The Fibonacci dimension of a graph

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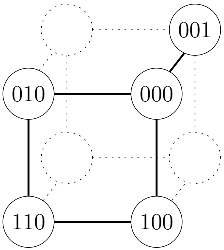
Partial cubes

- The hypercube Q_d of dimension d as a graph:
  - $V(Q_d) = \{\text{binary strings of length } d\}$.
  - Two vertices are adjacent if the strings differ in one position.
  - Distance between two vertices is Hamming distance of the strings.
  - each element of the string is a binary coordinate.
- Partial cubes: isometric subgraphs of hypercubes.



Isometric dimension

- The isometric dimension of a graph G, $\mathrm{idim}(G)$, is the smallest d such that G isometrically embeds into Q_d.
- $\mathrm{idim}(G)$ finite if and only if G is a partial cube.
- Attach binary strings to each vertex, so that the induced subgraph of Q_d and G have the same distances.
- $\mathrm{idim}(G)$ can be found in polynomial time via Θ-classes.
- There is one isometric embedding of G into $Q_{\mathrm{idim}(G)}$, up to
  - reordering the string coordinates.
  - switching the roles of 1 and 0 in a coordinate.

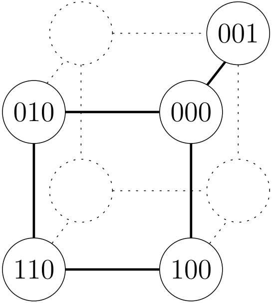
Lattice dimension

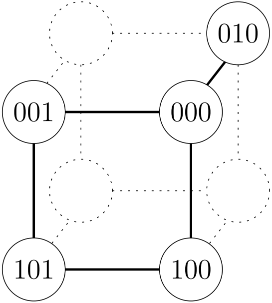
- The lattice dimension of a graph G, $\mathrm{ldim}(G)$, is the smallest d such that the graph embeds isometrically into $\mathbb{Z}^d$ with ℓ_1-metric.
- Attach $\mathbb{Z}$-strings to each vertex, so that the induced subgraph of $\mathbb{Z}^d$ and G have the same distances.
- $\mathrm{ldim}(G)$ finite iff $\mathrm{idim}(G)$ finite.
- The lattice dimension can be determined in polynomial time (Eppstein, 2005).

Fibonacci cubes

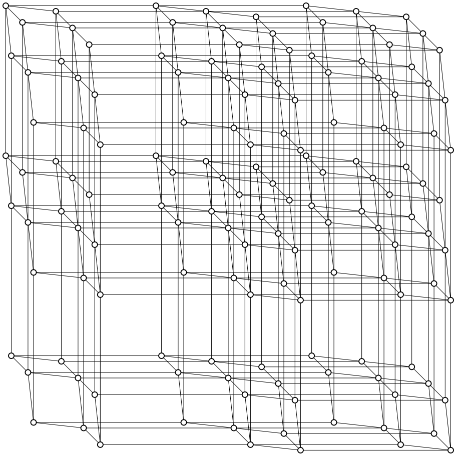
- Fibonacci string: binary string without two consecutive 1's.
  - 01010001 and 00100001 are Fibonacci strings.
  - 011000 and 001010110 are not Fibonacci strings.
- name comes from Zeckendorf's theorem.
- Fibonacci cube: Γ_d, $d \geq 1$: subgraph of Q_d induced by the Fibonacci strings of length d.
- appears in chemical graph theory: resonance of some certain hexagonal chains.
- used as network topology in parallel computation.
- "is G a Fibonacci cube?" is decidable in near-linear time.

Γ_3 in Q_3





The Fibonacci cube Γ_{10}



Fibonacci dimension

The Fibonacci dimension, $\mathrm{fdim}(G)$, is the smallest integer f such that G admits an isometric embedding into Γ_f.

Attach Fibonacci strings to each vertex, such that the induced subgraph of Γ_f and G have the same distances.

Relations to other dimensions

- $\mathrm{fdim}(G) < \infty$ iff $\mathrm{idim}(G) < \infty$ iff $\mathrm{ldim}(G) < \infty$.
- $\mathrm{idim}(G) \leq \mathrm{fdim}(G) \leq 2\,\mathrm{idim}(G) - 1$.
  - $G \to \Gamma_f \to Q_f$.
  - $(v \in G) \mapsto (abcde \in Q_d) \mapsto (a0b0c0d0e \in \Gamma_{2d-1})$.
- $\mathrm{ldim}(G) \leq \lceil \mathrm{fdim}(G)/2 \rceil$.
  - $\Gamma_f \to \mathbb{Z} \,\square\, \Gamma_{f-2}$ via

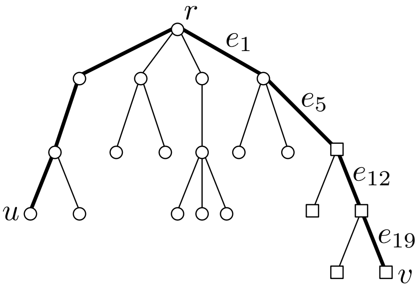
$$u \mapsto \begin{cases} (0, u^*) & \text{if } u = 01u^*; \\ (1, u^*) & \text{if } u = 00u^*; \\ (2, u^*) & \text{if } u = 10u^*. \end{cases}$$

- $\mathrm{fdim}(G) \leq \mathrm{idim}(G) + \mathrm{ldim}(G) - 1$.
- ...

Some Fibonacci dimensions

Proposition

For any tree T, $\mathrm{fdim}(T) = |E(T)|$.



$v \equiv 1000100000001000000010$

$u \equiv 0001000000010000000100$

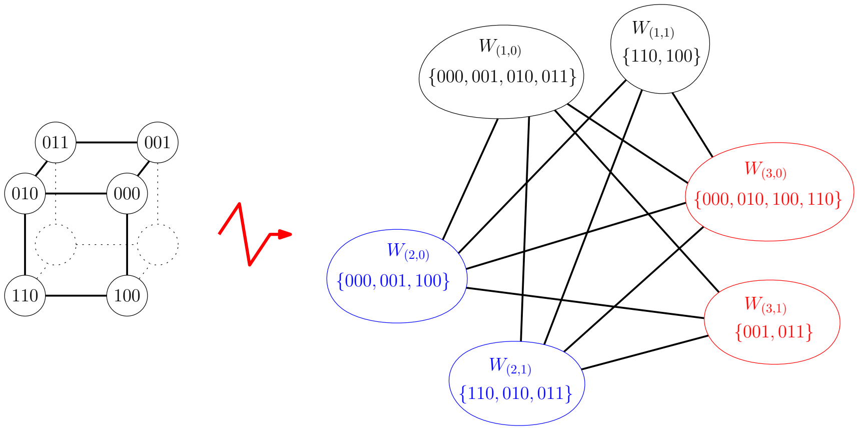
sibling of $v \equiv 1000100000001000000001$

Bad news

Theorem

It is NP-complete to decide if $\mathrm{idim}(G) = \mathrm{fdim}(G)$.

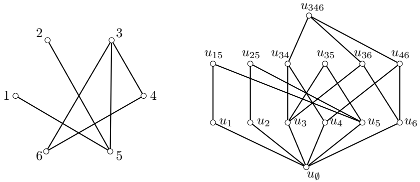
- $d = \mathrm{idim}(G)$.
- isometrically embed G in Q_d: attach binary labels of length d.
- the labels are unique up to permutations of coordinates and switching $0 \leftrightarrow 1$.
- Can we permute coordinates and switch $0 \leftrightarrow 1$ such that all labels are Fibonacci strings?
- This can be formalized using an associated graph $X(G) \ldots$.



011
001
010
000
110
100
$W_{(1,0)}$
$\{000, 001, 010, 011\}$
$W_{(1,1)}$
$\{110, 100\}$
$W_{(3,0)}$
$\{000, 010, 100, 110\}$
$W_{(2,0)}$
$\{000, 001, 100\}$
$W_{(3,1)}$
$\{001, 011\}$
$W_{(2,1)}$
$\{110, 010, 011\}$

More results

- Closely related to Hamiltonian path with weights 1 and 2.
- NP-hard to approximate fdim within $(741/740) - \varepsilon$.
- Approximable within $3/2$.
- $\mathrm{fdim}(G)$ can be determined in $O(2^k k^2 + k^2 n)$, where $k = \mathrm{idim}(G)$.
- $(1 + \varepsilon)$-approximation for simplex graphs.



$\Rightarrow$ approximating pairs of problems.

The end

- thanks
- thanks
- ...