moreover, let G be a finite group of automorphisms of the pair $(Z, \mathcal{V})$. Let Z^g denote the fixed point set of an element $g \in G$ and $\mathcal{N}^g$ is the normal bundle of Z^g in Z. Under the G–action, the complex vector bundle $\mathcal{N}^g$ has a decomposition

$$\mathcal{N}^g = \sum \mathcal{N}^g(\theta),$$

where $\mathcal{N}^g(\theta)$ is the subbundle on which g acts as $e^{i\theta}$. Then

$$\sum_p (-1)^p \mathrm{Trace}(g|H^p(Z, \mathcal{O}(\mathcal{V}))) = \left\{ \frac{\mathrm{ch}(\mathcal{V}|Z^g)(g)\mathcal{T}(Z^g)}{\prod_\theta(\prod_{j_\theta}(1 - e^{-x_{j_\theta} - i\theta})(\mathcal{N}^g(\theta)))} \right\} [Z^g],$$

where x_{j_θ} are Chern classes of $\mathcal{N}^g(\theta)$ and $\mathcal{T}(Z^g)$ is the Todd class of the tangent bundle on Z^g.

Note that the formula is written in the Grothendieck group $K(Z^g)$. The Grothendieck group is connected to the Chow ring $A(Z^g)$ in the following way: If Z^g is nonsingular, we can compute $K(Z^g)$ using only locally free sheaves ([17], Ex. II 6.10). This extends the definition of Chern classes from $A(Z^g)$ to $K(Z^g)$ and induces maps

$$\begin{aligned} c_t : K(Z^g) &\longrightarrow A(Z^g)[t] \\ ch : K(Z^g) &\longrightarrow A(Z^g) \otimes \mathbb{Q}. \end{aligned}$$

We will use the Atiyah-Singer formula in a simpler form. For $\mathcal{V}$ we take the trivial vector bundle, whose total Chern class is 1.

Theorem 2.3.4 *Let Z be a smooth projective threefold with a cyclic group action μ_r whose fixed locus is a curve C of genus g. Assume also that the group $\mu_r \ni \varepsilon$ acts in the normal direction of the curve by $\varepsilon^a, \varepsilon^{r-a}$ for some a coprime to r.*

If $r \neq 2$, the normal sheaf decomposes into $\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with the corresponding Chern classes $x_1 = c_1(\mathcal{N}_1)$, $x_2 = c_1(\mathcal{N}_2)$ in the Chow ring $A(C) \otimes \mathbb{Q}$. Then

$$\textstyle\sum_p (-1)^p \mathit{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) =$$

$$\tfrac{1-g}{(1-\varepsilon^{-a})(1-\varepsilon^a)} - \tfrac{\varepsilon^{-a}}{(1-\varepsilon^{-a})^2(1-\varepsilon^a)} \deg_C x_1 - \tfrac{\varepsilon^a}{(1-\varepsilon^{-a})(1-\varepsilon^a)^2} \deg_C x_2.$$

If $r = 2$,

$$\textstyle\sum_p (-1)^p \mathit{Trace}(-1 | H^p(Z, \mathcal{O}_Z)) = \tfrac{1-g}{4} - \tfrac{1}{8} \deg_C c_1(\mathcal{N}_{C/Z}).$$

Proof Every element $\varepsilon \in \mu_r$ fixes exactly a curve C, in other words $Z^\varepsilon = C$ for all ε. On a curve C with the canonical divisor K_C and genus g the following always holds

$$\mathrm{td}\mathcal{T}_C = 1 - \frac{1}{2} K_C \quad \text{and} \quad \deg K_C = 2(g-1).$$

The normal bundle $\mathcal{N}_{C/Z}$ of C in Z is of rank two, and is also independent on ε.

If $r \neq 2$, a and $r-a$ are different. Thus $\mathcal{N}_{C/Z}$ decomposes into two subbundles of rank one

$$\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2.$$

The group $\varepsilon \in \mu_r$ acts on each component of the normal bundle via $\varepsilon^a, \varepsilon^{r-a}$.

These considerations simplify the above formula. In the language of algebraic geometry it rewrites to

$$\sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) = \deg \left\{ \frac{1 \cdot \mathrm{td}(\mathcal{T}_C)}{(1-\varepsilon^{-a}e^{-x_1})(1-\varepsilon^{-(r-a)}e^{-x_2})} \right\}_1,$$

where x_i is the first Chern class of $\mathcal{N}_i$ for $i = 1, 2$; and $\{\ \}_1$ denotes the component of degree 1 in the Chow ring $A(C) \otimes \mathbb{Q}$.

In the Taylor series for $\frac{1}{1-\varepsilon^{-a}e^{-x_1}}$ we only need to consider the terms of degree ≤ 1. Thus,

$$\begin{aligned}
\tfrac{1}{1-\varepsilon^{-a}e^{-x_1}} &= 1 + \varepsilon^{-a}e^{-x_1} + \varepsilon^{-2a}e^{-2x_1} + \ldots \\
&= 1 + \varepsilon^{-a}(1-x_1) + \varepsilon^{-2a}(1-2x_1) + \ldots + \text{ terms of degree } \geq 2 \\
&= \tfrac{1}{1-\varepsilon^{-a}} - \tfrac{\varepsilon^{-a}}{(1-\varepsilon^{-a})^2}x_1 + \text{ terms of degree } \geq 2.
\end{aligned}$$

The same way we compute

$$\frac{1}{1-\varepsilon^a e^{-x_2}} = \frac{1}{1-\varepsilon^a} - \frac{\varepsilon^a}{(1-\varepsilon^a)^2}x_2 + \text{ terms of degree } \geq 2.$$

From comparing the factors in degree 1 we get

$$\begin{aligned}
\textstyle\sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) = \; & \tfrac{1-g}{(1-\varepsilon^{-a})(1-\varepsilon^a)} - \\
& \tfrac{\varepsilon^{-a}}{(1-\varepsilon^{-a})^2(1-\varepsilon^a)} \deg_C x_1 - \\
& \tfrac{\varepsilon^a}{(1-\varepsilon^{-a})(1-\varepsilon^a)^2} \deg_C x_2
\end{aligned}$$

This expression will play a crucial role in the following paragraphs.

For $r = 2$ the normal sheaf $\mathcal{N}_{C/Z}$ might not decompose. If it does, we have already proved that

$$\sum_p (-1)^p \mathrm{Trace}(-1 | H^p(Z, \mathcal{O}_Z)) = \frac{1-g}{4} + \frac{1}{8}(\deg_C x_1 + \deg_C x_2).$$

When $\mathcal{N}_{C/Z}$ does not decompose, the Atiyah–Singer equivariant RR has the form

$$\sum_p (-1)^p \mathrm{Trace}(-1|H^p(Z,\mathcal{O}_Z)) = \deg \left\{ \frac{1 \cdot \mathrm{td}(\mathcal{T}_C)}{(1+e^{-y_1})(1+e^{-y_2})} \right\}_1,$$

where $y_i = c_i(\mathcal{N}_{C/Z})$ for $i = 1, 2$. By comparing the factors in degree 1 we get

$$\sum_p (-1)^p \mathrm{Trace}(-1|H^p(Z,\mathcal{O}_Z)) = \frac{1-g}{4} - \frac{1}{8} \deg_C y_1.$$

$\square$

Remark 2.3.5 Note that we get the same formula for $r = 2$ whether $\mathcal{N}_{C/Z}$ decomposes or not. Therefore we can in all further computations assume that $\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$.

We will also prove the analogy to Theorem 2.3.4 for surfaces and points.

Lemma 2.3.6 *Let Z be a smooth projective threefold with a finite group G action that fixes exactly a surface S. Let the group element $g \in G$ act in the normal direction of the surface by ε^{-1} for some $\varepsilon \in \mu_r$.*

Then,

$$\sum_p (-1)^p \mathit{Trace}(g|H^p(Z,\mathcal{O}_Z)) =$$

$$\left[\tfrac{\varepsilon}{(1-\varepsilon)^3} - \tfrac{\varepsilon}{2(1-\varepsilon)^2} \right] S^3 + \tfrac{1}{2} \tfrac{\varepsilon}{(1-\varepsilon)^2} K_S \cdot S^2 + \tfrac{1}{12(1-\varepsilon)} (K_S^2 + c_2),$$

where S has the canonical divisor K_S and the second Chern class c_2.

Proof The Atiyah–Singer equivariant RR formula for this case rewrites to

$$\sum_p (-1)^p \mathrm{Trace}(g|H^p(Z,\mathcal{O}_Z)) = \deg \left\{ \frac{1 \cdot \mathrm{td}(\mathcal{T}_S)}{(1-\varepsilon e^{-c_1})} \right\}_2,$$

where c_1 is the first Chern class of the rank 1 normal bundle $\mathcal{N}$; and $\{\ \}_2$ denotes the component of degree 2 in the Chow ring $A(S) \otimes \mathbb{Q}$.

In the Taylor series for $\frac{1}{1-\varepsilon e^{-c_1}}$ we only need to consider the terms of degree ≤ 2. Thus,

$$\begin{aligned}\tfrac{1}{1-\varepsilon e^{-c_1}} &= 1+\varepsilon e^{-c_1}+\varepsilon^2 e^{-2c_1}+\dots\\ &= \tfrac{1}{1-\varepsilon}-\tfrac{\varepsilon}{(1-\varepsilon)^2}c_1+\left[\tfrac{\varepsilon}{(1-\varepsilon)^3}-\tfrac{\varepsilon}{2(1-\varepsilon)^2}\right]c_1^2+\text{ terms of degree }\geq 3.\end{aligned}$$

On a surface S with the canonical divisor K_S and the second Chern class c_2 the following always holds

$$\mathrm{td}\mathcal{T}_S = 1-\frac{1}{2}K_S+\frac{1}{12}(K_S^2+c_2).$$

Comparing the factors in degree 2 yields

$$\begin{aligned}\sum_p(-1)^p\mathrm{Trace}(g|H^p(Z,\mathcal{O}_Z)) &= \left[\frac{\varepsilon}{(1-\varepsilon)^3}-\frac{\varepsilon}{2(1-\varepsilon)^2}\right]c_1^2+\\ &\quad \frac{1}{2}\frac{\varepsilon}{(1-\varepsilon)^2}K_S\cdot c_1+\\ &\quad \frac{1}{12(1-\varepsilon)}(K_S^2+c_2).\end{aligned}$$

By the self–intersection formula $c_1(\mathcal{N}) = S\cdot S = S|_S$, thus

$$\begin{aligned}\sum_p(-1)^p\mathrm{Trace}(g|H^p(Z,\mathcal{O}_Z)) &= \left[\frac{\varepsilon}{(1-\varepsilon)^3}-\frac{\varepsilon}{2(1-\varepsilon)^2}\right]S^3+\\ &\quad \frac{1}{2}\frac{\varepsilon}{(1-\varepsilon)^2}K_S\cdot S^2+\\ &\quad \frac{1}{12(1-\varepsilon)}(K_S^2+c_2).\end{aligned}$$

$\square$

Lemma 2.3.7 *Let Z be a smooth projective threefold with a finite group G action that fixes exactly a point P. Let $g\in G$ act on the point by $\varepsilon^{a_1},\varepsilon^{a_2},\varepsilon^{a_3}$ for some a_1,a_2,a_3 coprime to r and $\varepsilon\in\mu_r$.*

Then

$$\textstyle\sum_p(-1)^p\,\mathit{Trace}(g|H^p(Z,\mathcal{O}_Z)) = \frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

Proof Write $dg_P \colon T_{Z,P} \to T_{Z,P}$ for the differential at point P fixed by the action of $g \equiv \varepsilon \in \mu_r$. Atiyah-Singer equivariant RR formula for points takes the simple form

$$\sum_p (-1)^p \mathrm{Trace}(g \mid H^p(Z, \mathcal{O}_Z)) =$$

$$\frac{1}{\det(1 - dg_P)} = \frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

□

2.3.3 The $\mu_r \oplus \mu_r$ group action

Definition 2.3.8 Let Z be a smooth projective threefold with the group $\mu_r \oplus \mu_r$ acting on it in a way that we now describe. For a chosen generator $\varepsilon \in \mu_r$ write

$$\mu_r \oplus \mu_r = \{(\varepsilon^\alpha, \varepsilon^\beta) \mid \alpha, \beta = 0, \ldots, r-1\}.$$

Let a, d_1 and d_2 be integers that are all coprime to r. Then the group element $(\varepsilon^\alpha, \varepsilon^\beta)$

- fixes a curve C, if $\alpha = \beta$.
  It acts in the normal direction of the curve by $\varepsilon^{a\alpha}, \varepsilon^{(r-a)\alpha}$.
- fixes a number of points P_λ on C, if $\alpha \neq 0$ and $\alpha \neq \beta$.
  It acts on each point P_λ by $\varepsilon^{a\alpha}, \varepsilon^{(r-a)\alpha}, \varepsilon^{d_2(\alpha-\beta)}$.
- fixes a number of points Q_ξ on C, if $\beta \neq 0$ and $\alpha \neq \beta$.
  It acts on each point Q_ξ by $\varepsilon^{a\beta}, \varepsilon^{(r-a)\beta}, \varepsilon^{d_1(\beta-\alpha)}$.
- fixes a number of surfaces S_λ such that $S_\lambda \cap C = P_\lambda$, if $\alpha = 0$.
  It acts in the normal direction of each S_λ by $\varepsilon^{-\beta d_2}$.
- fixes a number of surfaces T_ξ such that $T_\xi \cap C = Q_\xi$, if $\beta = 0$.
  It acts in the normal direction of each T_ξ by $\varepsilon^{-\alpha d_1}$.

From now on call such an action **Action ♠**.

Now write $\pi\colon Z \to X$ for the quotient map and let $\mathcal{L}_{i,j}$ be the (i,j)th eigensheaf of the $\mu_r \oplus \mu_r$ action on $\pi_*\mathcal{O}_Z$. Then

$$\pi_*\mathcal{O}_Z = \bigoplus_{i,j=0}^{r-1} \mathcal{L}_{i,j}$$

implies

$$H^p(Z, \mathcal{O}_Z) = \bigoplus_{i,j=0}^{r-1} H^p(X, \mathcal{L}_{i,j}).$$

The group action on any $f \in \mathcal{L}_{i,j}$ is given by $(\varepsilon^\alpha, \varepsilon^\beta)(f) = \varepsilon^{\alpha i+\beta j} \cdot f$, thus

$$\mathrm{Tr}((\varepsilon^\alpha, \varepsilon^\beta) : H^p(Z, \mathcal{O}_Z)) = \sum_{i,j=0}^{r-1} h^p(X, \mathcal{L}_{i,j}) \cdot \varepsilon^{\alpha i+\beta j}.$$

Moreover,

$$\sum_p (-1)^p \mathrm{Tr}((\varepsilon^\alpha, \varepsilon^\beta) : H^p(Z, \mathcal{O}_Z)) = \sum_{i,j=0}^{r-1} \chi(X, \mathcal{L}_{i,j}) \cdot \varepsilon^{\alpha i+\beta j}.$$

Observe that the Action ♠ on the curve C is a μ_r action, since

$$\mu_r \cong \{(\varepsilon^\alpha, \varepsilon^\alpha) \mid \alpha = 0, \ldots, r-1\} \subset \mu_r \oplus \mu_r.$$

The normal sheaf to C of genus g decomposes into $\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with the corresponding Chern classes $x_1 = c_1(\mathcal{N}_1),\ x_2 = c_1(\mathcal{N}_2)$. Thus by Theorem 2.3.4

$$\textstyle\sum_p (-1)^p \mathrm{Trace}((\varepsilon^\alpha, \varepsilon^\alpha) \mid H^p(Z, \mathcal{O}_Z)) =$$

$$\tfrac{1-g}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})} - \tfrac{\varepsilon^{-a\alpha}}{(1-\varepsilon^{-a\alpha})^2(1-\varepsilon^{a\alpha})} \deg_C x_1 - \tfrac{\varepsilon^{a\alpha}}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})^2} \deg_C x_2,$$

for every $1 \neq (\varepsilon^\alpha, \varepsilon^\alpha) \in \mu_r \oplus \mu_r$. Denote the right hand side of this equation by $A_{\alpha,\alpha}$.

If $\alpha \neq \beta$ are both nonzero, the element $(\varepsilon^\alpha, \varepsilon^\beta) \in \mu_r \oplus \mu_r$ fixes the points P_λ and Q_ξ. Then Lemma 2.3.7 implies

$$\textstyle\sum_p (-1)^p \mathrm{Trace}((\varepsilon^\alpha, \varepsilon^\beta) \mid H^p(Z, \mathcal{O}_Z)) =$$

$$\tfrac{\Lambda}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d_2(\alpha-\beta)})} + \tfrac{\Xi}{(1-\varepsilon^{-a\beta})(1-\varepsilon^{a\beta})(1-\varepsilon^{-d_1(\beta-\alpha)})},$$

where Λ and Ξ denote the number of points P_λ and Q_ξ respectively. Denote the right hand side of this equation by $A_{\alpha,\beta}$.

In the case $\alpha = 0$ and $\beta \neq 0$ the element $(1, \varepsilon^\beta) \in \mu_r \oplus \mu_r$ fixes the surfaces S_λ and the points Q_ξ. By Lemma 2.3.6

$$\sum_p (-1)^p \mathrm{Trace}((1, \varepsilon^\beta) \mid H^p(Z, \mathcal{O}_Z)) =$$

$$\sum_\lambda \left[\tfrac{\varepsilon^{d_2\beta}}{(1-\varepsilon^{d_2\beta})^3} - \tfrac{\varepsilon^{d_2\beta}}{2(1-\varepsilon^{d_2\beta})^2} \right] S_\lambda^3 + \tfrac{1}{2} \tfrac{\varepsilon^{d_2\beta}}{(1-\varepsilon^{d_2\beta})^2} K_{S_\lambda} \cdot S_\lambda^2 + \tfrac{1}{12(1-\varepsilon^{d_2\beta})} (K_{S_\lambda}^2 + c_2(S_\lambda))$$

$$+ \tfrac{\Xi}{(1-\varepsilon^{-a\beta})(1-\varepsilon^{a\beta})(1-\varepsilon^{-d_1\beta})}.$$

Denote this equation by $A_{0,\beta}$.

Similarly for $\alpha \neq 0$ and $\beta = 0$ the element $(\varepsilon^\alpha, 1) \in \mu_r \oplus \mu_r$ fixes the points P_λ and the surfaces T_ξ. We get

$$\sum_p (-1)^p \mathrm{Trace}((\varepsilon^\alpha, 1) \mid H^p(Z, \mathcal{O}_Z)) =$$

$$\sum_\xi \left[\tfrac{\varepsilon^{d_1\alpha}}{(1-\varepsilon^{d_1\alpha})^3} - \tfrac{\varepsilon^{d_1\alpha}}{2(1-\varepsilon^{d_1\alpha})^2} \right] T_\xi^3 + \tfrac{1}{2} \tfrac{\varepsilon^{d_1\alpha}}{(1-\varepsilon^{d_1\alpha})^2} K_{T_\xi} \cdot T_\xi^2 + \tfrac{1}{12(1-\varepsilon^{d_1\alpha})} (K_{T_\xi}^2 + c_2(T_\xi))$$

$$+ \tfrac{\Lambda}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d_2\alpha})}.$$

Denote this equation by $A_{\alpha,0}$.

Clearly, in the introduced notation $A_{0,0} = \chi(\mathcal{O}_Z)$.

Therefore, for every $1 \neq (\varepsilon^\alpha, \varepsilon^\beta) \in \mu_r \oplus \mu_r$ we get

$$\sum_{i,j=0}^{r-1} \chi(X, \mathcal{L}_{i,j}) \cdot \varepsilon^{\alpha i + \beta j} = A_{\alpha,\beta}.$$

In the case $(\varepsilon^\alpha, \varepsilon^\beta) = 1$ we get

$$\sum_{i,j=0}^{r-1} \chi(X, \mathcal{L}_{i,j}) = \chi(\mathcal{O}_Z).$$

The last two formulas can be considered as a linear system of r^2 equations in $\chi(X, \mathcal{L}_{i,j})$ and variable ε, given by Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & & \dots & \\ \dots & & \varepsilon^{\alpha i+\beta j} & \\ 1 & \dots & & \dots \end{pmatrix} \begin{pmatrix} \chi(X, \mathcal{L}_{0,0}) \\ \vdots \\ \chi(X, \mathcal{L}_{i,j}) \\ \vdots \end{pmatrix} = \begin{pmatrix} \chi(\mathcal{O}_Z) \\ \vdots \\ A_{\alpha,\beta} \\ \vdots \end{pmatrix}.$$

Since a Vandermonde matrix is invertible, we can solve the system

$$\frac{1}{r^2} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & & \dots & \\ \dots & & \varepsilon^{-\alpha i-\beta j} & \\ 1 & \dots & & \dots \end{pmatrix} \begin{pmatrix} \chi(\mathcal{O}_Z) \\ \vdots \\ A_{\alpha,\beta} \\ \vdots \end{pmatrix} = \begin{pmatrix} \chi(X, \mathcal{L}_{0,0}) \\ \vdots \\ \chi(X, \mathcal{L}_{i,j}) \\ \vdots \end{pmatrix}.$$

Write $\chi(X, \mathcal{L}_{0,0}) = \chi(\mathcal{O}_X)$ and eliminate $\chi(\mathcal{O}_Z)$ from the solution. We end up with

$$\chi(X, \mathcal{L}_{i,j}) = \chi(\mathcal{O}_X) + \frac{1}{r^2} \sum_{\alpha,\beta=0}^{r-1} (\varepsilon^{-\alpha i-\beta j} - 1) A_{\alpha,\beta}. \qquad \text{(E. vi)}$$

Insert $A_{\alpha,\beta}$ in this formula. Finally we get

$$\chi(X,\mathcal{L}_{i,j}) = \chi(\mathcal{O}_X)$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i-\alpha j}-1)\left(\tfrac{1-g}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})} - \tfrac{\varepsilon^{-a\alpha}}{(1-\varepsilon^{-a\alpha})^2(1-\varepsilon^{a\alpha})}\deg_C x_1 - \right.$$

$$\left. \tfrac{\varepsilon^{a\alpha}}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})^2}\deg_C x_2\right)$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\alpha=1,\beta=0;\text{ and }\alpha\neq\beta}^{r-1}(\varepsilon^{-\alpha i-\beta j}-1)\tfrac{\Lambda}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d_2(\alpha-\beta)})}$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\alpha=0,\beta=1;\text{ and }\alpha\neq\beta}^{r-1}(\varepsilon^{-\alpha i-\beta j}-1)\tfrac{\Xi}{(1-\varepsilon^{-a\beta})(1-\varepsilon^{a\beta})(1-\varepsilon^{-d_1(\beta-\alpha)})}$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\beta=1}^{r-1}(\varepsilon^{-\beta j}-1)\sum_\lambda\left(\left[\tfrac{\varepsilon^{d_2\beta}}{(1-\varepsilon^{d_2\beta})^3} - \tfrac{\varepsilon^{d_2\beta}}{2(1-\varepsilon^{d_2\beta})^2}\right]S_\lambda^3 + \tfrac{1}{2}\tfrac{\varepsilon^{d_2\beta}}{(1-\varepsilon^{d_2\beta})^2}K_{S_\lambda}\cdot S_\lambda^2 + \right.$$

$$\left.\tfrac{1}{12(1-\varepsilon^{d_2\beta})}(K_{S_\lambda}^2 + c_2(S_\lambda))\right)$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i}-1)\sum_\xi\left(\left[\tfrac{\varepsilon^{d_1\alpha}}{(1-\varepsilon^{d_1\alpha})^3} - \tfrac{\varepsilon^{d_1\alpha}}{2(1-\varepsilon^{d_1\alpha})^2}\right]T_\xi^3 + \tfrac{1}{2}\tfrac{\varepsilon^{d_1\alpha}}{(1-\varepsilon^{d_1\alpha})^2}K_{T_\xi}\cdot T_\xi^2 + \right.$$

$$\left.\tfrac{1}{12(1-\varepsilon^{d_1\alpha})}(K_{T_\xi}^2 + c_2(T_\xi))\right).$$

We can think of $\chi(\mathcal{O}_X)$ as RR–type expression in L_{ij}, where $L_{ij} = 0 \in \mathrm{Div}X \otimes \mathbb{Q}$ is the corresponding $\mathbb{Q}-$divisor. The rest of the formula is a sum of contributions coming from

- the curve C
- the points $\{P_\lambda\}$ and $\{Q_\xi\}$
- the surfaces $\{S_\lambda\}$ and $\{T_\xi\}$

which are the singularities of $\mathcal{L}_{i,j}$ on X.

In the sequel we prove two technical lemmas, which will simplify the above expressions.

Lemma 2.3.9 *Let C be a smooth curve in a smooth threefold Z. Then the first Chern class*

$$c_1(\mathcal{N}_{C/Z}) = K_C - K_Z|_C,$$

where K_Z is the canonical divisor of Z, K_C is the canonical divisor of the curve and $\mathcal{N}_{C/Z}$ its normal bundle.

Proof There exists a short exact sequence

$$0 \to \mathcal{T}_C \to i^*\mathcal{T}_Z \to \mathcal{N}_{C/Z} \to 0,$$

where $i\colon C \to Z$ is the inclusion. The corresponding Chern polynomials are

$$\begin{aligned}
&c_t(\mathcal{T}_C) = 1 - K_C t\\
&c_t(\mathcal{N}_{C/Z}) = 1 + c_1(\mathcal{N}_{C/Z})t + \dots\\
&c_t(\mathcal{T}_Z) = 1 - K_Z t + \dots
\end{aligned}$$

By comparing the coefficients of t in $c_t(i^*\mathcal{T}_Z) = c_t(\mathcal{T}_C)\cdot c_t(\mathcal{N}_{C/Z})$, we find

$$c_1(\mathcal{N}_{C/Z}) = K_C - K_Z|_C.$$

This in particular implies that

$$\deg c_1(\mathcal{N}_{C/Z}) = \deg K_C - \deg K_Z|_C = 2(g-1) - \deg K_Z|_C.$$

□

Lemma 2.3.10 *Let C be a smooth curve of genus g in a smooth threefold Z and $\varepsilon \in \mu_r$. Assume that the normal sheaf to C decomposes as $\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with $x_1 = c_1(\mathcal{N}_1),\ x_2 = c_1(\mathcal{N}_2)$. Then*

(1.) $$\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i}-1)\left(\frac{1-g}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})} - \frac{\varepsilon^{-a\alpha}}{(1-\varepsilon^{-a\alpha})^2(1-\varepsilon^{a\alpha})}\deg_C x_1 - \frac{\varepsilon^{a\alpha}}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})^2}\deg_C x_2\right)$$

$$= \frac{\overline{ik}(r-\overline{ik})}{4}\deg K_Z|_C + \frac{1}{12}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_C x_1 - \deg_C x_2)$$

and

(2.) $$\sum_{\alpha=1,\beta=0 \text{ and } \alpha\neq\beta}^{r-1}(\varepsilon^{-\alpha i}-1)\frac{1}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d(\alpha-\beta)})} =$$

$$-(r-1)\frac{\overline{ki}(r-\overline{ki})}{4},$$

where a, r, d, i, k are integers with a and d coprime to r, $\overline{}$ denotes the smallest residue mod r, and $\overline{ak} = 1$.

Proof (1.) First observe that

$$\tfrac{1-g}{(1-\varepsilon^{-a})(1-\varepsilon^{a})} - \tfrac{\varepsilon^{-a}}{(1-\varepsilon^{-a})^2(1-\varepsilon^{a})} \deg_C x_1 - \tfrac{\varepsilon^{a}}{(1-\varepsilon^{-a})(1-\varepsilon^{a})^2} \deg_C x_2 =$$

$$\tfrac{1}{(1-\varepsilon^{-a})(1-\varepsilon^{a})} \left(1 - g + \tfrac{\deg_C x_1}{2} + \tfrac{\deg_C x_1}{2}\right) -$$

$$\tfrac{1+\varepsilon^{-a}}{2(1-\varepsilon^{-a})^2(1-\varepsilon^{a})} \deg_C x_1 - \tfrac{1+\varepsilon^{a}}{2(1-\varepsilon^{-a})(1-\varepsilon^{a})^2} \deg_C x_2 =$$

$$\tfrac{1}{(1-\varepsilon^{-a})(1-\varepsilon^{a})} \left(1 - g + \tfrac{\deg_C x_1}{2} + \tfrac{\deg_C x_1}{2}\right) - \tfrac{1+\varepsilon^{a}}{2(1-\varepsilon^{-a})(1-\varepsilon^{a})^2}(\deg_C x_2 - \deg_C x_1)$$

for all $\varepsilon \in \mu_r - \{1\}$.

In the notation of Section 8 in [30]

$$\sum_{\varepsilon \neq 1} \frac{\varepsilon^{-i} - 1}{(1-\varepsilon^{a})(1-\varepsilon^{r-a})}$$

equals to $\sigma_i\big(\frac{1}{r}(r-a, a)\big) - \sigma_0\big(\frac{1}{r}(r-a, a)\big)$. This is by definition the same as $\sigma_{\overline{ki}}\big(\frac{1}{r}(-1, 1)\big) - \sigma_0\big(\frac{1}{r}(-1, 1)\big)$, where $ak \equiv 1 \bmod r$ and $\overline{}$ denotes the smallest residue mod r. Using Proposition (8.9) in [30] we get

$$-\frac{\overline{ik}(r - \overline{ik})}{2r}.$$

The same way

$$\textstyle\sum_{\varepsilon \neq 1} \tfrac{(\varepsilon^{-i}-1)(1+\varepsilon^{a})}{2(1-\varepsilon^{-a})(1-\varepsilon^{a})^2} =$$

$$\tfrac{1}{2} \textstyle\sum_{\varepsilon \neq 1} \left[-\tfrac{\varepsilon^{-i}-1}{(1-\varepsilon^{-a})(1-\varepsilon^{a})} + \tfrac{2(\varepsilon^{-i}-1)}{(1-\varepsilon^{-a})(1-\varepsilon^{a})^2} \right]$$

equals

$$-\frac{1}{2}\sigma_i\Big(\frac{1}{r}(a, r-a)\Big) + \frac{1}{2}\sigma_0\Big(\frac{1}{r}(a, r-a)\Big) + \sigma_i\Big(\frac{1}{r}(a, r-a, r-a)\Big) - \sigma_0\Big(\frac{1}{r}(a, r-a, r-a)\Big).$$

Using Proposition (8.10) in [30] we get

$$-\tfrac{1}{2}\sigma_{\overline{ki}}\big(\tfrac{1}{r}(1,-1)\big)+\tfrac{1}{2}\sigma_0\big(\tfrac{1}{r}(1,-1)\big)+\sigma_{r-\overline{ki}}\big(\tfrac{1}{r}(-1,1,1)\big)-\sigma_0\big(\tfrac{1}{r}(-1,1,1)\big)=$$

$$\tfrac{\overline{ki}(r-\overline{ki})}{4}-(r-\overline{ki})\tfrac{r^2-1}{12}+\textstyle\sum_{j=1}^{r-\overline{ki}-1}\tfrac{j(r-j)}{2}=\tfrac{1}{12}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki}).$$

These observations simplify the expression

$$\textstyle\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i}-1)\Big(\tfrac{1-g}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})}-\tfrac{\varepsilon^{-a\alpha}}{(1-\varepsilon^{-a\alpha})^2(1-\varepsilon^{a\alpha})}\deg_C x_1-$$

$$\tfrac{\varepsilon^{a\alpha}}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})^2}\deg_C x_2\Big)=$$

$$\textstyle\sum_{\mu_r\ni\varepsilon\neq 1}(\varepsilon^{-i}-1)\Big(\tfrac{1-g}{(1-\varepsilon^{-a})(1-\varepsilon^{a})}-\tfrac{\varepsilon^{-a}}{(1-\varepsilon^{-a})^2(1-\varepsilon^{a})}\deg_C x_1-\tfrac{\varepsilon^{a}}{(1-\varepsilon^{-a})(1-\varepsilon^{a})^2}\deg_C x_2\Big)$$

into

$$-\tfrac{\overline{ik}(r-\overline{ik})}{2}(1-g+\tfrac{\deg_C x_1}{2}+\tfrac{\deg_C x_1}{2})+$$

$$\tfrac{1}{12r}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_C x_1-\deg_C x_2).$$

By Lemma 2.3.9, the term $1-g+\frac{\deg_C x_1}{2}+\frac{\deg_C x_1}{2}=-\frac{1}{2}\deg K_Z|C$ which finishes the proof of (1.).

(2.) Since d and a are coprime to r

$$\textstyle\sum_{\alpha=1,\beta=0\text{ and}\alpha\neq\beta}^{r-1}(\varepsilon^{-\alpha i}-1)\tfrac{1}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d(\alpha-\beta)})}=$$

$$\textstyle\sum_{j=1}^{r-1}\tfrac{1}{1-\varepsilon^j}\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i}-1)\tfrac{1}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})}=$$

$$\textstyle\sum_{j=1}^{r-1}\tfrac{1}{1-\varepsilon^j}\left(\sigma_i\big(\tfrac{1}{r}(a,r-a)\big)-\sigma_0\big(\tfrac{1}{r}(a,r-a)\big)\right)=$$

$$-\tfrac{\overline{ik}(r-\overline{ik})}{2}\cdot\textstyle\sum_{j=1}^{r-1}\tfrac{1}{1-\varepsilon^j}=$$

$$-\tfrac{r-1}{2}\cdot\tfrac{\overline{ik}(r-\overline{ik})}{2}$$

□

2.3.4 Construction of an auxiliary cover

By Section 1.2 we can restrict ourselves to a threefold X embedded in a weighted projective space. In the neighbourhood of an A_{r-1} curve we will find two parameters t_1, t_2 such that $\sqrt[r]{t_1}, \sqrt[r]{t_2}$ define parameters in a cover Z. Then

$$X = Z/\mu_r \oplus \mu_r,$$

where $\mu_r \oplus \mu_r$ acts on $\sqrt[r]{t_1}, \sqrt[r]{t_2}$.

In the construction of the $\mu_r \oplus \mu_r$ cover we will need the following lemma.

Lemma 2.3.11 *Let*

$$X \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l)$$

with variables $x_1, \ldots, x_k, z_1, \ldots, z_l$ be a quasismooth well formed threefold with singularities of transverse type A_{r-1} along a curve C. Assume that $a_1, \ldots, a_k$ are not divisible by r and $b_1, \ldots, b_l$ are. The singularities along C are induced by the weights $b_1, \ldots, b_l$.

After reembedding $C \subset X$ in a larger weighted projective space if necessary, we can always assume that

$$\operatorname{hcf}\left(\frac{b_1}{r}, r\right) = \operatorname{hcf}\left(\frac{b_2}{r}, r\right) = 1$$

and

$$C \cap \{z_1 = 0\} \cap \{z_2 = 0\} = \emptyset.$$

Proof Denote by D the $\mathbb{Q}-$Cartier divisor given by $\mathcal{O}_X(1)$.

Since $b_1, \ldots, b_l$ induce the singularities along C that are all of type A_{r-1}, observe that $\operatorname{hcf}(b_1, \ldots, b_l) = r$. Let d be a positive number, coprime to r. By the Chinese Remainder Theorem there exist integers $\gamma_1, \ldots, \gamma_l$ such that

$$\sum_{i=1}^{l} \gamma_i b_i = dr.$$

By adding $\sum_{i=1}^{l} r^2 *_i b_i$ for suitable nonnegative integers $*_i$ to both sides of the equation, we get

$$\sum_{i=1}^{l} \gamma_i' b_i = dr + r^2 \sum_{i=1}^{l} *_i b_i,$$

where all γ_i' are nonnegative. This proves that $|(dr + r^2 \sum_{i=1}^{l} *_i b_i)D|$ has a section. Note that $d + r\sum_{i=1}^{l} *_i b_i$ is coprime to r.

By the above construction, we can always find d_1 and d_2 that are both coprime to r and $|d_1 rD|$ and $|d_2 rD|$ have sections. Sections $f_1 \in |d_1 rD|$ and $f_2 \in |d_2 rD|$ can be expressed as relations among $x_1, \ldots, x_k, z_1, \ldots, z_l$, namely $f_1 = p_1(x_i, z_j)$ and $f_2 = p_2(x_i, z_j)$.

Define

$$w_1 = p_1(x_i, z_j) \quad \text{and} \quad w_2 = p_2(x_i, z_j).$$

Then

$$C \subset X \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l)$$

with the variables $x_1, \ldots, x_k, z_1, \ldots, z_l$ is isomorphic to

$$C' \subset X' \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l, d_1 r, d_2 r),$$

with the variables $x_1, \ldots, x_k, z_1, \ldots, z_l, w_1, w_2$, where the corresponding ideals are

$$I_{C'} = I_C + < w_1 - p_1, w_2 - p_2 > \text{ and } I_{X'} = I_X + < w_1 - p_1, w_2 - p_2 > .$$

Each general section in $|d_1 rD|$ and $|d_2 rD|$ cuts C in a number of points. Pick two disjoint sections which fulfill the condition $C \cap \{w_1 = 0\} \cap \{w_2 = 0\} = \emptyset$. This finishes the proof. $\square$

Remark 2.3.12 Note that the reembedding in the proof really is necessary. Consider for example

$$X_{900} \subset \mathbb{P}(1, 29, 60, 90, 150).$$

Proposition 2.3.13 *Let*

$$X \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l)$$

be a quasismooth well formed threefold where $a_1, \ldots, a_k$ are not divisible by r and $b_1, \ldots, b_l$ are. Assume that $\mathcal{O}_X(1)$ contains a ${}_k\big(\frac{1}{r}(1,-1)\big)$ curve C of genus g, where the singularities along the curve are induced by the weights $b_1, \ldots, b_l$ and $(X, \mathcal{O}_X(1))$ contains no other singularities.

Then for all $m > 0$

$$\begin{aligned}\chi(X, \mathcal{O}_X(m)) &= \text{RR–type expression in } \mathcal{O}_X(m) + \\ & \tfrac{\overline{mk}(r-\overline{mk})}{4r^2} \deg K_Z|_{C_Z} + \\ & \tfrac{1}{12r^2}\overline{km}(r-\overline{km})(r-2\cdot\overline{km})(\deg_{C_Z} x_2 - \deg_{C_Z} x_1) - \\ & \Lambda(r-1)\tfrac{\overline{km}(r-\overline{km})}{4r^2} - \Xi(r-1)\tfrac{\overline{km}(r-\overline{km})}{4r^2},\end{aligned}$$

where

$$\begin{array}{ccc} C_Z & \subset & Z \\ & & \downarrow \\ C & \subset & X \end{array}$$

is a smooth auxiliary cover of X and the normal sheaf to C_Z decomposes into $\mathcal{N}_{C_Z/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with $x_1 = c_1(\mathcal{N}_1),\ x_2 = c_1(\mathcal{N}_2)$. The integers Λ and Ξ also depend on Z.

Proof Fix a generator $\varepsilon \in \mu_r$. We construct an auxiliary cover of X in the following way. Consider the global quotient of

$$\mathbb{P}\left(a_1, \ldots, a_k, \frac{b_1}{r}, \frac{b_2}{r}, b_3, \ldots, b_l\right)$$

with the variables $x_1, \ldots, x_k, z_1, \ldots, z_l$, by the group $\mu_r \oplus \mu_r \ni (\varepsilon^\alpha, \varepsilon^\beta)$ acting via

$$\underbrace{1, \ldots, 1}_{k}, \varepsilon^{-\alpha\frac{b_1}{r}}, \varepsilon^{-\beta\frac{b_2}{r}}, \underbrace{1, \ldots, 1}_{l-2}.$$

By Lemma 2.3.11 we can assume that $\frac{b_1}{r}$ and $\frac{b_2}{r}$ are both coprime to r. Thus, the invariants $x_1, \ldots, x_k,\ z_3, \ldots, z_l$ and z_1^r, z_2^r generate the coordinate ring of

$$\mathbb{P}\left(a_1, \ldots, a_k, \frac{b_1}{r}, \frac{b_2}{r}, b_3, \ldots, b_l\right) / \mu_r \oplus \mu_r$$

which is exactly

$$\mathbb{P}(a_1, \ldots, a_k, b_1, b_2, b_3 \ldots, b_l).$$

Write

$$\begin{array}{cccc} \pi\colon & \mathbb{P}\left(a_1, \ldots, a_k, \frac{b_1}{r}, \frac{b_2}{r}, b_3, \ldots, b_l\right) & \longrightarrow & \mathbb{P}\left(a_1, \ldots, a_k, b_1, b_2, b_3, \ldots, b_l\right) \\ & x_1, \ldots, x_k, z_1, z_2, z_3, \ldots, z_l & \longmapsto & x_1, \ldots, x_k, z_1^r, z_2^r, z_3, \ldots, z_l \end{array}$$

for the quotient map.

The threefold $X \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l)$ is defined by a set of equations and relations in variables

$$x_1, \ldots, x_k, t_1 = z_1^r, t_2 = z_2^r, z_3, \ldots, z_l.$$

The singularities along the curve C are induced by weights $b_1, \ldots, b_l$, thus

$$C = X \cap \{x_1 = \ldots = x_k = 0\}.$$

Now define

$$C_Z \subset Z \subset \mathbb{P}\left(a_1, \ldots, a_k, \frac{b_1}{r}, \frac{b_2}{r}, b_3, \ldots, b_l\right)$$

by the same equations and relations as

$$C \subset X \subset \mathbb{P}(a_1, \ldots, a_k, b_1, \ldots, b_l).$$

In other words, replace t_1 by z_1^r and t_2 by z_2^r in all equations and relations. By Lemma 2.3.11 $C \cap \{t_1 = t_2 = 0\} = \emptyset$. Thus, by the definition of C_Z also $C_Z \cap \{z_1 = z_2 = 0\} = \emptyset$, which implies that Z is smooth in the neighbourhood of C_Z.

The threefold Z inherits $\mu_r \oplus \mu_r = \{(\varepsilon^\alpha, \varepsilon^\beta) \mid \alpha, \beta = 0, \ldots, r-1\}$ action which we now describe:

- If $\alpha = \beta$ the action is

  $$\underbrace{1, \ldots, 1}_{k}, \varepsilon^{-\alpha\frac{b_1}{r}}, \varepsilon^{-\alpha\frac{b_2}{r}}, \underbrace{1, \ldots, 1}_{l-2},$$

  which is the same as

  $$\varepsilon^{\alpha a_1}, \ldots, \varepsilon^{\alpha a_k}, \underbrace{1, \ldots, 1}_{l}.$$

  It fixes C_Z and acts in the normal direction of the curve by $\varepsilon^{a\alpha}, \varepsilon^{(r-a)\alpha}$ for some a coprime to r.

- If $\alpha \neq 0$ and $\alpha \neq \beta$ it fixes a number of points given by $\{z_2 = 0\}$ on C_Z. The action is

  $$\varepsilon^{\alpha a_1}, \ldots, \varepsilon^{\alpha a_k}, 1, \varepsilon^{(\alpha-\beta)\frac{b_2}{r}}, \underbrace{1, \ldots, 1}_{l-2}$$

which acts on each point in Z by $\varepsilon^{a\alpha}, \varepsilon^{(r-a)\alpha}, \varepsilon^{(\alpha-\beta)\frac{b_2}{r}}$. Denote the number of these points by Λ.

- If $\beta \neq 0$ and $\alpha \neq \beta$ it fixes a number of points given by $\{z_1 = 0\}$ on C_Z. The action is

$$\varepsilon^{\beta a_1}, \ldots, \varepsilon^{\beta a_k}, \varepsilon^{(\beta-\alpha)\frac{b_1}{r}}, 1, \underbrace{1, \ldots, 1}_{l-2}$$

  which acts on each point in Z by $\varepsilon^{a\beta}, \varepsilon^{(r-a)\beta}, \varepsilon^{(\beta-\alpha)\frac{b_1}{r}}$. Denote the number of these points by Ξ.

- If $\alpha = 0$ it fixes surfaces given by $\{z_2 = 0\}$ on Z. The action is

$$\underbrace{1, \ldots, 1}_{k}, 1, \varepsilon^{-\beta\frac{b_2}{r}}, \underbrace{1, \ldots, 1}_{l-2},$$

  which acts in the normal direction by $\varepsilon^{-\beta\frac{b_2}{r}}$.

- If $\beta = 0$ it fixes surfaces given by $\{z_1 = 0\}$ on Z. The action is

$$\underbrace{1, \ldots, 1}_{k}, \varepsilon^{-\alpha\frac{b_1}{r}}, 1, \underbrace{1, \ldots, 1}_{l-2},$$

  which acts in the normal direction by $\varepsilon^{-\alpha\frac{b_1}{r}}$.

This is again an example of Action ♠ introduced in Definition 2.3.8.

Now write $\pi\colon Z \to X$ for the quotient map and let $\mathcal{L}_{i,j}$ be the (i,j)th eigensheaf of the action of $\mu_r \oplus \mu_r$ on $\pi_*\mathcal{O}_Z$.

Without loss of generality we can assume that the variables x_1 and x_2 in $\mathbb{P}$ induce the action on the normal direction of C. In other words, the corresponding weights are $a_1 = a \bmod r$ and $a_2 = -a \bmod r$. Observe that $\mathcal{O}_X(a_1)$ has a global section x_1. The divisor $D_a = \{x_1 = 0\}$ defines a divisorial subsheaf $\mathcal{O}_X(D_a) \subset k(X)$. On the open set $U_1 = \{t_1 \neq 0\}$ we can write

$$\frac{\sqrt[r]{t_1}^{a_1}}{x_1^{\frac{b_1}{r}}} \in \Gamma\left(U_1, \mathcal{O}_X\left(\frac{b_1}{r}D_a\right)\right),$$

and the same way on $U_2 = \{t_2 \neq 0\}$ we have

$$\frac{\sqrt[r]{t_2}^{a_1}}{x_1^{\frac{b_2}{r}}} \in \Gamma\left(U_2, \mathcal{O}_X\left(\frac{b_2}{r}D_a\right)\right).$$

This means that the sheaf $\mathcal{O}_X\left(\frac{b_1}{r}D_a\right)$ is generated by $\frac{\sqrt[r]{t_1}^{a_1}}{x_1^{\frac{b_1}{r}}}$ on U_1, and $\mathcal{O}_X\left(\frac{b_2}{r}D_a\right)$ is generated by $\frac{\sqrt[r]{t_2}^{a_1}}{x_1^{\frac{b_2}{r}}}$ on U_2. In order to simplify the notation substitute $t_1 = z_1^r$ and $t_2 = z_2^r$. On U_1, the function

$$\frac{z_1^{a_1}}{x_1^{\frac{b_1}{r}}} \in \mathcal{L}_{\overline{-a\frac{b_1}{r}},0},$$

since

$$(\varepsilon^\alpha, \varepsilon^\beta)\left(\frac{z_1^{a_1}}{x_1^{\frac{b_1}{r}}}\right) = \varepsilon^{-\alpha a\frac{b_1}{r}} \cdot \frac{z_1^{a_1}}{x_1^{\frac{b_1}{r}}}.$$

The same way on U_2, the function

$$\frac{z_2^{a_1}}{x_1^{\frac{b_2}{r}}} \in \mathcal{L}_{0,\overline{-a\frac{b_2}{r}}},$$

since

$$(\varepsilon^\alpha, \varepsilon^\beta)\left(\frac{z_2^{a_1}}{x_1^{\frac{b_2}{r}}}\right) = \varepsilon^{-\beta a\frac{b_2}{r}} \cdot \frac{z_2^{a_1}}{x_1^{\frac{b_2}{r}}}.$$

Thus we proved

$$\mathcal{O}_X\left(\frac{b_1}{r}D_a\right)\Big|_{U_1} \text{ has the same singularities as } \mathcal{L}_{\overline{-a\frac{b_1}{r}},0}|_{U_1}$$

and

$$\mathcal{O}_X\left(\frac{b_2}{r}D_a\right)\Big|_{U_2} \text{ has the same singularities as } \mathcal{L}_{0,\overline{-a\frac{b_2}{r}}}|_{U_2},$$

where $\overline{}$ denotes the smallest residue mod r.

Let m be an arbitrary positive integer. Since integers a, a_1 and $\frac{b_1}{r}, \frac{b_2}{r}$ are all coprime to r, we can find n_1, n_2 such that

$$n_1 \cdot a_1\frac{b_1}{r} = m \bmod r \quad \text{and} \quad n_2 \cdot a_1\frac{b_2}{r} = m \bmod r.$$

This implies:

$$\begin{array}{ll} & \mathcal{O}_X(m)|_{U_1} \\ \text{has the same singularities as} & \\ & \mathcal{O}_X(n_1 \cdot a_1 \frac{b_1}{r})|_{U_1} \cong \mathcal{O}_X(n_1 \frac{b_1}{r} D_a)|_{U_1}, \\ \text{which has the same singularities as} & \\ & \mathcal{L}_{\overline{-m},0}|_{U_1}. \end{array}$$

The first part of the statement holds, since the corresponding divisorial sheaves differ only by tensoring with an invertible sheaf. If $n_1 \cdot a_1 \frac{b_1}{r} = m + *r$ for some positive $*$, then

$$\mathcal{O}_X(m + *r) \cong \mathcal{O}_X(m) \otimes \mathcal{O}_X(\text{Cartier divisor}).$$

The second part of the statement is true since the section $\left(\frac{z_1^{a_1}}{x_1^{\frac{b_1}{r}}}\right)^{n_1}$ of $\mathcal{O}_X(n_1 \frac{b_1}{r} D_a)|_{U_1}$ lies in $\mathcal{L}_{\overline{-m},0}|_{U_1}$.

The same way we prove

$$\begin{array}{ll} & \mathcal{O}_X(m)|_{U_2} \\ \text{has the same singularities as} & \\ & \mathcal{O}_X(n_2 \cdot a_1 \frac{b_2}{r})|_{U_2} \cong \mathcal{O}_X(n_2 \frac{b_2}{r} D_a)|_{U_2}, \\ \text{which has the same singularities as} & \\ & \mathcal{L}_{0,\overline{-m}}|_{U_2}. \end{array}$$

Note also that $\mathcal{L}_{\overline{-m},0}|_{U_1 \cap U_2}$ has the same singularities as $\mathcal{L}_{0,\overline{-m}}|_{U_1 \cap U_2}$.

In (E. vi) on page 34 we proved

$$\chi(X, \mathcal{L}_{i,j}) = \chi(\mathcal{O}_X) + \frac{1}{r^2} \sum_{\alpha,\beta=0}^{r-1} (\varepsilon^{-\alpha i - \beta j} - 1) A_{\alpha,\beta}$$

for any Action ♠. Here we think of $\chi(\mathcal{O}_X)$ as RR–type expression in L_{ij}, where $L_{ij} = 0 \in \operatorname{Div} X \otimes \mathbb{Q}$ is the corresponding $\mathbb{Q}-$divisor. The rest of the formula is a sum of contributions coming from the singularities contained in $\mathcal{L}_{i,j}$.

This proves

$$\begin{array}{rcl} \chi(X, \mathcal{O}_X(m)) & = & \text{RR–type expression in } \mathcal{O}_X(m) + \\ & & \text{contributions from singularities of } \mathcal{L}_{\overline{-m},0}|_{U_1} + \\ & & \text{contributions from singularities of } \mathcal{L}_{0,\overline{-m}}|_{U_2} - \\ & & \text{contributions from singularities of } \mathcal{L}_{\overline{-m},0}|_{U_1 \cap U_2}. \end{array}$$

The following lemma will finish the proof of Proposition 2.3.13

Lemma 2.3.14 *The contributions from the singularities of $\mathcal{L}_{i,0}$ on $\{t_1 \neq 0\}$ are*

$$\tfrac{\overline{ik}(r-\overline{ik})}{4r^2}\deg K_Z|_{C_Z} + \tfrac{1}{12r^2}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_{C_Z} x_1 - \deg_{C_Z} x_2) -$$

$$\Lambda(r-1)\tfrac{\overline{ki}(r-\overline{ki})}{4r^2}.$$

The contributions from the singularities of $\mathcal{L}_{0,j}$ on $\{t_2 \neq 0\}$ are

$$\tfrac{\overline{jk}(r-\overline{jk})}{4r^2}\deg K_Z|_{C_Z} + \tfrac{1}{12r^2}\overline{kj}(r-\overline{kj})(r-2\cdot\overline{kj})(\deg_{C_Z} x_1 - \deg_{C_Z} x_2) -$$

$$\Xi(r-1)\tfrac{\overline{kj}(r-\overline{kj})}{4r^2}.$$

The contributions from the singularities of $\mathcal{L}_{i,0}$ on $\{t_1 \neq 0\} \cap \{t_2 \neq 0\}$ are

$$\tfrac{\overline{ik}(r-\overline{ik})}{4r^2}\deg K_Z|_{C_Z} + \tfrac{1}{12r^2}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_{C_Z} x_1 - \deg_{C_Z} x_2).$$

Proof By (E. vi) the contributions from the singularities of $\mathcal{L}_{i,0}$ are

$$\frac{1}{r^2}\sum_{\alpha,\beta=0}^{r-1}(\varepsilon^{-\alpha i}-1)A_{\alpha,\beta}.$$

On $\{t_1 \neq 0\}$ the contributions come from the curve C_Z, points $C_Z \cap \{z_2 = 0\}$ and surfaces $\{z_2 = 0\}$. By the definition of $A_{\alpha,\beta}$ this simplifies the above sum into

$$\tfrac{1}{r^2}\textstyle\sum_{\alpha=1}^{r-1}(\varepsilon^{-\alpha i}-1)\Big(\tfrac{1-g}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})} - \tfrac{\varepsilon^{-a\alpha}}{(1-\varepsilon^{-a\alpha})^2(1-\varepsilon^{a\alpha})}\deg_C x_1 - \tfrac{\varepsilon^{a\alpha}}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})^2}\deg_C x_2\Big)$$

$$+\tfrac{1}{r^2}\textstyle\sum_{\alpha=1,\beta=0;\text{ and }\alpha\neq\beta}^{r-1}(\varepsilon^{-\alpha i}-1)\tfrac{\Lambda}{(1-\varepsilon^{-a\alpha})(1-\varepsilon^{a\alpha})(1-\varepsilon^{-d_2(\alpha-\beta)})}$$

By Lemma 2.3.10 the sum can be written in a more compact form

$$\tfrac{\overline{ik}(r-\overline{ik})}{4r^2}\deg K_Z|_{C_Z} + \tfrac{1}{12r^2}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_{C_Z} x_1 - \deg_{C_Z} x_2) -$$

$$\Lambda(r-1)\tfrac{\overline{ki}(r-\overline{ki})}{4r^2}.$$

Exactly the same way we compute the contributions of $\mathcal{L}_{0,j}$ on $\{t_2 \neq 0\}$.

On $\{t_1 \neq 0\} \cap \{t_2 \neq 0\}$ the only contributions come from the curve C_Z and equal

$$\frac{\overline{ik}(r-\overline{ik})}{4r^2} \deg K_Z|_{C_Z} + \frac{1}{12r^2}\overline{ki}(r-\overline{ki})(r-2\cdot\overline{ki})(\deg_{C_Z} x_1 - \deg_{C_Z} x_2).$$

$\square$

2.3.5 RR does not depend on the auxiliary cover

We would like to express the quantities $\deg K_Z|_{C_Z}$, Λ and Ξ in Proposition 2.3.13 (depending on the auxiliary cover Z) with parameters of X.

In this construction we will need the following two lemmas.

Lemma 2.3.15 *Under the conditions of Proposition 2.3.13,*

$$\deg K_Z|_{C_Z} = r \cdot \deg K_X|_C + \deg R,$$

where R is the ramification divisor of a finite r to 1 morphism of curves $\pi|_{C_Z} \colon C_Z \to C$.

In particular

$$\deg K_Z|_{C_Z} - (r-1)(\Lambda + \Xi) = r \cdot \deg K_X|_C$$

Proof Consider the auxiliary cover

$$\begin{array}{ccccc}
C_Z & \subset & Z & \subset & \mathbb{P}\left(a_1,\dots,a_k,\frac{b_1}{r},\frac{b_2}{r},b_3,\dots,b_l\right) \\
\Big\downarrow{\scriptstyle \pi|_{C_Z}} & & \Big\downarrow{\scriptstyle \pi|_Z} & & \Big\downarrow{\scriptstyle \pi} \\
C & \subset & X & \subset & \mathbb{P}(a_1,\dots,a_k,b_1,b_2,b_3,\dots,b_l)
\end{array}$$

with the variables

$$x_1,\dots,x_k,z_1,z_2,z_3,\dots,z_l \longrightarrow x_1,\dots,x_k,t_1=z_1^r,t_2=z_2^r,z_3,\dots,z_l,$$

constructed in the proof of Proposition 2.3.13.

The variety X is an orbifold, as defined in Appendix A.2. As usual, restrict ourselves to a neighbourhood of C, which can be covered by local charts U_λ/G_λ. Every G_λ is isomorphic to $\mu_r \ni \varepsilon$ with action

$$x_\lambda,\ y_\lambda,\ z_\lambda \mapsto \varepsilon^a x_\lambda,\ \varepsilon^{r-a} y_\lambda,\ z_\lambda$$

on the local parameters of U_λ. Then,

$$\begin{array}{ccc} U_\lambda \subset \mathbb{C}^3 & \xrightarrow{q_\lambda} & U_\lambda/\mu_r \\ x_\lambda,\ y_\lambda,\ z_\lambda & \mapsto & x_\lambda^r,\ x_\lambda y_\lambda,\ y_\lambda^r, z_\lambda. \end{array}$$

The corresponding ring of invariants is

$$\mathbb{C}[x_\lambda^r, x_\lambda y_\lambda, y_\lambda^r, z_\lambda] \cong \mathbb{C}[u_\lambda, v_\lambda, w_\lambda, z_\lambda]/(u_\lambda w_\lambda = v_\lambda^r).$$

Observe that $dx_\lambda \wedge dy_\lambda \wedge dz_\lambda$ is a basis for ω_{U_λ} which is invariant under the group action. Now set $s = \frac{du_\lambda \wedge dv_\lambda \wedge dz_\lambda}{u_\lambda}$. We claim that s is a basis for ω_{U_λ/μ_r}. Indeed, by differentiating $u_\lambda = x_\lambda^r,\ v_\lambda = x_\lambda y_\lambda$ we get

$$q_\lambda^* s = r \cdot dx_\lambda \wedge dy_\lambda \wedge dz_\lambda.$$

Recall from Appendix A.2 that the sheaves ω_{U_λ/μ_r} glue together to define $\omega_X = \mathcal{O}_X(K_X)$.

There are two different types of points on C:

- points that have one of the coordinates t_1 or t_2 zero. Recall that there are $\Lambda + \Xi$ such points.
- points for which t_1 and t_2 are nonzero.

Let P be a point of the first type with a local chart $U/\mu_r \subset X$. Note that $\pi^{-1}(P)$ is only one point. Choose $V \subset \pi^{-1}(U/\mu_r) \subset Z$, a neighborhood of $\pi^{-1}(P)$ with local parameters x, y, t. Then

$$\begin{array}{cccc} \pi|_V: & V & \longrightarrow & U/\mu_r \\ & x, y, t & \mapsto & x^r, xy, y^r, t^r. \end{array}$$

Obviously the diagram

$$\begin{array}{ccccc} & & U & & x, y, z \\ & & \downarrow & & \downarrow \\ V & \longrightarrow & U/\mu_r & & x^r, xy, y^r, z \\ x, y, t & \mapsto & x^r, xy, y^r, t^r & & \end{array}$$

factors. The induced map

$$f\colon \quad \begin{array}{ccc} V & \longrightarrow & U \\ x, y, t & \mapsto & x, y, t^r \end{array}$$

is a r to 1 morphism with the ramification divisor $R = \{t = 0\}$. Its ramification index is r. By Lemma A.1.1

$$\omega_V \cong f^* \omega_U \otimes \mathcal{O}_R.$$

Let Q be a point of the second type with a local chart $U'/\mu_r \subset X$. Note that $\pi^{-1}(Q)$ contains r points. Denote by $\{V_i'\}_{i=1,\dots,r}$ the corresponding neighborhoods in $\pi^{-1}(U'/\mu_r) \subset Z$. In this case the induced map $f|_{V_i'}\colon V_i' \longrightarrow U'$ is an isomorphism, and

$$\omega_{V_i'} \cong \omega_{U'} \text{ or eqivalently } f^*\omega_{U'} \cong \omega_{\cup_i V_i'}.$$

Recall from the previous page that ω_{U_λ/μ_r} are defined by ω_{U_λ}. Gluing together different charts V_λ in Z and U_λ/μ_r in X yields

$$\omega_Z \cong \pi^* \omega_X \otimes \mathcal{O}_R,$$

where R is the ramification divisor of π supported on $\Lambda + \Xi$ surfaces given by $t_1 = 0$ or $t_2 = 0$. For the corresponding divisors on Z this implies

$$K_Z \sim \pi^* K_X + R.$$

Restrict these divisors to the curve C_Z. Then $\deg R|_{C_Z} = (r-1)(\Lambda + \Xi)$, since each ramification surface intersects C_Z in exactly one point with the ramification index r.

Next consider $\deg(\pi^* K_X)|_{C_Z}$. Since X is $\mathbb{Q}$–Gorenstein, a multiple of K_X can be moved away from the branched points on C. By definition tK_X intersects C in $t \deg K_X|_C$ points. Since $\pi|_{C_Z}\colon C_Z \to C$ is r to 1 and etale away from $R|_{C_Z}$, the divisor $\pi^* t K_X$ intersects C_Z in $r \cdot t \deg K_X|_C$ points. (Think of K_X set theoretically as a Weil divisor. Then $P \in \pi^* K_X \cap C_Z \Rightarrow \pi(P) \in K_X \cap C$. Conversely, for a generic $Q \in K_X \cap C \Rightarrow \pi^{-1}(Q)$ contains r points in $\pi^* K_X \cap C_Z$, since $\pi^{-1}(C) = C_Z$.) This finishes the proof of Lemma 2.3.15. $\square$

Lemma 2.3.16 *Under the conditions of Proposition 2.3.13*

$$\deg_{C_Z} x_2 - \deg_{C_Z} x_1 = 2r(\mathit{genus}(C) - 1) - r \deg K_X|_C - 2\deg_{C_Z} x_1.$$

Proof Recall that the auxiliary cover Z is smooth in the neighborhood of C_Z with $c_1(\mathcal{N}_{C_Z/Z}) = x_1 + x_2$. By Lemma 2.3.9

$$\deg c_1(\mathcal{N}_{C_Z/Z}) = \deg K_{C_Z} - \deg K_Z|_{C_Z} = 2(\text{genus}(C_Z) - 1) - \deg K_Z|_{C_Z}.$$

Now use Hurwitz theorem and Lemma 2.3.15

$$\deg c_1(\mathcal{N}_{C_Z/Z}) = \deg_{C_Z} x_2 + \deg_{C_Z} x_1 =$$

$$2r(\text{genus}(C) - 1) + \deg R - r \deg K_X|_C - \deg R.$$

From here we get

$$\deg_{C_Z} x_2 - \deg_{C_Z} x_1 = 2r(\text{genus}(C) - 1) - r \deg K_X|_C - 2\deg_{C_Z} x_1.$$

$\square$

Theorem 2.3.17 *Let X be a quasismooth and well formed threefold embedded in a weighted projective space and let D be a divisor on X given by $\mathcal{O}_X(1)$. Assume that the singularities of (X, D) consist of*

- *curves C of type ${}_k\big(\frac{1}{r}(1,-1)\big)$ and genus g with no dissident points,*
- *isolated singular points P.*

The Riemann–Roch formula for the divisorial sheaf $\mathcal{O}_X(mD)$ is

$$\begin{aligned} \chi(X, \mathcal{O}_X(mD)) &= \chi(\mathcal{O}_X) + \tfrac{1}{12} mD(mD - K_X)(2mD - K_X) + m\tfrac{D\cdot c_2(X)}{12} \\ &\quad + \textstyle\sum_C s_C(mD) + \mathcal{P}, \end{aligned}$$

where

$$\begin{aligned} s_C(mD) &= -m\tfrac{\overline{mk}(r-\overline{mk})}{2r} \deg D|_C + \tfrac{\overline{mk}(r-\overline{mk})}{4r} \deg K_X|_C \\ &\quad + \tfrac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r} \cdot \big(g - 1 - \tfrac{1}{2}\deg K_X|_C - \tfrac{\iota_C}{r}\big) \end{aligned}$$

and $\mathcal{P}$ only depends on the isolated points of X. The integer ι_C is an invariant of X in the neighbourhood of C.

Proof Every component of the singular locus gives a contribution to the Riemann–Roch formula which is a local analytic invariant. This means that the contributions can be computed separately on the local neighbourhood of each singularity.

Proposition 2.3.13, Lemma 2.3.15 and Lemma 2.3.16 together imply

$$\begin{aligned}\chi(X, \mathcal{O}_X(mD)) &= (\text{RR–type expression in } mD)+ \\ & \textstyle\sum_C s'_C(mD) + \mathcal{P},\end{aligned}$$

where

$$\begin{aligned}s'_C(mD) &= \tfrac{\overline{mk}(r-\overline{mk})}{4r}\deg K_X|_C \\ &+ \tfrac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r}\cdot\left(g-1-\tfrac{1}{2}\deg K_X|_C - \tfrac{1}{r}\deg_{C_Z} x_1\right)\end{aligned}$$

The existence of the RR formula proves that the integers $\iota_C := \deg_{C_Z} x_1$ do not depend on the auxiliary covers and are thus invariants of X.

The Riemann–Roch type expression in mD is just $\mathrm{ch}(mD)\cdot\mathrm{td}(\mathcal{T}_X)$. This contains the terms

$$\frac{1}{6}m^3D^3,\ \frac{1}{4}m^2D^2K_X,\ \frac{1}{12}mDK_X^2,\ 1-p_a \text{ and } \frac{1}{12}mD\cdot c_2(X).$$

Some multiple of D is a Cartier divisor, i.e. $|tD|$ defines a surface S for some t. Therefore D^3 can be computed away from the singularities. The same way D^2K_X and DK_X^2 can be computed away from the singularities. The arithmetic genus p_a is a birational invariant of variety and $1-p_a = \chi(\mathcal{O}_X)$.

On the other hand the Cartier divisor tD always intersects the singular locus C in a finite number of points. These singular points will induce additional contributions to our RR formula, which we now describe. Consider the short exact sequence

$$0 \to \mathcal{T}_S \to i^*\mathcal{T}_X \to \mathcal{N}_{S/X} \to 0,$$

where $i\colon S \to X$ is the inclusion. The corresponding Chern polynomials are

$$\begin{aligned}c_t(\mathcal{T}_S) &= 1 - K_S t + c_2(S)t^2, \\ c_t(\mathcal{N}_{S/Z}) &= 1 + c_1(\mathcal{N}_{S/X})t, \\ c_t(\mathcal{T}_X) &= 1 - K_X t + c_2(X)t^2 + c_3(X)t^3.\end{aligned}$$

Use the self–intersection formula $c_1(\mathcal{N}_{S/X}) = S \cdot S$ and the adjunction formula $K_S = S|_S + K_X|_S$ and then compare the coefficients of t^2. We get $c_2(S) - (S|_S)^2 - S|_S K_X|_S = c_2(X)|_S$, which is the same as

$$c_2(S) - S^3 - K_X S^2 = S \cdot c_2(X).$$

Recall that S was defined by tD and that S^3 and $K_X S^2$ can be computed away from the singular locus. The contributions to Riemann–Roch coming from the term $\frac{1}{12} mD \cdot c_2(X) = \frac{m}{t}\frac{1}{12} S \cdot c_2(X)$ will thus equal the contributions induced by $\frac{m}{t}\frac{1}{12} c_2(S)$ on S. Riemann–Roch for a surface with isolated singular points was computed by Miles Reid in [30]. Given the $\mathbb{Q}-$Cartier divisor $mD|_S$, the term $\frac{1}{12} c_2(S)$ contributes exactly the so called basket $-\frac{\overline{mk}(r-\overline{mk})}{2r}$ for each singular point. There is $t \cdot \deg D|_C$ points in the intersection $S \cap C$.

By these considerations, we can interpret the term $\frac{1}{12} mD \cdot c_2(X)$ appearing in $\mathrm{ch}(mD) \cdot \mathrm{td}(\mathcal{T}_X)$ as

$\frac{1}{12} mD \cdot c_2(X)$ computed on the nonsingular part of $X-$

$m \cdot \frac{\overline{mk}(r-\overline{mk})}{2r} \deg D|_C$.

Finally we get

RR–type expression in $mD =$

$\chi(\mathcal{O}_X) + \frac{1}{12} mD(mD - K_X)(2mD - K_X) + m \cdot \frac{D \cdot c_2(X)}{12} -$

$m \cdot \frac{\overline{mk}(r-\overline{mk})}{2r} \deg D|_C$

where $D^3,\ K_X D^2,\ K_X^2 D$ and $D \cdot c_2(X)$ are computed away from the singularities. $\square$

Observe that Theorem 2.3.17 and Corollary 2.2.5 indeed are identical for X whose curves of singularities contain no dissident points.

Now all that remains is to calculate the contributions from the singular points.

2.4 Contributions from singular points

We proved that it is enough to compute the contributions to RR from singular points on any model that contains such singularities. First we will show the existence of such projective varieties.

Proposition 2.4.1 *Fix positive integers s and a_1, a_2, a_3, and assume that a_i, a_j, s have no common factor for all different $i, j \in \{1, 2, 3\}$. There exists a smooth projective 3–fold Z together with $\mu_s = \mathbb{Z}/s\mathbb{Z}$ action that we now describe:*
Choose a generator $\varepsilon \in \mu_s$. The action fixes a number of points on which ε acts by

$$\varepsilon\colon\; z_1, z_2, z_3 \;\mapsto\; \varepsilon^{a_1} z_1, \varepsilon^{a_2} z_2, \varepsilon^{a_3} z_3.$$

If $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$*, these points lie on curves which are fixed by $\varepsilon^{\frac{s}{\alpha_i}}$. Here $\varepsilon^{\frac{s}{\alpha_i}} \in \mu_{\alpha_i}$ acts in the normal direction of each curve by*

$$\varepsilon^{\frac{s}{\alpha_i}}\colon\; z_j, z_k \;\mapsto\; \varepsilon^{\frac{s}{\alpha_i} a_j} z_j, \varepsilon^{\frac{s}{\alpha_i} a_k} z_k, \quad \textit{for } j, k \in \{1, 2, 3\} - \{i\}$$

and freely away from the curve.

Proof The assumption $\mathrm{hcf}(a_i, a_j, s) = 1$ implies that the action fixes no two dimensional stratum, thus the quotient variety is well formed. We imitate the proof of (8.4) in [30]. Choose an integer $l \geq 3$ and consider action of μ_s on $\mathbb{P}^{l+3}(1, 1, \ldots, 1)$ given by

$$x_1, x_2, x_3, x_4, \ldots, x_{l+4} \;\mapsto\; \varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2, \varepsilon^{a_3} x_3, x_4, \ldots, x_{l+4}.$$

This action fixes $\mathbb{P}^l = \{x_1 = x_2 = x_3 = 0\}$ and acts in the normal direction by

$$x_1, x_2, x_3 \;\mapsto\; \varepsilon^{a_1} x_1, \varepsilon^{a_2} x_2, \varepsilon^{a_3} x_3.$$

If $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$, the action is not free on $\mathbb{P}^{l+1} = \{x_j = x_k = 0\}$ (for $j, k \in \{1, 2, 3\} - \{i\}$), which is fixed by $\varepsilon^{\frac{s}{\alpha_i}}$. It acts in the normal direction by

$$\varepsilon^{\frac{s}{\alpha_i}}\colon\; x_j, x_k \;\mapsto\; \varepsilon^{\frac{s}{\alpha_i} a_j} x_j, \varepsilon^{\frac{s}{\alpha_i} a_k} x_k.$$

Another locus on which the action might not be free, is $\{x_4, \ldots, x_{l+4} = 0\}$. We will avoid this locus by defining

$$X \subset \mathbb{P}^{l+3}/\mu_s,$$

as a complete intersection of l general very ample divisors. Let Z be the inverse image of X under the quotient $\mathbb{P}^{l+3} \to \mathbb{P}^{l+3}/\mu_s$. Such Z clearly satisfies the conditions in the proposition. $\square$

Using the above notation, we additionally assume that

$$a_j + a_k = 0 \mod \alpha_i.$$

This implies that the singular locus of X equals to a number of dissident

$$A_{\alpha_i - 1} = \frac{1}{\alpha_i}(a_j, a_k)$$

curves, which intersect in dissident points of type $\frac{1}{s}(a_1, a_2, a_3)$.

Let now X be a projective threefold with a singularity of type $\frac{1}{s}(a_1, a_2, a_3)$, which is a quotient of a smooth projective variety Z under μ_s action as described in Proposition 2.4.1.

Write $\pi\colon Z \to X$ for the quotient map and let $\mathcal{L}_n$ be the nth eigensheaf of the action of μ_s on $\pi_*\mathcal{O}_Z$. Then

$$\pi_*\mathcal{O}_Z = \bigoplus_{n=0}^{s-1} \mathcal{L}_n$$

implies

$$H^p(Z, \mathcal{O}_Z) = \bigoplus_{n=0}^{s-1} H^p(X, \mathcal{L}_n).$$

The group action on any $f \in \mathcal{L}_n$ is given by $\varepsilon(f) = \varepsilon^n \cdot f$, thus

$$\mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{n=0}^{s-1} h^p(X, \mathcal{L}_n) \cdot \varepsilon^n.$$

Moreover,

$$\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z)) = \sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) \cdot \varepsilon^n.$$

In order to simplify the notation, denote $\sum_p (-1)^p \mathrm{Tr}(\varepsilon : H^p(Z, \mathcal{O}_Z))$ by A_ε. Then

$$\sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) \cdot \varepsilon^n = A_\varepsilon.$$

In the case $\varepsilon = 1$, we get

$$\sum_{n=0}^{s-1} \chi(X, \mathcal{L}_n) = \chi(\mathcal{O}_Z).$$

From now on fix a generator $\varepsilon \in \mu_s$. The last two formulas can be considered as a linear system of s equations in $\chi(X, \mathcal{L}_n)$ and variable ε, given by Vandermonde matrix,

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{s-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(s-1)} \\ \dots & \dots & \dots & \dots & \\ 1 & \varepsilon^{s-1} & \dots & \dots & \varepsilon \end{pmatrix} \begin{pmatrix} \chi(X, \mathcal{L}_0) \\ \chi(X, \mathcal{L}_1) \\ \chi(X, \mathcal{L}_2) \\ \vdots \\ \chi(X, \mathcal{L}_{s-1}) \end{pmatrix} = \begin{pmatrix} \chi(\mathcal{O}_Z) \\ A_\varepsilon \\ A_{\varepsilon^2} \\ \vdots \\ A_{\varepsilon^{s-1}} \end{pmatrix}.$$

Since the Vandermonde matrix is invertible, we can solve the system

$$\frac{1}{s} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \dots & \varepsilon \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \dots & \varepsilon^2 \\ \dots & \dots & \dots & \dots & \\ 1 & \varepsilon & \dots & \dots & \varepsilon^{s-1} \end{pmatrix} \begin{pmatrix} \chi(\mathcal{O}_Z) \\ A_\varepsilon \\ A_{\varepsilon^2} \\ \vdots \\ A_{\varepsilon^{s-1}} \end{pmatrix} = \begin{pmatrix} \chi(X, \mathcal{L}_0) \\ \chi(X, \mathcal{L}_1) \\ \chi(X, \mathcal{L}_2) \\ \vdots \\ \chi(X, \mathcal{L}_{s-1}) \end{pmatrix}.$$

Write $\chi(X, \mathcal{L}_0) = \chi(\mathcal{O}_X)$ and eliminate $\chi(\mathcal{O}_Z)$ from the solution. We end up with formula

$$\begin{aligned} \chi(X, \mathcal{L}_n) &= \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{j=1}^{s-1} (\varepsilon^{-jn} - 1) A_{\varepsilon^j} \qquad \text{(E. vii)} \\ &= \chi(\mathcal{O}_X) + \frac{1}{s} \sum_{\varepsilon \in \mu_s} (\varepsilon^{-n} - 1) A_\varepsilon. \end{aligned}$$

We have already computed terms A_ε for all $\varepsilon \in \mu_s$ using the Atiyah-Singer formula. If $\varepsilon^{a_i} \neq 1$ for all $i = 1, 2, 3$, then the fixed locus of ε is a number of points. By Lemma 2.3.7 we get

$$A_\varepsilon = \sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) = \frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

If, on the other hand $\varepsilon^{a_i} = 1$ for one of the $i = 1, 2, 3$, then the fixed locus of ε is a curve C_i. In this case $\mathrm{hcf}(a_i, s) = \alpha_i \neq 1$ and ε is an element of μ_{α_i}. The action in the normal direction equals

$$\varepsilon^{a_j}, \varepsilon^{a_k} = \varepsilon^{a_j}, \varepsilon^{-a_j}.$$

Thus, we can use Theorem 2.3.4 for the cyclic group μ_{α_i}. Let C_i have genus g_i, and let x_1, x_2 denote the first Chern classes of $\mathcal{N}_{C_i/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$. We get

$$A_\varepsilon = \sum_p (-1)^p \mathrm{Trace}(\varepsilon | H^p(Z, \mathcal{O}_Z)) =$$

$$\frac{1-g_i}{(1-\varepsilon^{-a_j})(1-\varepsilon^{a_j})} - \frac{\varepsilon^{-a_j}}{(1-\varepsilon^{-a_j})^2(1-\varepsilon^{a_j})} \deg_{C_i} x_1 - \frac{\varepsilon^{a_j}}{(1-\varepsilon^{-a_j})(1-\varepsilon^{a_j})^2} \deg_{C_i} x_2.$$

Since we assumed that a_i, a_j, s have no common divisor for all $i, j = 1, 2, 3$, we can split the sum over $\{\varepsilon \in \mu_s\}$ into four subsumes over (possibly empty) disjoint sets

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_i} \neq 1 \ \forall i = 1, 2, 3\},$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_1} = 1\} = \{\varepsilon \in \mu_{\alpha_1}\},$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_2} = 1\} = \{\varepsilon \in \mu_{\alpha_2}\} \text{ and}$$

$$\{\varepsilon \in \mu_s \text{ s.t. } \varepsilon^{a_3} = 1\} = \{\varepsilon \in \mu_{\alpha_3}\}.$$

This rewrites Formula (E. vii) into

$$\chi(X, \mathcal{L}_n) = \chi(\mathcal{O}_X)$$

$$+\frac{1}{s} \sum_{\varepsilon \in \mu_s,\ \varepsilon^{a_i} \neq 1 \ \forall i=1,2,3} (\varepsilon^{-n} - 1) \frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}$$

$$+\frac{1}{s} \sum_{\varepsilon \in \mu_{\alpha_1}} (\varepsilon^{-n} - 1) \left(\frac{1-g_1}{(1-\varepsilon^{-a_2})(1-\varepsilon^{a_2})} - \frac{\varepsilon^{-a_2}}{(1-\varepsilon^{-a_2})^2(1-\varepsilon^{a_2})} \deg_{C_1} x_1 - \frac{\varepsilon^{a_2}}{(1-\varepsilon^{-a_2})(1-\varepsilon^{a_2})^2} \deg_{C_1} x_2 \right)$$

$$+\frac{1}{s} \sum_{\varepsilon \in \mu_{\alpha_2}} (\varepsilon^{-n} - 1) \left(\frac{1-g_2}{(1-\varepsilon^{-a_3})(1-\varepsilon^{a_3})} - \frac{\varepsilon^{-a_3}}{(1-\varepsilon^{-a_3})^2(1-\varepsilon^{a_3})} \deg_{C_2} x_1 - \frac{\varepsilon^{a_3}}{(1-\varepsilon^{-a_3})(1-\varepsilon^{a_3})^2} \deg_{C_2} x_2 \right)$$

$$+\frac{1}{s} \sum_{\varepsilon \in \mu_{\alpha_3}} (\varepsilon^{-n} - 1) \left(\frac{1-g_3}{(1-\varepsilon^{-a_1})(1-\varepsilon^{a_1})} - \frac{\varepsilon^{-a_1}}{(1-\varepsilon^{-a_1})^2(1-\varepsilon^{a_1})} \deg_{C_3} x_1 - \frac{\varepsilon^{a_1}}{(1-\varepsilon^{-a_1})(1-\varepsilon^{a_1})^2} \deg_{C_3} x_2 \right).$$

By Lemma 2.3.10 (1) we can put the last formula in a more compact form

$\chi(X,\mathcal{L}_n)=\chi(\mathcal{O}_X)$

$+\frac{1}{s}\sum_{\varepsilon\in\mu_s,\ \varepsilon^{a_i}\neq 1\ \forall i=1,2,3}(\varepsilon^{-n}-1)\frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}$

$+\frac{\overline{nk_1}(\alpha_1-\overline{nk_1})}{4s}\deg K_Z|_{C_1}+\frac{1}{12s}\overline{nk_1}(\alpha_1-\overline{nk_1})(\alpha_1-2\cdot\overline{nk_1})(\deg_{C_1}x_1-\deg_{C_1}x_2)$

$+\frac{\overline{nk_2}(\alpha_2-\overline{nk_2})}{4s}\deg K_Z|_{C_2}+\frac{1}{12s}\overline{nk_2}(\alpha_2-\overline{nk_2})(\alpha_2-2\cdot\overline{nk_2})(\deg_{C_2}x_1-\deg_{C_2}x_2)$

$+\frac{\overline{nk_3}(\alpha_3-\overline{nk_3})}{4s}\deg K_Z|_{C_3}+\frac{1}{12s}\overline{nk_3}(\alpha_3-\overline{nk_3})(\alpha_3-2\cdot\overline{nk_3})(\deg_{C_3}x_1-\deg_{C_3}x_2),$

where $a_jk_i=1 \bmod \alpha_i$ and $\overline{}$ denotes the smallest residue mod α_i for pairs $(j,i)=(2,1),(3,2),(1,3)$ respectively.

We can think of $\chi(\mathcal{O}_X)$ as RR-type expression for L_n, where $L_n=0\in \mathrm{Div}X\otimes\mathbb{Q}$ is the corresponding $\mathbb{Q}-$divisor. The rest of the formula is a sum of contributions coming from

- dissident points of type $\frac{1}{s}(a_1,a_2,a_3)$,
- dissident curve of type $\frac{1}{\alpha_1}(a_2,a_3)$,
- dissident curve of type $\frac{1}{\alpha_2}(a_3,a_1)$,
- dissident curve of type $\frac{1}{\alpha_3}(a_1,a_2)$,

which are singularities of $\mathcal{L}_n$ on X.

Now we have everything needed to conclude the proof of the main theorem.

Proof of Theorem 1.4.2 By Corollary 2.2.5 we only need to add the contributions $\mathcal{P}$ from the singular points to the RR formula. These were computed above.

Let C be a $\frac{}{mk}\big(\frac{1}{r}(1,-1)\big)$ curve for $\mathcal{O}_X(mD)$ with dissident points

$$\{P_\lambda\mid\lambda\in\Lambda\}\text{ of types }\frac{}{mn_\lambda}\Big(\frac{1}{s_\lambda}(a_{1\lambda},a_{2\lambda},a_{3\lambda})\Big).$$

Note that for every λ there exists i such that $\mathrm{hcf}(a_{i\lambda}, s_\lambda) = r$, since $P_\lambda \in C$. Moreover, $mn_\lambda k_{i\lambda} = mk \bmod r$, where $k_{i\lambda}$ is defined by $a_{j\lambda}k_{i\lambda} = 1 \bmod r$ for $i \neq j$.

The contribution to RR from this locus is a sum of

$$\frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{6r^2}\left((r-2)(1-g)+\frac{r-2}{2}K_Y\,{}^CE_1\,{}^CE_2+{}^CE_{r-2}^2\,{}^CE_{r-1}\right)$$

by Corollary 2.2.5, and of

$$\tfrac{1}{12s_\lambda}\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})(\deg_{C_{i\lambda}}x_1-\deg_{C_{i\lambda}}x_2)$$
$$+\tfrac{1}{s_\lambda}\textstyle\sum_{\varepsilon\in\mu_{s_\lambda},\ \varepsilon^{a_{i\lambda}}\neq 1\ \forall i=1,2,3}(\varepsilon^{-mn_\lambda}-1)\frac{1}{(1-\varepsilon^{-a_{1\lambda}})(1-\varepsilon^{-a_{2\lambda}})(1-\varepsilon^{-a_{3\lambda}})}.$$

for every point P_λ. The first two rows of the sum can be written together as

$$\frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{12r^2\tau_C}N_C,$$

where $\tau_C = \frac{1}{r}\,\mathrm{lcm}\{s_\lambda\}$ is the index of C and the integer N_C is an invariant of X in the neighbourhood of C.

□

3 Calabi–Yau threefolds

3.1 The definition of a Calabi–Yau and its Hilbert series

A *Calabi–Yau threefold* is a normal projective threefold X with canonical Gorenstein singularities, satisfying $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. Calabi–Yaus with only terminal singularities are minimal models with Kodaira dimension 0. There is a huge, possibly infinite number of deformation types. A lot of research on Calabi–Yaus was done because of their connection with string theory (check Appendix A.3).

Corollary 3.1.1 *Let X be a Calabi–Yau threefold and let D be a $\mathbb{Q}$–Cartier divisor. Assume that (X, D) is quasismooth and well formed with the following singularities*

- *points Q of type ${}_n\big(\frac{1}{s}(a_1, a_2, a_3)\big)$,*
- *curves C of generic type ${}_k\big(\frac{1}{r}(1, -1)\big)$ with index τ_C.*

Then for all positive integers m,

$$\begin{aligned} h^0(X, \mathcal{O}_X(mD)) &= \tfrac{1}{6}m^3D^3 + m\tfrac{D\cdot c_2(X)}{12} \\ &\quad + \textstyle\sum_Q c_Q(mD) + \sum_C s_C(mD), \end{aligned}$$

where

$$c_Q(mD) = \frac{1}{s} \sum_{\substack{\varepsilon\in\mu_s,\\ \varepsilon^{a_i}\neq 1 \ \forall i=1,2,3}} (\varepsilon^{-nm} - 1)\frac{1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}$$

and

$$s_C(mD) = -m\frac{\overline{mk}(r-\overline{mk})}{2r}\deg D|_C + \frac{\overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})}{12r^2\tau_C}N_C.$$

Here $\overline{}$ denotes the smallest residue mod r and the integer N_C is an invariant of X in the neighbourhood of C.

Proof   This is a direct corollary of Theorem 1.4.2. By Kodaira vanishing $h^p(X, mD) = 0$ for all $p > 0$ and by Serre duality $\chi(\mathcal{O}_X) = 0$. Moreover, $K_X \sim 0$ by the definition of Calabi–Yau . $\square$

Remark 3.1.2 Observe that for a $\frac{1}{2}(1,1)$ curve of singularities the term involving N_C is 0. The Riemann–Roch formula for this special case was already proved in [32].

Theorem 3.1.3 *Let X be the same as in Corollary 3.1.1. The Hilbert series for X is*

$$\begin{aligned} P_X(t) &= 1 + \tfrac{1}{6} D^3 \cdot \tfrac{t^3+4t^2+t}{(1-t)^4} + \tfrac{c_2(X)\cdot D}{12} \cdot \tfrac{t}{(1-t)^2} \\ &\quad + \textstyle\sum_Q \tilde{P}_Q(t) + \sum_C \tilde{P}_C(t), \end{aligned}$$

where every curve C contributes

$$\begin{aligned} \tilde{P}_C(t) &= -\deg D|_C \left(\tfrac{1}{1-t^r} \textstyle\sum_{i=1}^{r-1} i \tfrac{\overline{ik}(r-\overline{ik})}{2r} t^i + \tfrac{rt^r}{(1-t^r)^2} \sum_{i=1}^{r-1} \tfrac{\overline{ik}(r-\overline{ik})}{2r} t^i \right) \\ &\quad + \tfrac{N_C}{12r^2\tau_c} \cdot \tfrac{1}{1-t^r} \textstyle\sum_{i=1}^{r-1} \overline{ik}(r-\overline{ik})(r-2\cdot\overline{ik}) t^i \end{aligned}$$

and every point Q contributes

$$\tilde{P}_Q(t) = \frac{1}{1-t^s} \sum_{i=1}^{s-1} c_Q(iD) t^i.$$

Here

$$c_Q(iD) = \frac{1}{s} \sum_{\substack{\varepsilon \in \mu_s, \\ \varepsilon^{a_j} \neq 1 \ \forall j=1,2,3}} \frac{\varepsilon^{-ni} - 1}{(1-\varepsilon^{-a_1})(1-\varepsilon^{-a_2})(1-\varepsilon^{-a_3})}.$$

Proof   The Hilbert series is defined by

$$P_X(t) = 1 + \sum_{m=1}^{\infty} h^0(X, \mathcal{O}_X(mD)) t^m.$$

Use the following trivial expansions:

- $\frac{t^3+4t^2+t}{(1-t)^4} = t + 2^3t^2 + 3^3t^3 + \ldots + m^3t^m + \ldots,$

- $\frac{t}{(1-t)^2} = t + 2t^2 + \ldots + mt^m + \ldots,$
- $\frac{1}{1-t^r} \sum_{i=1}^{r-1} ia_i t^i + \frac{rt^r}{(1-t^r)^2} \sum_{i=1}^{r-1} a_i t^i = a_1 t + 2a_2 t^2 + \ldots + ma_{\overline{m}} t^m + \ldots,$
- $\frac{1}{1-t^r} \sum_{i=1}^{r-1} a_i t^i = a_1 t + \ldots + a_{r-1} t^{r-1} + a_1 t^{r+1} + \ldots + a_{\overline{m}} t^m + \ldots,$

where $a_1, \ldots, a_{r-1}$ are arbitrary rational numbers, $a_0 = 0$ and $\overline{}$ denotes the smallest residue mod r. The formula for $P_X(t)$ follows directly from Corollary 3.1.1, as for $m \in \mathbb{Z}$ the factors

$$\overline{mk}(r - \overline{mk}) \quad \text{and} \quad \overline{mk}(r - \overline{mk})(r - 2 \cdot \overline{mk})$$

are r periodic and $c_Q(mD)$ are s periodic. $\square$

3.2 CYs in weighted projective spaces

The formula for the Hilbert series in Theorem 3.1.3 was constructed with the aim to generate new examples of Calabi–Yau threefolds. In this section we will describe a computer search for Calabi–Yaus embedded in weighted projective spaces.

3.2.1 Resolution of the graded ring

Many cases of projective algebraic varieties can be studied in terms of graded rings.

Consider a projectively normal threefold $X \subset \mathbb{P}(w_0, \ldots, w_n)$. The graded ring

$$R = R(X, \mathcal{O}_X(1))$$

over the polynomial ring $A = \mathbb{C}[x_0, \ldots, x_n]$ is a finite module over A. By the Hilbert syzygies theorem, there exists a finite free resolution

$$0 \to L_c \to \cdots \to L_1 \to A \to R \to 0,$$

where each $L_i = \bigoplus A(-b_{i,j})$ is a free graded module. We assume that R is Gorenstein, thus the length of the resolution equals to the codimension of X in $\mathbb{P}$.

Then the Hilbert series of R is

$$P(t) = \frac{Q(t)}{\prod_{k=0}^{n}(1-t^{w_k})},$$

where $Q(t)$ is the multiplied out Hilbert polynomial

$$Q(t) = \sum_{i,j}(-1)^i t^{b_{i,j}}.$$

The terms $b_{1,j}$ are the degrees of relations of R, the terms $b_{2,j}$ are the degrees of first syzygies and so on. Note that the polynomial $Q(t)$ does not entirely determine the resolution of R.

Lemma 3.2.1 *The degree of the c codimensional X in $\mathbb{P}$ equals*

$$\frac{\sum_{i,j}(-1)^i \binom{b_{i,j}}{c}}{\prod_{k=0}^{n} w_k}.$$

PROOF The degree of a variety is defined in the following way

$$\deg X = \lim_{m\to\infty} \frac{H^0(X, \mathcal{O}_X(m)) \cdot d!}{m^d},$$

where $d = n - c$ is the dimension of X. We can compute this limit from the Hilbert series

$$P_X(t) = \frac{Q(t)}{\prod_{k=0}^{n}(1-t^{w_k})} = \frac{(1-t)^{n-d}R_1(t)}{(1-t)^{n+1}R_2(t)},$$

where the polynomials $R_i(t)$ only have positive terms. This implies

$$\deg X = \lim_{m\to\infty} \binom{d+m}{d} \cdot \frac{R_1(1)}{R_2(1)} \cdot \frac{d!}{m^d} = \frac{R_1(1)}{R_2(1)}.$$

Obviously $R_2(1) = \prod_{k=0}^{n} w_k$, and $R_1(1)$ can be computed from $Q(t)$ by differentiating it c times and then inserting $t = 1$. $\square$

We can conclude more from the free resolution of R. The shape of R suggests the shape of the embedding.

In codimension 1, X can only be a hypersurface. If X has codimension 2, it is a complete intersection. In codimension 3, X is defined by the submaximal Pfaffians of some skew-symmetric matrix of odd size. This was proved by Buchsbaum and Eisenbud in [6]. Note that a 3×3 skew–symmetric matrix defines a complete intersection. For higher codimensions there are no similar theorems about the defining equations and relations of X. Thus, it is much harder to prove the existence of X. In Chapter 4 we construct some examples of Corti–Reid [8] varieties in high cosimensions. The technique that works well on concrete examples (check Section 4.2), is the theory of unprojections developed by Miles Reid. More on this can be found in [26] and [25].

3.2.2 The computer algorithm

Gavin Brown wrote a computer program in MAGMA which we now describe. The input data are:

- rational numbers $D^3,\ D \cdot c_2(X)$,
- integers a_1, a_2, a_3, s, n for each singular point
- integers $r,\ k,\ \tau_C,\ r\tau_C \cdot \deg D|_C$ and N_C for each curve of singularities,

which is equivalent to giving the data:

- integers $h^0(X, D)$ and $h^0(X, 2D)$,
- Points: $\left\{ {}_n\left(\frac{1}{s}(a_1, a_2, a_3)\right) \right\}$,
- Curves: $\left\{ \text{type} = {}_k\left(\frac{1}{r}(1, -1)\right),\ \deg D|_C,\ \tau_C \text{ and } N_C \right\}$.

In the first step the computer generates the Hilbert series $P(t)$ from Theorem 3.1.3 by using the input data. In the next step it finds a multiplied out Hilbert polynomial $Q(t)$. This is equivalent to finding weights $w_0, \ldots, w_n$ such that

$$\prod_{k=0}^{n}(1 - t^{w_k})P(t)$$

is a polynomial. As described in Subection 3.2.1, $Q(t)$ corresponds to the resolution of a graded ring R. This gives a candidate for X embedded in $\mathbb{P}(w_0, \ldots, w_n)$. Of course we need to check the existence of X in every separate case. Since $(X, \mathcal{O}_X(1))$ satisfies the Riemann–Roch formula it is likely to have the prescribed singularities.

Remarks 3.2.2

(I) Different types of singularities can give the same contributions to the Riemann-Roch formula and thus define the same Hilbert series.

(II) In practice it is difficult to "guess" the suitable rational numbers $D^3, D \cdot c_2(X)$ such that all coefficients in the generated series are integers. Instead we choose two integers for $h^0(X, D)$ and $h^0(X, 2D)$ as part of the input data. The terms D^3 and $D \cdot c_2(X)$ can then be evaluated from the RR formula.

(III) Note that $r\tau_C \cdot \deg D|_C$ is an integer, since $r\tau_C D$ is Cartier in the neighbourhood of C.

(IV) The contribution from a singular point of type $\frac{1}{s}(a_1, a_2, a_3)$ involves summing over the complex numbers $\varepsilon \in \mu_s,\ \varepsilon^{a_j} \neq 1\ \forall j = 1, 2, 3$. In order to avoid introducing complex numbers in the computer algorithm, we read off the contributions from a suitable model. We will briefly describe the method which was suggested by Miles Reid and is also part of Brown's program in Magma.

Consider $\mathbb{P}(a_1, a_2, a_3, s)$. As always assume that $\operatorname{hcf}(a_i, a_j, s) = 1$ for different $i, j = 1, 2, 3$ and denote $\operatorname{hcf}(s, a_i) = \alpha_i$. Its Hilbert series equals

$$P_{\mathbb{P}}(t) = \frac{1}{(1 - t^{a_1})(1 - t^{a_2})(1 - t^{a_3})(1 - t^s)}.$$

The coefficient in front of t^m in the Taylor extension of $P_{\mathbb{P}}(t)$ is $h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m))$. By Theorem 1.4.2,

$$\begin{aligned} h^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) &= \text{RR–type expression in } m + \\ &\ \text{contributions from the singularities.} \end{aligned}$$

The singularities of $(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ are the point $(0, 0, 0, 1)$ of type

$$_{s-1}\Big(\frac{1}{s}(a_1, a_2, a_3)\Big)$$

and for $i = 1, 2, 3$ and $j, k \neq i$

$$\begin{cases} \text{a curve of type } \frac{1}{\alpha_i}(a_j, a_k) & \text{if } \alpha_i \neq 1, \\ \text{an isolated point } \frac{1}{a_i}(a_j, a_k, s) & \text{if } \alpha_i = 1. \end{cases}$$

For

$$m = \frac{a_1a_2a_3}{\alpha_1\alpha_2\alpha_3},\ 2\frac{a_1a_2a_3}{\alpha_1\alpha_2\alpha_3}, \ldots, (s-1)\frac{a_1a_2a_3}{\alpha_1\alpha_2\alpha_3}$$

the contributions to RR will come only from the singularity $(0,0,0,1)$. Indeed, $\mathcal{O}_{\mathbb{P}}(\frac{a_1a_2a_3}{\alpha_1\alpha_2\alpha_3})$ can be moved away from the other singular points of $\mathbb{P}$. Since $\frac{a_1}{\alpha_1}$, $\frac{a_2}{\alpha_2}$, $\frac{a_3}{\alpha_3}$ and s are coprime, we get all $s-1$ periodic contributions, which is what we wanted to compute.

(IV) We can put no bounds on the invariants N_C, which are determined by the type of singularities along C and its embedding in X.

When an A_{r-1} curve C contains no dissident points ($\tau_C = 1$), we have computed N_C in two different ways. First, for the resolution $f\colon Y \to X$ which is crepant over the generic point of C

$$N_C = 2\left((r-2)(1-g) + {}^C E_{r-2}^2\, {}^C E_{r-1}\right)$$

by Corollary 2.2.5. Second, for the auxiliary cover $\pi\colon Z \to X$

$$N_C = 2\left(r(g-1) - \deg_{C_Z} x_1\right)$$

by Theorem 2.3.17. Recall that g is the genus of C, ${}^C E_i$ are the exceptional surfaces of f over C and $x_1 = c_1(\mathcal{N}_1)$ where $\mathcal{N}_{C/Z} = \mathcal{N}_1 \oplus \mathcal{N}_2$.

We will prove an interesting relation between the above invariants. Write e for the intersection ${}^C E_{r-2}^2\, {}^C E_{r-1}$ and $\bar{e}$ for ${}^C E_1\, {}^C E_2^2$. By the formula on page 19

$$e + \bar{e} = 2(r-2)(g-1).$$

In Section 2.3 we defined $\iota = \deg c_1(\mathcal{N}_1)$. Write $\bar{\iota}$ for $\deg_{C_Z} x_2 = \deg c_1(\mathcal{N}_2)$. In Lemma 2.3.16 we showed that

$$\iota + \bar{\iota} = \deg c_1(\mathcal{N}_{C_Z/Z}) = 2r(g-1).$$

By comparing the formulas for N_C we get

$$\frac{1}{r}((r-2)(1-g) + e) = -\left((1-g) + \frac{\iota}{r}\right).$$

Substituting e by $-2(r-2)(1-g) - \bar{e}$, we see that

$$\bar{e} = \iota + 2(1-g).$$

Similarly, substitute ι by $-2(1-g)r - \bar{\iota}$ to get

$$e = \bar{\iota} + 2(1-g).$$

4 Construction of examples

4.1 Complete intersections

The simplest examples of Calabi–Yau threefolds are complete intersections

$$X_{d_1,\ldots,d_l} \subset \mathbb{P}(b_1,\ldots,b_{l+4}),$$

where $d_1+\ldots+d_l = b_1+\ldots+b_{l+4}$. In this section we use the notation of [11].

4.1.1 Classification of singularities

Proposition 4.1.1 *Let X be a quasismooth and well formed Calabi–Yau defined as a complete intersection in a weighted projective space. Then* $\operatorname{Sing}(X)$ *consists of:*

- *points of type $\frac{1}{r}(a,b,c)$, where $a+b+c=0$ mod r.*
- *curves of transverse type $\frac{1}{r'}(a',b')$, where $a'+b'=0$ mod r'.*

Proof Since X is normal, $\operatorname{Sing} X$ consists of a number of isolated points and a number of (possibly intersecting) curves. The singularities of X are induced by the singularities of $\mathbb{P}$.

Consider first a point $P \in X$ of type $\frac{1}{r}(a,b,c)$. If P is an isolated point or a dissident point, it can only appear in a complete intersection of the form

$$X_{d_1,\ldots,d_l,e_1r,\ldots,e_kr} \subset \mathbb{P}(a_1,\ldots,a_{l+3},b_1r,\ldots,b_{k+1}r),$$

where d_i and a_j are not divisible by r. After a suitable permutation of the variables, we can assume

$$\begin{aligned} a_{l+1} &= a \bmod r,\\ a_{l+2} &= b \bmod r,\\ a_{l+3} &= c \bmod r. \end{aligned} \tag{E. viii}$$

Let $f_1,\ldots,f_l,g_1,\ldots,g_k$ be the defining equations of X, and let $x_1,\ldots,x_{l+3},y_1,\ldots,y_{k+1}$ be the coordinates of $\mathbb{P}$. Since X is quasismooth, we can use the implicit function theorem to express $x_1,\ldots,x_l$ out of $f_1,\ldots,f_l$. This means that after a suitable permutation of equations

$$f_i = x_i \cdot \prod_j y_j^{\alpha_j} + \text{ other monomials of } \deg d_i,$$

for $i = 1, \dots, l$. The corresponding degrees are

$$d_i = a_i + r*_i, \tag{E. ix}$$

where $*_i$ denote positive integers.

By the adjunction formula for Calabi–Yau threefolds we get

$$\begin{aligned} 0 &= d_1 + \ldots + d_l + e_1 r + \ldots + e_k r - \\ &\quad a_1 - \ldots - a_{l+3} - b_1 r - \ldots - b_{k+1} r. \end{aligned} \tag{E. x}$$

Then (E. viii), (E. ix) and (E. x) imply

$$a + b + c = 0 \bmod r.$$

Next consider a curve C of type $\frac{1}{r'}(a', b')$. Such a singularity can only appear in a complete intersection of the form

$$X_{d_1,\ldots,d_l,e_1 r',\ldots,e_k r'} \subset \mathbb{P}(a_1, \ldots, a_{l+2}, b_1 r', \ldots, b_{k+2} r'),$$

where d_i and a_j are not divisible by r'. We can assume

$$\begin{aligned} a_{l+1} &= a' \bmod r', \\ a_{l+2} &= b' \bmod r'. \end{aligned} \tag{E. xi}$$

Let $f_1, \ldots, f_l, g_1, \ldots, g_k$ be the defining equations of X, and let $x_1, \ldots, x_{l+2}, y_1, \ldots, y_{k+2}$ be the coordinates of $\mathbb{P}$. Since X is quasismooth, we can use the implicit function theorem on every point of C. This eliminates $x_1, \ldots, x_l$ out of $f_1, \ldots, f_l$. The remaining variables x_{l+1}, x_{l+2} are local parameters to the normal direction of C. This means that after a suitable permutation of equations

$$f_i = x_i \cdot \prod_j y_j^{\alpha_j} + \text{ other monomials of } \deg d_i,$$

for all $i = 1, \ldots, l$. The corresponding degrees are

$$d_i = a_i + r' *_i, \tag{E. xii}$$

where $*_i$ denote positive integers.

Use the adjunction formula for Calabi–Yaus together with (E. xi) and (E. xii) to get

$$a' + b' = 0 \bmod r'.$$

$\square$

4.1.2 The degree of curves

For a threefold X in a weighted projective space the sheaf $\mathcal{O}_X(1)$ defines a $\mathbb{Q}$–Cartier divisor D. By slight abuse of notation we call $\deg D|_C$ defined on page 8 the *degree of* C. In the case of complete intersections $\deg C$ can be computed by the following lemma.

Lemma 4.1.2 *Let X be a quasismooth complete intersection in a weighted projective space $\mathbb{P}$ containing an A_{r-1} curve C. Choose the defining equations of X that have degrees $d_1 \ldots d_l$ divisible by r. Similarly, let $a_1 \ldots a_{l+2}$ be the weights of $\mathbb{P}$ divisible by r. Then*

$$\deg C = \frac{d_1 \ldots d_l r}{a_1 \ldots a_{l+2}}.$$

Proof The singularities along C are induced by the weights of $\mathbb{P}$ that are divisible by r. Thus C can be expressed as a complete intersection $X_{d_1,\ldots,d_l} \subset \mathbb{P}(a_1, \ldots, a_{l+2})$, where $d_1, \ldots, d_l$ and $a_1, \ldots, a_{l+2}$ are divisible by r. Choose t big enough and containing the factor

$$\prod_{i=1}^{l+2} a_i.$$

Then $tD = \mathcal{O}_X(t)$ is Cartier and defines a surface H. The intersection $C \cap H$ equals to $X_{t,d_1,\ldots,d_l} \subset \mathbb{P}(a_1, \ldots, a_{l+2})$ and is a finite number of points. It is isomorphic to a quasismooth

$$X_{\frac{t}{r},\frac{d_1}{r},\ldots,\frac{d_l}{r}} \subset \mathbb{P}(\frac{a_1}{r}, \ldots, \frac{a_{l+2}}{r}).$$

By quasismoothness all intersection points lie outside the fundamental simplex of $\mathbb{P}$ and by the generalised Bezout theorem there is $\frac{td_1 \ldots d_l r}{a_1 \ldots a_{l+2}}$ of them. Finally the degree equals to

$$\deg C = \frac{d_1 \ldots d_l r}{a_1 \ldots a_{l+2}}.$$

□

4.1.3 Examples

Example 4.1.3 Using Fletcher's notation [11], a general hypersurface $X_{13} \subset \mathbb{P}(1,1,3,4,4)$ with coordinates x_1, x_2, y, z_1, z_2 is given by an equation

$$f = x_1^{13} + x_2^{13} + y^3 z_1 + y x_1^{10} + z_1^3 x_1 + z_2^3 x_2 + ...$$

It is easy to check that X_{13} is quasismooth. Thus the adjunction formula holds and X_{13} is Calabi–Yau . It contains a singular point $P = (0,0,1,0,0)$. Since $\frac{\partial f}{\partial z_1}(P) \neq 0$, we can use the implicit function theorem to express z_1 out of f. This means that x_1, x_2, z_2 are the local parameters for X_{13} around P. The group $\mu_3 \ni \varepsilon$ acts by

$$x_1, x_2, z_2 \sim \varepsilon x_1, \varepsilon x_2, \varepsilon^4 z_2.$$

Thus we proved that $(X_{13}, \mathcal{O}_X(1))$ contains P of type ${}_2\big(\frac{1}{3}(1,1,1)\big)$.

The threefold X_{13} also contains a curve C defined by $\{x_1 = x_2 = y = 0\}$, which is a copy of $\mathbb{P}^1$. Choose a point on C, e.g. $(0,0,0,0,1)$. Using the implicit function theorem again, we can choose x_1, y to be the local parameters on a hyperplane that contains the chosen point. The elements of $\varepsilon \in \mu_4$ act via $x_1, y \sim \varepsilon x_1, \varepsilon^3 y$. Thus every point along C is of transverse type $\frac{1}{4}(1,3)$.

Define D by $\mathcal{O}_X(1)$. Then

$$D^3 = \frac{13}{48}, \ \deg C = \frac{1}{4} \ \text{ and } \ 1 - g = 1.$$

We can also compute $E_2^2 E_3$, where E_i are the exceptional surfaces of the resolution along C :
In the toric variety $\mathbb{P}(1,1,3,4,4)$, the blow up along the A_3 curve corresponds

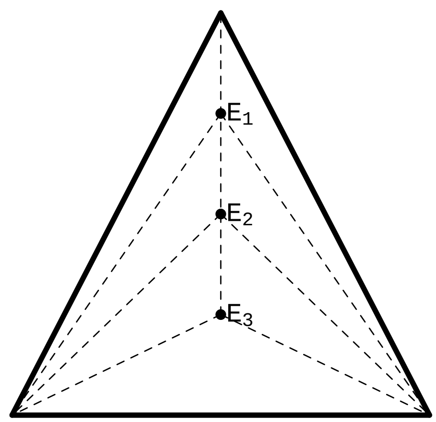

Figure III: Resolution of an A_3 curve

to Picture III. We add rays E_1, E_2, E_3 to a three dimensional facet. On the hypersurface $f^*(4D)$, the restrictions of E_2 and E_3 intersect once, thus $f^*(4D)E_2E_3 = 1$. This yields

$$\begin{array}{rcl} 3 & = & 4f^*(3D)E_2E_3 = 4f^*(D_3)E_2E_3 \quad = \\ & & (4D_3' + 3E_1 + 2E_2 + E_3)E_2E_3 \quad = \quad 2E_2^2E_3 + E_2E_3^2 \end{array}$$

On the other hand, as in the proof on page 19

$$E_2^2E_3 + E_2E_3^2 = 2(g-1) = -2.$$

Finally we get $E_2^2E_3 = 5$. Thus we computed $N_C = 14$.

The only missing input data is $c_2(X) \cdot D$. We can compute it directly from the RR formula for $m = 12$, since then there are no contributions. We get $\frac{1}{12}c_2(X) \cdot D = \frac{3}{2}$.

By Corollary 3.1.1 we get

$$h^0(X, mD) = \frac{13}{288}m^3 + \frac{3}{2}m + c_P(mD) - \frac{1}{32}m(4-\overline{m})\overline{m} + \frac{7}{6 \cdot 16}\overline{m}(4-\overline{m})(4-2\overline{m}),$$

where

$$c_P(mD) = \begin{cases} \frac{1}{9} & \text{if } m = 1 \text{mod} 3, \\ -\frac{1}{9} & \text{if } m = 2 \text{mod} 3, \\ 0 & \text{if } m = 0 \text{mod} 3. \end{cases}$$

It is easy to check that this yields the Hilbert series for X_{13}

$$P_X(t) = \frac{1-t^{13}}{(1-t)^2(1-t^3)(1-t^4)^2}.$$

Example 4.1.4 Consider the complete intersection $X_{10,15} \subset \mathbb{P}(2,3,5,5,5,5)$ with variables x, y, z_1, z_2, z_3, z_4. It is defined by two equations f, g of degrees 10 and 15 respectively. It is easy to verify that this is a quasismooth Calabi–Yau. We can avoid $(1,0,0,0,0,0)$ with f and $(0,1,0,0,0,0)$ with g. Therefore, the singular locus of $X_{10,15}$ contains only a curve C given by $X_{10,15} \cap \{x = y = 0\}$.

Choose an arbitrary point on C. The variables x, y are local parameters of a hypersurface around the chosen point. The group $\varepsilon \in \mu_5$ act on them by $x, y \sim \varepsilon^2 x, \varepsilon^3 y$. This means that $(X, \mathcal{O}_X(1))$ contains a singular curve of transverse type

$$_4\Big(\frac{1}{5}(2,3)\Big) = {}_2\Big(\frac{1}{5}(1,-1)\Big).$$

The degree of C is $\frac{6}{5}$ and the other invariants of $X_{10,15}$ are

$$D^3 = \frac{1}{25} \quad \text{and} \quad \frac{1}{12}c_2(X)\cdot D = \frac{5}{6}.$$

For $N_C = -6$ we get

$$h^0(X, mD) = \frac{1}{150}m^3 + \frac{5}{6}m - \frac{3}{25}m\cdot\overline{2m}(5-\overline{2m}) - \frac{1}{50}\overline{2m}(5-\overline{2m})(5-2\cdot\overline{2m}),$$

where $\overline{}$ is the smallest residue mod 5. This again yields the Hilbert series

$$P_X(t) = \frac{(1-t^{10})(1-t^{15})}{(1-t^2)(1-t^3)(1-t^5)^4}.$$

We continue by two examples generated by the computer search.

Example 4.1.5 For the input data we choose

- $h^0(X, D) = 2$ and $h^0(X, 2D) = 4$

- Points: ${}_4\big(\frac{1}{5}(1,1,3)\big)$
- Curves: type$=\frac{1}{3}(1,2)$, deg $=\frac{1}{3}$, index$=1$, $N_C=4$.

The output we get is

- Codimension: 2
- Weights: 1,1,2,3,3,5
- Multiplied out Hilbert polynomial: $t^{15}-t^8-t^7+1$

This data defines a complete intersection

$$X_{7,8}\subset \mathbb{P}(1,1,2,3,3,5).$$

The threefold $X_{7,8}$ is a quasismooth Calabi–Yau by Theorem 8.7 in [11]. It is easy to check that the only singularities are (0,0,0,0,0,1) of type ${}_4\big(\frac{1}{5}(1,1,3)\big)$ and a line of ${}_1\big(\frac{1}{3}(1,2)\big)$ transverse type.

Example 4.1.6 For the input data we choose

- $h^0(X,D)=1$ and $h^0(X,2D)=1$
- Points:
  1. ${}_5\big(\frac{1}{6}(1,2,3)\big)$,
  2. ${}_{13}\big(\frac{1}{14}(1,6,7)\big)$,
  3. ${}_{20}\big(\frac{1}{21}(1,6,14)\big)$,
- Curves:
  1. type$=\frac{1}{2}(1,1)$, deg $=\frac{1}{42}$, index$=21$,
  2. type$=\frac{1}{3}(1,2)$, deg $=\frac{1}{42}$, index $=14$, $N_C=55$,
  3. type$=\frac{1}{7}(1,6)$, deg $=\frac{1}{6}$, index $=6$, $N_C=35$,

The output we get is

- Codimension: 1
- Weights: 1,6,7,14,21

- Multiplied out Hilbert polynomial: $t^{49} - 1$

This data defines a Calabi–Yau hypersurface $X_{49} \subset \mathbb{P}(1,6,7,14,21)$ with variables $x_1, x_6, x_7, x_{14}, x_{21}$. It is easy to check that $(X, \mathcal{O}_X(1))$ is quasismooth and well formed and has the following singularities

- dissident curve $C_2 = \{x_1 = x_7 = 0\}$ of type $\frac{1}{2}(1,1)$,
- dissident curve $C_3 = \{x_1 = x_7 = x_{14} = 0\}$ of type $\frac{1}{3}(1,2)$,
- dissident curve $C_7 = \{x_1 = x_6 = 0\}$ of type $\frac{1}{7}(1,6)$,
- point $P_{2,3}$ of type ${}_5\big(\frac{1}{6}(1,2,3)\big)$ on the intersection of C_2 and C_3,
- point $P_{2,7}$ of type ${}_{13}\big(\frac{1}{14}(1,6,7)\big)$ on the intersection of C_2 and C_7,
- point $P_{3,7}$ of type ${}_{20}\big(\frac{1}{21}(1,6,14)\big)$ on the intersection of C_3 and C_7.

4.2 Pfaffians in codimension 3 and 4

4.2.1 The Type I unprojection

As discussed in Subsection 3.2.1, it is difficult to analyse the varieties in high codimension. Therefore we project them to lower codimension by the so called unprojection method. The same method can be used to construct new varieties. We will now briefly describe the Type I unprojection.

Let Y be a projective Gorenstein variety of codimension c in $\mathbb{P}(a_0, \dots, a_n)$, containing a projective Gorenstein variety Π of codimension $c+1$. Let the homogenous polynomials $h_1, \dots, h_l$ generate the ideal of Π and $f_1, \dots, f_k$ the ideal of Y. Then there exists a rational function t on Y with pole along Π. In other words, there exist homogeneous polynomials $m_1, \dots, m_l$ such that

$$t = \frac{m_1}{h_1} = \cdots = \frac{m_l}{h_l}.$$

Then, X defined by

$$f_1 = \cdots = f_k = th_1 - m_1 = \cdots = th_l - m_l = 0$$

is a projective Gorenstein variety of codimension $c+1$ in $\mathbb{P}(a_0, \dots, a_n, a_{n+1})$. Here $a_{n+1} = \operatorname{wt} t$. The construction $\varphi\colon Y \longrightarrow X$ is called the *Type I unprojection.*

Denote the point $(\underbrace{0,\dots,0}_{n+1},1) \in X$ by P and the locus

$$X \cap \{h_1 = \dots = h_l = 0\}$$

by Γ. The inverse of φ is the projection from P. Thus the restriction

$$\varphi \colon Y \setminus \Pi \longrightarrow X \setminus \Gamma$$

is an isomorphism.

4.2.2 Examples

In this subsection we use the standard notion of Pfaffians as in [1], [25] and [28].

Example 4.2.1 For the input data we choose

- $h^0(X,D) = 2$ and $h^0(X,2D) = 4$
- Points: ${}_4\big(\frac{1}{5}(1,1,3)\big)$
- Curves: type$= \frac{1}{3}(1,2)$, deg $= \frac{2}{3}$, $\tau_C = 1$, $N_C = 8$.

The output we get is

- Codimension: 3
- Weights: 1,1,2,3,3,3,5
- Multiplied out Hilbert polynomial: $-t^{18} + t^{12} + 2t^{11} + 2t^{10} - 2t^8 - 2t^7 - t^6 + 1$

This output data suggests that X is a threefold in $\mathbb{P}(1,1,2,3,3,3,5)$ defined by submaximal Pfaffians of a 5×5 skew-symmetric matrix M. There are five relations of degrees 6,7,7,8,8 and five syzygies of degrees 10,10,11,11,12. Denote the variables of $\mathbb{P}(1,1,2,3,3,3,5)$ by $x_1, x_2, y, z_1, z_2, z_3, t$. We fill in the entries of M as follows:

$$M = \begin{pmatrix} t & a_4 & b_3 & c_3 \\ & a_5 & b_4 & c_4 \\ & & z_2 & -z_1 \\ & & & y \end{pmatrix},$$

where the polynomials a_i, b_i, c_i have degrees

$$\begin{pmatrix} 5 & 4 & 3 & 3 \\ & 5 & 4 & 4 \\ & & 3 & 3 \\ & & & 2 \end{pmatrix}.$$

The corresponding Pfaffians are

$$\begin{aligned}
\mathrm{Pf}_1 &= a_5 y + b_4 z_1 + c_4 z_2, \\
\mathrm{Pf}_2 &= a_4 y + b_3 z_1 + c_3 z_2, \\
\mathrm{Pf}_3 &= ty - b_3 c_4 + c_3 b_4, \\
\mathrm{Pf}_4 &= -tz_1 - a_4 c_4 + c_3 a_5, \\
\mathrm{Pf}_5 &= tz_2 - a_4 b_4 + b_3 a_5.
\end{aligned}$$

When X is quasismooth and well formed the adjunction formula holds and has the form

$$\omega_X \cong \mathcal{O}_X((10+10+11+11+12)-(6+7+7+8+8)-(1+1+2+3+3+3+5)) = \mathcal{O}_X,$$

where the sum runs over the degrees of syzygies, relations and weights respectively. In this case X is a Calabi–Yau .

The last thing we need to check is whether X is quasismooth. It is enough to examine whether $\mathrm{Cone}(X)$ is smooth on each singular stratum of $\mathbb{P}(1,1,2,3,3,3,5)$.
point (0,0,1,0,0,0,0): Since some power of y appears in Pf_2, Pf_4 and Pf_5, this point is not contained in X.

point (0,0,0,0,0,0,1): The Pfaffians contain no monomial t^n for some n, thus this point lies in X. The rank of M at this point is 2 and by the implicit function theorem we can express y, z_1 and z_2 out of the equations. Indeed, on the affine piece $t = 1$ there are locally three linearly independent Pfaffians

$$\begin{aligned}
\mathrm{Pf}_3 &= y - b_3 c_4 + c_3 b_4, \\
\mathrm{Pf}_4 &= -z_1 - a_4 c_4 + c_3 a_5, \\
\mathrm{Pf}_5 &= z_2 - a_4 b_4 + b_3 a_5.
\end{aligned}$$

This means that x_1, x_2 and z_3 are the local parameters of X around the point. The group $\varepsilon \in \mu_5$ acts by

$$x_1, x_2, z_3 \sim \varepsilon x_1, \varepsilon x_2, \varepsilon^3 z_3,$$

which implies that the point is a quotient singularity of type $\frac{1}{5}(1,1,3)$.

stratum $(0,0,0,z_1,z_2,z_3,0)$: is defined by $\{x_1 = x_2 = y = t = 0\}$ and M restricted to this stratum equals

$$\begin{pmatrix} 0 & 0 & b_3 & c_3 \\ & 0 & 0 & 0 \\ & & z_2 & -z_1 \\ & & & 0 \end{pmatrix}.$$

A point of the form $(0,0,0,z_1,z_2,z_3,0)$ will lie in X iff rk $M|_{\{x_1=x_2=y=t=0\}} \leq 2$. But,

$$\text{rk } M|_{\{x_1=x_2=y=t=0\}} \leq 2 \iff \text{rk} \begin{pmatrix} b_3 & c_3 \\ z_2 & -z_1 \end{pmatrix} \leq 1.$$

This condition gives exactly the Pfaffian

$$\text{Pf}_2 = b_3 z_1 + c_3 z_2,$$

which defines a curve $C = X \cap \{x_1 = x_2 = y = t = 0\}$. Choose for example $b_3 = z_3$ and $c_3 = z_2$. At every point on the curve rk $M = 2$, thus Cone(X) is smooth along C. This finishes the proof that X is quasismooth.

Next we determine the type of singularities along C. For every point on C one of z_i is nonzero. First, let $z_1 \neq 0$. Use the implicit function theorem on

$$\text{Pf}_4 = -t z_1 - a_4 c_4 + c_3 a_5$$

to eliminate t. Assume that b_4 contains the monomial $z_1 x_1$. Then we can express x_1 out of

$$\text{Pf}_1 = a_5 y + b_4 z_1 + c_4 z_2.$$

This means that x_2 and y are the local transverse parameters of X around the points on C for which $z_1 \neq 0$. The group $\varepsilon \in \mu_3$ acts by

$$x_2, y \sim \varepsilon x_2, \varepsilon^2 y.$$

The same way we will choose the local parameters for $z_2 \neq 0$. Assume that c_4 contains $x_1 z_2$, and then eliminate x_1 out of Pf_1 and t out of

$$\mathrm{Pf}_5 = tz_2 - a_4 b_4 + z_3 a_5.$$

In the case $z_3 \neq 0$ we also assume that a_5 contains yz_3 and c_4 contains $x_1 z_3$. Then we express y out of Pf_5 and x_1 out of

$$\mathrm{Pf}_3 = ty - z_3 c_4 + c_3 b_4.$$

Thus x_2 and t are the local transverse parameters in the neighbourhood of $z_3 \neq 0$ with the group $\varepsilon \in \mu_3$ acting as before by

$$x_2, t \sim \varepsilon x_2, \varepsilon^2 t.$$

This implies that C contains quotient singularities of transverse type $\frac{1}{3}(1,2)$.

It is obvious that X is well formed. Indeed, $\mathbb{P}(1,1,2,3,3,3,5)$ is well formed and X contains no codimension 4 singular stratum of $\mathbb{P}$.

Example 4.2.2 For the input data we choose

- $h^0(X, D) = 2$ and $h^0(X, 2D) = 4$
- Points: ${}_4\big(\frac{1}{5}(1,1,3)\big)$
- Curves: type $= \frac{1}{3}(1,2)$, deg $= 1$, $\tau_C = 1$, $N_C = 12$.

The output is

- Codimension: 4
- Weights: 1,1,2,3,3,3,3,5
- Multiplied out Hilbert polynomial: $t^{21} - 3t^{15} - 3t^{14} - 3t^{13} + 2t^{12} + 6t^{11} + 6t^{10} + 2t^9 - 3t^8 - 3t^7 - 3t^6 + 1$

This data suggests that X is a threefold in $\mathbb{P}(1,1,2,3,3,3,3,5)$ and there are nine relations of degrees $6,6,6,7,7,7,8,8,8$ respectively. The resolution of the graded ring $R(X, \mathcal{O}_X(1))$ is

$$0 \to A(-21) \to \bigoplus\nolimits_3 A(-15) \bigoplus\nolimits_3 A(-14) \bigoplus\nolimits_3 A(-13) \to$$

$$\bigoplus_2 A(-12) \bigoplus_6 A(-11) \bigoplus_6 A(-10) \bigoplus_2 A(-9) \to$$
$$\bigoplus_3 A(-8) \bigoplus_3 A(-7) \bigoplus_3 A(-6) \to A \to R \to 0,$$
where $\bigoplus_i A$ denotes the direct sum of i copies of the same module.

We will prove the existence of X by the Type I unprojection. Consider

$$\Pi = \{y = z_1 = z_2 = z_3 = 0\} \subset Y \subset \mathbb{P}(1,1,2,3,3,3,3),$$

where $x_1, x_2, y, z_1, z_2, z_3, z_4$ are the coordinates in $\mathbb{P}(1,1,2,3,3,3,3)$ and Y is defined by a set of Pfaffians

$$\begin{array}{ll}
\mathrm{Pf}_1 = z_2 q_3 - z_3 q_2 + z_4 q_1 & \text{of degree } 7, \\
\mathrm{Pf}_2 = z_1 q_3 - z_2 q_2 + z_3 q_1 & \text{of degree } 7, \\
\mathrm{Pf}_3 = y q_3 - z_2 z_4 + z_3^2 & \text{of degree } 6, \\
\mathrm{Pf}_4 = y q_2 - z_1 z_4 + z_2 z_3 & \text{of degree } 6, \\
\mathrm{Pf}_5 = y q_1 - z_1 z_3 + z_2^2 & \text{of degree } 6.
\end{array}$$

These Pfaffians come from a 5×5 skew–symmetric matrix

$$M = \begin{pmatrix} y & z_1 & z_2 & z_3 \\ & z_2 & z_3 & z_4 \\ & & q_1 & q_2 \\ & & & q_3 \end{pmatrix},$$

where all polynomials q_i have degree 4. The existence of the codimension 3 Calabi–Yau Y can be checked the same way as in Example 4.2.1.

By the unprojection there exists a rational function t on Y with pole along Π. In other words, there exist homogeneous polynomials $m_1, \dots, m_4$ such that

$$t = \frac{m_1}{z_1} = \frac{m_2}{z_2} = \frac{m_3}{z_3} = \frac{m_4}{y}.$$

By [25] t has degree $\mathrm{wt}\, x_1 + \mathrm{wt}\, x_2 + \mathrm{wt}\, z_4 = 5$. We define X by

$$\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$$

of degrees $7, 7, 6, 6, 6$ respectively, and by

$$ty - m_4,\ tz_1 - m_1,\ tz_2 - m_2,\ tz_3 - m_3$$

of degrees $7, 8, 8, 8$.

In the next step we check that X has the expected quotient singularities. point (0,0,1,0,0,0,0,0): Since some power of y appears in Pf_3, Pf_4 and Pf_5, this point is not contained in X.

point (0,0,0,0,0,0,0,1): The equations contain no monomial of t^n for some n, thus this point lies in X. We will express y, z_1, z_2 and z_3 with the other parameters using the implicit function theorem. On the affine piece $t = 1$ we have linearly independent equations

$$y = m_4,\ z_1 = m_1,\ z_2 = m_2,\ z_3 = m_3.$$

This means that x_1, x_2 and z_4 are the local parameters of X around the point. The group $\varepsilon \in \mu_5$ acts by

$$x_1, x_2, z_4 \sim \varepsilon x_1, \varepsilon x_2, \varepsilon^3 z_4,$$

which implies that the point is a quotient singularity of type $\frac{1}{5}(1, 1, 3)$.

stratum $(0, 0, 0, z_1, z_2, z_3, z_4, 0)$: is defined by $\{x_1 = x_2 = y = t = 0\}$. Its intersection with X is given by

$$\begin{aligned}\mathrm{Pf}_3 &= -z_2 z_4 + z_3^2,\\ \mathrm{Pf}_4 &= -z_1 z_4 + z_2 z_3,\\ \mathrm{Pf}_5 &= -z_1 z_3 + z_2^2,\end{aligned}$$

which define a twisted cubic C. Choose an arbitrary point on C, e.g. one with $z_1 \neq 0$. Assume also that q_3 contains the monomial $z_1 x_1$. On the affine piece $z_1 = 1$ we can eliminate t from the equation $t = m_1$, and x_1 from Pf_2. Thus we get the local transverse parameters x_2, y for the neighbourhood of the chosen point. The group $\varepsilon \in \mu_3$ acts by

$$x_2, y \sim \varepsilon x_2, \varepsilon^2 y,$$

which implies that C contains quotient singularities of transverse type $\frac{1}{3}(1, 2)$.

The fact that X is quasismooth can be checked by the computer that generates the Jacobian matrix $J(X)$ and then checks if $\mathrm{Sing}(X) = \emptyset$. The singular locus of X is defined by the suitable minors of $J(X)$ and the ideal $I(X)$ of X.

It is obvious that X is well formed.

Example 4.2.3 For the input data we choose

- $h^0(X, D) = 3$ and $h^0(X, 2D) = 6$,
- Points: ${}_2\big(\frac{1}{3}(1,1,1)\big)$,
- Curves: type$= \frac{1}{4}(1,3)$, deg $= \frac{1}{4}$, $\tau_C = 1$, $N_C = 14$.

The output we get is

- Codimension: 3,
- Weights: 1,1,1,3,3,4,4,
- Multiplied out Hilbert polynomial: $-t^{17} + t^{12} + t^{11} + 2t^{10} + t^8 - t^9 - 2t^7 - t^6 - t^5 + 1$.

The output data suggests that X is a threefold in $\mathbb{P}(1,1,1,3,3,4,4)$. Let $x_1, x_2, x_3, y_1, y_2, z_1, z_2$ be the variables of $\mathbb{P}$. Then X is defined by the submaximal Pfaffians of

$$M = \begin{pmatrix} y_2 & a_2 & x_1 & x_2 \\ & a_6 & b_5 & c_5 \\ & & z_2 & -z_1 \\ & & & y_1 \end{pmatrix} \text{ with degrees } \begin{pmatrix} 3 & 2 & 1 & 1 \\ & 6 & 5 & 5 \\ & & 4 & 4 \\ & & & 3 \end{pmatrix},$$

where a_2, a_6, b_5 and c_5 are general polynomials of degrees $2, 6, 5, 5$ respectively. The corresponding Pfaffians are

$$\begin{aligned} \mathrm{Pf}_1 &= a_6 y_1 + b_5 z_1 + c_5 z_2, \\ \mathrm{Pf}_2 &= a_2 y_1 + x_1 z_1 + x_2 z_2, \\ \mathrm{Pf}_3 &= y_2 y_1 - c_5 x_1 + b_5 x_2, \\ \mathrm{Pf}_4 &= -y_2 z_1 - a_2 c_5 + a_6 x_2, \\ \mathrm{Pf}_5 &= y_2 z_2 - a_2 b_5 + a_6 x_1. \end{aligned}$$

It is easy to check that X is a quasismooth well formed Calabi–Yau . Its singularities consist of the projective line $\{x_1 = x_2 = x_3 = y_1 = y_2 = 0\}$ of transverse type A_3 and the isolated point $\{x_1 = x_2 = x_3 = y_1 = z_1 = z_2 = 0\}$ of type $\frac{1}{3}(1,1,1)$.

Example 4.2.4 For the input data we choose

- $h^0(X,D)=3$ and $h^0(X,2D)=6$
- Points:
  1. ${}_2\big(\frac{1}{3}(1,1,1)\big)$,
  2. ${}_8\big(\frac{1}{9}(1,3,5)\big)$,
- Curves: type$=\frac{1}{3}(1,2)$, $\deg=\frac{1}{9}$, $\tau_C=3$, $N_C=22$.

The output we get is

- Codimension: 3
- Weights: 1,1,1,3,3,5,9
- Multiplied out Hilbert polynomial: $-t^{23}+t^{17}+t^{15}+2t^{13}+t^{11}-t^{12}-2t^{10}-t^8-t^6+1$

This data suggests that X is a threefold in $\mathbb{P}(1,1,1,3,3,5,9)$ with the variables x_1,x_2,x_3,y_1,y_2,z,v. It is defined by the submaximal Pfaffians of a 5×5 skew-symmetric matrix

$$M=\begin{pmatrix} y_2 & a_7 & a_5 & x_1 \\ & b_7 & b_5 & x_2 \\ & & v & -z \\ & & & y_1 \end{pmatrix} \quad \text{with degrees} \quad \begin{pmatrix} 3 & 7 & 5 & 1 \\ & 7 & 5 & 1 \\ & & 9 & 5 \\ & & & 3 \end{pmatrix},$$

were a_i,b_i are general polynomials. The corresponding Pfaffians are

$$\begin{aligned}
\mathrm{Pf}_1 &= b_7y_1+b_5z+x_2v,\\
\mathrm{Pf}_2 &= a_7y_1+a_5z+x_1v,\\
\mathrm{Pf}_3 &= y_2y_1-a_5x_2+b_5x_1,\\
\mathrm{Pf}_4 &= -y_2z-a_7x_2+b_7x_1,\\
\mathrm{Pf}_5 &= y_2v-a_7b_5+a_5b_7.
\end{aligned}$$

The same way as in Example 4.2.1 we check that X is quasismooth and well formed. The only singular stratum is

$$\{x_1=x_2=x_3=z=0\}\cap X.$$

It is given by the equations $y_1y_2=0$, $vy_2=0$ and defines the isolated point $(0,0,0,0,1,0,0)$ and the line $C=\{y_2=0\}$. Note that C is an example of a dissident curve. It is easy to check that the type of singularities is the same as in the input data.

4.3 The weighted Grassmannian $\mathrm{wGr}(2,5)$

4.3.1 The definition of $\mathrm{wGr}(2,5)$

Let $\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$ be the 4×4 Pfaffians of the generic skew matrix

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}.$$

The affine variety $\mathrm{aGr}(2,5)$ is by definition

$$\mathrm{aGr}(2,5) = \mathrm{Spec}\ \mathbb{C}[x_{i,j}]/I,$$

where ideal I is generated by $\mathrm{Pf}_1, \dots, \mathrm{Pf}_5$.

Using the notation of [8], let $w = (w_1, \dots, w_5)$ denote a vector of half-integers such that $w_i + w_j \in \mathbb{Z}$ is positive for all i, j. Impose a $\mathbb{C}^*$ action on $\mathrm{aGr}(2,5)$ by

$$x_{ij} \longmapsto \lambda^{w_i + w_j} x_{ij}$$

and define the *weighted Grassmannian* by

$$\mathrm{wGr}(2,5) = \left(\mathrm{aGr}(2,5) - \{0\}\right)/\mathbb{C}^*.$$

Equivalently, we can define

$$\mathrm{wGr}(2,5) = \mathrm{Proj}\ \mathbb{C}[x_{ij}]/I,$$

where the ring $\mathbb{C}[x_{ij}]$ is graded by $\mathrm{wt}\, x_{ij} = w_i + w_j$.

The Hilbert series of a weighted Grassmannian is

$$P(t) = \frac{1 - \sum_{i=1}^5 t^{s-w_i} + \sum_{i=1}^5 t^{s+w_i} - t^{2s}}{\prod_{i,j}(1 - t^{w_i + w_j})},$$

where $s = w_1 + \cdots + w_5$. If $\mathrm{wGr}(2,5)$ is well formed, its canonical class equals

$$K_{\mathrm{wGr}} = \mathcal{O}(-2s).$$

Moreover, by Lemma 3.2.1

$$\deg \mathrm{wGr} = \mathcal{O}(1)^6 = \frac{\sum \binom{s-w_i}{3} - \sum \binom{s+w_i}{3} + \binom{2s}{3}}{\prod (w_i + w_j)}.$$

4.3.2 Quasilinear sections of $\mathrm{wGr}(2,5)$

Many codimension 3 Calabi-Yau threefolds can be represented as quasilinear sections of a weighted Grassmannian $\mathrm{wGr}(2,5)$ or a cone over $\mathrm{wGr}(2,5)$.

Definition 4.3.1 Write $\mathcal{C}\mathrm{wGr}(2,5)$ for the projective cone over $\mathrm{wGr}(2,5)$. This means that we add one extra variable to the homogeneous coordinate ring, which is not involved in any relation.

Let $w = (w_1, \ldots, w_5)$ be the defining vector of $\mathrm{wGr}(2,5)$. The section

$$X = \mathrm{wGr}(2,5) \cap (f_1) \cap \ldots \cap (f_l)$$

defined by general forms is a *quasilinear section* of $\mathrm{wGr}(2,5)$ if for every $k = 1, \ldots, l$ exist i, j such that $\deg f_k = w_i + w_j$. The same way we define the *quasilinear section* of $\mathcal{C}\mathrm{wGr}(2,5)$.

Lemma 4.3.2 *The codimension 3 threefolds from Examples 4.2.1, 4.2.3 and 4.2.4 can be represented as quasilinear sections of a cone over* $\mathrm{wGr}(2,5)$.

PROOF We claim that X from Example 4.2.3 can be represented as a quasilinear section in

$$\mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1^3, 2, 3^2, 4^2, 5^2, 6).$$

Choose

$$w = \left(-\frac{1}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}\right),$$

which defines as skew-symmetric matrix with weights

$$w_i + w_j = \begin{pmatrix} 3 & 2 & 1 & 1 \\ & 6 & 5 & 5 \\ & & 4 & 4 \\ & & & 3 \end{pmatrix}.$$

These are exactly the degrees of the elements in M, the defining matrix for X. Note that $h^0(X, D) = 3$ but M contains only two entries of degree 1. Thus we must add an extra variable of degree 1 to the homogeneous coordinate ring of $\mathrm{wGr}(2,5)$. This variable is not involved in any relation and thus defines the projective cone $\mathcal{C}\mathrm{wGr}(2,5)$.

We will prove that X is a general quasilinear section

$$(2) \cap (5)^2 \cap (6) \cap \mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1^3, 2, 3^2, 4^2, 5^2, 6).$$

At the beginning of this section we wrote

$$K_{\mathcal{C}\mathrm{wGr}} = \mathcal{O}(-2(-1/2 + 7/2 + 5/2 + 2 \times 3/2) - 1) = \mathcal{O}(-18).$$

By the adjunction formula the canonical class of the quasilinear section equals $K_X = \mathcal{O}(-18 + 2 + 2 \times 5 + 6)$, which is trivial.

The Hilbert series of $\mathcal{C}\mathrm{wGr}(2,5)$

$$P_{\mathcal{C}\mathrm{wGr}}(t) = \frac{-t^{17} + t^{12} + t^{11} + 2t^{10} + t^8 - t^9 - 2t^7 - t^6 - t^5 + 1}{(1-t)^3(1-t^2)(1-t^3)^2(1-t^4)^2(1-t^5)^2(1-t^6)}$$

yields the Hilbert series of X, which is the same as the computer output.

Also X from Example 4.2.4 can be represented as a quasilinear section in the 7 dimensional

$$\mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1^3, 3^2, 5^3, 7^2, 9).$$

Let

$$w = \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{2}, \frac{7}{2}, -\frac{1}{2}\right).$$

Then $w_i + w_j$ correspond to the degrees of the elements in M, which are

$$\begin{pmatrix} 3 & 7 & 5 & 1 \\ & 7 & 5 & 1 \\ & & 9 & 5 \\ & & & 3 \end{pmatrix}.$$

Since $h^0(X, D) = 3$ and M contains only two entries of degree 1, we must add an extra variable of degree 1 to the homogeneous coordinate ring of $\mathrm{wGr}(2,5)$. This defines the projective cone $\mathcal{C}\mathrm{wGr}(2,5)$.

We claim that X is a general quasilinear section

$$(5)^2 \cap (7)^2 \cap \mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1^3, 3^2, 5^3, 7^2, 9).$$

Since $K_{\mathcal{C}\mathrm{wGr}} = \mathcal{O}(-2(2 \times 3/2 + 11/2 + 7/2 - 1/2) - 1) = \mathcal{O}(-24)$, the adjunction formula implies that $\mathcal{O}_X(-24 + 2 \times 5 + 2 \times 7)$ is trivial.

The Hilbert series of $\mathcal{C}\mathrm{wGr}(2,5)$ is

$$P_{\mathcal{C}\mathrm{wGr}}(t) = \frac{-t^{23}+t^{17}+t^{15}+2t^{13}+t^{11}-t^{12}-2t^{10}-t^{8}-t^{6}+1}{(1-t)^3(1-t^3)^2(1-t^5)^3(1-t^7)^2(1-t^9)}.$$

Thus the multiplied out Hilbert polynomial of X is consistent with the computer output.

In Example 4.2.1 take $w = (2,3,2,1,1)$. This gives a $\mathrm{wGr}(2,5) \subset \mathbb{P}(2,3^4,4^3,5^2)$ defined by a skew-symmetric matrix with the same weights $w_i + w_j$ as

$$M = \begin{pmatrix} 5 & 4 & 3 & 3 \\ & 5 & 4 & 4 \\ & & 3 & 3 \\ & & & 2 \end{pmatrix}.$$

Its canonical class equals $K_{\mathrm{wGr}} = \mathcal{O}(-18)$. The threefold X can not be realised directly as a general quasilinear section in $\mathrm{wGr}(2,5)$. The point is that $h^0(X,D) = 2$, so the graded ring $R(X,D)$ has 2 generators of degree 1. On the other hand, the matrix M has no elements of weight 1. Therefore X is defined as a quasilinear section in

$$\mathcal{C}\mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1,1,2,3^4,4^3,5^2).$$

This means that we add two extra variables of degree 1 to the homogeneous coordinate ring, which are not involved in any relation.

Thus we proved that $X \subset \mathbb{P}(1,1,2,3,3,3,5)$ is isomorphic to

$$(3) \cap (4)^3 \cap (5) \cap \mathcal{C}\mathcal{C}\mathrm{wGr}(2,5) \subset \mathbb{P}(1,1,2,3^4,4^3,5^2).$$

Since $K_{\mathcal{C}\mathcal{C}\mathrm{wGr}} = \mathcal{O}(-20)$, the adjunction formula implies $K_X = \mathcal{O}_X(3 + 3 \times 4 + 5 - 20)$. This shows that X is indeed a Calabi–Yau . $\square$

4.3.3 The Hodge numbers

In this subsection we compute the Hodge numbers for Examples 4.2.3 and 4.2.4. The method is based on the Type I unprojection and was suggested by Balázs Szendrői. We apply the same method to Example 4.2.1 but we encounter worse singularities.

We start by proving two technical lemmas.

Lemma 4.3.3 *Let X be a quasilinear section in* $\mathrm{wGr}(2,5)$ *or* $\mathcal{C}\mathrm{wGr}(2,5)$, *which is quasismooth. Then* $\dim \mathrm{Pic}_{\mathbb{Q}} X = 1$.

Proof Consider $\mathrm{Gr}(2,5)$ embedded in an ordinary projective space $\mathbb{P}^9$ by the Plűcker embedding. Denote by $\mathrm{aGr}(2,5)$ the affine cone over $\mathrm{Gr}(2,5)$ in $\mathbb{A}^{10}$. By [17, Ex. II 6.3] there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}\,(\mathrm{Gr}) \longrightarrow \mathrm{Cl}\,(\mathrm{aGr}) \longrightarrow 0,$$

where the first arrow sends $1 \mapsto \mathcal{O}(1)$ and the second is the restriction to $\mathrm{aGr} - \{0\}$ followed by the inclusion in aGr. By [13, page 274] the sheaf $\mathcal{O}(1)$ generates $\mathrm{Pic}\,(\mathrm{Gr}(2,5))$. This means that the first map is surjective and

$$\mathrm{Cl}\,(\mathrm{aGr}) = \mathrm{Cl}\,(\mathrm{aGr} - \{0\}) = 0.$$

In other words, aGr is factorial.

Let X be a quasilinear section in $\mathrm{wGr}(2,5)$. Write $\mathrm{a}X$ for the affine cone over X. This is the complete intersection singularity given by the same (quasilinear) equations as X. Since X is quasismooth, $0 \in \mathrm{a}X$ is an isolated singularity. By Grothendieck theorem and Lefschetz principle (check [7, Lemma 3.5]), $\mathrm{a}X$ is factorial. Thus every Weil divisor on X can be represented as the quotient by $\mathbb{C}^*$ of a principal divisor on $\mathrm{a}X$, which is invariant under the $\mathbb{C}^*$ action.

The same proof works for quasilinear sections in $\mathcal{C}\mathrm{wGr}(2,5)$, since for every factorial ring R the ring $R[x]$ is also factorial. ☐

Lemma 4.3.4 *Let $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ be homogenous polynomials on $\mathbb{P}^2$ of degrees a, a', a'', b, b', b'' respectively, such that*

$$a - b = a' - b' = a'' - b''.$$

Consider a 0-dim scheme defined by

$$\mathrm{rank} \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{pmatrix} \leq 1.$$

For a generic choice of polynomials, this scheme will contain

$$aa' + b'b'' + ab''$$

points.

Proof Consider the matrix

$$\begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \end{pmatrix}$$

from the lemma. Its minors define a set S in $\mathbb{P}^2$ whose ring has resolution

$$\mathcal{O}_S \leftarrow \mathcal{O}_{\mathbb{P}^2} \leftarrow \mathcal{O}(-a-b') \oplus \mathcal{O}(-a-b'') \oplus \mathcal{O}(-a'-b'') \leftarrow$$

$$\leftarrow \mathcal{O}(-a-a'-b'') \oplus \mathcal{O}(-a-b'-b'') \leftarrow 0.$$

Thus, the Hilbert function for S is

$$P_S(t) = \frac{1 - t^{a+b'} - t^{a+b''} - t^{a'+b''} + t^{a+a'+b} + t^{a+b'+b''}}{(1-t)^3}.$$

Since S is of codimension 2 in $\mathbb{P}^2$, by Lemma 3.2.1 its degree equals

$$\begin{aligned} &-\tbinom{a+b'}{2} - \tbinom{a+b''}{2} - \tbinom{a'+b''}{2} + \tbinom{a+a'+b''}{2} + \tbinom{a+b'+b''}{2} \\ &= aa' + b'b'' + ab''. \end{aligned}$$

□

Let $h^{1,1}$ and $h^{2,1}$ be the Hodge numbers and e the Euler characteristic with compact support of a smooth Calabi–Yau as described in Appendix A.3.

Proposition 4.3.5 *Let X be the Calabi–Yau from Example 4.2.3 and $\hat{X}$ its desingularisation. Then*

$$h^{1,1}(\hat{X}) = 5,\ h^{2,1}(\hat{X}) = 73 \quad and \quad e(\hat{X}) = -136.$$

Proof We will divide the proof into 5 steps.

∘ Step 1: Project X to a codimension 2 complete intersection Y.

By the Type I unprojection, X can be constructed from a complete intersection $Y \subset \mathbb{P}(1,1,1,3,4,4)$, which we now describe. Denote

$$J = \begin{pmatrix} a_2 & x_1 & x_2 \\ a_6 & b_5 & c_5 \end{pmatrix},$$

where a_2, a_6, b_5, c_5 are general polynomials in $x_1, x_2, x_3, y_1, z_1, z_2$ of degrees $2, 6, 5, 5$ respectively. There exists an isomorphism

$$\varphi\colon Y \setminus \Pi \longrightarrow X \setminus \Gamma,$$

where

$$\Pi = \{y_1 = z_1 = z_2 = 0\} \subset Y = \left\{ J \begin{pmatrix} y_1 \\ z_1 \\ z_2 \end{pmatrix} = 0 \right\} \subset \mathbb{P}(1,1,1,3,4,4)$$

and $\Gamma \subset X$ is defined by

$$\operatorname{rank} J|_{\{y_1=z_1=z_2=0\}} \leq 1.$$

In other words, Y can be viewed as a codimension 2 complete intersection of degrees 5 and 9 in $\mathbb{P}(1,1,1,3,4,4)$ on which we impose the plane Π.

∘ Step 2: The Euler number of a general c.i. $Y_{5,9}$.

Denote by $Y_{5,9}$ a general quasismooth complete intersection in $\mathbb{P}(1,1,1,3,4,4)$, and by $\hat{Y}_{5,9}$ its MPCP–desingularisation. We use the notation of Appendix A.3. By [22] or A. Klemm's list of the codimension 2 complete intersections we find that

$$e(\hat{Y}_{5,9}) = 2(h^{1,1} - h^{2,1}) = 2(4 - 109) = -210.$$

The only singularities of $Y_{5,9}$ lie along the projective line $C = \{x_1 = x_2 = x_3 = y_1 = 0\}$. Then $\hat{Y}_{5,9}$ is obtained from $Y_{5,9}$ by resolving along C. The exceptional locus consists of 3 ruled surfaces E_1, E_2, E_3. All E_i are ruled over $C \cong \mathbb{P}^1$, thus $e(E_i) = e(\mathbb{P}^1) \times e(\mathbb{P}^1)$. The intersections $E_1 \cap E_2$ and $E_2 \cap E_3$ both meet in a copy of $\mathbb{P}^1$, whereas $E_1 \cap E_3 = \emptyset$. Therefore

$$e(E_1 \cup E_2 \cup E_3) = 3e(\mathbb{P}^1)^2 - 2e(\mathbb{P}^1) = 3 \cdot 4 - 2 \cdot 2 = 8.$$

By the definition of blow up

$$Y_{5,9} - C \;\cong\; \hat{Y}_{5,9} - \{E_1 \cup E_2 \cup E_3\}$$

and by additivity of the Euler number

$$e(Y_{5,9}) = e(\hat{Y}_{5,9}) - e(E_1 \cup E_2 \cup E_3) + e(C) = -210 - 8 + 2 = -216.$$

$\circ$ Step 3: The difference between $\operatorname{Sing} Y_{5,9}$ and $\operatorname{Sing} Y$.

By Bertini's theorem Y is quasismooth away from the base locus Π. Recall the defining equations of Y

$$\begin{aligned} f_5 &= a_2 y_1 + x_1 z_1 + x_2 z_2, \\ f_9 &= a_6 y_1 + b_5 z_1 + c_5 z_2. \end{aligned} \tag{E. xiii}$$

Since we can assume that a_6 contains y_1^2, the point $(0,0,0,1,0,0)$ does not lie in Y. On the other hand z_1 and z_2 induce a line of singularities of transverse type $\frac{1}{4}(1,3)$. Consider the affine piece $z_1 \neq 0$ and assume that b_5 contains $x_2 z_1$. We can eliminate x_1, x_2 out of f_5 and f_9. Thus x_3 and y_1 are the local transverse parameters around the curve with the $\varepsilon \in \mu_4$ group action

$$x_3, y_1 \;\sim\; \varepsilon x_3, \varepsilon y_1.$$

The same argument holds for $z_2 \neq 0$.

We proved that Y and $Y_{5,9}$ have the same singularities away from Π.

Next we determine the singularities of Y that lie on Π. The Jacobian of (E. xiii) restricted to Π equals

$$\left.\begin{pmatrix} 0 & 0 & 0 & a_2 & x_1 & x_2 \\ 0 & 0 & 0 & a_6 & b_5 & c_5 \end{pmatrix}\right|_{\Pi}.$$

Therefore,

$$\operatorname{Sing} Y|_{\Pi} = \operatorname{rank} J|_{\{y_1 = z_1 = z_2 = 0\}} \leq 1.$$

Thus $\operatorname{Sing} Y$ consists of 37 points on Π by Lemma 4.3.4.

Now we will determine the type of these singularities. For every point of Π one of x_1, x_2, x_3 is not zero. We will use the implicit function theorem on (E. xiii). On the affine piece $x_1 = 1$, we can express z_1 out of the first equation f_5. The quadratic monomials in the second equation f_9 come from $a_6 y_1 + c_5 z_2$. They give a quadratic

$$y_1(x_2 + x_3 + y_1 + z_2) + z_2(x_2 + x_3 + y_1 + z_2)$$

of rank 4. This tells that all singularities for which $x_1 \neq 0$ are simple nodes.

Exactly the same argument holds for $x_2 \neq 0$.

We can also assume that a_6 contains monomial x_3^6. Then, on the affine piece $x_3 = 1$ we can express y_1 out of f_9. The quadratic monomials in the first equation f_5 come from $x_1 z_1 + x_2 z_2$, which is clearly a quadratic of rank 4. Therefore also the singularities for which $x_3 \neq 0$ are simple nodes.

We proved that Y is a complete intersection of degrees 5 and 9 in $\mathbb{P}(1,1,1,3,4,4)$ with 37 simple nodes. It is well known [14], that every node adds 1 to the Euler characteristic. This is because nodes have very simple topology and each node has the sphere S^3 as its Milnor fibre. Therefore

$$e(Y) = e(Y_{5,9}) + 37 = -216 + 37 = -179.$$

∘ Step 4: Unproject from Y to compute $e(X)$.

By the Type I unprojection there exists an isomorphism

$$\varphi\colon Y \setminus \Pi \longrightarrow X \setminus \Gamma,$$

where X is defined by Y and an extra variable

$$y_2 = \frac{x_1 c_5 - x_2 b_5}{y_1} = \frac{x_2 a_6 - a_2 c_5}{z_1} = \frac{a_2 b_5 - x_1 a_6}{z_2}.$$

Note that these equations are indeed the defining Pfaffians of X in Example 4.2.3. Note also that Γ in $X \subset \mathbb{P}(1,1,1,3,3,4,4)$ is defined by

$$\operatorname{rank} J|_{\{y_1 = z_1 = z_2 = 0\}} \leq 1.$$

This inequality restricted to $\{y_2 = 0\} \subset X$ satisfies the conditions of Lemma 4.3.4. Thus Γ is a union of 37 lines that all meet in $(0,0,0,0,1,0,0)$ as shown on Figure IV.

Finally we can compute

$$e(X) = e(Y) - e(\Pi) + e(\Gamma) = -179 - 3 + 37 + 1 = -144.$$

∘ Step 5: The Hodge numbers of $\hat{X}$.

Let $\hat{X}$ be the MPCP-desingularisation of X. In Example 4.2.3 we prove that X is quasismooth and has a $\frac{1}{4}(1,3)$ transverse line C and a $\frac{1}{3}(1,1,1)$ singular point P.

Figure IV: The Type I unprojection $Y \to X$

The resolution of $C \cong \mathbb{P}^1$ induces three exceptional surfaces E_1, E_2, E_3. In STEP 2 we show that $e(E_1 \cup E_2 \cup E_3) = 8$. Note that P allows a crepant toric resolution. Its exceptional divisor E is a copy of $\mathbb{P}^2$. This yields

$$\begin{aligned} e(\hat{X}) &= e(X) - e(P) - e(C) + e(\mathbb{P}^2) + e(E_1 \cup E_2 \cup E_3) \\ &= -144 - 1 - 2 + 3 + 8 = -136. \end{aligned}$$

The desingularisation adds the divisors E_1, E_2, E_3 and E to the Picard group. Thus, by Lemma 4.3.3

$$h^{1,1}(\hat{X}) = \rho(\hat{X}) = 1 + 1 + 3 = 5.$$

From here we get

$$h^{2,1}(\hat{X}) = h^{1,1}(\hat{X}) - \frac{1}{2} e(\hat{X}) = 73.$$

$\square$

In the next example we will need the following lemma:

Lemma 4.3.6 *Let $\mathbb{P}(1,3,5,9)$ have the coordinates t_1, t_3, t_5, t_9. Denote by E the exceptional locus of the toric resolution of $C = \{t_1 = t_5 = 0\}$ and $(0,0,0,1) \in C$. Then $e(E) = 12$.*

PROOF The toric threefold $\mathbb{P}(1,3,5,9)$ can be described by the fan in $\mathbb{Z}^3$

spanned on the vectors

$$\begin{aligned} v_1 &= (-9,-5,-3), \\ v_3 &= (0,0,1), \\ v_5 &= (0,1,0), \\ v_9 &= (1,0,0). \end{aligned}$$

Note that $v_1 + 3v_3 + 5v_5 + 9v_9 = 0$.

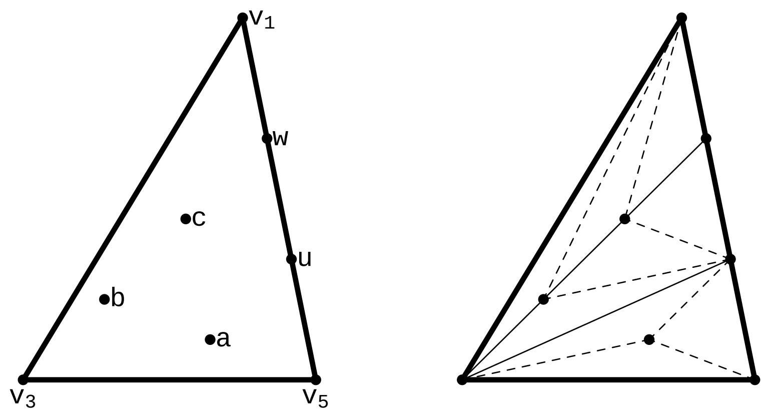

Figure V: Resolution of the dissident curve

We resolve C by adding rays

$$w = \tfrac{2}{3}v_1 + \tfrac{1}{3}v_5 = (-6,-3,-2),$$

$$u = \tfrac{1}{3}v_1 + \tfrac{2}{3}v_5 = (-3,-1,-1)$$

to the fan. These two vectors correspond to the exceptional divisors E_w and E_u. Next we add

$$a = \tfrac{1}{9}(v_1 + 3v_3 + 5v_5) = \tfrac{1}{3}(v_3 + v_5 + u) = (-1,0,0),$$

$$b = \tfrac{1}{9}(2v_1 + 6v_3 + v_5) = \tfrac{2}{3}v_3 + \tfrac{1}{3}w = (-2,-1,0),$$

$$c = \tfrac{1}{9}(4v_1 + 3v_3 + 2v_5) = \tfrac{1}{3}v_3 + \tfrac{2}{3}w = (-4,-2,-1),$$

which correspond to the divisors E_a, E_b, E_c. Note that this is a crepant resolution.

In order to compute the Euler number $e(E_w \cup E_u \cup E_a \cup E_b \cup E_c)$, we need to understand what are these divisors topologically and how do they intersect. Using the terminology of [12], every divisor is determined by its Star. In $\mathbb{Z}^2$, this corresponds to the fans on Figure VI.

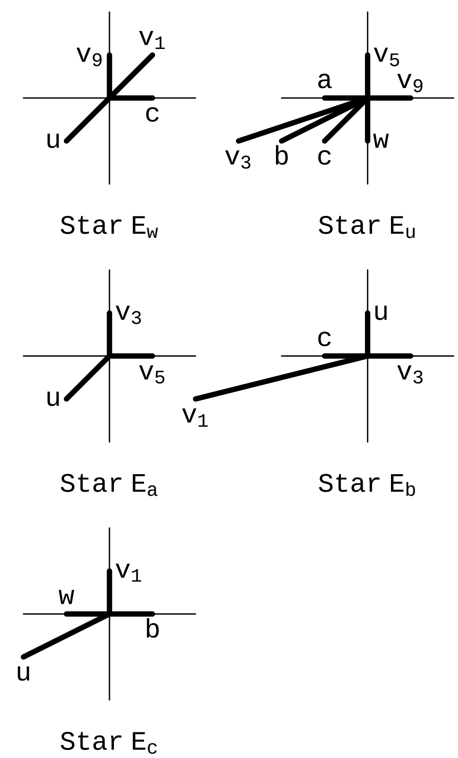

Figure VI: Fans of the exceptional surfaces

The exceptional surfaces form the configuration on Figure VII

Every nontrivial intersection of two exceptional surfaces is a copy of $\mathbb{P}^1$ with the Euler number 2. Observe also that all exceptional divisors are toric surfaces. By [12, Section 3.2.], the Euler number of a compact toric variety defined by a fan F in $\mathbb{Z}^n$ equals to the number of n-dimensional cones in F.

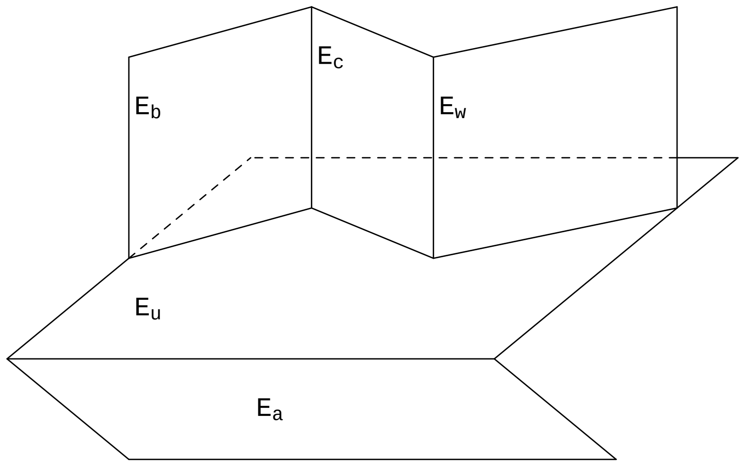

Figure VII: Configuration of the exceptional surfaces

Thus

$$
\begin{aligned}
&e(E_w \cup E_u \cup E_a \cup E_b \cup E_c) = \\
&\\
&e(E_w) + e(E_u) + e(E_a) + e(E_b) + e(E_c) - \\
&e(E_u \cap E_a) - e(E_u \cap E_b) - e(E_u \cap E_c) - \\
&e(E_u \cap E_w) - e(E_b \cap E_c) - e(E_c \cap E_w) + \\
&e(E_u \cap E_b \cap E_c) + e(E_u \cap E_c \cap E_w) = \\
&\\
&4 + 7 + 3 + 4 + 4 - (2 + 2 + 2 + 2 + 2 + 2) + 2 = 12.
\end{aligned}
$$

□

Proposition 4.3.7 *Let X be the Calabi–Yau from Example 4.2.4 and $\hat{X}$ its desingularisation. Then*

$$h^{1,1}(\hat{X}) = 7,\ \ h^{2,1}(\hat{X}) = 92 \quad \textit{and} \quad e(\hat{X}) = -170.$$

Proof We will use the same method as in Proposition 4.3.5

∘ Step 1: Project X to a codimension 2 complete intersection Y.

By the Type I projection we project X to $Y \subset \mathbb{P}(1,1,1,3,5,9)$. Write

$$J = \begin{pmatrix} a_7 & a_5 & x_1 \\ b_7 & b_5 & x_2 \end{pmatrix},$$

where a_i, a_6, b_j are general polynomials in x_1, x_2, x_3, y_1, z, v and define

$$Y = \left\{ J \begin{pmatrix} y_1 \\ z \\ v \end{pmatrix} = 0 \right\}.$$

There exists an isomorphism

$$\varphi \colon Y \setminus \Pi \longrightarrow X \setminus \Gamma,$$

where

$$\Pi = \{y_1 = z = v = 0\} \subset Y \subset \mathbb{P}(1,1,1,3,4,4),$$

and $\Gamma \subset X$ is defined by

$$\operatorname{rank} J|_{\{y_1 = z = v = 0\}} \leq 1.$$

The singular threefold Y is a codimension 2 complete intersection in $\mathbb{P}(1,1,1,3,4,4)$. It is defined by two equations of degrees 10 on which we impose Π.

∘ Step 2: The Euler number of a general c.i. $Y_{10,10}$.

Denote by $Y_{10,10}$ a generic quasismooth complete intersection in $\mathbb{P}(1,1,1,3,5,9)$, and by $\hat{Y}_{10,10}$ its MPCP-desingularisation. The only singularities of $Y_{10,10}$ lie along the projective line $C = \{x_1 = x_2 = x_3 = z = 0\}$. It is easy to see that C is a curve of transverse type $\frac{1}{3}(1,2)$ with a $\frac{1}{9}(1,3,5)$ dissident point on it. From [22] we read

$$e(\hat{Y}_{10,10}) = 2(h^{1,1} - h^{2,1}) = 2(6 - 138) = -264.$$

By the definition of desingularisation

$$\hat{Y}_{10,10} - E \;\cong\; Y_{10,10} - C,$$

where E denotes the exceptional locus. Lemma 4.3.6 implies

$$e(Y_{10,10}) = e(\hat{Y}_{10,10}) - e(E) + e(C) = -264 - 12 + 2 = -274.$$

$\circ$ STEP 3: The difference between $\operatorname{Sing} Y_{10,10}$ and $\operatorname{Sing} Y$.

The defining equations of Y are

$$\begin{aligned} f_{10} &= a_7 y_1 + a_5 z + x_1 v \\ g_{10} &= b_7 y_1 + b_5 z + x_2 v. \end{aligned} \tag{E. xiv}$$

By Bertini's theorem Y is quasismooth away from the base locus Π. We use the implicit function theorem on (E. xiv) to find the local transverse parameters in the neighbourhood of the curve $\{x_1 = x_2 = x_3 = z = 0\} \subset Y$. This shows that Y has a curve of transverse type $\frac{1}{3}(1,2)$ with a dissident point $\frac{1}{9}(1,3,5)$. These singularities are the same as the singularities of the generic $Y_{10,10}$.

Additionally, Y has singularities on Π. They are given by

$$\operatorname{Sing} Y|_{\Pi} = \operatorname{rank} J|_{\{y_1 = z = v = 0\}} \leq 1.$$

By Lemma 4.3.4 we get 47 points on Π.

Next we will prove that all 47 points are nodes. For every point on Π one of the x_1, x_2, x_3 is not zero. We will use the implicit function theorem on the defining equations (E. xiv). On the affine piece $x_1 = 1$, we can express v out of f_{10}. This gives hypersurface singularities, which are determined by the quadratic monomials in g_{10}. They define a quadric

$$y_1(x_2 + x_3 + y_1 + z) + z(x_2 + x_3 + y_1 + z)$$

of rank 4.

Exactly the same argument holds for $x_2 \neq 0$.

For $x_3 \neq 0$ we can assume that a_7 contains x_3^7. On the affine piece $x_3 = 1$ we can express y_1 out of f_{10}. The quadratic monomials in g_{10} come from $z(x_1 + x_2 + z) + x_2 v$, which is again a quadric of rank 4.

We proved that Y is a complete intersection in $\mathbb{P}(1,1,1,3,5,9)$ with 47 simple nodes. Therefore

$$e(Y) = e(Y_{10,10}) + 47 = -274 + 47 = -227.$$

○ Step 4: Unproject from Y to compute $e(X)$.

Recall the Type I unprojection and the isomorphism

$$\varphi \colon Y \setminus \Pi \longrightarrow X \setminus \Gamma,$$

where X is constructed from Y by adding

$$y_2 = \frac{x_2 a_5 - x_1 b_5}{y_1} = \frac{x_1 b_7 - x_2 a_7}{z} = \frac{a_7 b_5 - a_5 b_7}{v}$$

and $\Gamma \subset X \subset \mathbb{P}(1,1,1,3,3,5,9)$ is defined by

$$\operatorname{rank} J|_{\{y_1 = z = v = 0\}} \leq 1.$$

For $\{y_2 = 0\}$ in X this inequality satisfies the conditions of Lemma 4.3.4. Thus Γ is a union of 47 lines that all meet in $(0,0,0,0,1,0,0)$.

Finally we get

$$e(X) = e(Y) - e(\Pi) + e(\Gamma) = -227 - 3 + 47 + 1 = -182.$$

○ Step 5: The Hodge numbers of $\hat{X}$.

In Example 4.2.3 we proved that $\operatorname{Sing}(X)$ contains a $\frac{1}{3}(1,1,1)$ singular point P and a $\frac{1}{3}(1,2)$ curve C with a dissident point Q.

The line C has a resolution with the exceptional locus E that was described in Lemma 4.3.6. The point P also allows a crepant toric resolution whose exceptional divisor E_P is a copy of $\mathbb{P}^2$. For the resolved $\hat{X}$ this yields

$$\begin{aligned} e(\hat{X}) &= e(X) - e(P) - e(C) + e(\mathbb{P}^2) + e(E) \\ &= -182 - 1 - 2 + 3 + 12 = -170. \end{aligned}$$

The desingularisation introduces the divisors E_w, E_u, E_a, E_b, E_c and E_P. Thus

$$h^{1,1}(\hat{X}) = 1 + 1 + 5 = 7.$$

From here we get

$$h^{2,1}(\hat{X}) = h^{1,1}(\hat{X}) - \frac{1}{2} e(\hat{X}) = 92.$$

□

Lemma 4.3.8 *In Example 4.2.1 the complete intersection Y obtained by the Type I unprojection contains worse singularities than simple nodes.*

Proof Consider X from Example 4.2.1. It is quasismooth and has a $\frac{1}{5}(1,1,3)$ isolated point $Q = (0,0,0,0,0,0,1)$ and a $\frac{1}{3}(1,2)$ transverse curve C. We can obtain X by the Type I unprojection from a complete intersection Y. Define $Y \subset \mathbb{P}(1,1,2,3,3,3)$ with variables $x_1, x_2, y, z_1, z_2, z_3$ by

$$\Pi = \{y = z_1 = z_2 = 0\} \subset Y = \left\{ \begin{pmatrix} a_4 & b_3 & c_3 \\ a_5 & b_4 & c_4 \end{pmatrix} \begin{pmatrix} y \\ z_1 \\ z_2 \end{pmatrix} = 0 \right\},$$

where a_i, b_j, c_k are generic homogeneous polynomials. The singular threefold Y can be viewed as a codimension 2 complete intersection of degrees 6 and 7 in $\mathbb{P}(1,1,2,3,3,3)$ on which we impose $\Pi \cong \mathbb{P}(1,1,3)$. Then X is the unprojection of Y by adding the parameter

$$t = \frac{b_3c_4 - b_4c_3}{y} = \frac{c_3a_5 - a_4c_4}{z_1} = \frac{a_4b_4 - b_3a_5}{z_2}$$

of degree 5. The defining equations of X are exactly the Pfaffians in Example 4.2.1.

Denote by $Y_{6,7}$ a general quasismooth complete intersection in $\mathbb{P}(1,1,2,3,3,3)$. The only singularities of $Y_{6,7}$ lie on the quadric $C = \{x_1 = x_2 = y = 0\}$. Indeed, by an equation of degree 6 we can avoid the $\frac{1}{2}$ singular point of $\mathbb{P}$. Moreover, the singularities along C are of the transverse type $\frac{1}{3}(1,2)$.

By Bertini's theorem it is easy to see that Y and $Y_{6,7}$ have the same singularities away from the base locus Π.

In the next step we determine the singularities of Y that lie on Π. Denote

$$\begin{pmatrix} a_4 & b_3 & c_3 \\ a_5 & b_4 & c_4 \end{pmatrix}$$

by J. Then

$$\operatorname{Sing} Y|_\Pi = \operatorname{rank} J|_{\{y=z_1=z_2=0\}} \leq 1.$$

We can not use Lemma 4.3.4 in this case since $\Pi \cong \mathbb{P}(1,1,3)$. Instead we use the computer to count the number of solutions for the equations defined by the minors of J. If we choose a_i, b_j, c_k in the variables x_1, x_2, z_3 general enough, we get 15 points on Π. Observe that $P = (0,0,1)$ is always one of them. Note also also that

$$P = \Pi \cap C.$$

The situation is shown on Figure VIII.

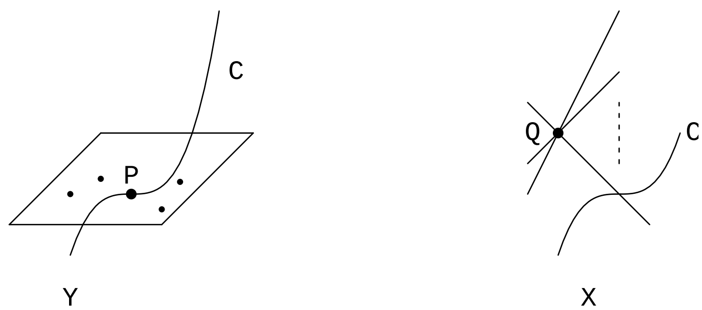

Figure VIII: Type I unprojection $Y \to X$

We will prove that 14 of these points are nodes on Y, whereas P is a worse singularity. For every point on Π one of x_1, x_2, z_3 is not zero. We can use the implicit function theorem on

$$\begin{aligned} a_4 y + b_3 z_1 + c_3 z_2 &= 0 \\ a_5 y + b_4 z_1 + c_4 z_2 &= 0. \end{aligned} \tag{E. xv}$$

Assume that a_4 contains the monomial x_1^4. Then, use the implicit function theorem on the affine piece $x_1 = 1$, to express y out of the first equation. The quadratic monomials in the second equation come from $a_6 y_1 + c_5 z_2$. They give a quadric

$$z_1(x_2 + z_1 + z_2 + z_3) + z_2(x_2 + z_1 + z_2 + z_3)$$

of rank 4. This tells that all singularities for which $x_1 \neq 0$ are simple nodes.

Exactly the same argument holds for $x_2 \neq 0$.

We can also assume that $b_3 = z_3$. At P we set $z_3 = 1$ and eliminate z_1 out of the first equation in (E. xv). The quadratic part of the second equation is $y^2 + (x_1 + x_2)z_2$, which is a quadric of rank 3. Therefore we also need to consider the cubic monomials

$$y^2 + (x_1 + x_2)z_2 + x_1^2 y + x_1 x_2 y + x_2^2 y + y^2 z_2 + x_1 z_2^2 + x_2 z_2^2.$$

We proved that P is a singularity of type

$$\{p(x_1, x_2, y, z_2) = 0\} \subset \mathbb{A}^4/\mu_3,$$

where p denotes the above cubic equation and $\varepsilon \in \mu_3$ acts by

$$x_1, x_2, y, z_2 \mapsto \varepsilon x_1, \varepsilon x_2, \varepsilon^2 y, z_2.$$

$\square$

4.4 Weighted orthogonal Grassmannians

In this section we discuss some examples of codimension 5 Calabi–Yau threefolds. The ambient spaces of these varieties will be weighted orthogonal Grassmannians, which are weighted projective analogs of the homogeneous orthogonal Grassmannian $\mathrm{OGr}(5,10)$.

4.4.1 Definitions and the spinor embedding

The construction we recall in this section was introduced in [8].

Consider $V = \mathbb{C}^{10}$ equipped with a nondegenerate quadratic form q. Choose a basis

$$e_1, \ldots, e_5, f_1, \ldots, f_5$$

for V such that

$$q = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad V = U \oplus U^\vee,$$

where $U = \langle e_1, \ldots, e_5 \rangle$ and $U^\vee = \langle f_1, \ldots, f_5 \rangle$. A vector space $F \subset V$ is *isotropic* if q is identically zero on F. Define the *orthogonal Grassmannian* $\mathrm{OGr}(5,10)$ by

$$\mathrm{OGr}(5,10) = \left\{ F \in \mathrm{Gr}(5,V) \;\middle|\; \begin{array}{l} F \text{ is isotropic and} \\ \dim F \cap U \text{ is odd} \end{array} \right\}.$$

A 5-plane F in the neighbourhood of U has a basis of five vectors in V. Thus it can be written as a matrix (I, M), where M and identity I are 5×5 matrices. Observe that F is isotropic for q if and only if M is a skew symmetric matrix. Therefore, an affine neighbourhood of U in $\mathrm{OGr}(5,10)$ is given by (I, M), where M is a skew 5×5 matrix.

$\mathrm{OGr}(5,10)$ can be covered by 16 affine pieces. Each one is obtained from the affine piece (I, M), by swapping evenly many of the columns in I with the corresponding columns in M.

Define the space of spinor

$$S^+ = \mathbb{C} \oplus \bigwedge^2 U \oplus \bigwedge^4 U.$$

For an element (x, M, P) in S^+, where $x \in \mathbb{C}$ we introduce notation

$$M = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

This defines the *spinor embedding*

$$\mathrm{OGr}(5, 10) \hookrightarrow \mathbb{P}(S^+),$$

where (x, M, P) satisfy the 10 equations

$$xP = \mathrm{Pf}\, M, \quad \text{and} \quad MP = 0.$$

More precisely, these equations are:

$$\begin{aligned} N_1 &= xx_1 - x_{23}x_{45} + x_{24}x_{35} - x_{25}x_{34}, \\ N_2 &= xx_2 + x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}, \\ N_3 &= xx_3 - x_{12}x_{45} + x_{14}x_{25} - x_{15}x_{24}, \\ N_4 &= xx_4 + x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}, \\ N_5 &= xx_5 - x_{12}x_{34} + x_{13}x_{24} - x_{14}x_{23}, \end{aligned}$$

$$\begin{aligned} N_{-1} &= x_2x_{12} + x_3x_{13} + x_4x_{14} + x_5x_{15}, \\ N_{-2} &= -x_1x_{12} + x_3x_{23} + x_4x_{24} + x_5x_{25}, \\ N_{-3} &= -x_1x_{13} - x_2x_{23} + x_4x_{34} + x_5x_{35}, \\ N_{-4} &= -x_1x_{14} - x_2x_{24} - x_3x_{34} + x_5x_{45}, \\ N_{-5} &= -x_1x_{15} - x_2x_{25} - x_3x_{35} - x_4x_{45}. \end{aligned}$$

The same as for a weighted projective space, we can assign weights to x, x_{ij}, x_k. Using representation theory (check Section 4 in [8] for the proof), we can introduce the following numerology. Let $w_1, \ldots, w_5$ be weights of $U = \mathbb{C}^5$ and u a further overall weight on S^+. Define $s = w_1 + \ldots + w_5$. Then the generators have weights

$$\begin{aligned} \mathrm{wt}\, x &= u \\ \mathrm{wt}\, x_{ij} &= u + w_i + w_j \\ \mathrm{wt}\, x_i &= u + s - w_i. \end{aligned}$$

The weighted orthogonal Grassmannian wOGr$(5,10)$ has the Hilbert numerator

$$Q(t) = 1 - t^{s+2u}Q_V + t^{2s+3u}Q_{S^-} - t^{2s+5u}Q_{S^+} + t^{3s+6u}Q_V - t^{4s+8u},$$

where

$$\begin{aligned} Q_V &= \textstyle\sum_{i=1}^5 t^{w_i} + t^{-w_i}, \\ Q_{S^+} &= 1 + \textstyle\sum_{i,j} t^{w_i+w_j} + \sum_i t^{s-w_i}, \\ Q_{S^-} &= 1 + \textstyle\sum_{i,j} t^{-w_i-w_j} + \sum_i t^{-s+w_i}. \end{aligned}$$

The corresponding Hilbert series is

$$P_{\mathrm{wOGr}}(t) = \frac{Q(t)}{(1-t^u)\cdot\prod_i(1-t^{u+s-w_i})\cdot\prod_{ij}(1-t^{u+w_i+w_j})}.$$

If wOGr$(5,10)$ is well formed, its canonical class equals

$$K_{\mathrm{wOGr}} = \mathcal{O}(-4s-8u).$$

4.4.2 Examples in codimension 5

In this subsection we construct new examples of Calabi–Yau threefolds as quasilinear sections of a fixed weighted orthogonal Grassmannian.

Choose the weights

$$(w_1, w_2, w_3, w_4, w_5) = (0,0,1,1,1) \text{ and } u = 1,$$

defining wOGr$(5,10)$. Using the notation of Subsection 4.4.1, we get

- $s = 3$,
- the coordinates x and x_{12} have weight 1,
- the coordinates x_{13}, x_{14}, x_{15} and x_{23}, x_{24}, x_{25} have weight 2,
- the coordinates x_{34}, x_{35}, x_{45} and x_3, x_4, x_5 have weight 3,
- the coordinates x_1 and x_2 have weight 4.

The spinor embedding takes $\mathrm{wOGr}(5,10)$ into $\mathbb{P}(1^2,2^6,3^6,4^2)$, with the defining equations $N_1,\dots,N_5,\ N_{-1},\dots,N_{-5}$ of degrees $5,5,4,4,4,\ 5,5,6,6,6$ respectively.

Its Hilbert function equals

$$P_{\mathrm{wOGr}}(t) =$$

$$\frac{1-3t^4-4t^5-t^6+6t^7+6t^8+2t^9-2t^{11}-6t^{12}-6t^{13}+t^{14}+4t^{15}+3t^{16}-t^{20}}{(1-t)^2(1-t^2)^6(1-t^3)^6(1-t^4)^2}$$

and the canonical class is $K_{\mathrm{wOGr}} = \mathcal{O}(-20)$.

Example 4.4.1 For the input data we choose

- $h^0(X,D) = 2$ and $h^0(X,2D) = 6$
- Points: /
- Curves: type$=\frac{1}{3}(1,2)$, deg $= 1$, index$= 1$, $N_C = -6$.

The output we get is

- Codimension: 5
- Weights: 1,1,2,2,2,3,3,3,3
- Multiplied out Hilbert polynomial: $1-3t^4-4t^5-t^6+6t^7+6t^8+2t^9-2t^{11}-6t^{12}-6t^{13}+t^{14}+4t^{15}+3t^{16}-t^{20}$

We will prove that the output defines a general quasilinear section

$$X = \mathrm{wOGr}(5,10) \cap (2)^3 \cap (3)^2 \cap (4)^2$$

in the chosen $\mathrm{wOGr}(5,10)$.

The fact that X is a Calabi–Yau can be checked in two ways. Either $X \subset \mathbb{P}(1^2,2^3,3^4)$ with the adjunction number 20 yields $-2\times 1-3\times 2-4\times 3+20 = 0$, or the adjunction formula for $(2)^3\cap(3)^2\cap(4)^2 \subset \mathrm{wOGr}(5,10)$ gives $K_X = \mathcal{O}_X(-20+3\times 2+2\times 3+2\times 4) \sim 0$.

In the next paragraph we study the cyclic quotient singularities of X, induced by the singularities of $\mathbb{P}$. The Calabi–Yau X contains no $\frac{1}{2}$ or $\frac{1}{4}$

singularities, since there are enough equations of even degrees. Next consider C defined by the weights of degree 3 in $\mathbb{P}$. More precisely, C is given by

$$\begin{array}{lrcl} N_{-3}: & x_4x_{34}+x_5x_{35} & = & 0, \\ N_{-4}: & -x_3x_{34}+x_5x_{45} & = & 0, \\ N_{-5}: & -x_3x_{35}-x_4x_{45} & = & 0, \end{array} \quad \text{and} \quad (3)^2: \ l_1 = l_2 = 0,$$

where l_1 and l_2 are two linear equations in $x_3, x_4, x_5,\ x_{34}, x_{35}, x_{45}$. Rewrite the above as

$$\operatorname{rank}\begin{pmatrix} x_3 & x_4 & x_5 \\ x_{45} & x_{35} & x_{34} \end{pmatrix} \leq 1 \quad \text{and} \quad l_1 = l_2 = 0,$$

which proves that C is a twisted cubic.

Choose a $\mathbb{Q}-$Cartier divisor D such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(1)$. Then $\deg D|_C = 1$.

We claim that C is of transverse type $\frac{1}{3}(1,2)$. For example, a point on C at which $x_3 \neq 0$ has the local transverse parameters x_{12} and x_{24}. Indeed, by the implicit function theorem we can express x out of N_3, $\ x_{13}, x_{23}$ out of N_{-1}, N_{-2} respectively; and the variables x_{14}, x_{15}, x_{25} and x_1, x_2 out of the linear equations corresponding to $(2)^3$ and $(4)^2$. The same way we find the local parameters for any other point on C.

Recall the formula in Corollary 3.1.1. The data $h^0(X,D) = 2,\ h^0(X,2D) = 6$ are equivalent to

$$D^3 = \frac{5}{3} \quad \text{and} \quad D \cdot c_2(X) = 26.$$

Then

$$h^0(X, mD) = \frac{5}{18}m^3 + \frac{13}{6}m + \alpha_m,$$

where

$$\alpha_m = \begin{cases} -\frac{1}{3}m - \frac{1}{9} & \text{if } m = 1 \bmod 3, \\ -\frac{1}{3}m + \frac{1}{9} & \text{if } m = 2 \bmod 3, \\ 0 & \text{if } m = 0 \bmod 3. \end{cases}$$

From this we calculate the Hilbert function of X

$$P_X(t) = \frac{1-3t^4-4t^5-t^6+6t^7+6t^8+2t^9-2t^{11}-6t^{12}-6t^{13}+t^{14}+4t^{15}+3t^{16}-t^{20}}{(1-t)^2(1-t^2)^3(1-t^3)^4},$$

which is what we predicted on page 104.

Example 4.4.2 For the input data we choose

- $h^0(X, D) = 1$ and $h^0(X, 2D) = 6$
- Points: $3 \times {}_1\left(\frac{1}{3}(1,1,1)\right)$
- Curves: type$= \frac{1}{2}(1,1)$, deg $= \frac{7}{2}$, index$= 1$.

The output we get is

- Codimension: 5
- Weights: 1,2,2,2,2,2,3,3,3
- Multiplied out Hilbert polynomial: $1 - 3t^4 - 4t^5 - t^6 + 6t^7 + 6t^8 + 2t^9 - 2t^{11} - 6t^{12} - 6t^{13} + t^{14} + 4t^{15} + 3t^{16} - t^{20}$

This output suggests a general quasilinear section in the chosen wOGr(5, 10)

$$X = \mathrm{wOGr}(5,10) \cap (1) \cap (2) \cap (3)^3 \cap (4)^2.$$

Use the adjunction formula for $(1) \cap (2) \cap (3)^3 \cap (4)^2 \subset \mathrm{wOGr}(5,10)$ to compute $K_X = \mathcal{O}(-20 + 1 + 2 + 3 \times 3 + 2 \times 4) \sim 0$. Thus X is a Calabi–Yau.

We will study the cyclic quotient singularities of X, induced by the weights of $\mathbb{P}$. We can avoid the weights of degree 4 by two equations $(4)^2$, thus there are no $\frac{1}{4}$ singularities.

On the other hand, the weights of degree 2 determine a singular locus C. More precisely, C is defined by the equations involving the variables of even degrees:

$$\begin{array}{lrcl} N_3: & x_{14}x_{25} - x_{15}x_{24} & = & 0, \\ N_4: & -x_{13}x_{25} + x_{15}x_{23} & = & 0, \\ N_5: & x_{13}x_{24} - x_{14}x_{23} & = & 0, \\ & & & \\ N_{-3}: & -x_1x_{13} - x_2x_{23} & = & 0, \\ N_{-4}: & -x_1x_{14} - x_2x_{24} & = & 0, \\ N_{-5}: & -x_1x_{15} - x_2x_{25} & = & 0, \end{array} \quad \text{and} \quad (2) \cap (4)^2 : \ l = q_1 = q_2 = 0,$$

where l is a linear equation of degree 2 and q_1, q_2 are equations of degree 4 which are linear in x_1, x_2 and quadratic in $x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}$. The above system can be written into

$$\operatorname{rank}\begin{pmatrix} x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \end{pmatrix} \leq 1,$$

$$(x_1, x_2) \cdot \begin{pmatrix} x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \end{pmatrix} = 0,$$

$$l = q_1 = q_2 = 0.$$

This proves that C is a curve of transverse type $\frac{1}{2}(1,1)$.

We will also compute the degree of C. Let a divisor D be given by $\mathcal{O}_X(1)$. The divisor $2D$ is Cartier in the neighbourhood of C and has a section, defined by an equation $l' = 0$ of degree 2. Observe that

$$\operatorname{rank}\begin{pmatrix} x_{13} & x_{14} & x_{15} \\ x_{23} & x_{24} & x_{25} \end{pmatrix} \leq 1 \bigcap \{l = l' = 0\} \tag{E. xvi}$$

is a twisted cubic which can be parametrically given by:

$$\operatorname{rank}\begin{pmatrix} 1 & a & a^2 \\ a & a^2 & a^3 \end{pmatrix} \leq 1.$$

Now eliminate the variables x_1, x_2 of weight 4 out of q_1, q_2 and represent them by a. For $i = 1, 2$ we get $x_i = r_i(a)$, where r_i is a polynomial of degree 6 in a. The last equation we need to consider is $(x_1, x_2) \cdot (x_{13}, x_{23}) = 0$. This equals to

$$(r_1(a), r_2(a)) \cdot (1, a) = 0,$$

which has degree 7 and thus has 7 solutions. Thus we proved that $\deg D|_C = \frac{7}{2}$.

The weights of degree 3 define a locus given by the equations

$$\begin{array}{rrcl} N_{-3}: & x_4x_{34} + x_5x_{35} & = & 0, \\ N_{-4}: & -x_3x_{34} + x_5x_{45} & = & 0, \quad \text{and} \quad (3)^3: \ l_1 = l_2 = l_3 = 0, \\ N_{-5}: & -x_3x_{35} - x_4x_{45} & = & 0, \end{array}$$

where l_1, l_2 and l_3 are linear in $x_3, x_4, x_5,\ x_{34}, x_{35}, x_{45}$. The same way as in Example 4.4.1 this defines a twisted cubic intersected by a hyperplane. Denote the intersecting points by P_1, P_2, P_3.

Next we determine the type of singularities at these points. Assume that $x_4 \neq 0$ at P_1. By the implicit function theorem we can express x out of N_4 and x_{14}, x_{24} out of N_{-1}, N_{-2}. We can also eliminate $x_{12},\ x_{13},\ x_1, x_2$ from the equations $(1),\ (2),\ (4)^2$. Thus, the variables x_{15}, x_{23}, x_{25} of weights 2 are the local parameters in the neighbourhood of P_1. Exactly the same computation determines the type of P_2 and P_3. We proved that P_1, P_2, P_3 are of type

$$_2\Big(\frac{1}{3}(2,2,2)\Big) = {}_1\Big(\frac{1}{3}(1,1,1)\Big).$$

From $h^0(X,D) = 1$ and $h^0(X,2D) = 6$ we calculate

$$D^3 = \frac{5}{4} \text{ and } D \cdot c_2(X) = 24.$$

Then by Corollary 3.1.1

$$h^0(X, mD) = \frac{5}{24}m^3 + 2m + \alpha_m + 3 \cdot \beta_m,$$

where

$$\alpha_m = \begin{cases} -\frac{1}{4} \cdot m \cdot \frac{7}{2} & \text{if } m = 1 \bmod 2, \\ 0 & \text{if } m = 0 \bmod 2, \end{cases}$$

$$\beta_m = \begin{cases} \frac{1}{3} \sum_{\varepsilon \in \mu_3} \frac{\varepsilon^{-1}-1}{(1-\varepsilon^{-1})^3} = -\frac{1}{9} & \text{if } m = 1 \bmod 3, \\ \frac{1}{3} \sum_{\varepsilon \in \mu_3} \frac{\varepsilon^{-2}-1}{(1-\varepsilon^{-1})^3} = \frac{1}{9} & \text{if } m = 1 \bmod 3, \\ 0 & \text{if } m = 0 \bmod 3. \end{cases}$$

Here α_m are the contributions from the singularities along C and β_m are the basket contributions from P_1, P_2, P_3.

It is easy to check that this yields the Hilbert function $P_X(t)$.

Example 4.4.3 For the input data we choose

- $h^0(X, D) = 2$ and $h^0(X, 2D) = 7$
- Points: $_3\big(\frac{1}{4}(2,3,3)\big)$
- Curves: type $= \frac{1}{2}(1,1)$, deg $= \frac{7}{4}$, index $= 2$.

The output we get is

- Codimension: 5

- Weights: 1,1,2,2,2,2,3,3,4
- Multiplied out Hilbert polynomial: $1-3t^4-4t^5-t^6+6t^7+6t^8+2t^9-2t^{11}-6t^{12}-6t^{13}+t^{14}+4t^{15}+3t^{16}-t^{20}$

The output suggests the following interesting example. Consider a general quasilinear section

$$X = \mathrm{wOGr}(5,10) \cap (2)^2 \cap (3)^4 \cap (4).$$

By the adjunction formula for $(2)^2 \cap (3)^4 \cap (4) \subset \mathrm{wOGr}(5,10)$ we get $K_X = \mathcal{O}(-20+2\times 2+4\times 3+4) \sim 0$. Thus X is a Calabi–Yau .

Next we will analyse the cyclic quotient singularities of X which are induced by the weights of $\mathbb{P}$. Consider the equations involving weights of degree 3. They define a cubic intersected with two hyperplanes.This defines an empty set, thus there are no $\frac{1}{3}$ singularities.

There is only one equation of degree 4, which is linear in x_1 and x_2. Therefore X contains a singular point P of index $\frac{1}{4}$. Note that P lies on a curve C determined by the equations

$$\begin{array}{lrcl} N_3: & x_{14}x_{25}-x_{15}x_{24} & = & 0, \\ N_4: & -x_{13}x_{25}+x_{15}x_{23} & = & 0, \\ N_5: & x_{13}x_{24}-x_{14}x_{23} & = & 0, \\ & & & \\ N_{-3}: & -x_1x_{13}-x_2x_{23} & = & 0, \\ N_{-4}: & -x_1x_{14}-x_2x_{24} & = & 0, \\ N_{-5}: & -x_1x_{15}-x_2x_{25} & = & 0, \end{array} \quad \text{and} \quad (2)^2 \cap (4): \ l_1 = l_2 = q = 0,$$

where l_1, l_2 are linear equations of degree 2, and q is a degree 4 equation that is linear in x_1, x_2 and quadratic in $x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}$.

Next we determine the type of P. Assume that $x_1 \neq 0$ at P. By the implicit function theorem we can express x, x_{12} out of equations N_1, N_{-2} and x_{13}, x_{14}, x_{15} out of N_{-3}, N_{-4}, N_{-5}. Moreover, we can eliminate x_{23}, x_{24} from the equations $(2)^2: \ l_1 = l_2 = 0$ and x_3, x_4, x_5, x_{34} from the four equations $(3)^4$. The remaining variables are x_{25}, x_{35}, x_{45} of degrees $2,3,3$ respectively, which are the local parameters for P. This proves that P is of type

$$_3\Big(\frac{1}{4}(2,3,3)\Big) = {}_1\Big(\frac{1}{4}(2,1,1)\Big).$$

Thus C is an example of a dissident $\frac{1}{2}(1,1)$ curve.

In the last step we compute $\deg D|_C$. As before, let $D = \mathcal{O}_X(1)$. Then $4D$ is Cartier along C. The intersection number $C \cap |4D|$ is determined by (E. xvi) on page 107 and therefore equals 7. Thus

$$\deg D|_C = \frac{7}{4}.$$

From $h^0(X, D) = 2,\ h^0(X, 2D) = 7$ and Corollary 3.1.1 we get

$$h^0(X, mD) = \frac{5}{16}m^3 + \frac{9}{4}m + \alpha_m + \beta_m,$$

where

$$\alpha_m = \begin{cases} -\frac{1}{4} \cdot m \cdot \frac{7}{4} & \text{if } m = 1 \bmod 2, \\ 0 & \text{if } m = 0 \bmod 2, \end{cases}$$

$$\beta_m = \begin{cases} \frac{1}{4} \sum_{\varepsilon \in \mu_4 - \{1\},\ \varepsilon^2 \neq 1} \frac{\varepsilon^{-1} - 1}{(1-\varepsilon^{-1})^2(1-\varepsilon^{-2})} = -\frac{1}{8} & \text{if } m = 1 \bmod 4, \\ \frac{1}{4} \sum_{\varepsilon \in \mu_4 - \{1\},\ \varepsilon^2 \neq 1} \frac{\varepsilon^{-2} - 1}{(1-\varepsilon^{-1})^2(1-\varepsilon^{-2})} = 0 & \text{if } m = 2 \bmod 4, \\ \frac{1}{4} \sum_{\varepsilon \in \mu_4 - \{1\},\ \varepsilon^2 \neq 1} \frac{\varepsilon^{-3} - 1}{(1-\varepsilon^{-1})^2(1-\varepsilon^{-2})} = \frac{1}{8} & \text{if } m = 3 \bmod 4, \\ 0 & \text{if } m = 0 \bmod 4. \end{cases}$$

Here α_m are contributions from the transverse singularities along C and β_m are the basket contributions from the dissident point $P \in C$.

This yields the expected Hilbert function.

5 Outstanding questions

Question 1 How precisely does the input data for the Riemann–Roch formula determine the output model?

The basket contributions to RR coming from different singular points can cancel out. Consider for example ${}_1\big(\frac{1}{3}(1,1,1)\big)$ and ${}_2\big(\frac{1}{3}(1,1,1)\big)$. On the other hand, a ${}_k\big(\frac{1}{r}(1,-1)\big)$ curve C always contributes the term

$$m\frac{\overline{mk}(r-\overline{mk})}{2r}\deg D|_C$$

which is linear in m. Since $\deg D|_C$ is positive, the contributions from curves can not cancel. Note also that

$$f(m) := \overline{mk}(r-\overline{mk}) \quad \text{and} \quad g(m) := \overline{mk}(r-\overline{mk})(r-2\cdot\overline{mk})$$

are both periodic mod r and $f(m) = f(-m)$, $g(m) = -g(-m)$.

Question 2 How can we compute the Hodge numbers for the Calabi–Yaus in the thesis?

There exist many papers on the Hodge numbers of Calabi–Yau complete intersections in toric varieties. The essence of all proofs are [10] and [15], which compute the Hodge numbers of hypersurfaces in a torus. Concrete formulas and algorithms can be found in [9] and [18, Formula 4.21.], and some interesting examples are in [22], [20].

In Section 4.3 we compute the Hodge numbers of some Calabi–Yau quasilinear sections in $\mathrm{wGr}(2,5)$ by using the Type I unprojection. This method can not be used in general as we encounter worse singularities (Lemma 4.3.8).

Can we compute the Milnor fibre of the singularities that are worse than nodes? Can we use different types of unprojections to avoid the bad singularities?

Question 3 An interesting problem would be to find the mirror pairs for the Calabi–Yaus constructed in this thesis.

It is often possible to construct new Calabi–Yau threefolds from old ones by degenerating the complex structure and then resolving the singularities. In Section 4.2 we explain how this can be done by using the method of Type I unprojection. As suggested in [24], the method of projection–unprojection should also be used to produce new mirrors from old. The mirrors are known for complete intersections in weighted projective spaces [18] or straight Grassmannians [5].

Question 4 Do there exist birational maps between Examples 4.2.3 and 4.2.4, and the Calabi–Yau complete intersections in toric ambient varieties with the same Hodge numbers?

Question 5 In Proposition 4.1.1 we classified all possible singularities of quasismooth well formed Calabi–Yau complete intersections. The same result holds for a codimension 3 Calabi–Yau, which is constructed from a codimension 2 c.i. by the Type I unprojection.

Can we determine all possible quotient singularities of Calabi–Yaus in weighted projective spaces?

Question 6 For an A_{r-1} curve C in a quasismooth and well formed threefold X there exists a unique crepant resolution (Lemma 2.2.2). It is a sequence of $\lfloor \frac{r}{2} \rfloor$ blow–ups and induces $r-1$ different exceptional surfaces.

If X is not quasismooth and well formed the exceptional surfaces can form a different configuration. It can be proved that another configuration exists only when r is even. Moreover, it consists of $\frac{r}{2}$ exceptional surfaces; one is ruled over C and the rest are ruled over a double cover of C.

The interesting fact is that the Riemann-Roch formula remains the same.

Can we use RR to construct smooth 3–folds with more complicated configurations of surfaces [33]?

Question 7 What is the general orbifold Riemann–Roch formula? Is it possible to state and prove the formula without using the Atiyah–Singer equivariant RR?

Appendix

A.1 The Riemann–Hurwitz formula

Lemma A.1.1 *Let $f\colon S \longrightarrow T$ be a finite separable morphism of smooth threefolds. Then*

$$K_S \sim f^*K_T + R,$$

where R is the ramification divisor on S.

PROOF There exists an exact sequence

$$f^*\Omega_T \longrightarrow \Omega_S \longrightarrow \Omega_{S/T} \longrightarrow 0. \tag{E. xvii}$$

Since S and T are smooth, Ω_S and $f^*\Omega_T$ are locally free sheaves of rank 3. $\Omega_{S/T}$ is a torsion sheaf that measures the difference between Ω_S and Ω_T. It can be locally defined as $\operatorname{div}(J)$, where

$$J = \frac{\partial(s_1, s_2, s_3)}{\partial(t_1, t_2, t_3)}$$

is determinant of the Jacobian matrix. Therefore the support of $\Omega_{S/T}$ equals to the ramification divisor R on S. In other words, considering R as a closed subscheme of S, its structure sheaf $\mathcal{O}_R$ is isomorphic to $\Omega_{S/T}$.

We claim that the sequence (E. xvii) is short exact. Since the sheaves $f^*\Omega_T$ and Ω_S are torsion free and have the same rank, we only need to prove that the map $f^*\Omega_T \longrightarrow \Omega_S$ is surjective at the generic point of S. But since $k(S)$ is separable over $k(T)$, the sheaf $\Omega_{S/T}$ is zero at the generic point of S by II, 8.6.A in [17].

Take the highest exterior powers in the exact sequence

$$0 \longrightarrow f^*\Omega_T \longrightarrow \Omega_S \longrightarrow \mathcal{O}_R \longrightarrow 0$$

and since f^* commutes with $\bigwedge$, we get

$$\omega_S \cong f^*\omega_T \otimes \mathcal{O}_R.$$

Moreover, $K_S \sim f^*K_T + R$, which ends the proof.

A.2 Orbifolds

Definition A.2.1 A variety X of dimension n is an *orbifold* if every point has a neighborhood analytically equivalent to $0 \in U/G$, where $G \subset \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup with no complex reflections other than identity and $U \subset \mathbb{C}^n$ is a $G-$stable neighborhood of the origin.

The condition that G has no complex reflections is equivalent to X being quasismooth.

Definition A.2.2 Let

$$X = \begin{pmatrix} U_\lambda \\ \downarrow \\ U_\lambda/G_\lambda \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} V_\xi \\ \downarrow \\ V_\xi/H_\xi \end{pmatrix}$$

be orbifolds. An *orbifold morphism* is a morphism of varieties $f \colon X \to Y$ that (after refining U_λ/G_λ if necessary) lifts to

$$\begin{array}{lrcll} & f_\# \colon G_\lambda & \longrightarrow & H_\xi & \text{a homomorphism of groups} \\ \text{and} & & & & \\ & f_{\lambda\xi} \colon U_\lambda & \longrightarrow & V_\xi & \qquad\qquad\text{a map of covers} \end{array}$$

so that for all $g \in G_\lambda$ and $u \in U_\lambda$

$$f_{\lambda\xi}(gu) = f_\#(g)(f_{\lambda\xi}(u)).$$

The main "rule" in the theory of orbifolds is that over $\mathbb{Q}$ they behave like manifolds. The above definitions show that the theories which only depend on the local analytic nature of varieties can be developed on the smooth cover U, and then suitably adjusted under the finite map $U \to U/G$. In all cases that we consider, G will be isomorphic to μ_r for some r.

For example, we can define intersection theory, Hodge theory etc.

There are two equivalent ways of defining ω_X on an orbifold X :

1. $\omega_X = j_*(\Omega^n_{X^0})$, where $j \colon X^0 \hookrightarrow X$ is the inclusion of the smooth locus of X.

2. The sections on a local chart U/G are defined to be the $G-$invariant sections on U, which means $\Gamma(U/G, \omega_X) = \Gamma(U, \omega_U)^G$. There is a natural way to define the sheaf by glueing together different charts.

Since orbifolds are Cohen-Macaulay, ω_X is the dualising sheaf of X.

We restrict ourselves to Gorenstein orbifolds which have at worst canonical singularities.

Remark A.2.3 A morphism of varieties $f\colon X \to Y$, where X is smooth and Y an orbifold, does not automatically lift to a morphism of orbifolds. For example consider the inclusion

$$\begin{array}{ccc} & & \mathbb{C}^2 = \text{Spec } \mathbb{C}[x,y] \\ & & \downarrow \\ \mathbb{C} = \text{Spec } \mathbb{C}[z] & \longrightarrow & \mathbb{C}^2/\mu_2 = \text{Spec } \mathbb{C}[u,v,w]/(uw = v^2) \end{array}$$

defined by

$$\begin{array}{ccc} \mathbb{C}[u,v,w]/(uw=v^2) & \longrightarrow & \mathbb{C}[z] \\ u & \mapsto & z \\ v & \mapsto & 0 \\ w & \mapsto & 0. \end{array}$$

The quotient by μ_2 is given by $u = x^2, v = xy, w = y^2$. The diagram lifts to

$$\text{Spec } \mathbb{C}[z] \otimes_{\mathbb{C}[u,v,w]/(uw=v^2)} \mathbb{C}[x,y] \cong \text{Spec } \mathbb{C}[x,y]/(xy, y^2)$$

which is not smooth.

A.3 The Hodge numbers and mirror pairs

Topological field theories for Calabi–Yau threefolds predict the following property of mirror manifolds:
If two smooth Calabi–Yau threefolds V and V^* form a *mirror pair*, then their Hodge numbers must satisfy the relations

$$h^{p,q}(V) = h^{3-p,q}(V^*), \quad \text{for} \;\; 0 \leq p,q \leq 3.$$

In particular, for the Euler numbers we get

$$e(V) = -e(V^*).$$

Since V is a Calabi–Yau threefold,

$$\begin{array}{rcl} h^{1,0} & = & h^1(V, \mathcal{O}_V) = 0, \\ h^{2,0} & = & h^2(V, \mathcal{O}_V) = 0, \\ h^{3,0} & = & h^{0,3} = h^0(V, \Omega^3_V) = h^0(V, \mathcal{O}_V) = 1. \end{array}$$

Thus we only need to consider $h^{1,1}, h^{1,2}$ and the Euler number

$$e = 2(h^{1,1} - h^{1,2}).$$

It is well known that $\dim \operatorname{Pic}_{\mathbb{Q}} V = \dim H^2(V, \mathbb{Q}) = h^{1,1}(V)$.

By [10], the cohomology with compact support $H_c^k(X, \mathbb{Q})$ of a quasismooth projective variety X carries a natural Hodge structure which coincides with the classical one. This implies that

- $e(X) = \sum_{i \in I} e(X_i)$, if X is a finite disjoint union of locally closed subvarieties $\{X_i\}_{i \in I}$;
- $e(X) = e(Y) \times e(F)$ for a locally trivial bundle $f\colon X \to Y$ with fibre F.

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