Corrigendum to “Algorithms for graphs of bounded treewidth via orthogonal range searching”

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In our paper [1] we have the following statement and start of proof:

Lemma 1 (Lemma 3 in [1]). *Let $k \geq 1$ be a constant. Given a graph G with $n > k+1$ vertices and treewidth at most k, we can find in linear time a subset $A \subseteq V(G)$ of vertices such that:*

(i) *A has between $\frac{n}{k+1}$ and $\frac{nk}{k+1}$ vertices;*

(ii) *A has at most k portals;*

(iii) *adding edges between the portals of A does not change the treewidth of G.*

Proof. Consider a tree decomposition $(\{X_i \mid i \in I\}, T)$ of G with width k. We next transform it into another tree decomposition where the tree has maximum degree $k+1$ and where any two adjacent bags differ by at least one vertex. This transformation can be done as follows. Firstly, we add vertices to each bag X_i, $i \in I$, while keeping property (iii) in Definition 1, until each bag has exactly $k+1$ elements. Secondly, we contract any edge $ij \in E(T)$ whenever $X_i = X_j$. It now holds $X_i \neq X_j$ for any two nodes i, j of T. Finally, for each node i in T of degree at least $k+2$ we create $k+1$ new bags $Y_{i_0}, Y_{i_2}, \ldots, Y_{i_k}$, where each new bag is a different proper subset of X_i with k elements, remove the edges of T between i and its neighbors Γ_i, add edges to T between i and i_j for $j = 0, \ldots, k$, and add for each $i' \in \Gamma_i$ an edge between $X_{i'}$ and some X_{i_j} with the property $X_{i'} \cap X_i \subset X_{i_j}$. This finishes the transformation. With a slight abuse of notation, we keep using $(\{X_i \mid i \in I\}, T)$ for the resulting tree decomposition of G.

[...] □

There are two problems here. The first problem is that the statement should assume that $k \geq 2$. Indeed, for $k = 1$, the graph is a tree and the first item of the statement tells that A should have between $\frac{n}{1+1}$ and $\frac{n \cdot 1}{1+1}$ vertices, that is, exactly $n/2$ vertices. This cannot be, for example, when n is odd. However, this is not just an issue of parity, but a structural problem. For example, consider 3 stars with m vertices each, and add one new vertex with edges to each of the centers of the stars. The resulting tree has $3m+1$ vertices and there is no set A that would have *roughly* half of the vertices and one portal. See Figure 1.

The case when the graph G has treewidth $k = 1$ is special. In that case, the graph is a tree, and it is folklore that one can obtain a set A that has between $n/3$ and $2n/3$ vertices and portal of size 1.

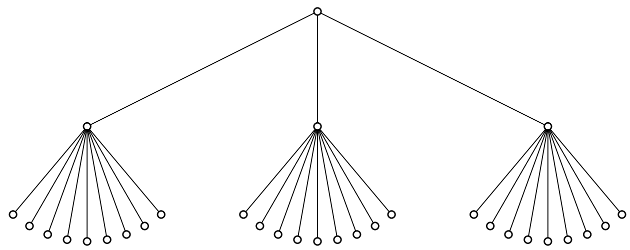

Figure 1: Example showing that for $k = 1$ we cannot get A with one portal and $|A|$ approximately $n/2$.

The second problem is that the first paragraph of the proof does not achieve what it claims because the bags Y_{i_j} may have unbounded degree. See Figure 2 for an example. The tree T'' would be the outcome of the transformation.

Here is a corrected statement and proof. In the statement, we only change that $k \geq 2$ is needed. The case of $k = 1$, when the graph is a tree, should be treated separately.

Lemma 2 (Corrected version of Lemma 3 in [1])**.** *Let $k \geq 2$ be a constant. Given a graph G with $n > k+1$ vertices and treewidth at most k, we can find in linear time a subset $A \subseteq V(G)$ of vertices such that:*

(i) *A has between $\frac{n}{k+1}$ and $\frac{nk}{k+1}$ vertices;*

(ii) *A has at most k portals;*

(iii) *adding edges between the portals of A does not change the treewidth of G.*

Proof. Consider a tree decomposition $(\{X_i \mid i \in I\}, T)$ of G with width k. This means that $|X_i| \leq k+1$ for all $i \in I$. With a slight abuse of notation, we denote the vertices of T sometimes by the index i and sometimes by the bag X_i. We next transform the tree decomposition into another tree decomposition where the tree has maximum degree $k+1$ and where *the intersection of any two adjacent bags has at most k vertices*[1]. This transformation has a few steps. Firstly, we add vertices to each bag X_i, $i \in I$, while maintaining a tree decomposition, until each bag has exactly $k+1$ elements. Secondly, we contract any edge $ij \in E(T)$ whenever $X_i = X_j$. Let $(\{X'_i \mid i \in I'\}, T')$ be the resulting tree decomposition. It now holds $X'_i \neq X'_j$ and $|X'_i| = k+1$ for each two distinct nodes i, j of T'.

For each node i in T', we do the following transformation. Let Γ'_i be the neighbors of i in T'. We create $k+1$ new bags $Y_{i_0}, Y_{i_1}, \ldots, Y_{i_k}$, where each new bag is a different proper subset of X'_i with $|X'_i| - 1 = k$ elements, remove the edges of T' between i and its neighbors Γ'_i, add edges to T' between i and i_j for $j = 0, \ldots, k$, and add for each $\ell \in \Gamma'_i$ an edge between X'_ℓ and some X'_{i_j} with the property $X'_\ell \cap X'_i \subset X'_{i_j}$. See Figure 2 for an example of this transformation. This is the transformation in the proof of Lemma 3 in [1]. As it can be seen in the example and we have mentioned before, the degree of the bags Y_{i_j} can be arbitrarily large, and thus something else has to be done. Let $(\{X''_i \mid i \in I''\}, T'')$ be the resulting tree decomposition.

For each $i \in I''$, we have $|X''_i| = k$ or $|X''_i| = k+1$. Moreover, each $i \in I''$ with $|X''_i| = k+1$ has precisely $k+1$ neighbors in T'', while each $i \in I''$ with $|X''_i| = k$ may have arbitrarily large degree in T''. We replace each $i \in I''$ with $|X''_i| = k$ by a tree of maximum degree 3 making copies of X''_i, as it is often used to reduce the maximum degree of a tree. One precise way to do this

[1] This is the key difference in the proof.

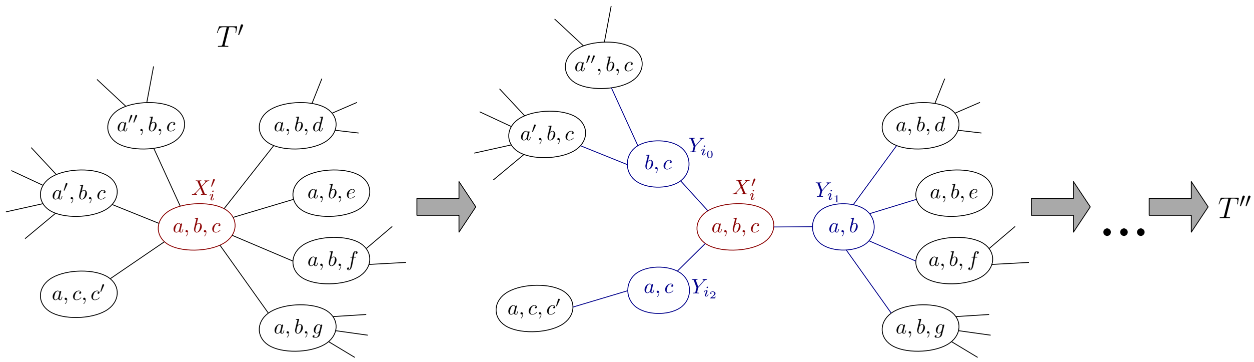

Figure 2: Example showing one step of the transformation (locally) from T' to T''. In this example $k = 2$.

is to do the following for each $i \in I''$ with $|X''_i| = k$ and degree $d > 3$ in T''. Let $\{i_1, \ldots, i_d\}$ be the neighbors of i in T''. We create new nodes $j_1, \ldots, j_d$ and add them to I'', we create new bags $Z_{j_1}, \ldots, Z_{j_d} = X''_i$, and the new tree is obtained from T'' by removing i (and its incident edges), and adding the edges $\{X''_{i_1} Z_{j_1}, \ldots, X''_{i_d} Z_{j_d}\} \cup \{Z_{j_1} Z_{j_2}, Z_{j_2} Z_{j_3}, \ldots, Z_{j_{d-1}} Z_{j_d}\}$. See Figure 3 for an example. With this local operation, we ensure that we have degree at most 3 for all bags X''_i with $|X''_i| = k$. The degree of bags X''_i with $|X''_i| = k + 1$ remains unaltered.

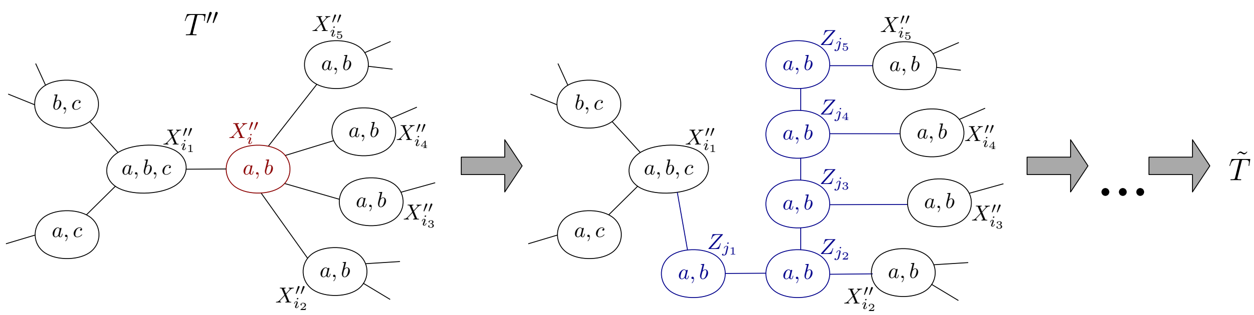

Figure 3: Example showing one step of the transformation (locally) from T'' to $\tilde{T}$. In this examples $k = 2$.

Let $(\{\tilde{X}_i \mid i \in \tilde{I}\}, \tilde{T})$ be the resulting tree decomposition of G. The tree $\tilde{T}$ has maximum degree $\max\{k + 1, 3\} = k + 1$ (here it is relevant that $k \geq 2$) and, for each edge $ij \in E(\tilde{T})$, we have $|\tilde{X}_i \cap \tilde{X}_j| \leq k$. The transformation can be done in linear time. (The size of the decomposition grows as a function of k, but k is constant.)

From this point we can continue with the proof given for Lemma 3 in [1]. □

The rest of the results given in [1] remain valid. In fact, the rest of that paper assumes treewidth at least 2, for which the lemma was correct (but its proof was not satisfactory).

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References

[1] S. Cabello and C. Knauer. Algorithms for graphs of bounded treewidth via orthogonal range searching. *Comput. Geom.*, 42(9):815–824, 2009.