On the Transversality of Restricted Linear Systems

Anita Buckley and Marina Zompatori

June 4, 2002

ABSTRACT

In this paper we prove a generalization of the Transversality Lemma first proved by Hirschowitz [5] and later by Ciliberto and Miranda [1].We plan to use this lemma to calculate the dimensions of linear systems of plane curves with homogeneous vanishing conditions, using a degeneration technique developed by Ciliberto and Miranda [1].

Introduction

In order to study linear systems of plane curves of the type $\mathcal{L}_d(m^n)$, i.e. systems of plane curves of degree d going through n points in general position with multiplicity at least m at every point, it is often useful to consider degenerations of the plane itself, of the bundle $\mathcal{O}(d)$ and of the position of the points. One possible approach is to degenerate the plane by blowing up general points on the central fiber of $\mathbb{P}^2 \times \Delta$, where Δ is a disc centered at the origin as follows.

Consider $V = \mathbb{P}^2 \times \Delta$ and its projections $p_1 : V \to \Delta$ and $p_2 : V \to \mathbb{P}^2$. Denote $V_t = \mathbb{P}^2 \times \{t\}$. Consider n general points in the plane V_0 and blow V up at these points. We get a new threefold X equipped with maps $f : X \to V$, and $\pi = p_1 \circ f : X \to \Delta$, which gives a flat family of surfaces over Δ. If we denote by X_t the fiber of π over $t \in \Delta$, then for $t \neq 0$, X_t is just a plane $\mathbb{P}^2$, whereas X_0 consists of the proper transform of V_0, which we denote by Y, and of n copies of $\mathbb{P}^2$. The n disjoint copies of $\mathbb{P}^2$ are the exceptional divisors and each of them intersects Y along an exceptional line. Using this degeneration with an appropriate degeneration of the line bundle $\mathcal{O}(d)$, we see then that the system we want to study is now obtained by glueing together curves from the different $\mathbb{P}^2$s and curves from Y.

[1]We wish to thank the organizers of Pragmatic 2001 for sponsoring our stay in Catania, Rick Miranda and Ciro Ciliberto for introducing us to problems on linear systems and for many invaluable conversations. We would also like to thank Dan Abramovich, who suggested and explained the use of monodromy arguments in the proof of the transversality lemma, and Balázs Szendrői for corrections and comments.

To understand the dimension of the resulting system, which will be a specialization of $\mathcal{L}_d(m^n)$, we need to calculate the dimension of the intersection of linear systems on Y and the systems on the planes.

If we think about the vector spaces of polynomials whose projectivizations give the systems of curves described above, the problem becomes then finding the dimension of the intersection of a vector space W, which in our application corresponds to the restriction of curves on Y onto the exceptional lines, with a sum of vector spaces $U_1 \oplus \ldots \oplus U_n$, where each U_i is the restriction onto exceptional line E_i of the system on the corresponding $\mathbb{P}^2$. This intersection takes place in $V_1 \oplus \ldots \oplus V_n$, where the projectivization of each V_i is the linear system of divisors of some degree k on $\mathbb{P}^1$.

In our application, all the copies of $\mathbb{P}^2$ in X, are obtained as exceptional divisors from blow-ups at points in general position. In this setting, monodromy considerations give us symmetry conditions on the space W, which, in a more general case, have to be assumed.

We need the notion of *transversal intersection*, which we now define.

Definition 1. *Suppose V_1, V_2 are two subspaces of a vector space V. Then we say that V_1 and V_2 intersect transversally if either* $\dim(V_1 \cap V_2) = 0$ *or* $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(V)$.

Hirschowitz [5] and later Ciliberto and Miranda [1] proved the following result known as The Transversality Lemma.

Proposition 2. *Let $G = GL(2, \mathbb{C})$ be the automorphism group of $\mathbb{C}^2$. Let V be the vector space whose projectivization is the linear system of divisors of degree k on $\mathbb{P}^1$. Note that G acts naturally on V, and on linear subspaces of V of any dimension. Then for any two linear subspaces W and U of V there is an element $g \in G$ such that W meets gU transversally.*

The purpose of this paper is to generalize the above result to an arbitrary number of vector spaces using an inductive argument.

Let V_i, for $i = 1, \ldots, n$ be isomorphic vector spaces whose projectivization is the linear system of divisors of degree k, and for each i let U_i be a subspace of V_i. The same group $G = GL(2, \mathbb{C})$ is acting on every V_i. Consider a vector space W which is a subspace of $V_1 \oplus \ldots \oplus V_n$ and let $p_i : W \to V_i$ be the projection map to each component. The situation is shown in the following diagram of vector spaces:

$$\begin{array}{ccccccccccc} & & U_1 & \oplus & U_2 & \oplus & \cdots & \oplus & U_{n-1} & \oplus & U_n \\ & & \cap & & \cap & & & & \cap & & \cap \\ W & \hookrightarrow & V_1 & \oplus & V_2 & \oplus & \cdots & \oplus & V_{n-1} & \oplus & V_n \end{array} \qquad (\spadesuit)$$

Our aim is to find elements $g_1, \ldots, g_n \in G$ such that W and $g_1U_1 \oplus \ldots \oplus g_nU_n$ meet transversally in $V_1 \oplus \ldots \oplus V_n$. This will not be possible for general W, U_i and V_i. The result will hold if the vector spaces satisfy a *quasi-symmetric* condition which we now define.

Definition 3. *The diagram (♠) is* <u>*quasi-symmetric*</u> *if it satisfies the following two properties:*
(a) *if there exist an integer r, $1 \leq r < n$, and elements $g_1, \ldots, g_r$ in G such that*

$$(p_1, \ldots, p_r)W \cap (g_1U_1 \oplus \ldots \oplus g_rU_r) = \{0\}$$

then there exist $g'_1, \ldots, g'_{r+1}$ in G such that

$$(p_1, \ldots, p_{r+1})W \cap (g'_1U_1 \oplus \ldots \oplus g'_{r+1}U_{r+1}) = \{0\}$$

as well;
(b) *suppose that for $1 \leq r < n$, there is an r-tuple of elements $g_1, \ldots, g_r$ such that the intersection*

$$(p_1, \ldots, p_r)W \cap (g_1U_1 \oplus \ldots \oplus g_rU_r)$$

is transversal and nontrivial. Define

$$W' = (p_1, \ldots, p_r)^{-1}[(p_1, \ldots, p_r)W \cap (g_1U_1 \oplus \ldots \oplus g_rU_r)]$$

and assume that for some g_{r+1} the intersection

$$p_{r+1}W' \cap g_{r+1}U_{r+1} = \{0\}$$

then there exist $g'_1, \ldots, g'_{r+1}$ in G such that

$$(p_1, \ldots, p_{r+1})W \cap (g'_1U_1 \oplus \ldots \oplus g'_{r+1}U_{r+1}) = \{0\}$$

1 Proof of Transversality Lemma for any n

Theorem 4. *If the diagram (♠) satisfies the quasi-symmetric condition, then there exist elements $g_1, \ldots, g_n \in G$ such that W and*

$$g_1U_1 \oplus \cdots \oplus g_nU_n$$

meet transversally in $V_1 \oplus \ldots \oplus V_n$.

The aim of this section is to prove the Theorem 4. We prove it using induction on n. The case $n = 1$ is given by the Proposition 2. Its proof can be found in [1] on page 199.

Assume that the statement holds for $n-1$. If

$$(p_1, \dots p_{n-1})W \cap (g_1U_1 \oplus \dots \oplus g_{n-1}U_{n-1}) = \{0\},$$

then by condition (a) the intersection of

$$W \text{ with } g'_1U_1 \oplus \dots \oplus g'_nU_n,$$

for some $g'_1 \dots, g'_n \in G$, is also $\{0\}$. In this case we get transversal intersection.

Next define

$$W_1 = (p_1, \dots, p_{n-1})^{-1}[(p_1, \dots p_{n-1})W \cap g_1U_1 \oplus \dots \oplus g_{n-1}U_{n-1}].$$

Note that W_1 equals

$$W \cap (g_1U_1 \oplus \dots \oplus g_{n-1}U_{n-1} \oplus V_n) =$$

$$\{(v_1, \dots, v_n) \in W \mid v_i \in g_iU_i \ \forall \ i = 1, \dots, n-1\}.$$

Its dimension is given by the following formula,

$$\begin{aligned} \dim W_1 \quad &= \qquad (1) \\ \dim(\ker(p_1, \quad \dots, \quad p_{n-1})) &+ \dim((p_1, \dots p_{n-1})W \cap g_1U_1 \oplus \dots \oplus g_{n-1}U_{n-1}). \end{aligned}$$

By applying the Transversality Lemma for $n = 1$, (Proposition 2), to $p_n(W_1)$ and U_n, subspaces of V_n, we can find an element $g_n \in G$ such that $p_n(W_1)$ and g_nU_n meet transversally. Suppose they meet trivially, then by condition (b)there exist $g'_1, \dots, g'_n \in G$ such that

$$W \cap (g'_1U_1 \oplus \dots \oplus g'_nU_n) = \{0\}.$$

So in the case $p_n(W_1) \cap g_nU_n = \{0\}$, we again get the desired transversality.

The only case we still need to analyze is when the above transversal intersections are non trivial.

Define

$$W_2 = (p_n|_{W_1})^{-1}(p_n(W_1) \cap g_nU_n)$$

which equals to $W \cap (g_1U_1 \oplus \ldots \oplus g_nU_n)$. Its dimension is given by the following expression,

$$\dim W_2 = \dim \ker(p_n|_{W_1}) + \dim(p_n(W_1) \cap g_nU_n). \tag{2}$$

Observe also that $\dim(W \cap (g_1U_1 \oplus \ldots \oplus g_nU_n)) = \dim W_2$. By transversality in the case $n = 1$, having already examined the case of trivial intersection, we have

$$\dim(p_n(W_1) \cap g_nU_n) = \dim p_n(W_1) + \dim U_n - \dim V_n.$$

By formula (2), we get the following identity,

$$\dim(W_2) = \dim \ker(p_n|_{W_1}) + \dim(p_n(W_1)) + \dim(U_n) - \dim(V_n).$$

Now note that $\dim(\ker(p_n|_{W_1})) + \dim(p_n(W_1)) = \dim W_1$. Therefore $\dim W_2 = \dim W_1 + \dim U_n - \dim V_n$. By formula (1), we derive the following expression for $\dim W_2$,

$$\begin{aligned}\dim W_2 \quad = \quad & \dim \ker(p_1, \ldots p_{n-1}) \\ & + \dim((p_1, \ldots p_{n-1})(W) \cap g_1U_1 \oplus \ldots \oplus g_{n-1}U_{n-1}) \\ & + \dim U_n - \dim(V_n).\end{aligned}$$

Since we have already considered the case

$$(p_1, \ldots p_{n-1})W \cap (g_1U_1 \oplus \ldots \oplus g_{n-1}U_{n-1}) = \{0\},$$

we can now assume that this intersection is not zero and apply the transversality lemma in case $n - 1$. This gives

$$\dim \ker(p_1, \ldots p_{n-1}) + \dim(p_1, \ldots p_{n-1})(W) + \dim U_1 + \ldots + \dim U_n - \dim V_1 - \ldots - \dim V_n.$$

Finally, $\dim \ker(p_1, \ldots p_{n-1}) + \dim(p_1, \ldots p_{n-1})(W)$ add up to $\dim(W)$ and the expression above reduces to

$$\dim W + \dim U_1 + \ldots \dim U_n - \dim V_1 - \ldots - \dim V_n.$$

We have $\dim(W \cap (g_1U_1 \oplus \ldots \oplus g_nU_n)) = \dim W_2$ and is therefore equal to

$$\dim W + \dim U_1 + \ldots \dim U_n - \dim V_1 - \ldots - \dim V_n.$$

This implies that the two spaces W and $g_1U_1 \oplus \ldots \oplus g_nU_n$ intersect transversally in $V_1 \oplus \ldots \oplus V_n$, which completes the proof. $\square$

2 Application to systems of plane curves.

As we described in the Introduction and more precisely in [7], we need the generalized transversality lemma in order to study the dimension of linear systems of plane curves of some degree with base points of equal multiplicity.

By a degeneration of the plane and of the linear system itself, as explained in [7], we get the following diagram of vector spaces.

$$\begin{array}{ccccccccc} & & \mathbf{L}_k(m^{n_2}) & \oplus & \mathbf{L}_k(m^{n_2}) & \oplus \cdots \oplus & \mathbf{L}_k(m^{n_2}) & \\ & & \downarrow r_1 & & \downarrow r_2 & & \downarrow r_{n_1} & \quad (\ast) \\ \mathbf{L}_d(k^{n_1}) & \xrightarrow{\rho_Y} & H^0(\mathcal{O}_{E_1}(k)) & \oplus & H^0(\mathcal{O}_{E_2}(k)) & \oplus \cdots \oplus & H^0(\mathcal{O}_{E_{n_1}}(k)). & \end{array}$$

Here $\mathbf{L}_k(m^{n_2})$ is the linear system of plane curves of degree k with n_2 generic base points all of multiplicity m and similarly $\mathbf{L}_d(k^{n_1})$ denotes the linear system of plane curves of degree d with n_1 base points in general position all having the same multiplicity k.

This is how these systems arise in the application we are interested in: the environment we're in consists of the union of a variety Y, obtained as the proper transform of the plane blown up at n_1 generic points, and n_1 copies of the projective plane $\mathbb{P}_i$, obtained as exceptional divisors in our construction. Each $\mathbb{P}_i$ intersects Y along a line E_i. We not only degenerate the plane but also the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$, by pulling it back to the variety obtained by blowing up and twisting by $\mathcal{O}(kY)$. The system whose dimension we want to compute contains exactly the curves that are deformations of plane curves in $\mathbf{L}_d(m^{n_1 n_2})$. It is given as a fibered product of the systems $\mathbf{L}_k(m^{n_2})$ on $\mathbf{P}_i$ and the system $\mathbf{L}_d(k^{n_1})$ on Y. The restrictions of these systems to the lines E_i gives divisors of degree k on the line. In other words, the system is obtained by "pasting" together the system from Y and the systems from the planes $\mathbf{P}_i$ along the lines E_i. Details about this construction, which is a degeneration technique introduced by C. Ciliberto and R. Miranda in order to study systems of plane curves, can be found in [7].

By restricting to exceptional lines E_i we therefore get the following diagram

$$\begin{array}{ccccccccccc} & & U_1 & \oplus & U_2 & \oplus & \cdots & \oplus & U_{n_1-1} & \oplus & U_{n_1} \\ & & \cap & & \cap & & & & \cap & & \cap \\ W & \hookrightarrow & V_1 & \oplus & V_2 & \oplus & \cdots & \oplus & V_{n_1-1} & \oplus & V_{n_1} \end{array},$$

on which we want to use the transversality lemma.

3 Systems of plane curves satisfy quasi-symmetric condition

Our goal is to apply the transversality lemma to the above diagram. We need to show that it satisfies the quasi-symmetric condition.

Note that for a general number of points in $\mathbf{P}^2$, there exists no automorphism on $\mathbf{P}^2$ permuting these points. In order to get the symmetric group action, we need to enlarge our ambient set-up and use monodromy.

Consider the product of n_1 copies of the plane, $\mathbf{P}^2 \times \ldots \times \mathbf{P}^2$, and its subset $D := \{(x_1, \ldots, x_{n_1}) \mid \exists 1 \leq i < j \leq n_1 \text{ such that } x_i = x_j\}$. Then denote $B = (\mathbf{P}^2 \times \ldots \times \mathbf{P}^2) \setminus D$.

Next consider the product $B \times \mathbf{P}^2 \times \Delta$, where Δ is a disc centered at the origin. Such product comes equipped with a projection to B which has sections $\sigma_1, \ldots, \sigma_{n_1}$.

We blow up $B \times \mathbb{P}^2 \times \Delta$ along these sections and thus obtain the space $\mathcal{X}$. Such space comes naturally with projection $\pi_1 : \mathcal{X} \to \Delta$. Consider $\mathcal{X}_0 := \pi_1^{-1}(0)$, a reducible scheme that consists of the exceptional subschemes $\mathcal{P}_1, \ldots, \mathcal{P}_{n_1}$ and of $\mathcal{Y}$, which is the direct transform of $B \times \mathbb{P}^2 \times \{0\}$.

$$
\begin{array}{ccccc}
\mathcal{Y} \cup_1^{n_1} \mathcal{P}_i = \mathcal{X}_0 & \subset & \mathcal{X} & \xrightarrow{\pi_1} & \Delta \\
 & & \big\downarrow \text{blow-up} & & \\
\big\downarrow f & & B \times \mathbf{P}^2 \times \Delta & & \\
 & & \downarrow & & \\
B & = & B & &
\end{array}
$$

Also denote by $\mathcal{E}_i$, the bundle $\mathcal{P}_i \cap \mathcal{Y}$.

Remark: Note that the fiber over each $b \in B$ is the same construction we describe in [7], where we just take $\mathbb{P}^2 \times \Delta$ and blow it up at n_1 points in the central fiber. In the same construction, we also need to degenerate the bundle $\mathcal{O}_{\mathbb{P}^2}(d)$, by pulling it back to the three-fold obtained by blowing up and twisting it with $\mathcal{O}(Y)$. We proceed in a similar way now in our more general situation.

Let η be the natural map $\eta : \mathcal{X} \to \mathbb{P}^2$.

Let $\mathcal{F}$ be the following sheaf on $\mathcal{X}$,

$$\mathcal{F} = \eta^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}_{\mathcal{X}}(k\mathcal{Y}).$$

If we restrict $\mathcal{F}$ to the central fiber $\mathcal{X}_0$, we obtain a sheaf $\mathcal{F}(d,k)$ such that $\mathcal{F} \mid_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}}(d\eta^* L - \sum_{i=1}^{n_1} k\mathcal{E}_i)$ and $\mathcal{F} \mid_{\mathcal{P}_i} = \mathcal{O}_{\mathcal{P}_i}(k)$.

Denote by f the map from $\mathcal{X}_0$ to B and consider the following sheaves on B,

$$\mathcal{W} := f_*\mathcal{F}(d,k)\mid_{\mathcal{Y}},\;\; \mathcal{U}_i := f_*\mathcal{F}(d,k)\mid_{\mathcal{P}_i} \text{ and } \mathcal{V}_i := f_*\mathcal{F}(d,k)\mid_{\mathcal{E}_i}.$$

Our situation is now described by the following diagram of bundles over B,

$$\begin{array}{ccccccccccc} & & \mathcal{U}_1 & \oplus & \mathcal{U}_2 & \oplus & \cdots & \oplus & \mathcal{U}_{n_1-1} & \oplus & \mathcal{U}_{n_1} \\ & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} & & & & \downarrow{\scriptstyle r_{n_1-1}} & & \downarrow{\scriptstyle r_{n_1}} \\ \mathcal{W} & \xrightarrow{r} & \mathcal{V}_1 & \oplus & \mathcal{V}_2 & \oplus & \cdots & \oplus & \mathcal{V}_{n_1-1} & \oplus & \mathcal{V}_{n_1}, \end{array}$$

where $r,\ r_i$ are restrictions to $\mathcal{E}_i$.

The global sections of these sheaves over $b \in B$ are,

$$\mathcal{W}_b = H^0(Y_b, \mathcal{O}(d\eta^*L - \sum kE_{i,b})),\ V_{i,b} = H^0(E_{i,b}, \mathcal{O}_{\mathbb{P}^1(k)}) \text{ and } \mathcal{U}_{i,b} = H^0(P_{i,b}, \mathcal{O}_{\mathbb{P}^2}(k)),$$

where we also require that the sections in $\mathcal{U}_{i,b}$ vanish with multiplicity m at each of n_2 generically chosen points.

Here Y_b, $E_{i,b}$ and $\mathbb{P}_{i,b}$ are the fibers over $b \in B$ of $\mathcal{Y}$, $\mathcal{E}_i$ and $\mathcal{P}_i$ respectively. Note from the construction that Y_b is the proper transform of the plane blown up at n_1 generic points, and $\mathbb{P}_{i,b}$ is a projective plane which intersects Y_b along a projective line $E_{i,b}$. Using the terminology of linear systems this can be expressed as

$$\mathcal{W}_b = \mathbf{L}_d(k^{n_1}), V_{i,b} = H^0(\mathcal{O}_{E_1}(k)) \text{ and } \mathcal{U}_{i,b} = \mathbf{L}_k(m^{n_2}).$$

So over each $b \in B$, we have the construction that arises in diagram $(*)$ on page 6.

Recall that we want to calculate the dimension of the intersection in $\bigoplus_1^{n_1} V_{i,b}$. Restricting to $E_{i,b}$ leaves us with the following inclusions:

$$\begin{array}{ccccccccccc} & & U_{1,b} & \oplus & U_{2,b} & \oplus & \cdots & \oplus & U_{n_1-1,b} & \oplus & U_{n_1,b} \\ & & \cap & & \cap & & & & \cap & & \cap \\ W_b & \hookrightarrow & V_{1,b} & \oplus & V_{2,b} & \oplus & \cdots & \oplus & V_{n_1-1,b} & \oplus & V_{n_1,b}. \end{array}$$

Next we describe group action on it: Let $\mathcal{G}$ be a family of groups $G := GL_2(\mathbb{C})$ over B defined by

$$Aut_B(f_*\mathcal{F}(0,1)\mid_{\mathcal{E}_1}) \times \ldots \times Aut_B(f_*\mathcal{F}(0,1)\mid_{\mathcal{E}_{n_1}}.$$

The symmetric group S_{n_1} acts on $\mathcal{G}$ by acting both on B and on the copies of GL_2, so that if

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n_1 \\ \sigma_1 & \sigma_2 & \ldots & \sigma_{n_1} \end{pmatrix}$$

is an element of S_{n_1}, then $(b,g) = (b_1, \ldots, b_{n_1}; g_1, \ldots, g_{n_1})$ is sent to $\sigma(b,g) = (b_{\sigma_1}, \ldots, b_{\sigma_{n_1}}; g_{\sigma_1}, \ldots, g_{\sigma_{n_1}})$.

The construction of bundles $\mathcal{U}_i$ and $\mathcal{W}$ and the symmetric group action induce isomorphisms

$$U_{\sigma_i,(b_1,\ldots,b_{n_1})} \cong U_{i,(b_{\sigma_1},\ldots,b_{\sigma_{n_1}})} \text{ and } p_{\sigma_i} W_{(b_1,\ldots,b_{n_1})} \cong p_i W_{(b_{\sigma_1},\ldots,b_{\sigma_{n_1}})},$$

where p_i is the restriction map $p_i : W \to V_i$.

The aim of this construction is to prove that vector spaces U_i, V_i, W satisfy quasi symmetric condition defined in 3. We first check condition (a): Choose $\sigma = \begin{pmatrix} 1 & 2 & \ldots & r & r+1 & \ldots & n_1 \\ 2 & 3 & \ldots & r+1 & 1 & \ldots & n_1 \end{pmatrix}$. If there exists $(b,g) \in \mathcal{G}$ such that

$$(p_1, \ldots, p_r) W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_r U_{r,b}) = 0$$

then there exists an open set $\mathcal{A} \subset \mathcal{G}$ with the same property. Choose an element $(b_1, \ldots, b_{n_1}; g_1, \ldots, g_{n_1})$ in the intersection of open sets $\mathcal{A} \cap \sigma^{-1}(\mathcal{A})$. Then

$$(p_1, \ldots, p_r) W_{\sigma(b)} \cap (g_{\sigma_1} U_{1,\sigma(b)} \oplus \ldots \oplus g_{\sigma_r} U_{r,\sigma(b)}) = 0$$

since $\sigma(b,g) \in \mathcal{A}$. But this is isomorphic to

$$(p_{\sigma_1}, \ldots, p_{\sigma_r}) W_b \cap (g_{\sigma_1} U_{\sigma_1,b} \oplus \ldots \oplus g_{\sigma_r} U_{\sigma_r,b}) = (p_2, \ldots, p_{r+1}) W_b \cap (g_2 U_{2,b} \oplus \ldots \oplus g_{r+1} U_{r+1,b}) = 0.$$

The obvious inclusion

$$\begin{aligned} &(p_1, \ldots, p_{r+1}) W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_{r+1} U_{r+1,b}) \subseteq \\ &((p_1, \ldots, p_r) W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_r U_{r,b})) \oplus ((p_2, \ldots, p_{r+1}) W_b \cap (g_2 U_{2,b} \oplus \ldots \oplus g_{r+1} U_{r+1,b})) = 0 \end{aligned}$$

proves condition (a).

For condition (b) assume that there exists $(b,g) \in \mathcal{G}$ for which

$$(p_1, \ldots, p_r) W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_r U_{r,b})$$

is transversal and nonzero, and

$$p_{r+1} W' \cap g_{r+1} U_{r+1,b} = 0.$$

Recall that W' is defined to be

$$(p_1, \ldots, p_r)^{-1}[p_1, \ldots, p_r W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_r U_{r,b})].$$

Then there exists an open set $\mathcal{C} \subseteq \mathcal{G}$ with the same property. $\sigma^{-1}(\mathcal{C})$ is also open. For every element $(b_1, \ldots, b_{n_1}; g_1, \ldots, g_{n_1})$in the intersection $\mathcal{C} \cap \sigma^{-1}(\mathcal{C})$ we get

$$p_{r+1} W'' \cap g_{\sigma_{r+1}} U_{r+1,\sigma(b)} = 0,$$

where

$$W'' = (p_1, \ldots, p_r)^{-1}[p_1, \ldots, p_r W_{\sigma(b)} \cap (g_{\sigma_1} U_{1,\sigma(b)} \oplus \ldots \oplus g_{\sigma_r} U_{r,\sigma(b)})].$$

This is true since $\sigma(b, g) \in \mathcal{C}$.

Using above isomorphisms we get

$$W'' \cong (p_{\sigma_1}, \ldots, p_{\sigma_r})^{-1}[p_{\sigma_1}, \ldots, p_{\sigma_r} W_b \cap (g_{\sigma_1} U_{\sigma_1,b} \oplus \ldots \oplus g_{\sigma_r} U_{\sigma_r,b})],$$

which we denote by $\ddot{W}$. Clearly $p_{r+1} W'' \cong p_{\sigma_{r+1}} \ddot{W}$ and $g_{\sigma_{r+1}} U_{r+1,\sigma(b)} \cong g_{\sigma_{r+1}} U_{\sigma_{r+1},b}$ This implies

$$p_{\sigma_{r+1}} \ddot{W} \cap g_{\sigma_{r+1}} U_{\sigma_{r+1},b} \cong p_{r+1} W'' \cap g_{\sigma_{r+1}} U_{r+1,\sigma(b)} = 0.$$

Finally take $(b, g) \in \mathcal{C} \cap \sigma^{(1)}(\mathcal{C}) \cap \ldots \cap \sigma^{(r)}(\mathcal{C})$ where

$$\sigma^{(i)} = \begin{pmatrix} 1 & 2 & \ldots & i & \ldots & r+1 & \ldots & n_1 \\ 1 & 2 & \ldots & r+1 & \ldots & i & \ldots & n_1 \end{pmatrix}.$$

For every element

$$w \in W_b \cap (g_1 U_{1,b} \oplus \ldots \oplus g_{r+1} U_{r+1,b})$$

we get $p_{r+1} w \in p_{r+1} W' \cap g_{r+1} U_{r+1,b}$ which is by assumption 0. On the other hand $p_i w \in p_i \ddot{W} \cap g_i U_{i,b}$ which is again 0, since $\sigma^{(i)} \in \mathcal{C}$. We proved that

$$w = (p_1, \ldots, p_{r+1}) w = 0.$$

This verifies condition (b) in the definition of quasi-symmetry.

References

[1] C. Ciliberto and R. Miranda, *Degenerations of Planar Linear Systems*, J. Reine Angew. Math. **501** (1998), 191–220.

[2] C. Ciliberto and R. Miranda, *Linear Systems of Plane Curves with Base Points of Equal Multiplicity*, Transactions Amer. Math. Soc. **352** (2000), 4037–4050.

[3] R. Miranda, *Linear Systems of Plane Curves*, Notices of the AMS (2) **46** (1999), 192–202.

[4] L. Evain, *La Fonction de Hilbert de la réunion de* 4^h *points génériques de* $\mathbb{P}^2$ *de même multiplicité*, J. of Alg. Geom. **8** (1999).

[5] A. Hirschowitz, *Existence de faisceaux réflexive de rang deux sur* $\mathbb{P}^3$ *a bonne cohomologie*, Inst. Hautes Ét. Sci. **66** (1988), 104–137.

[6] A. Hirschowitz, *Une Conjecture pour la Cohomologie des Diviseurs sur les Surfaces Rationelles Génériques*, J. Reine Angew. Math. **397** (1989), 208–213.

[7] A. Buckley and M.Zompatori, *Linear Systems of Plane Curves with a Composite Number of Base Points of Equal Multiplicity*, 2002

Anita Buckley
Mathematics Institute, University of Warwick
Coventry CV4 7AL, United Kingdom
E-mail address: `mocnik@maths.warwick.ac.uk`
and
Marina Zompatori
Department of Mathematics, Boston University
Boston MA 02215, Unites States
E-mail address: `marinaz@math.bu.edu`