On the b-chromatic number of regular graphs*

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Abstract

The b-chromatic number of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class contains at least one vertex that has a neighbor in each of the other color classes. We prove that every d-regular graph with at least $2d^3$ vertices has b-chromatic number $d+1$, that the b-chromatic number of an arbitrary d-regular graph with girth $g=5$ is at least $\lfloor \frac{d+1}{2} \rfloor$ and that every d-regular graph, $d \geq 6$, with diameter at least d and with no 4-cycles admits a b-coloring with $d+1$ colors.

Key words: b-chromatic number, size, girth, diameter

AMS subject classification (2010): 05C15

1 Introduction and preliminaries

The b-*chromatic number* $\varphi(G)$ of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class has a vertex adjacent to at least one vertex in each of the other color classes. Such a coloring is called a b-*coloring*. Let G be a graph and c a b-coloring of G. If $x \in V(G)$ has a neighbor in each other color class we will say that x *realizes color* $c(x)$ and that $c(x)$ is *realized on* x. We can easily imagine the color classes as different communities, where every community i has a representative (a vertex realizing color i) that is able to communicate with all the others communities.

This concept was introduced in 1999 by Irving and Manlove [14] who proved that determining $\varphi(G)$ is NP-hard in general, but can be computed in polynomial time for trees. Kratochvíl, Tuza, and Voigt [23] further showed that determining $\varphi(G)$ is NP-hard already for bipartite graphs. Corteel, Valencia-Pabon, and Vera [7] proved that there is no constant $\epsilon > 0$ for which the b-chromatic number can be approximated within a factor of $(120/113)-\epsilon$ in polynomial time, unless P=NP.

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From the combinatorial perspective, b-colorings have been studied for different families of graphs. Effantin and Kheddouci [8, 9, 10] studied the b-chromatic number of powers of paths, cycles, complete binary trees, and complete caterpillars. Hoáng and Kouider [13] introduced and studied the concept of b-perfect graphs. Kouider and Zaker [22] gave several bounds for $\varphi(G)$ depending on the clique number and the clique partition number of G. Barth, Cohen, and Faik [3] discussed the b-spectrum of graphs. More recent work has considered cographs and P_4-sparse graphs [5], Kneser graphs [15], vertex-deleted subgraphs [12], and Mycielskians [1, 2]. The b-chromatic number of the Cartesian product of general graphs was studied in [20, 21], while the Cartesian product of complete graphs is considered in [6]. In [17] three standard products were considered: the strong, the lexicographic, and the direct product. Tight lower and upper bounds were determined. The lower bounds can be derived from the b-chromatic number of factors of the product. It is shown that there is no upper bound with respect to the b-chromatic number of the factors for the Cartesian, the strong, the lexicographic, and the direct product.

A trivial upper bound for $\varphi(G)$ is $\Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G. Let $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$ be the degree sequence of G. The parameter $m(G) = \max\{i \mid d(v_i) \geq i - 1\}$ is an improved upper bound for $\varphi(G)$; see Irving and Manlove [14]. Regular graphs play an important role when studying the b-chromatic number because for every regular graph G the equality $m(G) = \Delta(G) + 1$ holds. In this paper we study the b-chromatic number of regular graphs.

Our results. Kouider and El Sahili [19] and Kratochvíl, Tuza, and Voigt [23] proved that $\varphi(G) = \Delta(G) + 1$ for any d-regular graph G with at least d^4 vertices. This means that for a given d there are only a finite number of d-regular graphs G with $\varphi(G) < d + 1$. It is known [16] that there are precisely four cubic graphs with $\varphi(G) < \Delta(G) + 1$, one of them being the Petersen graph. All the exceptions have no more then 10 vertices. In this paper we reduce the bound of d^4 vertices to $2d^3$ vertices. Thus, any d-regular graph with at least $2d^3$ vertices has b-chromatic number $d + 1$. This result is derived in Section 2.

El Sahili and Kouider [11] asked whether it is true that every d-regular graph G with girth at least 5 satisfies $\varphi(G) = d + 1$. The question was solved for (i) d-regular graphs with girth at least 6 [18]; (ii) d-regular graphs with girth 5 and no C_6 [11]; (iii) d-regular graphs different from the Petersen graph with girth at least 5 and $d \leq 6$ [4]. However, there are regular graphs with small b-chromatic number, like for example the complete bipartite graph $K_{n,n}$. We show that the b-chromatic number of d-regular graphs with girth $g = 5$ is at least $\left\lfloor \frac{d+1}{2} \right\rfloor$. Therefore, for graphs of girth at least 5 the parameter $\varphi(G)$ is bounded from below by a linear function of the degree. This result is derived in Section 3.

In Section 4 we show that the b-chromatic number of d-regular graphs with diameter at least d and no 4-cycles is $d + 1$. This result provides another sufficient condition for regular graphs to achieve maximum b-chromatic number. We conclude in Section 5.

Approach. Henceforth, G will denote a d-regular graph. For any positive integer m we will use the notation $[m] = \{1, \ldots, m\}$.

We construct b-colorings using variations of the following technique. Assume we have a partial coloring that realizes some of the colors. Let s be an uncolored vertex. To decide if the partial coloring can be extended so that s realizes color i, we construct a bipartite graph H where one partition consists of the neighborhood $N(s) = N_G(s)$ of s, the other partition consists of the colors $[d+1] \setminus \{i\}$, and we have an edge between vertex u and color c whenever coloring u with c does not conflict with the current partial coloring. If the graph H has a perfect matching M, then we can extend the partial coloring so that vertex s realizes color

i. This is achieved assigning color i to s, and assigning to each vertex of $N(s)$ the color matched via M. To decide if H has a perfect matching we can then use Hall's theorem.

2 Size

We will use the following result to extend the number of realized colors in a partial coloring. See Fig. 1.

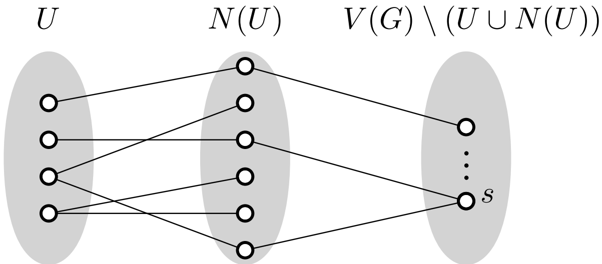

Figure 1: Schematic situation in Lemma 2.1.

Lemma 2.1. *Assume we have a partial coloring of G assigning colors from $[d+1]$ to $U \subset V(G)$. Let $s \in V(G) \setminus (U \cup N(U))$. If for each $j \in [d-1]$*

$$|\{v \in N(s) \mid v \text{ is adjacent to at least } j \text{ vertices of } U\}| \leq d - j\,,$$

then we can extend the partial coloring such that an arbitrary color of the set $[d+1]$ is realized on vertex s.

Proof. Let $c : U \to [d+1]$ be the partial coloring of G. Without loss of generality, let $d+1$ be the color we wish to realize on s. Consider a bipartite graph H, where one partition consists of $N_G(s)$ and the other of colors $[d]$. Vertex $v \in N_G(s)$ is adjacent to color $a \in [d]$ in H if and only if vertex v can receive color a; that is, if $a \notin c(N_G(v) \cap U)$ (Fig. 2).

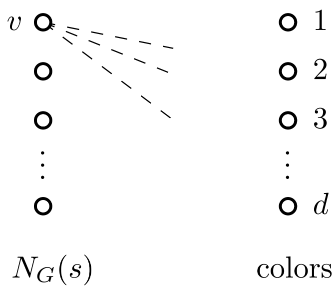

Figure 2: Graph H

We have

$$\deg_H(v) \;=\; d - |c(N_G(v) \cap U)| \;\geq\; d - |N_G(v) \cap U|.$$

With Hall's theorem we shall prove that within the given assumptions there exists a perfect matching in H. Take any $T \subseteq N_G(s)$. If $|T| = 1$, then $|N_H(T)| \geq 1$ because $s \notin U$. If $|T| \geq 2$, the assumptions of the lemma imply

$$|\{v \in N_G(s) \mid |N_G(v) \cap U| \geq d - |T| + 1\}| \;\leq\; d - (d - |T| + 1) \;=\; |T| - 1,$$

which means that T has at least one vertex t with $|N_G(t) \cap U| \leq d - |T|$. Therefore

$$\begin{aligned} |N_H(T)| \;&\geq\; \max_{t\in T}\{\deg_H(t)\} \;=\; d - \min_{t\in T}\{|c(N_G(t)\cap U)|\} \\ &\geq d - \min_{t\in T}\{|N_G(t)\cap U|\} \;\geq\; d - (d - |T|) \;=\; |T|. \end{aligned}$$

We conclude by Hall's theorem that the graph H has a perfect matching M. We can extend the partial coloring c to $U \cup \{s\} \cup N_G(s)$ assigning to s color $d+1$ and assigning to each vertex $v \in N_G(s)$ the color given by the matching M. In this partial coloring, vertex s realizes color $d+1$. ◻

Theorem 2.2. *Let G be a d-regular graph with at least $2d^3$ vertices. Then $\varphi(G) = d+1$.*

Proof. Let $|V(G)| \geq 2d^3$. The proof will iteratively construct a partial coloring of G in such a way that, at the end of iteration z ($z = 1, 2, \ldots, d+1$), we have a partial coloring that realizes the colors of $[z]$. In the partial coloring, only vertices that realize a color and their neighborhoods are colored. We start with a partial coloring that does not color any vertex.

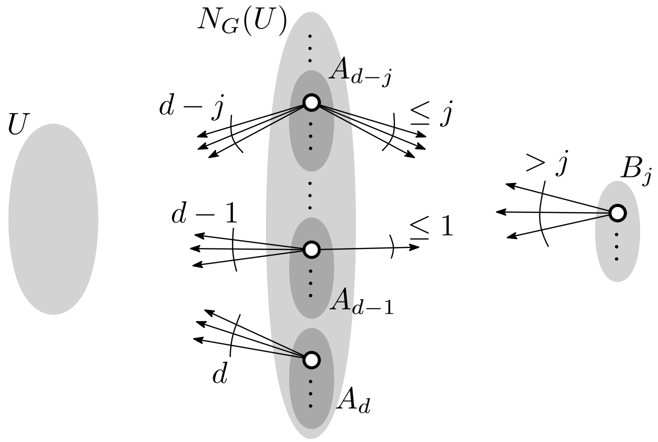

Figure 3: Schema showing the sets A_i and B_j in the proof of Theorem 2.2.

Consider an iteration $z \in [d+1]$. Let U be the set of vertices that were already colored. For $z = 1$ we have $U = \emptyset$. We can assume by induction that the colors $[z-1]$ are already realized by the partial coloring of U. We want to find a vertex where to realize color z. The size of U is at most $d(d+1)$ because it contains $z - 1 \leq d$ vertices realizing some color and their neighborhoods.

For each $i \in [d]$, let A_i be the set of vertices from $V(G) \setminus U$ that are adjacent to exactly i vertices of U. See Fig. 3. Note that

$$N(U) = \bigcup_{i=1}^{d} A_i\,.$$

Next we double count the edges $(U, N(U))$ between U and $N(U)$. On the one hand, $(U, N(U))$ contains exactly $\sum_i i \cdot |A_i|$ edges. On the other hand, since the vertices of U that realize a color do not contribute any edge to $(U, N(U))$, there are at most $(z-1)d \leq d^2$ vertices in U that are adjacent to $N(U)$. Using that each one of them has at most $d-1$ neighbors in $N(U)$ because they have at least one neighbor within U, we conclude that

$$\sum_{i=1}^{d} i \cdot |A_i| \;=\; |(U, N(U))| \;\leq\; d^2(d-1). \tag{1}$$

It also follows that

$$|N(U)| \le d^2(d-1).$$

For each $j \in [d-1]$, let

$$B_j = \{v \notin U \cup N(U) \mid |N(v) \cap (A_d \cup A_{d-1} \cup \ldots \cup A_{d-j})| > j\},$$

and define

$$B = \bigcup_{j=1}^{d-1} B_j\,.$$

Assume that there is a vertex s that does not belong to $U \cup N(U) \cup B$. Then, for each $j \in [d-1]$, the vertex s does not belong to B_j, which means that $|N(s) \cap (A_d \cup A_{d-1} \cup \ldots \cup A_{d-j})| \le j$. Setting $j' = d-j$, we see that, for each $j' \in [d-1]$, the neighborhood $N(s)$ has at most $d-j'$ vertices which are adjacent to at least j' vertices of U. By Lemma 2.1, color z can be realized on vertex s.

We have to determine the minimum size of $V(G)$ in order to ensure that $U \cup N(U) \cup B$ is not the whole $V(G)$. Double counting the edges between B_j and $(A_d \cup A_{d-1} \cup \ldots \cup A_{d-j})$, and noting that a vertex of A_i has at most $d-i$ neighbors outside $U \cup N(U)$, we obtain

$$j \cdot |B_j| < \sum_{i=d-j}^{d} (d-i) \cdot |A_i| = \sum_{i=1}^{j} i \cdot |A_{d-i}|,$$

and thus

$$|B_j| < \frac{1}{j} \sum_{i=1}^{j} i \cdot |A_{d-i}|.$$

This leads to

$$\begin{aligned}
|B| \le \sum_{j=1}^{d-1} |B_j| &< \sum_{j=1}^{d-1} \frac{1}{j} \sum_{i=1}^{j} i \cdot |A_{d-i}| \\
&= \sum_{i=1}^{d-1} i \cdot |A_{d-i}| \sum_{j=i}^{d-1} \frac{1}{j} \\
&\le \sum_{i=1}^{d-1} i \cdot |A_{d-i}| \frac{d-i}{i} \\
&\le \sum_{i=1}^{d-1} (d-i) \cdot |A_{d-i}| \\
&\le d^2(d-1),
\end{aligned}$$

where in the last step we have used (1).

Joining the bounds we see

$$|U| + |N(U)| + |B| < d(d+1) + d^2(d-1) + d^2(d-1) = 2d^3 - d^2 + d.$$

Therefore, if $|V(G)| \ge 2d^3 - d^2 + d$, we can find in every iteration z a vertex s on which color z can be realized. When $|V(G)| \ge 2d^3$ this condition holds. As usual, after iteration $z = d+1$ we can color the rest of the graph G greedily. $\square$

Next we show that the bound $2d^3$ can not be lowered below d^2 in general. For this purpose we first introduce the definition of lexicographic product. The *lexicographic product* of graphs G and H, denoted by $G[H]$, has vertex set $V(G) \times V(H)$ and two vertices are adjacent if (i) they are adjacent in the first coordinate, or (ii) they are equal in the first and adjacent in the second coordinate. From the definition it is obvious that the lexicographic product is not commutative.

Let $d \geq 2$ be an even positive integer and let $S_{d/2}$ be an independent set on $\frac{d}{2}$ vertices. The graph $C_{2d}\left[S_{d/2}\right]$, where C_{2d} is a cycle on $2d$ vertices, is a d-regular graph with exactly $2d \cdot \frac{d}{2} = d^2$ vertices. Fig. 4 shows the 4-regular graph $C_8\left[S_2\right]$. We prove that its b-chromatic number is at most d.

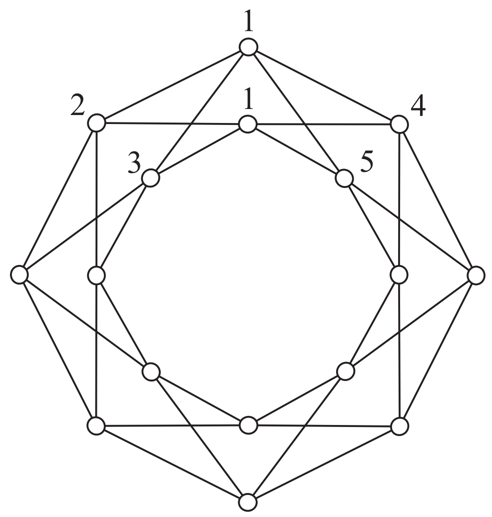

Figure 4: Lexicographic product $C_8\left[S_2\right]$

Proposition 2.3. *Let $d \geq 2$ be an even positive integer. Then $\varphi\left(C_{2d}\left[S_{d/2}\right]\right) \leq d$.*

Proof. The vertices of $C_{2d}\left[S_{d/2}\right]$ whose first coordinate is the same form a so-called $S_{d/2}$-fiber, which is a copy of the independent set $S_{d/2}$. There are $2d$ different $S_{d/2}$-fibers, and they form a partition of the vertex set of the graph.

Consider a b-coloring of graph $C_{2d}\left[S_{d/2}\right]$ with $\varphi\left(C_{2d}\left[S_{d/2}\right]\right)$ colors. Consider the vertex realizing color i and let F be the $S_{d/2}$-fiber that contains it. See for example the case $i = 1$ for the graph $C_8\left[S_2\right]$ in Fig. 4. Colors $[d+1] \setminus \{i\}$ must then be used on the two neighboring $S_{d/2}$-fibers. However, this implies that all the vertices of F must have color 1. Thus none of the vertices of F or the two neighboring $S_{d/2}$-fibers can realize any of the colors $[d+1] \setminus \{i\}$. This means that the $S_{d/2}$-fibers which have a vertex that realizes a color must be at least at distance two apart. Since there are only $2d$ $S_{d/2}$-fibers, at most d colors can be realized, and thus $\varphi\left(C_{2d}\left[S_{d/2}\right]\right) \leq d$. $\square$

With this result we have given a d-regular graph, for any even d, that has d^2 vertices and whose b-chromatic number is not $d + 1$.

3 Girth

The complete bipartite graphs $K_{d,d}$ are d-regular graphs whose b-chromatic number is always 2, independently of d. Our objective in this section is to show that this can not happen for

d-regular graphs of girth 5 because their b-chromatic number must grow with the value of d. We start with a technical lemma that will also be used in Section 4.

Lemma 3.1. *Let H be a bipartite graph with partitions U and V such that $|U| = |V|$, let u^* be a vertex of U, and let v^* be a vertex V. If $\deg_H(x) \geq |U|/2$ for all $x \in U \cup V \setminus \{u^*, v^*\}$ and $\deg_H(u^*), \deg_H(v^*) > 0$, then H has a perfect matching.*

Proof. Let $T \subseteq U$ a nonempty subset of U. We will show that $|N_H(T)| \geq |T|$. We have four cases to consider.

Case 1: $|T| = 1$. In this case $T = \{u\}$ for some vertex $u \in U$. Since $\deg_H(u) > 0$ we have $|N_H(T)| = \deg_H(u) \geq 1 = |T|$.

Case 2: $1 < |T| \leq \frac{|U|}{2}$. Then

$$|N_H(T)| \geq \max_{x \in T}\{\deg_H(x)\} \geq \frac{|U|}{2} \geq |T|\,.$$

Case 3: $\frac{|U|}{2} < |T| \leq |U| - 1$. For all $x \in V \setminus \{v^*\}$ we have $\deg_H(x) \geq \frac{|U|}{2}$ and therefore $N_H(x) \cap T \neq \emptyset$. This means that $V \setminus \{v^*\} \subseteq N_H(T)$, which implies $|N_H(T)| \geq |V| - 1 = |U| - 1 \geq |T|$.

Case 4: $|T| = |U|$. Because $\deg_H(x) > 0$ for all $x \in V$, it follows that $|N_H(T)| = |V| = |T|$.

Combining all the cases and using Hall's theorem completes the proof. □

Theorem 3.2. *Let G be a d-regular graph with girth 5. Then $\varphi(G) \geq \left\lfloor \frac{d+1}{2} \right\rfloor$.*

Proof. Consider a vertex v_1, and denote its neighbors by v_i, $i \in \{2, \ldots, d+1\}$. We define $V_i = N(v_i) \setminus \{v_1\}$. Since G has girth 5, the sets V_i and V_j are disjoint for each $i \neq j$ in the range $\{2, \ldots, d+1\}$. See Fig. 5, where the edges adjacent to v_i, $i \in [d+1]$, are included, but there may be some additional edges between the other vertices that are not shown in the figure. Note that a vertex of V_i may be adjacent to at most one vertex of V_j, as otherwise G would contain a 4-cycle.

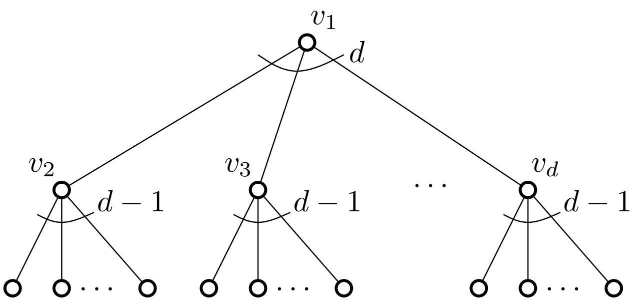

Figure 5: Girth 5 subgraph

We construct a partial coloring of G. Firstly, we color vertex v_i with color i for each $i \in [d+1]$. Then, we iteratively consider $i = 2, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor$, and distribute colors $\{2, \ldots, d+1\} \setminus \{i\}$ to the vertices V_i in such a way that we realize color i on vertex v_i.

Assume we are on the i-th iteration. We have already colored $N(v_j)$ for all $j < i$. Consider the bipartite graph H_i, where one partition consists of vertices V_i and the other of colors $\{2, \ldots, d+1\} \setminus \{i\}$. In the graph H_i, vertex $u \in V_i$ is adjacent to color $c \in \{2, \ldots, d+1\} \setminus \{i\}$ if and only if vertex u can be colored with color c in the current partial coloring of G.

We next argue that $\deg_{H_i}(x) \geq d - i$ for each $x \in V(H)$. Since in G each vertex $u \in V_i$ is adjacent to at most one vertex in each V_j, it follows that u has at most $i - 1$ neighbors that are already colored. Therefore, $\deg_{H_i}(u) \geq d-1-(i-1) = d-i$ for all $u \in V_i$. Similarly, for $j < i$ each color c appears at most once in each V_j because the partial coloring realizes color j in v_j. Therefore, each color c is forbidden for at most $i-1$ vertices of V_i, and we have $\deg_{H_i}(c) \geq d-1-(i-1) = d-i$.

Since the bipartite graph H_i has minimum degree $d-i$ and each class of the partition has $d-1$ vertices, Lemma 2.1 guarantees that H_i has a perfect matching, provided that

$$\min\left\{\deg_{H_i}(x) \mid x \in V(H_i)\right\} \geq d-i \geq \frac{d-1}{2}.$$

This condition is fulfilled when

$$i \leq d - \frac{d-1}{2} = \frac{d+1}{2}.$$

Therefore, at each iteration we can find a matching in H_i, and use it to color V_i such that v_i realizes color i.

When we finish the iteration $i = \left\lfloor \frac{d+1}{2} \right\rfloor$, we have a partial $(d+1)$-coloring of G where colors $\{1, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor\}$ are realized. We color the rest of the graph with the greedy algorithm using colors $\{1, \ldots, d+1\}$. The resulting coloring is not necessarily a proper b-coloring. However, we can convert it into a proper b-coloring with a stepwise recoloring, as follows. Set the color set $S = [d+1]$. While the current coloring is not a proper b-coloring with colors S, we choose a color $c \in S$ that is not realized, recolor any vertex of color c with another color of the set $S \setminus \{c\}$, and remove c from the set S. Thus, at each step we reduce the size of S by one and obtain a new coloring using the colors $S \setminus \{c\}$. It is obvious that when the procedure finishes, we have a proper b-coloring of G with colors S. Moreover, the colors $\{1, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor\}$ are realized throughout the recoloring process, and therefore we obtain a b-coloring with at least $\left\lfloor \frac{d+1}{2} \right\rfloor$ colors. □

4 Diameter

In this section we show that a d-regular graph with large diameter and no 4-cycles has b-chromatic number $d+1$, thus providing another sufficient condition to attain maximum b-chromatic number.

Theorem 4.1. *Let G be a d-regular graph with no 4-cycles and* $\operatorname{diam}(G) \geq d$*. Then* $\varphi(G) = d+1$.

Proof. Let $d \geq 6$. Since diam(G) $\geq$ d, there exists a path $v_1 v_2 \ldots v_d, v_{d+1}$ (Fig. 6) that satisfies $d(v_1, v_{d+1}) = d$. Let $V_1 = N(v_1) \setminus \{v_2\}$, $V_{d+1} = N(v_{d+1}) \setminus \{v_d\}$, and $V_i = N(v_i) \setminus \{v_{i-1}, v_{i+1}\}$, $i \in \{2, \ldots, d\}$. The size of V_1 and V_{d+1} is $d-1$ and the size of V_i, $i \in \{2, \ldots, d\}$, is $d-2$. To reduce the number of boundary cases, we define $V_j = \emptyset$ for any $j \notin [d+1]$.

Since G has no 4-cycle and $v_1 v_2 \ldots v_d, v_{d+1}$ is a shortest path, the vertices of V_i have the following properties:

(a) $|V_{i-1} \cap V_i| \leq 1$. When $V_{i-1} \cap V_i$ is nonempty, we denote by u_i its unique vertex.

(b) $V_i \cap V_j = \emptyset$, for each $j \leq i-2$. Here we use that there are no 4-cycles for $j = i-2$ and the shortest path property for $j < i-2$.

(c) Any vertex of V_i is adjacent to at most one vertex of V_{i-2} and at most one vertex of V_{i-3}.

(d) No vertex of V_i is adjacent to any vertex of V_j, for each $j \leq i-4$ or $j = i-1$.

(e) If $V_i \cap V_{i+1}$ is nonempty, then u_{i+1} is not adjacent to any vertex of V_j, for each $j \leq i-3$ or $j = i-1$.

We will construct a $(d+1)$-b-coloring of G such that vertex v_i realizes color i for each $i \in [d+1]$. We start assigning color i to vertex v_i, for each $i \in [d+1]$. The vertices of $V_1, V_2, \ldots$ will be colored inductively such that, when we finish coloring V_i, the following inductive statement holds: vertex v_j realizes color j for each $j \in [i]$, and if $V_i \cap V_{i+1}$ is nonempty, then u_{i+1} has a color distinct from $i+2$.

We start coloring the vertices of V_1 with colors $\{3, \ldots, d+1\}$ injectively, such that, if $V_1 \cap V_2$ is not empty, then u_2 gets assigned color $d+1 \neq 3$.

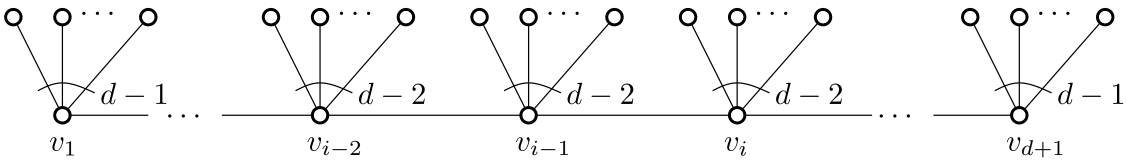

Figure 6: Path $v_1 v_2 \ldots v_d, v_{d+1}$

Consider now the case $i \geq 2$, where we want to color V_i. Because of properties (a) and (b), at most one vertex of V_i is already colored, namely the unique vertex u_i that may be in $V_{i-1} \cap V_i$. Let C_i be the set of colors $[d+1] \setminus \{i-1, i, i+1\}$. Note that $|C_i| = |V_i| \geq d-2$, with equality when $i \neq d+1$. If the vertices of V_i are colored injectively with the colors C_i, then vertex v_i will realize color i. Note that we have to respect the color already assigned to u_i, if such vertex exists.

Consider a bipartite graph H_i where one partition is V_i and the other partition is C_i. The graph H_i contains an edge between a vertex v and a color c if coloring v with color c does not conflict with the current coloring. In particular, if the vertex u_i exists, then it has degree 1 in H_i. Furthermore, if $V_i \cap V_{i+1}$ is nonempty, we remove from H_i the edge between vertex u_{i+1} and color $i+2$, to avoid that u_{i+1} could get color $i+2$.

We next look at the degrees in H_i for vertices in V_i. Because of properties (c) and (d), each vertex from $V_i \setminus (V_{i+1} \cup V_{i-1})$ has at most three colored neighbors: v_i, one from V_{i-2}, and one from V_{i-3}. Since color $i+1$ is not in C_i, we conclude that each vertex of $V_i \setminus (V_{i+1} \cup V_{i-1})$ has degree at least $|C_i| - 2$ in H_i. If $V_i \cap V_{i+1}$ is nonempty, then the vertex u_{i+1} has at most three colored neighbors because of properties (c) and (e): v_i, v_{i+1}, and a vertex from V_{i-2}. Since colors i and $i+1$ are not in C_i and we removed from H_i the edge between u_{i+1} and $i+2$, it follows that u_{i+1} also has degree at least $|C_i| - 2$ in H_i. If $V_{i-1} \cap V_i$ is nonempty, then the vertex u_i has degree one in H_i. We conclude that any vertex of $V_i \setminus V_{i-1}$ has degree at least $|C_i| - 2$ in H_i, while u_i has degree at least one.

We next look at the degrees in H_i for colors in C_i. Each color $c \in C_i$ appears at most once in V_{i-2} and at most once in V_{i-3} because each vertex v_j realizes a color, for $j \leq i-1$. Because of a symmetric version of property (c), each vertex of V_{i-3} or V_{i-2} has at most one neighbor in V_i. This means that each color of $C_i \setminus \{i+2\}$ is forbidden for at most two vertices of V_i, and has degree at least $|C_i| - 2$ in H_i. Color $i+2$ may be additionally forbidden for vertex u_{i+1}, if such vertex exists, because we removed that edge from H_i. We conclude that any color of $C_i \setminus \{i+2\}$ has degree at least $|C_i| - 2$ in H_i, while color $i+2$ has degree at least $|C_i| - 3$.

By Lemma 3.1, if $|C_i| - 2 \geq |C_i|/2$ and $|C_i| - 3 > 0$, then we can injectively color the vertices of $V_i \setminus V_{i-1}$ with the colors of C_i. This last condition is equivalent to $|C_i| \geq 4$, which is fulfilled when $d \geq 6$. Moreover, since H_i does not contain the edge between vertex u_{i+1} and color $i+2$, the inductive statement holds. When we finish coloring V_{d+1}, the inductive statement implies that vertex v_j realizes color j for all $j \in [d+1]$. The rest of the graph can be colored greedily using $d+1$ colors. □

One might think that the assumption about 4-cycles is forced. However, the graph $C_{2d}\left[S_{d/2}\right]$ considered in Section 2, where d is even, is d-regular and has diameter d, but its b-chromatic number is not $d+1$.

5 Concluding remarks

We have shown two sets of sufficient conditions that force a d-regular graph to have b-chromatic number $d+1$. One of them concerns the size of the graph and the other the diameter. We have also seen that girth 5 forces large b-chromatic number. We think that progress in the following problems would be of interest:

- We conjecture that there exists a constant $c > 1$ such that any d-regular graph with at least $c \cdot d^2$ vertices has b-chromatic number $d+1$. Perhaps a probabilistic method would be useful for improving our bound in Section 2.
- Petersen graph is the only known d-regular graph with girth 5 whose b-chromatic number is not $d+1$. It is tempting to think that this may be the only exception [4, 11]. A weaker conjecture is that the b-chromatic number should be at least d, which would improve our results from Section 3. In this case d-regular graphs with girth 5 would have only two possibilities for the b-chromatic number, namely d or $d+1$.
- Are there any other sufficient conditions to ensure that the b-chromatic number of a d-regular graph is $d+1$?

References

[1] R. Balakrishnan, S. Francis Raj, Bounds for the b-chromatic number of the Mycielskian of some familes of graphs, manuscript.

[2] R. Balakrishnan, S. Francis Raj, Bounds for the b-chromatic number of $G - v$, manuscript.

[3] D. Barth, J. Cohen, T. Faik, On the b-continuity property of graphs, Discrete Appl. Math. 155 (2007) 1761–1768.

[4] M. Blidia, F. Maffray, Z. Zemir, On b-colorings in regular graphs, Discrete Appl. Math. 157 (2009) 1787–1793.

[5] F. Bonomo, G. Durán, F. Maffray, J. Marenco, M. Valencia-Pabon, On the b-coloring of cographs and P_4-sparse graphs, Graphs Combin. 25 (2009) 153–167.

[6] F. Chaouche, A. Berrachedi, Some bounds for the b-chromatic number of a generalized Hamming graphs, Far East J. Appl. Math. 26 (2007) 375–391.

[7] S. Corteel, M. Valencia-Pabon, J-C. Vera, On approximating the b-chromatic number, Discrete Appl. Math. 146 (2005) 106–110.

[8] B. Effantin, The b-chromatic number of power graphs of complete caterpillars, J. Discrete Math. Sci. Cryptogr. 8 (2005) 483–502.

[9] B. Effantin, H. Kheddouci, The b-chromatic number of some power graphs, Discrete Math. Theor. Comput. Sci. 6 (2003) 45–54.

[10] B. Effantin, H. Kheddouci, Exact values for the b-chromatic number of a power complete k-ary tree, J. Discrete Math. Sci. Cryptogr. 8 (2005) 117–129.

[11] A. El Sahili, M. Kouider, About b-colourings of regular graphs, Res. Rep. 1432, LRI, Univ. Orsay, France, 2006.

[12] S. Francis Raj, R. Balakrishnan, Bounds for the b-chromatic number of vertex-deleted subgraphs and the extremal graphs (extended abstract), Electron. Notes Discrete Math. 34 (2009) 353—358.

[13] C. T. Hoáng, M. Kouider, On the b-dominating coloring of graphs, Discrete Appl. Math. 152 (2005) 176–186.

[14] R. W. Irving, D. F. Manlove, The b-chromatic number of a graph, Discrete Appl. Math. 91 (1999) 127–141.

[15] R. Javadi, B. Omoomi, On b-coloring of the Kneser graphs, Discrete Math. 309 (2009) 4399–4408.

[16] M. Jakovac, S. Klavžar, The b-chromatic number of cubic graphs, Graph Combin. 26 (2010) 107–118.

[17] M. Jakovac, I. Peterin, On the b-chromatic number of strong, lexicographic, and direct product, manuscript.

[18] M. Kouider, b-chromatic number of a graph, subgraphs and degrees, Res. Rep. 1392, LRI, Univ. Orsay, France, 2004.

[19] M. Kouider, A. El Sahili, About b-colouring of regular graphs, Rapport de Recherche No 1432, CNRS-Université Paris Sud–LRI.

[20] M. Kouider, M. Mahéo, Some bounds for the b-chromatic number of a graph, Discrete Math. 256 (2002) 267–277.

[21] M. Kouider, M. Mahéo, The b-chromatic number of the Cartesian product of two graphs, Studia Sci. Math. Hungar. 44 (2007) 49–55.

[22] M. Kouider, M. Zaker, Bounds for the b-chromatic number of some families of graphs, Discrete Math. 306 (2006) 617–623.

[23] J. Kratochvíl, Zs. Tuza, M. Voigt, On the b-chromatic number of graphs, In WG '02: Graph-Theoretic Concepts Comput. Sci., Lecture Notes in Comput. Sci. 2573 (2002) 310–320.