

# ELEMENTARY TRANSFORMATIONS OF PFAFFIAN REPRESENTATIONS OF PLANE CURVES

ANITA BUCKLEY

FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, SLOVENIA

ABSTRACT. Let  $C$  be a smooth curve in  $\mathbb{P}^2$  given by an equation  $F = 0$  of degree  $d$ . In this paper we consider elementary transformations of linear pfaffian representations of  $C$ . Elementary transformations can be interpreted as actions on a rank 2 vector bundle on  $C$  with canonical determinant and no sections, which corresponds to the cokernel of a pfaffian representation. Every two pfaffian representations of  $C$  can be bridged by a finite sequence of elementary transformations. Pfaffian representations and elementary transformations are constructed explicitly. For a smooth quartic, applications to Aronhold bundles and theta characteristics are given.

## 1. INTRODUCTION

Let  $k$  be an algebraically closed field and  $C$  a nonsingular curve defined by an irreducible polynomial  $F(x_0, x_1, x_2)$  of degree  $d$  in  $\mathbb{P}^2$ . A *linear pfaffian representation* of  $C$  is a  $2d \times 2d$  skew-symmetric matrix

$$A = \begin{bmatrix} 0 & L_{12} & L_{13} & \cdots & L_{12d} \\ -L_{12} & 0 & L_{23} & \cdots & L_{22d} \\ -L_{13} & -L_{23} & 0 & & \\ \vdots & \vdots & & \ddots & \vdots \\ -L_{12d} & -L_{22d} & & \cdots & 0 \end{bmatrix}$$

with linear forms  $L_{ij} = a_{ij}^0 x_0 + a_{ij}^1 x_1 + a_{ij}^2 x_2$  such that

$$\text{Pf } A(x_0, x_1, x_2) = cF(x_0, x_1, x_2) \text{ for some } c \in k, c \neq 0.$$

Its cokernel is a rank 2 vector bundle on  $C$ . Throughout the paper we equate the notion of vector bundles and locally free sheaves. In particular, we equate line bundles with invertible sheaves and denote their inverse by  $^{-1}$ .

Two pfaffian representations  $A$  and  $A'$  are *equivalent* if there exists  $X \in \text{GL}_{2d}(k)$  such that

$$A' = XAX^t.$$

There is a one to one correspondence between linear pfaffian representations (up to equivalence) of  $C$  and rank 2 vector bundles (up to isomorphism) on  $C$  with certain properties. This well known result is summed up in the following theorem of Beauville [3, Corollary 2.4].

**Theorem 1.1.** *Let  $C$  be a smooth plane curve defined by a polynomial  $F$  of degree  $d$  and let  $\mathcal{E}$  be a rank 2 bundle on  $C$  with determinant  $\mathcal{O}_C(d-1)$  and  $H^0(C, \mathcal{E}(-1)) = 0$ .*

---

1991 *Mathematics Subject Classification.* 15B57; 14Q05; 14H60.

*Key words and phrases.* Pfaffian representations, elementary transformations, linear matrices, polynomials in projective plane, vector bundles,

Then there exists a  $2d \times 2d$  skew-symmetric linear matrix  $A$  with  $\text{Pf } A = F$  and an exact sequence

$$(1) \quad 0 \rightarrow \bigoplus_{i=1}^{2d} \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{A} \bigoplus_{i=1}^{2d} \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{E} \rightarrow 0.$$

Conversely, let  $A$  be a linear skew-symmetric  $2d \times 2d$  matrix with  $\text{Pf } A = F$ . Then its cokernel is a rank 2 bundle with  $\det \mathcal{E} \cong \mathcal{O}_C(d-1)$  and  $H^0(C, \mathcal{E}(-1)) = 0$ .

The condition  $H^0(C, \mathcal{E}(-1)) = 0$  implies that  $\mathcal{E}(-1)$  and  $\mathcal{E}$  are semistable.

Using this, in [5] all linear pfaffian representations of  $C$  (up to equivalence) were found and related to the moduli space  $M_C(2, \omega_C)$  of semistable rank 2 vector bundles on  $C$  with canonical determinant. In particular, pfaffian representations of  $C$  can be parametrised by the open set  $M_C(2, \omega_C) - \{\mathcal{K} : h^0(C, \mathcal{K}) > 0\}$ . The standard notation for  $\{\mathcal{K} : h^0(C, \mathcal{K}) > 0\}$  is  $\Theta_{2, \omega_C}$ , since Drezet and Narasimhan [9] proved that it is a Cartier divisor in the moduli space. The properties of the moduli space were extensively studied in [20], [24] and [23]; for example it is an irreducible, normal projective variety and for  $C$  of genus  $g \geq 2$  it has dimension  $3(g-1)$ .

Study of pfaffian representations is strongly related to and motivated by determinantal representations. A linear determinantal representation of  $C$  is a  $d \times d$  matrix  $M(x_0, x_1, x_2)$  of linear forms such that

$$\det M(x_0, x_1, x_2) = cF(x_0, x_1, x_2) \text{ for some } c \in k, c \neq 0.$$

Determinantal representations  $M$  and  $M'$  are equivalent if there exists  $X, Y \in \text{GL}_d(k)$  such that

$$M' = XMY.$$

By [26] all linear determinantal representations of  $C$  (up to equivalence) can be parametrised by the open set in the Jacobian variety

$$\{\text{line bundle } \mathcal{L} : \deg \mathcal{L} = \frac{1}{2}d(d-3), h^0(C, \mathcal{L}) = 0\}.$$

Determinantal representations can be seen as a special case of pfaffian representations. Indeed, every determinantal representation  $M$  induces a decomposable pfaffian representation

$$(2) \quad \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix}.$$

The corresponding cokernel equals  $\text{Coker } M \oplus (\text{Coker}^{-1} M \otimes \mathcal{O}(d-1))$  as described in [5] and [7]. In the moduli space, decomposable pfaffian representations correspond to an open subset of the singular locus of  $M_C(2, \omega_C)$ . Note that the equivalence relation is also well defined since

$$\begin{bmatrix} 0 & XMY \\ -(XMY)^t & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & Y^t \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} X^t & 0 \\ 0 & Y \end{bmatrix}.$$

Elementary transformations of determinantal representations were introduced by M. S. Livsic and Kravitsky in operator theory. Vinnikov et al generalised these ideas in [25], [2] using notions of vessels and Cauchy kernels. An explicit and complete description of elementary transformations of determinantal representations can be found in [27]. Elementary transformations of vector bundles are due to Maruyama [18]. For a modern exposition and proofs we refer to Abe [1]. The most traditional example of elementary transformations is that of ruled surfaces (i.e., of rank 2 bundles). A good reference on ruled surfaces is [12, Chapter V. 2].

Throughout the paper we will be using the following properties of the wedge product. It is well known that  $\bigwedge^2 k^{2d}$  can be identified with  $2d \times 2d$  skew-symmetric matrices. Let  $\Omega_n$  denote the set of vectors  $\sum_{i=1}^n w_i \wedge w'_i$  in  $\bigwedge^2 k^{2d}$ , where  $\dim\{w_1, \dots, w_n, w'_1, \dots, w'_n\} = 2n$ . Elements of  $\Omega_n$  are said to have *irreducible length*  $n$  since they can be written as a sum of  $n$  and not less than  $n$  pure nonzero products. In [17] it is shown that  $\Omega_n$  is isomorphic to the set of all rank  $2n$  skew-symmetric matrices. The isomorphism equals

$$(3) \quad \sum_{i=1}^{2d} \alpha_i e_i \wedge \sum_{j=1}^{2d} \beta_j e_j \mapsto \sum_{i,j=1}^{2d} (\alpha_i \beta_j - \alpha_j \beta_i) (E_{ij} - E_{ji}),$$

where  $\{e_1, \dots, e_{2d}\}$  is the standard basis for  $k^{2d}$  and  $\{E_{ij}\}$  is the standard basis for  $2d \times 2d$  matrices. Note that these are the Plücker coordinates in  $\text{Gr}(2, 2d)$ .

A brief outline of the paper is the following. In section 2 we present pfaffian representations in canonical forms. This enables us to write an algorithm which computes all the pfaffian representations of  $C$  and thus gives a description of  $M_C(2, \omega_C)$ . For suitable choices of vectors in the cokernel bundles discussed in Section 3, we define elementary transformations of pfaffian representations in Section 4. In Section 6 elementary transformations of pfaffian representations are related to elementary transformations of vector bundles. The main Theorem 7.3 in Section 7 proves that any two pfaffian representations can be bridged by a finite sequence of elementary transformations of Type I and II. In other words, we can explicitly construct all pfaffian representations of  $C$  from a given one (for example, from a decomposable representation with symmetric blocks induced by one of the  $2^{g-1}(2^g + 1)$  even theta characteristics). Section 8 is an exposition on plane quartics. Concrete examples and algorithms for computing Aronhold bundles and theta characteristics are given.

## 2. THE CANONICAL FORM

Canonical forms play an important role in explicit descriptions of the moduli space  $M_C(2, \omega_C)$ . In the sequel we outline an algorithm for such computation.

Let  $A = x_0 A_0 + x_1 A_1 + x_2 A_2$  be a pfaffian representation of  $C$ . We can always assume that after a projective change of coordinates  $C$  intersects the line  $L : x_0 = 0$  in distinct points  $P_1 = (0, p_1, 1), \dots, P_d = (0, p_d, 1)$ . We will prove that  $A$  is equivalent to a representation in the canonical form.

**Proposition 2.1.** *For every pfaffian representation  $A$  of  $C$  there exists a basis of  $k^{2d}$  in which  $A$  has the canonical form*

$$(4) \quad A = x_1 \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} - x_2 \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & D_d \end{bmatrix} + x_0 A_0,$$

where

$$I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} 0 & p_i \\ -p_i & 0 \end{bmatrix}.$$

*Proof.* As above, let  $P_i = (0, p_i, 1)$  be  $d$  distinct points in the intersection of  $F(x_0, x_1, x_2) = 0$  with the line  $L : x_0 = 0$ . By restricting to  $L$ , we obtain the pencil of skew-symmetric matrices

$$A_L = x_1 A_1 + x_2 A_2$$

with  $\text{Pf } A_L = F|_L = F(0, x_1, x_2) = \prod_{i=1}^d (x_1 - p_i x_2)$ . Since all the matrices in the representation are skew-symmetric,  $\text{Coker } A$  and  $\text{Ker } A$  define the same rank 2 vector bundle  $\mathcal{E}$ . For  $\mathcal{E}$  and the point  $P_i$ , let  $\mathcal{E}(P_i)$  denote the residued vector space, which means the kernel of  $p_i A_1 + A_2$ . Thus  $\mathcal{E}(P_i)$ ,  $i = 1, \dots, d$  are 2-dimensional subspaces in  $k^{2d}$ . In the proof of [5, Proposition 3.4] we showed that  $h^0(C, \mathcal{E}(-1)) = 0$  implies the following important fact: the union of bases of the vector spaces  $\mathcal{E}(P_i)$  span the whole space  $k^{2d}$ . In this basis  $A_L$  is equivalent to

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_d \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} 0 & x_1 - p_i x_2 \\ -x_1 + p_i x_2 & 0 \end{bmatrix}.$$

No  $2 \times 2$  block  $J_i$  is identically equal to 0, otherwise the rank of  $A$  at  $P_i$  would be at most  $2d - 4$ .  $\square$

Let  $A$  be a representation of  $C$  and  $\mathcal{E}$  its cokernel. Choose the basis of  $k^{2d}$

$$\cup_{i=1}^d \{2\text{-dimensional basis of } \mathcal{E}(P_i)\}.$$

and denote by  $P \in \text{GL}(2d, k)$  the change of basis matrix. Then  $P$  acts on the representation by  $A \mapsto P A P^t$ . Thus  $A$  is equivalent to a representation in the canonical form by Proposition 2.1.

In [5] we established a one to one correspondence between the pfaffian representations (up to equivalence) of  $C$  and the open set

$$(5) \quad M_C(2, \omega_C) - \{\mathcal{K} : h^0(C, \mathcal{K}) > 0\}.$$

It thus suffices to find all pfaffian representations of  $C$  in the canonical form, which will yield a set of equations describing the open subset (5) in the moduli space.

There are  $d(2d - 1)$  parameters in the representation (4), namely the entries of the  $2d \times 2d$  skew-symmetric matrix  $A_0 = [a_{ij}^0]$ . Since  $\text{Pf } A$  equals  $F$ , we get  $\frac{1}{2}d(d+1)$  relations among  $a_{ij}^0$ . Indeed, every monomial  $x_0^\alpha x_1^\beta x_2^\gamma$ ,  $\alpha \neq 0$ ,  $\alpha + \beta + \gamma = d$  in  $F$  gives one equation. By the implicit function theorem we are left with  $d(2d-1) - \frac{1}{2}d(d+1) = \frac{3}{2}d(d-1)$  parameters  $a_{ij}^0$ . Recall that pfaffian representations are equivalent under the action

$$A \mapsto R A R^t, \quad R \in \text{GL}(2d, k).$$

By a suitable  $R$  we can further reduce the number of parameters in  $A_0$ . Of course we only consider  $R$  whose action preserves the canonical form. This way we will reduce the number of equivalent representations in each equivalence class to 1. The following lemma is an elementary exercise in linear algebra.

**Lemma 2.2.** *The action  $A \mapsto R A R^t$  preserves the canonical form of the first two matrices in the representation if and only if  $R$  equals*

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & R_d \end{bmatrix},$$

where every  $R_i$  is an invertible  $2 \times 2$  matrix with determinant 1.

Matrix  $R$  in Lemma 2.2 acts on the third matrix in the representation by

$$R A_0 R^t = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & R_i \begin{bmatrix} a_{2i-1,2j-1}^0 & a_{2i-1,2j}^0 \\ a_{2i,2j-1}^0 & a_{2i,2j}^0 \end{bmatrix} R_j^t & & & \\ & & & \ddots & & \\ & & & & & 0 \end{bmatrix}, \quad i, j = 1, \dots, d.$$

In other words, view  $A_0$  as  $d \times d$  matrix of  $2 \times 2$  blocks. Then  $R$  acts on  $i, j$ -th block by  $* \mapsto R_i * R_j^t$ . In every  $R_i$  we have a choice of 3 independent parameters since its determinant is 1. Thus each  $R_i$  reduces the number of  $a_{ij}^0$ 's by 3.

For a general  $F(x_0, x_1, x_2)$  this sequence of reductions can be performed explicitly. We are left with

$$\begin{aligned} 3(g-1) &= \frac{3}{2}d(d-3) = \\ d(2d-1) & \text{ [number of parameters } a_{ij}^0 \text{ in } A \text{ in the canonical form]} - \\ \frac{1}{2}d(d+1) & \text{ [relations since Pf } A = F] - \\ 3d & \text{ [parameters reduced by the action of equivalence relation } R \cdot A \cdot R^t] \end{aligned}$$

independent variables  $a_{ij}^0$ . As expected, this represents the open subset (5) of the moduli space in  $\mathbb{P}^{d(2d-1)-1}$ .

**Remark 2.3.** In the above considerations we could take any other canonical form of the matrix pair  $A_1, A_2$ . For example, the equivalence relation action  $Q A Q^t$  of

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 1 & -1 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}$$

brings the first two matrices in (4) into

$$(6) \quad \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

where  $D$  is the diagonal matrix  $\{p_1, \dots, p_d\}$ . This canonical form is particularly useful since it naturally includes all the decomposable representations (2). The same canonical form was obtained in [13], where canonical forms for matrix pairs were classified purely by the methods of linear algebra.

### 3. TANGENTS

In this section we explain how tangents or lines through two points  $\lambda, \mu$  on  $C$  can be read from pfaffian representations. Moreover, for any pfaffian representation  $A$  we relate vectors in  $\text{Coker } A(\lambda)$  and  $\text{Coker } A(\mu)$ .

Since our aim is to do concrete calculations, we describe the correspondence in Theorem 1.1 explicitly: Denote by  $\text{Pf}^{ij} A$  the pfaffian of the  $(2d-2) \times (2d-2)$  skew-symmetric matrix obtained by removing the  $i$ th and  $j$ th rows and columns

from  $A$ . It is easy to prove the following analogue of the Jacobi's formula for the derivative of determinants:

$$(7) \quad \frac{\partial F(x_0, x_1, x_2)}{\partial x_k} = \sum_{i,j} a_{ij}^k (-1)^{i+j} \text{Pf}^{ij} A(x_0, x_1, x_2).$$

If for some  $x = (x_0, x_1, x_2) \in C$  all  $2d - 2$  pfaffian minors vanish, then  $x$  must be a singular point of  $F$ . By our assumption  $F$  is smooth, thus  $\text{rank } A(x) \geq 2d - 2$  for all  $x \in C$ . Rank of skew-symmetric matrices is even and  $\det A = F^2 = 0$ , thus equality holds. Therefore  $\text{Coker } A(x)$  defines a rank 2 bundle over  $C$ . Define the *pfaffian adjoint* of  $A$  to be the skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & (-1)^{i+j} \text{Pf}^{ij} A & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix},$$

with  $(-1)^{i+j} \text{Pf}^{ij} A$  being its  $(i, j)$ th entry for  $1 \leq i < j \leq 2d$ . Again, by analogy with determinants, the following holds

$$(8) \quad \tilde{A} A = \text{Pf} A \text{Id}_{2d}.$$

More properties and linear algebra of pfaffians can be found in [11, Appendix D].

Since  $\text{Pf} A = cF$ , the cokernel can be obtained from  $\tilde{A}$  by using (8). Indeed, for any point  $x \in C$  every column (or row) of  $\tilde{A}(x)$  is in  $\text{Coker } A(x)$ . Since  $A(x)$  is a skew-symmetric matrix with linear entries, we have

$$\text{Coker } A(x) \cong \text{Ker } A(x)$$

which we denote by  $\mathcal{E}(x)$ . We will often view  $\mathcal{E}$  as a  $\mathbb{P}^1$ -bundle, or equivalently as a ruled surface  $\mathbb{P}\mathcal{E}$ .

**Lemma 3.1.** *Every representation of  $C$  yields tangents and lines through  $\lambda, \mu \in C$  in the following way:*

- Let  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$  be independent vectors. Then  $u_\mu^t A(x) v_\lambda$  is either identically 0 or defines a line through  $\lambda$  and  $\mu$ .
- Let  $u_\lambda, v_\lambda \in \mathcal{E}(\lambda)$  be independent vectors. Then  $u_\lambda^t A(x) v_\lambda = 0$  is an equation of the tangent line at  $\lambda \in C$ .

*Proof.* The first assertion is obvious as  $u_\mu^t A(x) v_\lambda$  is linear and equals 0 at the points  $\lambda = (\lambda_0, \lambda_1, \lambda_2)$  and  $\mu = (\mu_0, \mu_1, \mu_2)$ .

The linear span  $\mathcal{L}in\{u_\lambda, v_\lambda\}$  equals  $\mathcal{E}(\lambda)$ . By (8),  $\tilde{A}$  is a rank 2 skew-symmetric matrix at the points of  $C$ . Using the Plücker coordinates in (3),  $u_\lambda \wedge v_\lambda$  equals to a multiple of  $\tilde{A}(\lambda)$ . Then by (7)

$$\begin{aligned} \sum_{k=0}^2 x_k \frac{\partial F}{\partial x_k}(\lambda) &= \sum_{k=0}^2 x_k \sum_{i,j} a_{ij}^k (-1)^{i+j} \text{Pf}^{ij} A(\lambda) \\ &= \sum_{i,j} (a_{ij}^0 x_0 + a_{ij}^1 x_1 + a_{ij}^2 x_2) (-1)^{i+j} \text{Pf}^{ij} A(\lambda) \\ &= \text{Trace} [A(x) \cdot \tilde{A}(\lambda)], \end{aligned}$$

which equals to a nonzero multiple of  $\text{Trace} [A(x) \cdot (u_\lambda \wedge v_\lambda)] = 2 u_\lambda^t A(x) v_\lambda$ . In particular we proved that  $u_\lambda^t A(x) v_\lambda$  is not identically 0.  $\square$

$\mathcal{E}(x)$  can be viewed as pairs of independent vectors  $[u_x, v_x]$  modulo right-hand-side multiplications by invertible  $2 \times 2$  matrices. Then  $T, T' \in \text{GL}(2, k)$  can be chosen so that

$$T [u_\lambda, v_\lambda]^t A(x) [u_\mu, v_\mu] T' = T \begin{bmatrix} u_\lambda^t A(x) u_\mu & u_\lambda^t A(x) v_\mu \\ v_\lambda^t A(x) u_\mu & v_\lambda^t A(x) v_\mu \end{bmatrix} T'$$

equals either

$$L(x) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L(x) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad L(x) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where  $L(x)$  is the defining equation of the line through  $\lambda$  and  $\mu$ . This enables us to relate points on  $C$ . A pair of points  $\lambda, \mu$  on  $C$  with respect to representation  $A(x)$  is either:

- *inadmissible* if  $u_\mu^t A(x) v_\lambda \equiv 0$  on  $\mathbb{P}^2$  for all  $v_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ ,  $u_\mu \in \mathbb{P}\mathcal{E}(\mu)$ ;
- *semiadmissible* if there exists a unique pair of vectors  $u_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ ,  $u_\mu \in \mathbb{P}\mathcal{E}(\mu)$  such that  $w_\mu^t A(x) u_\lambda \equiv 0$  and  $u_\mu^t A(x) w_\lambda \equiv 0$  for all  $w_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ ,  $w_\mu \in \mathbb{P}\mathcal{E}(\mu)$ ;
- *admissible* if for every  $v_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$  there exists exactly one  $v_\mu \in \mathbb{P}\mathcal{E}(\mu)$  for which  $v_\mu^t A(x) v_\lambda \equiv 0$  on the whole  $\mathbb{P}^2$ .

For admissible  $\lambda, \mu$  we thus obtain a one to one correspondence between vectors in  $\mathbb{P}\mathcal{E}(\lambda)$  and  $\mathbb{P}\mathcal{E}(\mu)$ . Namely,  $v_\lambda$  corresponds to  $v_\mu$  iff  $v_\mu^t A(x) v_\lambda \equiv 0$ . Consider next a semiadmissible pair  $\lambda, \mu \in C$  and  $v_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ ,  $v_\mu \in \mathbb{P}\mathcal{E}(\mu)$ . We claim that  $v_\mu^t A(x) v_\lambda \equiv 0$  if and only if either  $v_\lambda = u_\lambda$  or  $v_\mu = u_\mu$ . Indeed, assume that  $v_\lambda \neq u_\lambda$ , then by linearity  $v_\mu^t A(x) w_\lambda \equiv 0$  for any  $w_\lambda \in \mathcal{L}in\{v_\lambda, u_\lambda\} = \mathcal{E}(\lambda)$ . Thus by uniqueness in the definition  $v_\mu = u_\mu$ .

In the proof of Lemma 3.1 we showed

**Corollary 3.2.** *A bitangent line through  $\lambda, \mu \in C$  is defined by either  $u_\mu^t A(x) v_\lambda$ ,  $u_\lambda^t A(x) v_\lambda$  or  $u_\mu^t A(x) v_\mu$ .*

**Corollary 3.3.** *Let  $u_\lambda, v_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ . Then  $u_\lambda^t A(x) v_\lambda \equiv 0$  if and only if  $u_\lambda = v_\lambda$ .*

This means that  $\lambda, \lambda$  can be always considered as an admissible pair of points on  $C$ .

#### 4. ELEMENTARY TRANSFORMATIONS

In this section we define *elementary transformations of pfaffian representations* of the curve  $C : F(x_0, x_1, x_2) = 0$  in  $\mathbb{P}^2$ , which is not necessarily smooth or irreducible. The standard notation of vessels [2] will be used.

Choose the coordinates of  $F$  so that the line  $\{x_0 = 0\}$  and the curve intersect in  $d$  distinct smooth points. For the sake of clearer notation we move to the affine plane. Consider a linear pfaffian representation

$$\text{Pf}(y_1 \sigma_2 - y_2 \sigma_1 + \gamma) = c f(y_1, y_2) \quad c \neq 0,$$

where  $y_1 = \frac{x_1}{x_0}$ ,  $y_2 = \frac{x_2}{x_0}$  are the affine coordinates  $f(y_1, y_2) = F(1, y_1, y_2)$  and  $\sigma_1, \sigma_2, \gamma$  are  $2d \times 2d$  skew-symmetric matrices. Denote

$$\text{Coker}(y_1 \sigma_2 - y_2 \sigma_1 + \gamma) \cong \text{Ker}(y_1 \sigma_2 - y_2 \sigma_1 + \gamma)$$

by  $\mathcal{E}(y_1, y_2)$ .

Pick distinct regular affine points  $\lambda = (\lambda_1, \lambda_2)$  and  $\mu = (\mu_1, \mu_2)$  on  $C$ . Then each  $\mathcal{E}(\lambda)$ ,  $\mathcal{E}(\mu)$  is a 2 dimensional vector space in  $k^{2d}$ . For all  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$

$$(9) \quad \begin{aligned} (\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)v_\lambda &= 0 \\ (\mu_1\sigma_2 - \mu_2\sigma_1 + \gamma)u_\mu &= 0 \end{aligned}$$

and

$$(10) \quad \begin{aligned} (y_1\sigma_2 - y_2\sigma_1 + \gamma)v_\lambda &= ((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1)v_\lambda \\ (y_1\sigma_2 - y_2\sigma_1 + \gamma)u_\mu &= ((y_1 - \mu_1)\sigma_2 - (y_2 - \mu_2)\sigma_1)u_\mu \end{aligned}$$

hold. Thus

$$(11) \quad v_\lambda^t(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_\mu = 0 \quad \text{and} \quad v_\lambda^t(\mu_1\sigma_2 - \mu_2\sigma_1 + \gamma)u_\mu = 0,$$

which implies

$$(\lambda_1 - \mu_1)v_\lambda^t\sigma_2u_\mu = (\lambda_2 - \mu_2)v_\lambda^t\sigma_1u_\mu.$$

In other words, for any pair of complex parameters  $t_1, t_2$ ,

$$(12) \quad \frac{1}{t_1(\lambda_1 - \mu_1) + t_2(\lambda_2 - \mu_2)} v_\lambda^t(t_1\sigma_1 + t_2\sigma_2)u_\mu$$

is constant whenever the denominator is nonzero. Denote this constant by  $K_{v_\lambda u_\mu}$ .

The pair of vectors  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$  is called *admissible* if  $K_{v_\lambda u_\mu}$  is not 0. Define

$$\begin{aligned} \tilde{\gamma} &= \gamma + \frac{1}{K_{v_\lambda u_\mu}} (\sigma_1(u_\mu v_\lambda^t + v_\lambda u_\mu^t)\sigma_2 - \sigma_2(u_\mu v_\lambda^t + v_\lambda u_\mu^t)\sigma_1) \\ &= \gamma + \frac{1}{K_{v_\lambda u_\mu}} (\sigma_1(u_\mu v_\lambda^t + v_\lambda u_\mu^t)\sigma_2 - (\sigma_1(u_\mu v_\lambda^t + v_\lambda u_\mu^t)\sigma_2)^t), \end{aligned}$$

and

$$\begin{aligned} \bar{\gamma} &= \gamma + 2\rho (\sigma_1 v_\lambda v_\lambda^t \sigma_2 - \sigma_2 v_\lambda v_\lambda^t \sigma_1) \\ &= \gamma + 2\rho (\sigma_1 v_\lambda v_\lambda^t \sigma_2 - (\sigma_1 v_\lambda v_\lambda^t \sigma_2)^t), \end{aligned}$$

where  $u_\mu v_\lambda^t$ ,  $v_\lambda u_\mu^t$  and  $v_\lambda v_\lambda^t$  are rank 1 matrices and  $\rho$  is an arbitrary constant. It is obvious that  $\tilde{\gamma} = -\tilde{\gamma}^t$  and  $\bar{\gamma} = -\bar{\gamma}^t$  are skew-symmetric.

In the sequel the following properties of the wedge product will be useful: for any skew-symmetric matrix  $\sigma$  and vectors  $w, v$  hold  $(\sigma w)^t = -w^t \sigma$ ,  $w^t \sigma v = -v^t \sigma w$  and  $w \wedge v = 2(wv^t - vw^t)$ .

Using the above, we can rewrite  $\tilde{\gamma}$  into

$$(13) \quad \tilde{\gamma} = \gamma - \frac{1}{2K_{v_\lambda u_\mu}} \sigma_1 u_\mu \wedge \sigma_2 v_\lambda + \frac{1}{2K_{v_\lambda u_\mu}} \sigma_2 u_\mu \wedge \sigma_1 v_\lambda$$

and  $\bar{\gamma}$  into

$$(14) \quad \bar{\gamma} = \gamma + \rho \sigma_2 v_\lambda \wedge \sigma_1 v_\lambda.$$

**Theorem 4.1.** *Let  $y_1\sigma_2 - y_2\sigma_1 + \gamma$  be a representation of  $C$  and  $\mathcal{E}(y_1, y_2)$  its cokernel. Choose  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$  such that  $K_{v_\lambda u_\mu} \neq 0$ . Then  $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$  and  $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$  are pfaffian representations of  $C$  since*

$$\text{Pf}(y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}) = \text{Pf}(y_1\sigma_2 - y_2\sigma_1 + \gamma) = \text{Pf}(y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}).$$



*Proof.* For any  $T$  and skew-symmetric  $S$  holds

$$\text{Pf}(TST^t) = \det T \text{Pf} S$$

by [11, Appendix D]. Recall also that for any two square matrices  $T', T''$

$$\det(\text{Id} + T'T'') = \det(\text{Id} + T''T').$$

Therefore proving the equality

$$(15) \quad \left( \text{Id} + \frac{(t_1\sigma_1 + t_2\sigma_2)u_\mu v_\lambda^t}{K_{v_\lambda u_\mu}(t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2))} \right) (y_1\sigma_2 - y_2\sigma_1 + \gamma) \left( \text{Id} - \frac{v_\lambda u_\mu^t(t_1\sigma_1 + t_2\sigma_2)}{K_{v_\lambda u_\mu}(t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2))} \right) \\ \parallel \\ \left( \text{Id} - \frac{(t_1\sigma_1 + t_2\sigma_2)v_\lambda u_\mu^t}{K_{v_\lambda u_\mu}(t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2))} \right) (y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}) \left( \text{Id} + \frac{u_\mu v_\lambda^t(t_1\sigma_1 + t_2\sigma_2)}{K_{v_\lambda u_\mu}(t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2))} \right)$$

will imply the first statement. (15) equals

$$y_1\sigma_2 - y_2\sigma_1 + \gamma + \frac{1}{2K_{v_\lambda u_\mu}}(y_1\sigma_2 - y_2\sigma_1 + \gamma)v_\lambda \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)}u_\mu = \\ y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma} - \frac{1}{2K_{v_\lambda u_\mu}}(y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma})u_\mu \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)}v_\lambda.$$

From (12) we obtain that

$$-\frac{1}{2K_{v_\lambda u_\mu}}(\sigma_1 u_\mu \wedge \sigma_2 v_\lambda - \sigma_2 u_\mu \wedge \sigma_1 v_\lambda) u_\mu = \\ \frac{1}{K_{v_\lambda u_\mu}}(\sigma_1 u_\mu v_\lambda^t \sigma_2 - \sigma_2 v_\lambda u_\mu^t \sigma_1 - \sigma_2 u_\mu v_\lambda^t \sigma_1 + \sigma_1 v_\lambda u_\mu^t \sigma_2) u_\mu = \\ \frac{1}{K_{v_\lambda u_\mu}}(\sigma_1 u_\mu v_\lambda^t \sigma_2 u_\mu - \sigma_2 u_\mu v_\lambda^t \sigma_1 u_\mu) = (\lambda_2 - \mu_2)\sigma_1 u_\mu - (\lambda_1 - \mu_1)\sigma_2 u_\mu.$$

Analogously

$$\frac{1}{2K_{v_\lambda u_\mu}}(\sigma_1 u_\mu \wedge \sigma_2 v_\lambda - \sigma_2 u_\mu \wedge \sigma_1 v_\lambda) v_\lambda = (\lambda_2 - \mu_2)\sigma_1 v_\lambda - (\lambda_1 - \mu_1)\sigma_2 v_\lambda.$$

Together with (10) this implies

$$(16) \quad \begin{aligned} (\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma})u_\mu &= 0 \\ (\mu_1\sigma_2 - \mu_2\sigma_1 + \tilde{\gamma})v_\lambda &= 0 \end{aligned}$$

and

$$(17) \quad \begin{aligned} (y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma})v_\lambda &= ((y_1 - \mu_1)\sigma_2 - (y_2 - \mu_2)\sigma_1)v_\lambda \\ (y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma})u_\mu &= ((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1)u_\mu \end{aligned}$$

In other words,  $u_\mu \in \tilde{\mathcal{E}}(\lambda)$  and  $v_\lambda \in \tilde{\mathcal{E}}(\mu)$ , where  $\tilde{\mathcal{E}}(y_1, y_2) = \text{Coker}(y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma})$ .

Now it is easy to verify that

$$\begin{aligned} ((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1)v_\lambda \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)}u_\mu &= \\ -\sigma_1 u_\mu \wedge \sigma_2 v_\lambda + \sigma_2 u_\mu \wedge \sigma_1 v_\lambda - \\ ((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1)u_\mu \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)}v_\lambda. \end{aligned}$$

which together with (10) and (17) finishes the proof of (15).

The second statement will be proved if we show  
(18)

$$\begin{aligned} & (y_1\sigma_2 - y_2\sigma_1 + \gamma) \\ & \parallel \\ & \left( \text{Id} - 2\rho \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)} v_\lambda v_\lambda^t \right) (y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}) \left( \text{Id} + 2\rho v_\lambda v_\lambda^t \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)} \right) \\ & \parallel \\ & y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma} - \rho (y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}) v_\lambda \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)} v_\lambda. \end{aligned}$$

Note that

$$\begin{aligned} (\sigma_2 v_\lambda \wedge \sigma_1 v_\lambda) v_\lambda &= \\ (-\sigma_2 v_\lambda v_\lambda^t \sigma_1 + \sigma_1 v_\lambda v_\lambda^t \sigma_2) v_\lambda &= 0 \end{aligned}$$

implies

$$\begin{aligned} (19) \quad (\lambda_1\sigma_2 - \lambda_2\sigma_1 + \bar{\gamma}) v_\lambda &= 0 \\ (y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}) v_\lambda &= ((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1) v_\lambda. \end{aligned}$$

This means that  $v_\lambda \in \bar{\mathcal{E}}(\lambda)$ , where  $\bar{\mathcal{E}}(y_1, y_2)$  denotes the cokernel of  $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ .

The above and the obvious equality

$$\begin{aligned} -\sigma_2 v_\lambda \wedge \sigma_1 v_\lambda &= \\ -((y_1 - \lambda_1)\sigma_2 - (y_2 - \lambda_2)\sigma_1) v_\lambda \wedge \frac{t_1\sigma_1 + t_2\sigma_2}{t_1(y_1 - \lambda_1) + t_2(y_2 - \lambda_2)} v_\lambda. \end{aligned}$$

finish the proof of (18).  $\square$

By performing *elementary transformations* of  $y_1\sigma_2 - y_2\sigma_1 + \gamma$ , we constructed new pfaffian representations of  $C$ :

**Definition 4.2.**

- The *Type I elementary transformation* based on the admissible vectors  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$  is defined by  $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$ .
- The *Type II elementary transformation* based on  $v_\lambda \in \mathcal{E}(\lambda)$  and the constant  $\rho \neq 0$  is defined by  $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$ .

The fact that  $v_\lambda \in \tilde{\mathcal{E}}(\mu)$ ,  $u_\mu \in \tilde{\mathcal{E}}(\lambda)$  and  $v_\lambda \in \bar{\mathcal{E}}(\lambda)$  implies the following:

**Corollary 4.3.** *The Type I elementary transformation of  $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$  based on  $u_\mu \in \tilde{\mathcal{E}}(\lambda)$ ,  $v_\lambda \in \tilde{\mathcal{E}}(\mu)$  brings us back to  $y_1\sigma_2 - y_2\sigma_1 + \gamma$ . The same way the Type II elementary transformation of  $y_1\sigma_2 - y_2\sigma_1 + \bar{\gamma}$  based on  $v_\lambda \in \bar{\mathcal{E}}(\lambda)$  and  $-\rho$  brings us back to  $y_1\sigma_2 - y_2\sigma_1 + \gamma$ .*

It can be easily seen that the Type I and II elementary transformations are special rank 2 cases of "the concrete interpolation problem for meromorphic bundle maps" studied in [2]: For a given array of  $m$  constants  $\rho_i \in k$ ,  $m$  distinct points  $\lambda^i \in C$  and vectors  $w_i \in \mathcal{E}(\lambda^i)$  define the *Type CONINT* elementary transformation of  $y_1\sigma_2 - y_2\sigma_1 + \gamma$  by

$$\tilde{\gamma} = \gamma + \sigma_1 w \Gamma^{-1} w^t \sigma_2 - \sigma_2 w \Gamma^{-1} w^t \sigma_1,$$

where  $w = [w_1, \dots, w_m]$  and  $-\Gamma$  equals the  $m \times m$  symmetric matrix with  $\rho_i$ 's along the diagonal and  $K_{w_i w_j}$  at the  $(i < j)$ -th position. Then  $y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}$  is indeed a pfaffian representation of  $C$  since  $y_1\sigma_2 - y_2\sigma_1 + \gamma = Z^t(y) (y_1\sigma_2 - y_2\sigma_1 + \tilde{\gamma}) Z(y)$  for

$$Z(y) = \text{Id} + w \text{Diagonal} [t_1(y_1 - \lambda_1^i) + t_2(y_2 - \lambda_2^i)]_{i=1 \dots m}^{-1} \Gamma^{-1} w^t (t_1\sigma_1 + t_2\sigma_2).$$

An easy exercise in linear algebra shows that

$$Z^{-1}(y) = \text{Id} - w \Gamma^{-1} \text{Diagonal}[t_1(y_1 - \lambda_1^i) + t_2(y_2 - \lambda_2^i)]_{i=1 \dots m}^{-1} w^t (t_1 \sigma_1 + t_2 \sigma_2)$$

and  $\det Z(y) = \det Z^{-1}(y) = 1$ . Observe that the only condition for  $\tilde{\gamma}$  to exist is that  $\Gamma$  is invertible.

In Section 7 we will show that any two pfaffian representations of  $C$  can be bridged by a finite sequence of Type I and Type II elementary transformations. Thus it is enough to restrict our study to these two types.

## 5. ADMISSIBLE ARRAYS

We briefly relate the definitions of admissible pairs of points and admissible pairs of vectors. Let

$$\begin{aligned} A(x_0, x_1, x_2) &= x_1 A_1 + x_2 A_2 + x_0 A_0 \\ &= x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma \end{aligned}$$

be a representation of  $C$  with the cokernel  $\mathcal{E}(x)$ . As before, let  $v_\lambda \in \mathcal{E}(\lambda)$ ,  $u_\mu \in \mathcal{E}(\mu)$  for distinct  $\lambda, \mu \in C$ . We claim that

$$K_{v_\lambda u_\mu} = 0 \text{ if and only if } v_\lambda^t (x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma) u_\mu \equiv 0 \text{ for all } x \in \mathbb{P}^2.$$

Indeed,  $K_{v_\lambda u_\mu} = 0$  implies  $v_\lambda^t \sigma_1 u_\mu = v_\lambda^t \sigma_2 u_\mu = 0$  by (12) and moreover  $v_\lambda^t \gamma u_\mu = 0$  by (11). The converse is obvious.

We mention few more implications of the definition of  $K_{v_\lambda u_\mu}$ :

- i) If  $\lambda, \mu$  are admissible points on  $C$  then for every  $v_\lambda \in \mathcal{E}(\lambda)$  there exists exactly one  $v_\mu \in \mathcal{E}(\mu)$  for which  $K_{v_\lambda v_\mu} = 0$ .
- ii) For any  $\lambda$  and  $v_\lambda \in \mathcal{E}(\lambda)$  there exists  $\mu$  and  $u_\mu \in \mathcal{E}(\mu)$  such that  $v_\lambda, u_\mu$  are admissible vectors.
- iii) Let  $v_\lambda \in \mathcal{E}(\lambda)$ . Given any vector  $u$  such that  $K_{v_\lambda u} \neq 0$  there exist at most  $d$  distinct points  $\mu^j \in C$  for which  $u \in \mathcal{E}(\mu^j)$  and  $v_\lambda, u$  are admissible vectors.

Claim i) is obvious. For ii) recall that  $\text{Span}\{\text{Coker } A(\mu); \mu \in C\} = k^{2d}$ . If  $\mu$  didn't exist it would thus hold  $v_\lambda^t \sigma_1 u = v_\lambda^t \sigma_2 u = 0$  for all  $u \in k^{2d}$ . This contradicts  $\dim\{v_\lambda^t \sigma_1, v_\lambda^t \sigma_2\}^\perp = 2d - 2$ . Claim iii) will follow from (12). For any  $t \in k$

$$\mu_i^j = \lambda_i - \frac{v_\lambda^t \sigma_i t u}{K_{v_\lambda t u}}, \quad i = 1, 2$$

we obtain the affine coordinates of  $\mu^j$ . But the defining polynomial of  $C$  evaluated in  $(1, \mu_1^j, \mu_2^j)$  has at most  $d$  solutions for  $\frac{t}{K_{v_\lambda t u}}$ .

**Corollary 5.1.** *For every  $\lambda \in C$  there are at most finitely many  $\mu$  that do not form an admissible pair of points.*

## 6. ELEMENTARY TRANSFORMATIONS ACT ON THE COKERNEL BUNDLE OF REPRESENTATION

In this section we relate elementary transformations of vector bundles with the elementary transformations of pfaffian representations considered in Section 4.

**Definition 6.1.** Let  $\mathcal{E}$  be a rank 2 vector bundle over a smooth and irreducible  $C$ . Take an effective reduced divisor  $Z$  on  $C$  and consider the canonical surjection

$$\mathcal{E} \rightarrow k(Z) \rightarrow 0,$$

where  $k(Z)$  is a skyscraper sheaf at  $Z$ , i.e. rank 1  $\mathcal{O}_Z$ -module. Its kernel is a rank 2 vector bundle on  $C$  called the *elementary transformation of  $\mathcal{E}$  at  $Z$* . We denote it by  $\mathcal{E}' = \text{elem}_Z(\mathcal{E})$ .

On  $C$  it is equivalent to consider:

- (a) Ruled surface  $\pi : S \rightarrow C$  together with a base-point-free unisecant complete linear system  $|H|$ ;
- (b) Rank 2 vector bundle  $\mathcal{E}$  over  $C$  for which  $S = \mathbb{P}\mathcal{E}$  and  $\mathcal{E} \cong \pi_*\mathcal{O}_{\mathbb{P}\mathcal{E}}(H)$ ;
- (c) Linearly normal scroll  $R$  obtained as the image of the birational map  $\phi_H : S \rightarrow R \subset \mathbb{P}^N$  defined by  $|H|$ .

Analogously we can define elementary transformation at a point  $x \in C$  on each of the above:

- (a) On the ruled surface  $S$  we choose a point  $s \in \pi^{-1}(x)$ . Denote by  $B$  the blow-up of  $S$  at  $s$ . By Castelnuovo theorem we can contract the starting fibre  $\pi^{-1}(x)$  in  $B$  and obtain a new ruled surface  $\pi' : S' \rightarrow C$ ;
- (b)  $\mathcal{E}' = \text{elem}_x(\mathcal{E})$  is obtained as the elementary transformation of  $\mathcal{E}$  at the divisor  $x$  on  $C$ , i.e. as the kernel of  $\mathcal{E} \rightarrow k(x) \rightarrow 0$ ;
- (c) Pick a point  $r = \phi_H(s)$  on the scroll  $R$  such that  $\pi(s) = x$ . Projection from  $r$  yields a scroll  $R' \subset \mathbb{P}^{N-1}$ .

Fuentes and Pedreira [10] checked that these definitions are compatible, namely  $S' = \mathbb{P}\mathcal{E}'$ . Moreover,  $R'$  is the image of  $S'$  under the birational map defined by  $|H'|$ , where  $H' = \nu^*(H) - \pi^{-1}(x)$ . Here  $\nu : S' \rightarrow S$  denotes the inverse of the elementary transformation of  $S$  at  $s$ . In particular  $\mathcal{E}' \cong \pi'_*\mathcal{O}_{\mathbb{P}\mathcal{E}'}(H')$ .

The inverse of an elementary transformation of a ruled surface is again an elementary transformation: if  $S'$  is elementary transformation of  $S$  at  $s$ , then  $S$  is elementary transformation of  $S'$  at  $s'$ , where  $s'$  is the contraction of (the exceptional divisor of the blow-up)  $\cap$  (the original fibre  $\pi^{-1}(x)$ )  $\in B$ .

Up to tensoring line bundles, we can view the elementary transformation of a vector bundle in Definition 6.1 at  $Z = x^1 + \dots + x^m$  as an elementary transformation of  $\mathbb{P}\mathcal{E}$  at  $m$  points  $s_i \in \pi^{-1}(x^i)$ . More precisely, there exists a skyscraper sheaf  $k(Z)'$  that fits into the commutative diagram

$$(20) \quad \begin{array}{ccc} \mathcal{E} \otimes \mathcal{O}_C(-Z) & & \\ \downarrow g & & \\ \mathcal{E}' \xrightarrow{e} \mathcal{E} \rightarrow k(Z) & & \\ \downarrow & & \\ k(Z)' & & \end{array}$$

Here  $\mathcal{E} \otimes \mathcal{O}_C(-Z)$  is the following elementary transformation of  $\mathcal{E}'$  at  $Z$ : Take a point  $x \in C$  and free basis  $\{X'_1, X'_2\}$ ,  $\{X_1, X_2\}$  for  $\mathcal{E}'(x)$ ,  $\mathcal{E}(x)$  respectively. Denote by  $Y_i = e(X'_i) = e_{1i}X_1 + e_{2i}X_2$ ,  $i = 1, 2$  the image of the morphism  $e$ . At every  $x^i \in C$ ,  $i = 1, \dots, m$  the determinant of  $e$  is 0, thus projectively  $Y_1 = Y_2 = s_i \in \pi^{-1}(x^i)$  are the points of the blow up of  $\mathbb{P}\mathcal{E}$ . Locally  $k(Z) = kX_1 \oplus kX_2 / kY_1 + kY_2$ , therefore  $k(Z)$  is 0 whenever  $e$  is invertible and is a rank 1 bundle over the points  $x^i$ . Analogously  $g$ , given by the  $2 \times 2$  adjoint matrix of  $e$ , defines an elementary transformation of  $\mathcal{E}'$ . It is easy to check that it equals  $\mathcal{E} \otimes \mathcal{O}_C(-Z)$ . For  $x^i \in C$ ,  $i = 1, \dots, m$  the points of the blow up of  $\mathbb{P}\mathcal{E}'$  equal  $g_{11}X'_1 + g_{21}X'_2 = g_{12}X'_1 + g_{22}X'_2$ . Observe that  $e(g_{11}X'_1 + g_{21}X'_2) = 0$ . On the level of ruled surfaces  $e^{-1}$  and  $g^{-1}$  induce elementary transformations of  $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}_C(-Z)) = \mathbb{P}\mathcal{E}$  and  $\mathbb{P}\mathcal{E}'$  that are inverse to each other.

In the sequel we establish the explicit relation between elementary transformations of pfaffian representations and elementary transformations of the corresponding cokernel vector bundles. We use the notation of Definition 4.2. Again we work in projective coordinates  $x = (x_0, x_1, x_2)$  in  $\mathbb{P}^2$ . For any parameters  $t_1, t_2$  we define matrices with rational elements  $T(x), S(x), P(x), R(x)$  and  $Q(x)$  by:

$$\begin{aligned} T(x) &= \text{Id} + \frac{x_0}{K_{v_\lambda u_\mu}(t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0))} (t_1 \sigma_1 + t_2 \sigma_2) u_\mu v_\lambda^t, \\ S(x) &= \text{Id} + \frac{x_0}{K_{v_\lambda u_\mu}(t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0))} u_\mu v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2), \\ P(x) &= \text{Id} + \frac{x_0}{K_{v_\lambda u_\mu}(t_1(x_1 - \mu_1 x_0) + t_2(x_2 - \mu_2 x_0))} v_\lambda u_\mu^t (t_1 \sigma_1 + t_2 \sigma_2), \\ R(x) &= \text{Id} + \frac{x_0}{K_{v_\lambda u_\mu}(t_1(x_1 - \mu_1 x_0) + t_2(x_2 - \mu_2 x_0))} (t_1 \sigma_1 + t_2 \sigma_2) v_\lambda u_\mu^t \\ \text{and } Q(x) &= \text{Id} + \frac{2\rho x_0}{t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0)} v_\lambda v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2). \end{aligned}$$

**Theorem 6.2.** *Let  $C$  be a curve defined by  $F = \text{Pf}(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma) = 0$  and let  $x_1 \sigma_2 - x_2 \sigma_1 + x_0 \tilde{\gamma}$ ,  $x_1 \sigma_2 - x_2 \sigma_1 + x_0 \bar{\gamma}$  be elementary transformations of Type I and II respectively. Denote by  $\mathcal{E}(x), \tilde{\mathcal{E}}(x), \bar{\mathcal{E}}(x)$  the corresponding cokernels that are rank 2 vector bundles over  $C$ . The relating morphisms in the below diagrams can be expressed by elementary transformations of vector bundles*

$$\begin{array}{ccc} & \tilde{\mathcal{E}} & \\ S \uparrow & R^t \downarrow & \\ & \mathcal{E}' & \\ P \uparrow & T^t \downarrow & \\ & \mathcal{E} & \end{array} \quad \text{and} \quad \begin{array}{c} \bar{\mathcal{E}} \\ \uparrow \\ Q \\ \downarrow \\ \mathcal{E} \end{array}.$$

*Proof.* We will prove that  $S(x)P(x)$  maps  $\mathcal{E}(x)$  into  $\tilde{\mathcal{E}}(x)$  and  $Q(x)$  maps  $\mathcal{E}(x)$  into  $\bar{\mathcal{E}}(x)$ .

First consider Type I transformations. Observe that for  $x$  different from  $\lambda$  and  $\mu$  (12) implies

$$\begin{aligned} P(x)T^t(x) &= T^t(x)P(x) = \text{Id} \quad \text{and} \\ S^t(x)R(x) &= R(x)S^t(x) = \text{Id}. \end{aligned}$$

Using this, rewrite (15) into

$$T(x)(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma)T^t(x) = S^t(x)(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \tilde{\gamma})S(x),$$

or equivalently

$$(21) \quad R(x)T(x)(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma) = (x_1 \sigma_2 - x_2 \sigma_1 + x_0 \tilde{\gamma})S(x)P(x).$$

Therefore

$$(22) \quad S(x)P(x)\mathcal{E}(x) \subseteq \tilde{\mathcal{E}}(x).$$

Transposing (21) and multiplying by the inverse matrices yields the reverse inclusions. This means that the rational matrix function  $S(x)P(x)$  maps the vector bundle corresponding to the starting representation to the cokernel bundle of the new representation of Type I.

Similarly,  $\text{Id} + \frac{2\rho x_0}{t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0)} (t_1 \sigma_1 + t_2 \sigma_2) v_\lambda v_\lambda^t$  is the inverse of  $Q^t(x)$  for  $x \neq \lambda$ . For Type II elementary transformations, (18) becomes

$$Q^t(x)^{-1}(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma) = (x_1 \sigma_2 - x_2 \sigma_1 + x_0 \bar{\gamma})Q(x)$$

and

$$(23) \quad Q(x) \mathcal{E}(x) = \bar{\mathcal{E}}(x).$$

Thus the rational matrix function  $Q(x)$  maps  $\mathcal{E}(x)$  to the cokernel bundle  $\bar{\mathcal{E}}(x)$  of the representation obtained by the Type II transformation.

An attentive reader will notice that the matrices  $S(x), T(x), P(x), R(x), Q(x)$  also depend on parameters  $t_1, t_2$ . But for  $x$  such that  $t_1(x_1 - \lambda_1 x_0) + t_2(x_2 - \lambda_2 x_0) \neq 0$ ,  $t_1(x_1 - \mu_1 x_0) + t_2(x_2 - \mu_2 x_0) \neq 0$  and for every  $\varepsilon(x) \in \mathcal{E}(x)$ ,  $\tilde{\varepsilon}(x) \in \tilde{\mathcal{E}}(x)$ ,  $\bar{\varepsilon}(x) \in \bar{\mathcal{E}}(x)$  the following vectors

$$P(x)\varepsilon(x), \quad \tilde{\varepsilon}^t(x)T(x), \quad S(x)\varepsilon(x), \quad \tilde{\varepsilon}^t(x)R(x) \quad \text{and} \quad Q(x)\varepsilon(x), \quad Q^{-1}(x)\bar{\varepsilon}(x)$$

are independent of  $t_1, t_2$ . This is a corollary of (10), (17) and (19) since

$$\begin{aligned} 0 &= u_\mu^t(x_1\sigma_2 - x_2\sigma_1 + x_0\gamma)\varepsilon(x) = u_\mu^t((x_1 - \mu_1x_0)\sigma_2 - (x_2 - \mu_2x_0)\sigma_1)\varepsilon(x), \\ 0 &= \tilde{\varepsilon}^t(x)(x_1\sigma_2 - x_2\sigma_1 + x_0\tilde{\gamma})u_\mu = \tilde{\varepsilon}^t(x)((x_1 - \lambda_1x_0)\sigma_2 - (x_2 - \lambda_2x_0)\sigma_1)u_\mu, \\ 0 &= v_\lambda^t(x_1\sigma_2 - x_2\sigma_1 + x_0\gamma)\varepsilon(x) = v_\lambda^t((x_1 - \lambda_1x_0)\sigma_2 - (x_2 - \lambda_2x_0)\sigma_1)\varepsilon(x), \\ 0 &= \tilde{\varepsilon}^t(x)(x_1\sigma_2 - x_2\sigma_1 + x_0\tilde{\gamma})v_\lambda = \tilde{\varepsilon}^t(x)((x_1 - \mu_1x_0)\sigma_2 - (x_2 - \mu_2x_0)\sigma_1)v_\lambda \\ \text{and } 0 &= v_\lambda^t(x_1\sigma_2 - x_2\sigma_1 + x_0\tilde{\gamma})\bar{\varepsilon}(x) = v_\lambda^t((x_1 - \lambda_1x_0)\sigma_2 - (x_2 - \lambda_2x_0)\sigma_1)\bar{\varepsilon}(x). \end{aligned}$$

Indeed, rewrite the top equality into

$$(24) \quad \frac{u_\mu^t \sigma_1 \varepsilon(x)}{x_1 - \mu_1 x_0} = \frac{u_\mu^t \sigma_2 \varepsilon(x)}{x_2 - \mu_2 x_0}.$$

The only factor in  $P(x)\varepsilon(x)$  depending on  $t_1, t_2$  is

$$\frac{u_\mu^t(t_1\sigma_1 + t_2\sigma_2)\varepsilon(x)}{t_1(x_1 - \mu_1x_0) + t_2(x_2 - \mu_2x_0)}.$$

By making the cross products of the above fractions, it can be immediately verified, that this factor equals to (24) and is thus independent from  $t_1, t_2$ . The independency of other cases can be computed analogously.

Hence  $P(x), R(x), S(x), T(x)$  and  $Q(x)$  do not depend on  $t_1, t_2$  when restricted to  $\mathcal{E}(x)$ ,  $\tilde{\mathcal{E}}(x)$  and  $\bar{\mathcal{E}}(x)$  respectively. Moreover, by using (12) for the admissible pair of vectors  $v_\lambda \in \mathcal{E}(\lambda) \cap \tilde{\mathcal{E}}(\mu)$ ,  $u_\mu \in \mathcal{E}(\mu) \cap \tilde{\mathcal{E}}(\lambda)$ , we get

$$(25) \quad P(\lambda)v_\lambda = 0, \quad v_\lambda^t T(\mu) = 0, \quad u_\mu^t R(\lambda) = 0, \quad S(\mu)u_\mu = 0.$$

Next we analyse how elementary transformations act on rational sections of the cokernel bundles. Let  $q(x)$  be a *rational section* of  $\mathcal{E}(x)$ . This is a  $2d$ -tuple of rational functions along  $C$  with the property  $q(x) \in \mathcal{E}(x)$ . For example, by (8) it can be obtained by dividing a column in the adjoint matrix of the representation by a degree  $d - 1$  polynomial. Then by (22) and (23)

$$(26) \quad \tilde{q}(x) = S(x)P(x)q(x) \quad \text{and} \quad \bar{q}(x) = Q(x)q(x)$$

are rational sections of  $\tilde{\mathcal{E}}(x)$  and  $\bar{\mathcal{E}}(x)$  respectively. Additionally, denote by  $q'(x) = P(x)q(x) = R^t(x)\tilde{q}(x)$  sections in the auxiliary bundle  $\mathcal{E}'(x) = P(x)\mathcal{E}(x) = R^t(x)\tilde{\mathcal{E}}(x)$ . Proposition 6.3 will discuss the order of these rational sections at various points on  $C$ . This will describe  $P, S$  and  $Q$  in terms of elementary transformations of vector bundles considered in (20). In other words,

- $P$  at  $\lambda$  equals the morphism in the diagram

$$\begin{array}{c} \mathcal{E}' \otimes \mathcal{O}_C(-\lambda) \\ \downarrow \\ \mathcal{E} \xrightarrow{P} \mathcal{E}' \rightarrow k(\lambda). \end{array}$$

$\mathcal{E}' \otimes \mathcal{O}_C(-\lambda)$  (the induced elementary transformation of  $\mathcal{E}$  at  $\lambda$ ) is by (25) obtained through the blow-up of  $v_\lambda \in \mathbb{P}\mathcal{E}(\lambda)$ ;

- $P$  at  $\mu$  equals the elementary transformation of  $\mathcal{E}$  at  $\mu$  in the diagram

$$\begin{array}{c} \mathcal{E} \otimes \mathcal{O}_C(-\mu) \\ \downarrow \\ \mathcal{E}' \xrightarrow{T} \mathcal{E} \rightarrow k(\mu). \end{array}$$

Again by (25),  $\mathcal{E} \otimes \mathcal{O}_C(-\mu)$  is obtained through the blow-up of  $v_\lambda \in \mathbb{P}\mathcal{E}'(\mu)$ .

The same way  $S$  consists of the following:

- $S$  at  $\lambda$  equals the elementary transformation of  $\mathcal{E}'$  at  $\lambda$  in the diagram

$$\begin{array}{c} \mathcal{E}' \otimes \mathcal{O}_C(-\lambda) \\ \downarrow \\ \tilde{\mathcal{E}} \xrightarrow{R} \mathcal{E}' \rightarrow k(\lambda), \end{array}$$

where  $\mathcal{E}' \otimes \mathcal{O}_C(-\lambda)$  is obtained through the blow-up of  $u_\mu \in \mathbb{P}\tilde{\mathcal{E}}(\lambda)$ ;

- $S$  at  $\mu$  equals the morphism in the diagram

$$\begin{array}{c} \tilde{\mathcal{E}} \otimes \mathcal{O}_C(-\mu) \\ \downarrow \\ \mathcal{E}' \xrightarrow{S} \tilde{\mathcal{E}} \rightarrow k(\mu), \end{array}$$

where  $\tilde{\mathcal{E}} \otimes \mathcal{O}_C(-\mu)$  is obtained through the blow-up of  $u_\mu \in \mathbb{P}\mathcal{E}'(\mu)$ .

On the other hand,  $Q$  and  $Q^{-1}$  are described by the following diagram of elementary transformations at  $\lambda \in \text{Pic } C$ :

$$\begin{array}{ccccc} \bar{\mathcal{E}} \otimes \mathcal{O}_C(-\lambda) & & \mathcal{E} \otimes \mathcal{O}_C(-\lambda) & & \\ & \searrow & \swarrow & & \\ k(\lambda) \longleftarrow \bar{\mathcal{E}} \longleftarrow & \mathcal{E}'' & \longrightarrow \mathcal{E} & \longrightarrow & k(\lambda), \end{array}$$

where  $\mathcal{E}''$  is an elementary transformation of both  $\mathcal{E}$  and  $\bar{\mathcal{E}}$ . This will conclude the proof of Theorem 6.2. Figure [1] describes  $P, S$  and  $Q$  in terms of elementary transformations of ruled surfaces.  $\square$

**Proposition 6.3.** *Let  $q(x)$  be a rational section of  $\mathcal{E}(x)$  and  $o_\xi(q)$  its order at  $\xi$  and let  $\tilde{q}(x), \bar{q}(x), q'(x)$  be sections of  $\tilde{\mathcal{E}}(x), \bar{\mathcal{E}}(x), \mathcal{E}'(x)$  respectively.*

*If  $\xi$  is different from  $\lambda$  and  $\mu$  then  $o_\xi(q) = o_\xi(q') = o_\xi(\tilde{q}) = o_\xi(\bar{q})$ .*

*If  $\xi = \lambda$  then:*

*$o_\lambda(q') = o_\lambda(q) + 1$  iff  $q$  at  $\lambda$  passes through  $v_\lambda$ , otherwise  $o_\lambda(q) = o_\lambda(q')$ ;*

*$o_\lambda(\tilde{q}) = o_\lambda(\bar{q}) + 1$  iff  $\tilde{q}$  at  $\lambda$  passes through  $u_\mu$ , otherwise  $o_\lambda(\tilde{q}) = o_\lambda(\bar{q})$ ;*

*$o_\lambda(q) = o_\lambda(\bar{q})$  iff  $q$  and  $\bar{q}$  at  $\lambda$  both pass through  $v_\lambda$ , otherwise either  $o_\lambda(q) = o_\lambda(\bar{q}) + 1$  or  $o_\lambda(\bar{q}) = o_\lambda(q) + 1$ .*

*If  $\xi = \mu$  then:*

*$o_\mu(q) = o_\mu(q') + 1$  iff  $q'$  at  $\mu$  passes through  $v_\lambda$ , otherwise  $o_\mu(q) = o_\mu(q')$ ;*

*$o_\mu(\tilde{q}) = o_\mu(q') + 1$  iff  $q'$  at  $\mu$  passes through  $u_\mu$ , otherwise  $o_\mu(\tilde{q}) = o_\mu(q')$ .*

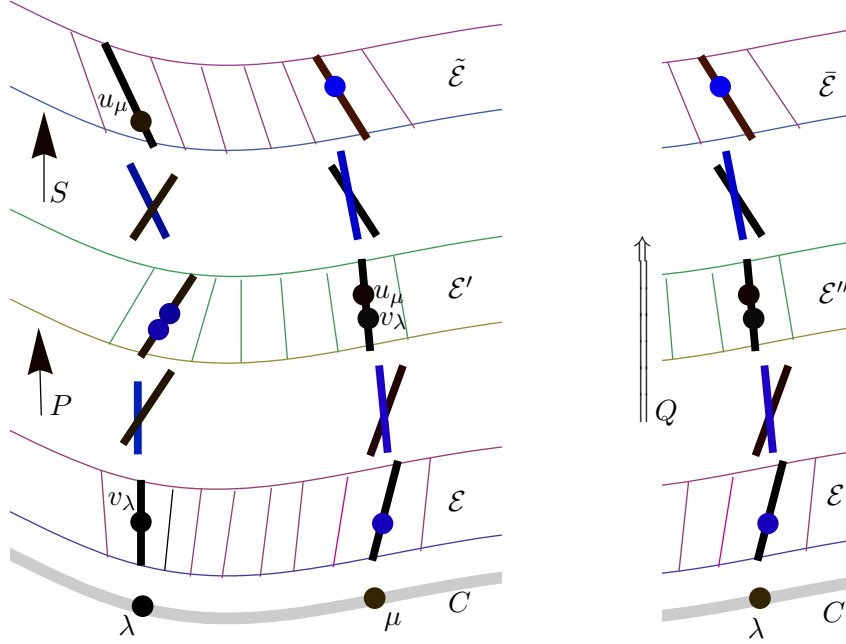


FIGURE 1. Elementary transformations of Type I and II

*Proof.* Pick a local parameter  $t$  at  $\xi = (\xi_0, \xi_1, \xi_2) \in C$  and expand  $x$  and  $q(x)$  into the power series with center  $\xi$ . We can assume that  $\xi_0 = 1$  and  $\frac{\partial F}{\partial x_0}(\xi) = 0$ . Thus

$$x(t) = \left( 1 + \cdots, \xi_1 + t \frac{\partial F}{\partial x_1}(\xi) + \cdots, \xi_2 + t \frac{\partial F}{\partial x_2}(\xi) + \cdots \right),$$

$$q(x) = c_o t^o + c_{o+1} t^{o+1} + \cdots,$$

where  $o = o_\xi(q)$  and  $\cdots$  denote higher order terms. By definition  $c_o \neq 0$ . Since  $(x_1 \sigma_2 - x_2 \sigma_1 + x_0 \gamma) q(x) = 0$  for all  $x \in C$ , in terms of the local parameter  $t$  we get  $t^o ((\xi_1 + \cdots) \sigma_2 - (\xi_2 + \cdots) \sigma_1 + (\xi_0 + \cdots) \gamma) (c_o + c_{o+1} t + \cdots) \equiv 0$  which implies

$$c_o \in \mathcal{E}(\xi).$$

Analogously we expand

$$q'(x) = c_{o'} t^{o'} + \cdots, \quad \tilde{q}(x) = c_{\tilde{o}} t^{\tilde{o}} + \cdots, \quad \bar{q}(x) = c_{\bar{o}} t^{\bar{o}} + \cdots,$$

where  $o' = o_\xi(q')$ ,  $\tilde{o} = o_\xi(\tilde{q})$ ,  $\bar{o} = o_\xi(\bar{q})$  and  $c_{\tilde{o}} \in \tilde{\mathcal{E}}(\xi)$ ,  $c_{\bar{o}} \in \bar{\mathcal{E}}(\xi)$ .

Let  $\xi \neq \lambda, \mu$ . Then  $S(x), P(x)$  and  $Q(x)$  are defined at  $\xi$  and can be expanded along  $t$ . Thus (26) gives

$$q'(x) = P(\xi) c_o t^o + \cdots, \quad \tilde{q}(x) = S(\xi) P(\xi) c_o t^o + \cdots \quad \text{and} \quad \bar{q}(x) = Q(\xi) c_o t^o + \cdots.$$

This proves  $o' \geq o$ ,  $\tilde{o} \geq o$  and  $\bar{o} \geq o$ . The power series for  $q'(x)$ ,  $\tilde{q}(x)$ ,  $\bar{q}(x)$  and  $q(x) = T^t(x) q'(x)$ ,  $q(x) = T^t R^t(x) \tilde{q}(x)$ ,  $q(x) = Q(x)^{-1} \bar{q}(x)$  analogously imply  $o \geq o'$ ,  $o \geq \tilde{o}$ ,  $o \geq \bar{o}$ .

Let now  $\xi = \lambda$ .  $P(x)$  and  $R(x)$  are defined at  $\lambda$  since  $\lambda \neq \mu$ . By the considerations in the proof of (25) we notice that  $P(\lambda) c_o = 0$  if and only if  $c_o$  is a multiple of  $v_\lambda$  and that  $R^t(\lambda) c_{\tilde{o}} = 0$  if and only if  $c_{\tilde{o}}$  is a multiple of  $u_\mu$ . Like above,  $q'(x) = P(\lambda) c_o t^o + \cdots$  implies  $o' \geq o$ . Moreover, the strict inequality holds iff  $c_o$  is a multiple of  $v_\lambda$  (i.e.,  $q$  at  $\lambda$  passes through  $v_\lambda$ ). The same way  $q'(x) = R^t(\lambda) c_{\tilde{o}} t^{\tilde{o}} + \cdots$  implies  $o' \geq \tilde{o}$  and the strict inequality holds iff  $\tilde{q}$  at  $\lambda$  passes through  $u_\mu$ .



Recall that  $\lambda = (1, \lambda_1, \lambda_2)$  is a regular point on  $C$ , therefore one of  $\frac{\partial F}{\partial x_1}(\lambda)$ ,  $\frac{\partial F}{\partial x_2}(\lambda)$  must be nonzero. Then

$$T(x) = \text{Id} + \frac{1}{t K_{v_\lambda u_\mu} \left( t_1 \left( \frac{\partial F}{\partial x_1}(\lambda) + \cdots \right) + t_2 \left( \frac{\partial F}{\partial x_2}(\lambda) + \cdots \right) \right)} (t_1 \sigma_1 + t_2 \sigma_2) u_\mu v_\lambda^t,$$

$$S(x) = \text{Id} + \frac{1}{t K_{v_\lambda u_\mu} \left( t_1 \left( \frac{\partial F}{\partial x_1}(\lambda) + \cdots \right) + t_2 \left( \frac{\partial F}{\partial x_2}(\lambda) + \cdots \right) \right)} u_\mu v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2)$$

imply

$$q(x) = T^t(x) q'(x) = \frac{-1}{K_{v_\lambda u_\mu} \left( t_1 \frac{\partial F}{\partial x_1}(\lambda) + t_2 \frac{\partial F}{\partial x_2}(\lambda) \right)} v_\lambda u_\mu^t (t_1 \sigma_1 + t_2 \sigma_2) c_{o'} t^{o'-1} + \cdots,$$

$$\tilde{q}(x) = S(x) q'(x) = \frac{1}{K_{v_\lambda u_\mu} \left( t_1 \frac{\partial F}{\partial x_1}(\lambda) + t_2 \frac{\partial F}{\partial x_2}(\lambda) \right)} u_\mu v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2) c_{o'} t^{o'-1} + \cdots.$$

This proves  $o \geq o' - 1$  and  $\bar{o} \geq o' - 1$ .

Apply the same arguments to

$$\bar{q}(x) = Q(x) q(x) = \frac{2\rho}{t_1 \frac{\partial F}{\partial x_1}(\lambda) + t_2 \frac{\partial F}{\partial x_2}(\lambda)} v_\lambda v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2) c_o t^{o-1} + \cdots,$$

$$q(x) = Q(x)^{-1} \bar{q}(x) = \frac{-2\rho}{t_1 \frac{\partial F}{\partial x_1}(\lambda) + t_2 \frac{\partial F}{\partial x_2}(\lambda)} v_\lambda v_\lambda^t (t_1 \sigma_1 + t_2 \sigma_2) c_{\bar{o}} t^{\bar{o}-1} + \cdots,$$

which proves  $\bar{o} \geq o - 1$  and  $o \geq \bar{o} - 1$ . Observe also that at least one of  $q, \bar{q}$  passes through  $v_\lambda$ , which is equivalent to  $c_o$  or  $c_{\bar{o}}$  being a multiple of  $v_\lambda$ . More precisely,  $o = \bar{o}$  iff  $q$  and  $\bar{q}$  at  $\lambda$  both pass through  $v_\lambda$ ,  $\bar{o} = o - 1$  iff  $c_o \neq v_\lambda \in \mathbb{P}\mathcal{E}$  and  $o = \bar{o} - 1$  iff  $c_{\bar{o}} \neq v_\lambda \in \mathbb{P}\mathcal{E}$ .

It remains to consider the point  $\xi = \mu$ . Since  $T^t(x)$  and  $S(x)$  are defined at  $\mu$ ,  $q(x) = T^t(x) q'(x)$  and  $\tilde{q}(x) = S(x) q'(x)$  imply  $o \geq o'$  and  $\bar{o} \geq o'$ . Moreover, by (25)  $o > o'$  iff  $q'$  passes through  $v_\lambda$  and  $\bar{o} > o'$  iff  $q'$  passes through  $u_\mu$ . Since  $P(x)$  and  $R^t(x)$  have poles at  $\mu$ ,  $q'(x) = P(x) q(x) = R^t(x) \tilde{q}(x)$  imply  $o' \geq o - 1$  and  $o' \geq \bar{o} - 1$ .  $\square$

## 7. PFAFFIANS ARISING FROM DECOMPOSABLE VECTOR BUNDLES

The *Kummer variety*  $\mathcal{K}_C$  of  $C$  is by definition the quotient of the Jacobian  $JC$  by the involution  $\mathcal{L} \mapsto \mathcal{L}^{-1} \otimes \mathcal{O}_C(d-3)$ . For  $g \geq 3$  it is standard [15], that  $M_C(2, \mathcal{O}_C(d-3))$  is singular along the Kummer variety. Note that the canonical line bundle  $\omega_C \cong \mathcal{O}_C(d-3)$ , by the adjunction formula for degree  $d$  curves in  $\mathbb{P}^2$ .

In [5, Theorem 5.1] we established a one to one correspondence between decomposable vector bundles in  $M_C(2, \mathcal{O}_C(d-3)) - \Theta_{2, \mathcal{O}_C(d-3)}$  and the open subset of Kummer variety  $\mathcal{K}_C - W$ , where  $W = \{\text{line bundles of degree } \frac{1}{2}d(d-3) \text{ with no sections}\}$ . On the other hand, by [5, Section 5], the cokernel  $\mathcal{E}$  of a pfaffian representation is decomposable if and only if it is of the form

$$\mathcal{E} \cong \text{Coker } M \oplus ((\text{Coker } M)^{-1} \otimes \mathcal{O}_C(d-1))$$

for  $M$  a determinantal representation of  $C$ . The line bundle  $\text{Coker } M$  is of degree  $\frac{1}{2}d(d-1)$  with  $H^0(C, \text{Coker } M \otimes \mathcal{O}_C(-1)) = 0$  and  $\mathcal{E}$  is the cokernel of the *decomposable matrix*

$$\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \text{ mentioned in (2) .}$$

Denote  $\mathcal{L} = \text{Coker } M \otimes \mathcal{O}_C(-1)$  and  $\mathcal{L}_t = \text{Coker } M^t \otimes \mathcal{O}_C(-1)$  that both have degree  $\frac{1}{2}d(d-3)$ . Then

$$\mathcal{L} \oplus \mathcal{L}_t \in M_C(2, \mathcal{O}_C(d-3)) - \Theta_{2, \mathcal{O}_C(d-3)}$$

and by the above

$$\mathcal{L}_t \cong \mathcal{L}^{-1} \otimes \mathcal{O}_C(d-3),$$

which explains the involution on the Kummer variety. The involution appears, because in general  $M$  and  $M^t$  are not equivalent determinantal representations, but

$$\begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & M^t \\ -M & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^t & 0 \end{bmatrix} \begin{bmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{bmatrix}$$

are equivalent pfaffian representations.

The proof of Theorem 6.2 and [27, Theorem 4] imply the following

**Corollary 7.1.** *Let  $\mathcal{E}$  be the decomposable cokernel of a decomposable pfaffian representation and let  $D = \text{div Coker } M$  and  $D' = \text{div Coker } M^t$ . Pick admissible vectors of the form*

$$\begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathcal{E}(\lambda), \quad \begin{bmatrix} 0 \\ u \end{bmatrix} \in \mathcal{E}(\mu)$$

and perform the Type I elementary transformation. Then

$$\begin{aligned} \mathcal{E} &\cong \mathcal{O}_C(D) \oplus \mathcal{O}_C(D') \\ &\quad P \downarrow \\ \mathcal{E}' &\cong \mathcal{O}_C(D + \lambda - \mu) \oplus \mathcal{O}_C(D') \\ &\quad S \downarrow \\ \tilde{\mathcal{E}} &\cong \mathcal{O}_C(D + \lambda - \mu) \oplus \mathcal{O}_C(D' + \mu - \lambda). \end{aligned}$$

**Example 7.2.** Recall the explicit calculation of pfaffian representations in the canonical form (6) in Remark 2.3. Using local parameters and the implicit function theorem, it is easy to see that by setting the entries  $a_{ij}^0$  for  $1 \leq i < j \leq d$  and  $d \leq i < j \leq 2d$  in  $A_0$  to zero, we obtain the singular locus of the set of pfaffian representations of  $C$  in the canonical form (6). Note that these representations are decomposable and non-equivalent to each other. Moreover, Vinnikov's explicit parametrisation of determinantal representations of  $C$  in [26] proves that these are all the decomposable pfaffian representations.

We will conclude this section by

**Theorem 7.3.** *From any given pfaffian representation of  $C$  we can build all the nonequivalent pfaffian representations of  $C$  by finite sequences of Type I and Type II elementary transformations.*

*Proof.* A finite sequence of Type I and Type II elementary transformations gives us a new pfaffian representation of  $C$  that is in general not equivalent to the starting one, since the cokernel bundles  $\mathcal{E}(x)$ ,  $\tilde{\mathcal{E}}(x)$ ,  $\bar{\mathcal{E}}(x)$  are in general not isomorphic. The auxiliary bundles  $\mathcal{E}'(x)$  and  $\mathcal{E}''(x)$  have determinants different to  $\mathcal{O}_C(d-1)$  and thus are not cokernels of pfaffian representations.

Pick the cokernel bundle  $\mathcal{E}$  of  $x_1\sigma_2 - x_2\sigma_1 + x_0\gamma$ . We will assume that the representation is in the canonical form (6).

In the first step we bridge  $\mathcal{E}$  with a decomposable vector bundle by applying a finite number of Type II elementary transformations. A finite sequence of  $m$  Type II



be embedded as a Coble quartic hypersurface in  $\mathbb{P}^7$  with singularities along the 3-dimensional Kummer variety  $\mathcal{K}_C$ . For references check [19], [15], [4].

In this section we establish the connection between two distinguished kinds of rank 2 vector bundles on  $C$ , namely decomposable bundles corresponding to even theta characteristics and indecomposable Aronhold bundles.

A *theta characteristic* of  $C$  is a line bundle  $\mathcal{L}_\vartheta$  with the property

$$\mathcal{L}_\vartheta^{\otimes 2} \cong \omega_C \cong \mathcal{O}_C(1).$$

A theta characteristic is called even (odd) if  $\dim H^0(C, \mathcal{L}_\vartheta)$  is even (odd). By [7] there are exactly 36 even theta characteristics on a smooth plane quartic. Since  $C$  is not hyperelliptic no even theta characteristic vanishes, which means  $H^0(C, \mathcal{L}_\vartheta) = 0$ . Therefore

$$H^0(C, \mathcal{L}_\vartheta \oplus \mathcal{L}_\vartheta) = 0 \quad \text{and} \quad \det(\mathcal{L}_\vartheta \oplus \mathcal{L}_\vartheta) \cong \omega_C.$$

In our notation

$$\mathcal{L}_\vartheta \oplus \mathcal{L}_\vartheta \in M_C(2, \mathcal{O}_C(1)) - \Theta_{2, \mathcal{O}_C(1)}$$

and thus every even theta characteristic induces a decomposable pfaffian representation of  $C$ . By Section 7 the corresponding pfaffian representations are

$$\begin{bmatrix} 0 & M_\vartheta \\ -M_\vartheta & 0 \end{bmatrix},$$

where  $\mathcal{L}_\vartheta = \text{Coker } M_\vartheta \otimes \mathcal{O}_C(-1)$  and  $M_\vartheta = M_\vartheta^t$ .

**Example 8.1.** An easy computation in Wolfram Mathematica shows that, if we add  $a_{i,d+j}^0 = a_{j,d+i}^0$  for  $1 \leq i < j \leq d$  to the equations describing the decomposable representations of Example 7.2 in the canonical form (6), we get exactly 36 solutions. As expected, they are the 36  $M_\vartheta$ .

These considerations are generalised to the following proposition.

**Proposition 8.2.** *For a line bundle  $\mathcal{L}$  on a nonsingular plane quartic  $C$  with  $H^0(C, \mathcal{L}) = 0$  the following are equivalent:*

- $\mathcal{L}$  is an even theta characteristics on  $C$ ,
- $\mathcal{L} \cong \mathcal{L}^{-1} \otimes \mathcal{O}_C(1)$ ,
- $\mathcal{L} = \text{Coker } M \otimes \mathcal{O}_C(-1)$  where  $M$  is a symmetric determinantal representation of  $C$  with the property  $\text{Coker } M \cong \text{Coker } M^t$ .

To every symmetric determinantal representation  $M_\vartheta$  of  $C$  one can associate a net of quadrics  $\mathbb{M}_\vartheta$  in  $\mathbb{P}^3$ . The base locus of  $\mathbb{M}_\vartheta$  consists of 8 points  $b_1, \dots, b_8$ , called the Cayley octad. We refer to [7, Theorem 6.3.2] that  $b_i M_\vartheta b_j$  for distinct  $i, j = 1, \dots, 8$  define the 28 bitangents to  $C$ , arranged in Aronhold sets. A recent result by Lehavi [16] shows, that the set of bitangents uniquely determines  $C$ . Moreover, there is a bijection between the 28 odd theta characteristics  $\mathcal{L}_{\vartheta_{ij}}$  and bitangents. Any even theta characteristic different from  $\mathcal{L}_\vartheta$  can be represented by the divisor class

$$(27) \quad \vartheta_{i,jkl} = \vartheta_{ij} + \vartheta_{ik} + \vartheta_{il} - K_C \text{ for distinct } i, j, k, l.$$

Next we define Aronhold bundles on  $C$  following [22]. Given  $\mathcal{J} \in \text{Pic}^1 C$  we define the 3-dimensional projective space  $\mathbb{P}(\mathcal{J}) := \mathbb{P} \text{Ext}^1(\omega_C \mathcal{J}^{-1}, \mathcal{J}) = |\omega_C^2 \mathcal{J}^{-2}|^*$ . A point in  $\mathbb{P}(\mathcal{J})$  defines an isomorphism class of extensions

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{K} \longrightarrow \omega_C \mathcal{J}^{-1} \longrightarrow 0.$$

On  $C$  pick the following data:

- an even theta characteristic  $\mathcal{L}_\vartheta$ ,
- a line bundle  $\mathcal{J}_\vartheta \in \text{Pic}^1 C$  such that  $\mathcal{J}_\vartheta^2 = \mathcal{L}_\vartheta$ ,
- a base point  $b$  of the net of quadrics  $\mathbb{M}_\vartheta$ .

The stable (thus indecomposable) rank 2 bundle with canonical determinant  $\mathcal{O}_C(1)$  defined by the point  $b \in \mathbb{P}(\mathcal{J}_\vartheta)$  is called the *Aronhold bundle*  $\mathcal{K}_{b,\vartheta}$ . Up to 2-torsion points of  $JC$ , the bundles  $\mathcal{K}_{b,\vartheta}$  are in 1-to-1 correspondence with the 288 unordered Aronhold sets.

We mention a useful characterisation of Aronhold bundles: Let  $\mathcal{K}$  be a stable noneffective rank 2 vector bundle with canonical determinant. By [14] the set of line subbundles of maximal degree has cardinality 8

$$\{\mathcal{J} \in \text{Pic}^1 C : \mathcal{J} \hookrightarrow \mathcal{K}, \text{ i.e. } h^0(\mathcal{J}^{-1} \otimes \mathcal{K}) > 0\} = \{\mathcal{J}_1, \dots, \mathcal{J}_8\}.$$

Since  $\mathcal{K} \in \mathbb{P}(\mathcal{J}_i)$  for  $i = 1, \dots, 8$ , there exist 28 effective divisors  $D_{ij}$  of degree 2 satisfying

$$(28) \quad \mathcal{J}_i \otimes \mathcal{J}_j = \mathcal{O}_C(D_{ij}) \quad \text{for distinct } i, j = 1, \dots, 8.$$

Conversely,  $\mathcal{K}$  is uniquely determined by eight line bundles with property (28) by [6]. Finally,  $\mathcal{K}$  is an Aronhold bundle if and only if the 28 effective divisors  $D_{ij}$  correspond to bitangents on  $C$ .

Ottaviani [21] gives a nice description of the Aronhold invariant as a pfaffian which we briefly recall. The *Aronhold invariant* of plane cubics is a quartic equation of  $\sigma_3(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ , which deals with the condition to express an equation of a plane cubic as the sum of three cubics. Here

$$\sigma_3(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) = \text{Zariski closure of } \{g_1^3 + g_2^3 + g_3^3 ; g_i \text{ linear forms}\}$$

denotes the 3-secant of the Veronese variety. Explicitly, the Aronhold invariant evaluated in

$$w_{000}x^3 + w_{111}y^3 + w_{222}z^3 + 6w_{012}xyz + 3w_{001}x^2y + 3w_{002}x^2z + 3w_{011}xy^2 + 3w_{022}xz^2 + 3w_{112}y^2z + 3w_{122}yz^2$$

equals  $\text{Pf } Ar$  for

$$(29) \quad Ar = \begin{bmatrix} 0 & w_{222} & -w_{122} & 0 & w_{112} & 0 & w_{022} & -w_{012} \\ & 0 & w_{022} & w_{122} & -w_{012} & -w_{022} & 0 & w_{002} \\ & & 0 & -w_{112} & 0 & w_{012} & -w_{002} & 0 \\ & & & 0 & -w_{111} & 0 & -w_{012} & w_{011} \\ & & & & 0 & -w_{011} & w_{001} & 0 \\ & & & & & 0 & w_{002} & -w_{001} \\ & & & & & & 0 & w_{000} \\ & & & & & & & 0 \end{bmatrix}.$$

On the other hand recall the *Scorza map* between plane quartics [8]:

$$F \mapsto \text{the Clebsch covariant quartic } S(F)$$

which is by definition

$$F \mapsto \text{polar cubic } P_x(F) \text{ at } x \in \mathbb{P}^2 \mapsto \text{Aronhold invariant}(P_x(F)).$$

Note that in this notation the coefficients  $w_{ijk}$  of the cubic  $P_{(x_0, x_1, x_2)}(F)$  are linear in  $x_0, x_1, x_2$ . By [8, Section 7] the curve  $S(F)$  carries a unique even theta characteristic  $\vartheta$ , more precisely, the Scorza map

$$F \mapsto (S(F), \vartheta)$$

is an injective birational isomorphism and the natural projection to the first component is an unramified covering of degree 36.

**Proposition 8.3.** *From the Aronhold pfaffian representation of  $S(F)$  we can recover the unique theta characteristic on  $S(F)$ .*

*Proof.* Our main tool will be the Scorza correspondence relating points on  $S(F)$ . On a nonsingular projective curve  $X$  of genus  $g > 0$  and a non-effective theta characteristic  $\vartheta$  we introduce *the Scorza correspondence*

$$R_\vartheta := \{(x, y) \in X \times X : h^0(\vartheta + x - y) > 0\}.$$

Dolgachev [7] proved that for  $X$  non-hyperelliptic,  $R_\vartheta$  are the only symmetric correspondences of type  $(g, g)$  without united points and some valence. For a correspondence  $\mathcal{R}$  define  $\mathcal{R}(x) = \{y \in X : (x, y) \in \mathcal{R}\}$ .  $\mathcal{R}$  has *valence*  $v$  if the divisor class of  $\mathcal{R}(x) + vx$  does not depend on  $x$ . Therefore, on  $S(F)$  there are 36 symmetric correspondences of type  $(3, 3)$ .

We are able to explicitly determine which points on  $S(F)$  are  $R_\vartheta$  related in two essentially different ways:

- (i) from a symmetric determinantal representation of  $S(F)$ ,
- (ii) from an Aronhold pfaffian representation of  $S(F)$ .

The two ways which induce the same Scorza correspondence on  $S(F)$  will relate the Aronhold pfaffian representation with the unique theta characteristic.

In step (i) we will use  $M_\vartheta$ , the symmetric determinantal representation of  $S(F)$  from Proposition 8.2. By definition  $\lambda, \mu \in S(F)$  are NOT  $R_\vartheta$  related if and only if  $h^0(\vartheta + \lambda - \mu) = 0$ . This means that  $\mathcal{O}_{S(F)}(\vartheta + \lambda - \mu) \oplus \mathcal{O}_{S(F)}(\vartheta + \mu - \lambda)$  has canonical determinant and no sections. Therefore, after tensoring by  $\mathcal{O}_{S(F)}(1)$ , it equals the cokernel of another decomposable pfaffian representation of  $S(F)$ . By Corollary 7.1 it is obtained from

$$(\mathcal{L}_\vartheta \oplus \mathcal{L}_\vartheta) \otimes \mathcal{O}_{S(F)}(1) \cong \text{Coker } M_\vartheta \oplus \text{Coker } M_\vartheta$$

by the Type I elementary transformation based on the admissible vectors

$$\begin{bmatrix} v \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u \end{bmatrix}, \text{ where } v \in \text{Coker } M_\vartheta(\lambda), u \in \text{Coker } M_\vartheta(\mu).$$

The definition of admissible vectors (12) thus proves that  $\lambda R_\vartheta \mu$  if and only if

$$v^t M_\vartheta(x) u \equiv 0 \text{ for all } v \in \text{Coker } M_\vartheta(\lambda), u \in \text{Coker } M_\vartheta(\mu), x \in \mathbb{P}^2.$$

Step (ii): Given an Aronhold pfaffian representation (29), we can retrieve  $F(x, y, z)$  from  $w_{ijk}(x_0, x_1, x_2)$  by integrating

$$P_{(x_0, x_1, x_2)}(F) = x_0 \frac{\partial F}{\partial x}(x, y, z) + x_1 \frac{\partial F}{\partial y}(x, y, z) + x_2 \frac{\partial F}{\partial z}(x, y, z).$$

The definition of the Aronhold invariant implies that, for any  $\lambda \in S(F)$  there exist linear forms  $g_1, g_2, g_3$  such that  $P_\lambda(F) = g_1^3 + g_2^3 + g_3^3$ . This defines another  $(3, 3)$  correspondence without united points on  $S(F)$ , which must by the above equal to some  $R_\vartheta$ :  $\lambda, \mu \in S(F)$  are *related* if the second polar  $P_{\lambda, \mu}(F) = g_i^2$  for some  $i = 1, 2, 3$ . Obviously  $\mu$  equals one of the vertices of the polar triangle spanned by the lines  $g_1, g_2, g_3$ . In [8, Theorem 7.8] Dolgachev and Kanev gave a beautiful construction of  $F$  from the polar triangles in  $S(F)$  and thus reconstructed  $F$  from  $(S(F), \vartheta)$ .  $\square$

**Remark 8.4.** Pauly's construction [22, §4.2] assigns to every stable noneffective  $\mathcal{K}$  with canonical determinant a net of quadrics whose bitangents correspond to  $D_{ij}$  in (28). This gives another proof of Proposition 8.3 since the Aronhold bundle  $\mathcal{K}_{b,\vartheta}$  induces exactly the net of quadrics  $\mathbb{M}_\vartheta$ . We are however not able to implement this construction explicitly.

**Corollary 8.5.** *Denote by  $R_\vartheta(\lambda)$  the polar triangle to  $\lambda \in S(F)$ . By [8],  $R_\vartheta(\lambda) - \lambda$  equals the divisor class of  $\mathcal{L}_\vartheta$  and is thus independent of  $\lambda$ . Then all the symmetric determinantal representations of  $S(F)$  can be obtained from  $M_\vartheta$  by a sequence of three Type I elementary transformations, by applying the second part of Proof 7.3 on the divisor  $\vartheta_{i,jkl} - \vartheta = R_{\vartheta_{i,jkl}}(\lambda) - \lambda - R_\vartheta(\lambda) + \lambda = R_{\vartheta_{i,jkl}}(\lambda) - R_\vartheta(\lambda)$ .*

**Example 8.6.** The Scorza map sends  $F = x^4 + x^3y - y^4 - yz^3 + 107^{1/3}xy^2z$  to

$$S(F) = \text{Pf}[Ar] = \begin{matrix} 27x_0^3x_1 - 432x_0x_1^3 - x_1^4 - 72 \cdot 107^{1/3}x_0^2x_1x_2 - \\ 9 \cdot 107^{1/3}x_0x_1^2x_2 + 81 \cdot 107^{-1/3}x_0^2x_2^2 - 108x_0x_2^3 - 27x_1x_2^3 \end{matrix}$$

for  $Ar$  defined in (29) with

$$\begin{matrix} w_{000} = 4x_0 + x_1, & w_{001} = x_0, & w_{011} = 1/3x_2, & w_{111} = -4x_1, & w_{002} = 0, \\ w_{012} = 1/3x_1, & w_{112} = 1/3x_0, & w_{022} = 0, & w_{122} = -x_2, & w_{222} = -x_1. \end{matrix}$$

Following the proof of Proposition 8.3 we will compute the unique theta characteristic on  $S(F)$ . We calculate in **Wolfram Mathematica** to precision  $10^{-10}$ . For  $\lambda = (1, 0, \frac{3}{4}107^{-1/3}) \in S(F)$  we get  $P_\lambda(F) = g_1^3 + g_2^3 + g_3^3$  for

$$\begin{matrix} g_1 = (4x + y)(-0.198' - 0.344'i), \\ g_2 = (0.002' - 2.089'i)y + (-0.181' + 0.104'i)z, \\ g_3 = (0.002' + 2.089'i)y + (-0.181' - 0.104'i)z, \end{matrix}$$

which is explicitly obtained from the equality  $\det \text{Hess}(P_\lambda(F)) = g_1g_2g_3$ . The intersections

$$\begin{matrix} \mu^1 = g_2 \cap g_3 = (1, 0, 0), \\ \mu^2 = g_1 \cap g_3 = (1, -4, -20.034' + 34.609'i), \\ \mu^3 = g_1 \cap g_2 = (1, -4, -20.034' - 34.609'i) \end{matrix}$$

determine the polar triangle  $R(\lambda)$  of  $\lambda$ . This proves that  $\lambda$  is in relation with  $\mu^1$ ,  $\mu^2$  and  $\mu^3$  on  $S(F)$ .

On the other hand it is easy to compute all the 36 symmetric determinantal representations of  $S(F)$ . For example, for

$$M_\vartheta = \begin{matrix} x_1 \text{Id}_4 - x_2 \text{Diag}[0, -3, 3(-1)^{1/3}, -3(-1)^{2/3}] + \\ x_0 \begin{bmatrix} 4 & -24.296' & 23.685' + 0.336'i & -23.685' + 0.336'i \\ & \frac{428}{3} - 107^{1/3} & -141.449' + 2.004'i & 141.449' + 2.004'i \\ & & \frac{428}{3} - 107^{1/3}(-1)^{2/3} & -145.099' \\ & & & \frac{428}{3} + 107^{1/3}(-1)^{1/3} \end{bmatrix} \end{matrix}$$

we have

$$\begin{matrix} v_\lambda = [-0.006 - 0.009i, -0.335 - 0.482i, -0.571 + 0.04i, -0.236 + 0.521i]^t \in \text{Coker } M_\vartheta(\lambda), \\ v_{\mu^1} = [0, -0.543 - 0.164i, -0.419 + 0.404i, 0.124 + 0.569i]^t \in \text{Coker } M_\vartheta(\mu^1), \\ v_{\mu^2} = [0.602 - 0.73i, -0.186 - 0.025i, -0.124 + 0.147i, 0.062 + 0.172i]^t \in \text{Coker } M_\vartheta(\mu^2), \\ v_{\mu^3} = [0.613 + 0.72i, -0.185 + 0.028i, -0.059 + 0.173i, 0.127 + 0.145i]^t \in \text{Coker } M_\vartheta(\mu^3). \end{matrix}$$

We check that

$$v_\lambda^t M_\vartheta(x_0, x_1, x_2) v_{\mu^i} = 0 \text{ for any } (x_0, x_1, x_2) \in \mathbb{P}^2, i = 1, 2, 3.$$

This proves that the corresponding  $\mathcal{L}_\vartheta$  is the unique theta characteristic on  $S(F)$  that we were looking for.

The question, whether the unique even theta characteristic on  $S(F)$  can be recovered from the cokernel bundle, was first asked by Ottaviani [21, Remark 2.3]. Proposition 8.3 gives a positive answer with a concrete construction done in Example 8.6. However, for no even theta characteristic  $\mathcal{L}_\vartheta$  on  $S(F)$  holds

$$h^0(\text{Coker } Ar \otimes \mathcal{O}_{S(F)}(-1) \otimes \mathcal{L}_\vartheta^{-1}) > 0.$$

Indeed, if  $\text{Coker } Ar \otimes \mathcal{O}_{S(F)}(-1) \otimes \mathcal{L}_\vartheta^{-1}$  is effective, it is not stable since it has trivial determinant. Then  $\text{Coker } Ar$  is also not stable and thus isomorphic to a direct sum of two line bundles. This is in contradiction with  $Ar$  being an indecomposable representation of  $S(F)$ .

## REFERENCES

- [1] T. Abe. *The elementary transformation of vector bundles on regular schemes*, **359** (9) Transactions of the American Mathematical Society (2007), 4285-4295.
- [2] J. A. Ball and V. Vinnikov. *Zero-pole interpolation for matrix meromorphic functions on a compact Riemann surface and a matrix Fay trisecant identity*, **121**(4) American Journal of Mathematics, (1999), 841-888.
- [3] A. Beauville. *Determinantal Hypersurfaces*, Michigan Math. J. **48** (2000), 39–63.
- [4] A. Beauville. *Vector bundles on curves and generalized theta functions: recent results and open problems*, Complex Algebraic Geometry, MSRI Publications **28** (1995), 17–33.
- [5] A. Buckley and T. Košir. *Plane Curves as Pfaffians*, eprint arXiv:math/0805.2831v1.
- [6] I. Choe, J. Choy and S. Park. *Maximal line subbundles of stable bundles of rank 2 over an algebraic curve*, Geom. Dedicata **125** (2007), 191–202.
- [7] I. Dolgachev. *Topics in classical algebraic geometry*, Lecture Notes <http://www.math.lsa.umich.edu/~idolga/lecturenotes.html>.
- [8] I. Dolgachev and V. Kanev. *Polar covariants of plane cubics and quartics*, Adv. Math. **98** (1993), 216–301.
- [9] J.M. Drezet and M.S. Narasimhan. *Groupe de Picard des varietes de modules de fibres semi-stables sur les courbes algebriques*, Invent. Math. **97** (1989), 53–94.
- [10] L. Fuentes and M. Pedreira. *The projective theory of ruled surfaces*, Note Mat. (1) **24** (2005), 25-63.
- [11] W. Fulton and P. Pragacz. *Shubert varieties and degeneracy loci*, Lecture notes in Mathematics **1689**, Springer-Verlag, 1998.
- [12] R. Hartshorne. *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, 1977.
- [13] P. Lancaster and L. Rodman. *Canonical forms for symmetric / skew-symmetric real matrix pairs under strict equivalence and congruence*, Linear Algebra Appl. **406** (2005), 1–76.
- [14] H. Lange and M.S. Narasimhan. *Maximal subbundles of rank 2 vector bundles on curves*, Math. Ann. **266** (1984) 55–72.
- [15] Y. Laszlo. *A propos de l'espace des modules des fibres de rang 2 sur une courbe*, Math. Annalen **299** (1994), 597–608.
- [16] D. Lehavi. *Any smooth plane quartic can be reconstructed from its bitangents*, Israel J. Math. **146** (2005), 371-379.
- [17] M. Marcus and R. Westwick. *Linear maps on skew-symmetric matrices: The invariance of elementary symmetric functions*, Pacific J. Math. (3) **14** (1960).
- [18] M. Maruyama. *On a family of algebraic vector bundles*, Number Theory, Algebraic Geometry and Commutative Algebra, Kinokuniya (1973), 95–149.
- [19] M.S. Narasimhan and S. Ramanan.  *$2\Theta$  linear systems on Abelian varieties*, Vector bundles on algebraic varieties, Oxford University Press (1987), 415–427.
- [20] P. E. Newstead. *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, 1978.
- [21] G. Ottaviani. *An invariant regarding Waring's problem for cubic polynomials*, Nagoya Math. J. **193** (2009), 95–110.
- [22] C. Pauly. *Self-Duality of Coble's Quartic Hypersurface and Applications*, Michigan Math. J. **50** (2002), 551–574.
- [23] J. Le Potier. *Lectures on Vector Bundles*, Cambridge studies in advanced mathematics **54**, Cambridge University Press, 1997.



- [24] C. S. Seshadri. *Fibres vectoriels sur les courbes algebriques*, Asterisque **96**, 1982.
- [25] A. Shapiro and V. Vinnikov. *Rational transformations of algebraic curves and elimination theory*, eprint arXiv:math/0507233.
- [26] V. Vinnikov. *Complete description of determinantal representations of smooth irreducible curves*, Lin. Alg. Appl., **125** (1989), 103–140.
- [27] V. Vinnikov. *Elementary transformations of determinantal representations of algebraic curves*, Lin. Alg. Appl., **135** (1990), 1–18.

DEPARTMENT OF MATHEMATICS, JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA, TEL: +38631810619,  
FAX: +38612517281

*E-mail address:* `anita.buckley@fmf.uni-lj.si`