Many distances in planar graphs

Sergio Cabello
IMFM
Ljubljana, Slovenia

supported by EC/FP6 Marie Curie Fellowship

Overview

- the problem: many distances in graphs
- new result
- previous results
- toolbox: Klein; Frederickson + Miller
- main ideas for the new result

Many distances in graphs

G a graph with edge lengths

$\Rightarrow$distance function $d_G(u, v)$

k-many distances in G:

Given G and pairs $(u_1, v_1), \ldots, (u_k, v_k)$,
compute $d_G(u_1, v_1), \ldots, d_G(u_k, v_k)$

we: planar graphs; $n = |V(G)| = \Theta(|E(G)|)$

Many distances in graphs

Given planar G and pairs $(u_1, v_1), \ldots, (u_k, v_k)$,
compute $d_G(u_1, v_1), \ldots, d_G(u_k, v_k)$

Natural problem

Many distances in graphs

Given planar G and pairs $(u_1, v_1), \ldots, (u_k, v_k)$,
compute $d_G(u_1, v_1), \ldots, d_G(u_k, v_k)$

Natural problem

Personal motivation: Shortest non-contractible cycle for graphs embedded on surface of genus g

Thm [with Mohar]: Let $\tilde{n} = O(g^{O(g)} n)$. Finding a shortest non-contractible cycle can be reduced in $O(\tilde{n})$ time to: computing $O(\tilde{n})$ distances in a planar graph with $O(\tilde{n})$ vertices.

New result

Thm: The k-many distances in planar graphs can be solved in $O^*(n^{2/3}k^{2/3} + n^{4/3})$ time

New result

Thm: The k-many distances in planar graphs can be solved in $O^*(n^{2/3}k^{2/3} + n^{4/3})$ time

Improvement for $k \in (n^{5/6}, n^2/\log^6 n)$

For $k = n$: $O^*(n^{4/3})$ time.

Obvious open problem: in $O^*(n + k)$ time?

New result

Thm: The k-many distances in planar graphs can be solved in $O^*(n^{2/3}k^{2/3} + n^{4/3})$ time

Idea: data structure for distances + queries to it

Separators. Topology.

Not used: the problem is offline.

Previous results

Who	k-many distances	For $k = n$
Djidjev	$O(n^{3/2} + k^{1/2}n)$ $O(n^{3/2} + k^{1/3}n^{4/3})$ $O^*(n^{5/3} + k^{1/2}n)$	$O(n^{3/2})$
Fakcharoenphol, Rao	$O^*(n + kn^{1/2})$	$O^*(n^{3/2})$
Henzinger et al	$O(kn)$	$O(n^2)$
Here	$O^*(n^{4/3} + n^{2/3}k^{2/3})$	$O(n^{4/3})$

Frederickson: APSP in $O(n^2)$ time

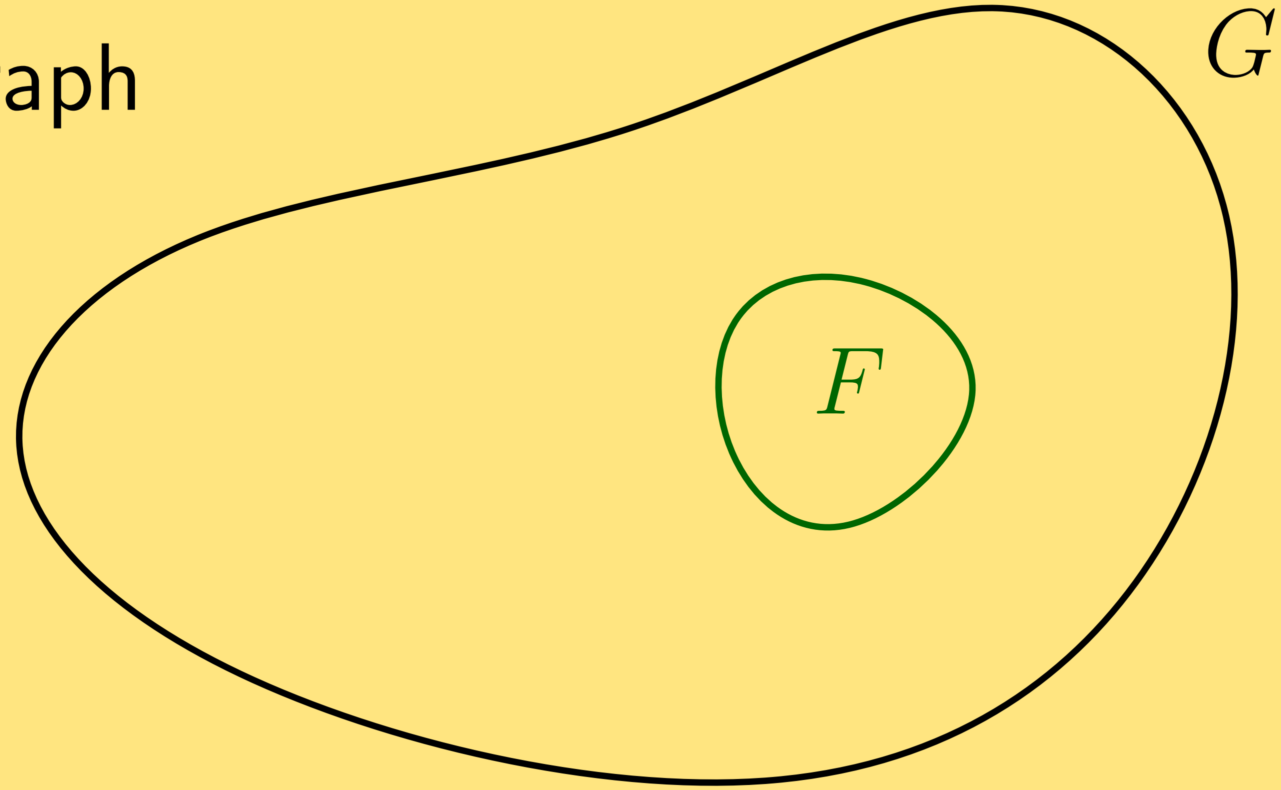
Chen, Xu: $f(n,p) = O^*(n^{5/3} + k^{1/2}n)$

Overview

- ~~the problem: many distances in graphs~~
- ~~new result~~
- ~~previous results~~
- toolbox: Klein; Frederickson + Miller
- main ideas for the new result

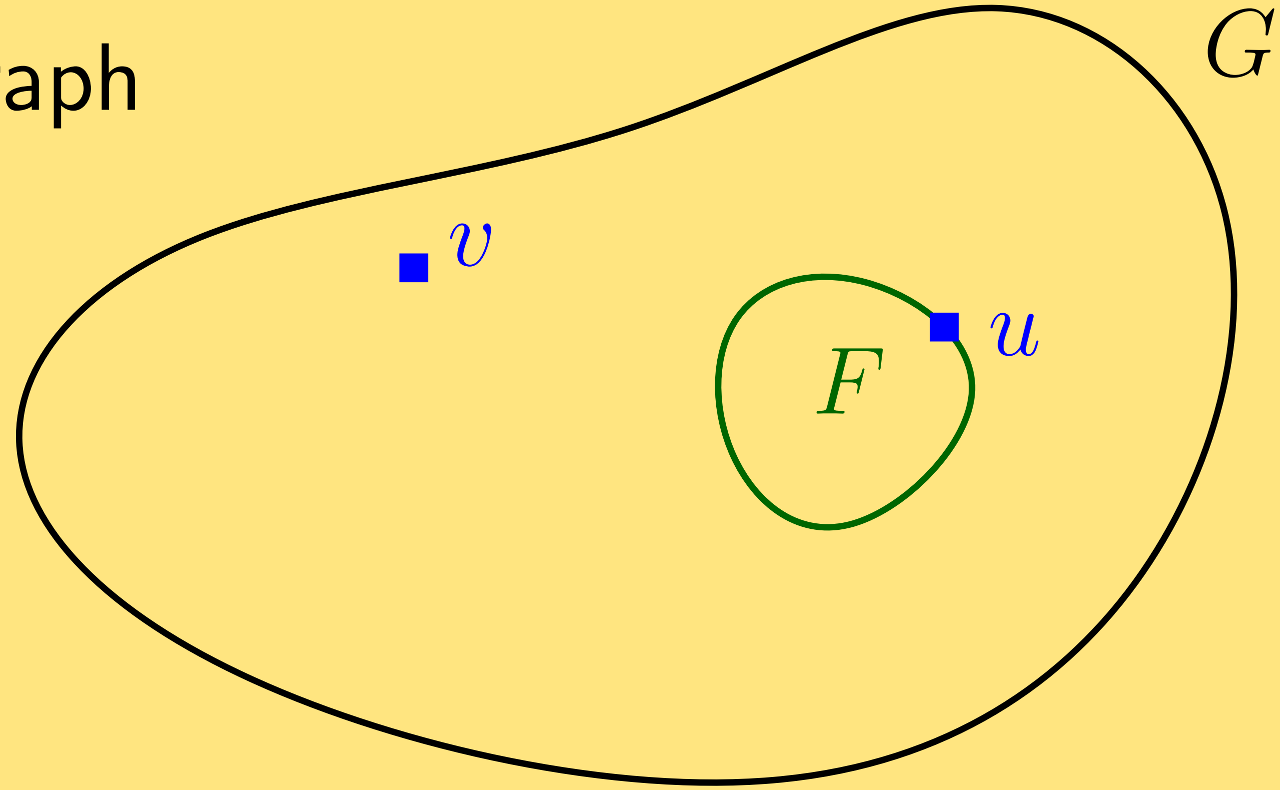
Toolbox: Klein'05

G a planar embedded graph
F a fixed face of G



Toolbox: Klein'05

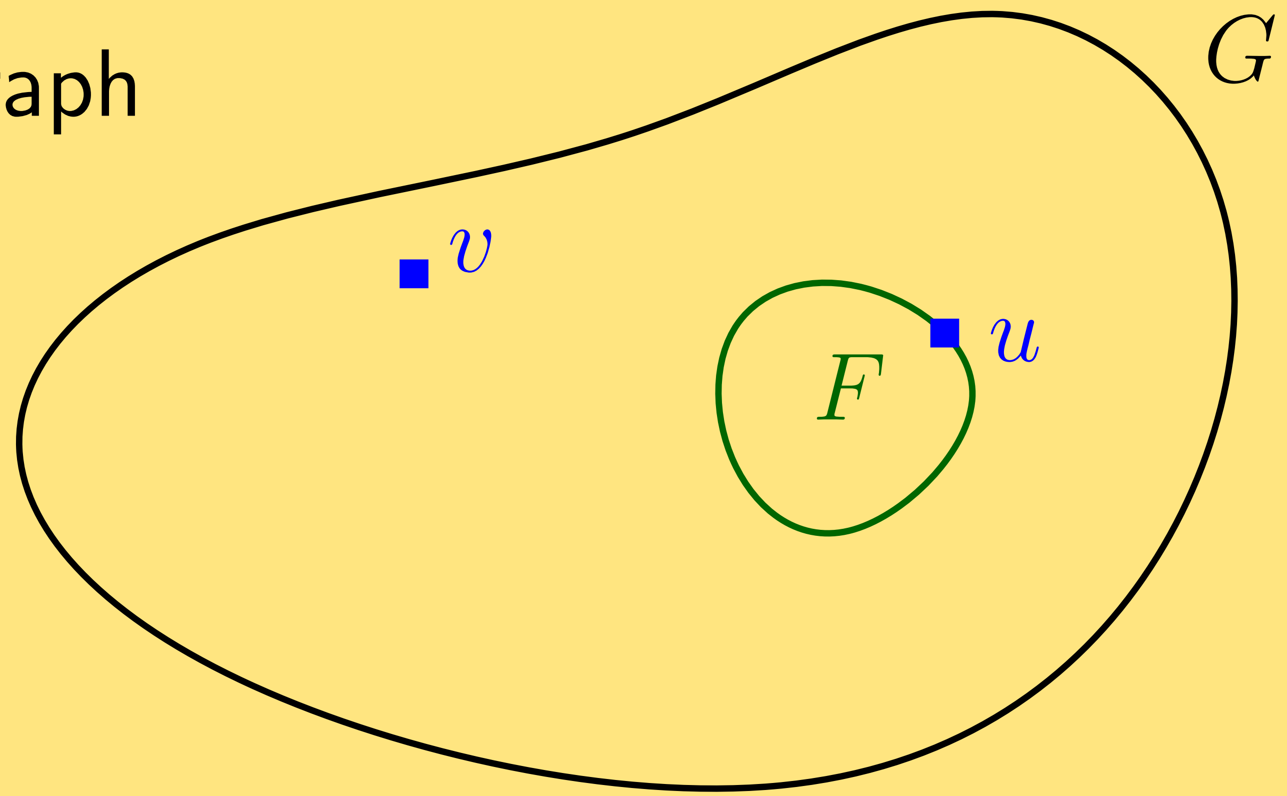
G a planar embedded graph
F a fixed face of G



Thm: Data structure for queries: "$d_G(u, v)$ with $u \in F$?"
$O(n \log n)$ preprocessing time, $O(\log n)$ time per query.

Toolbox: Klein'05

G a planar embedded graph
F a fixed face of G

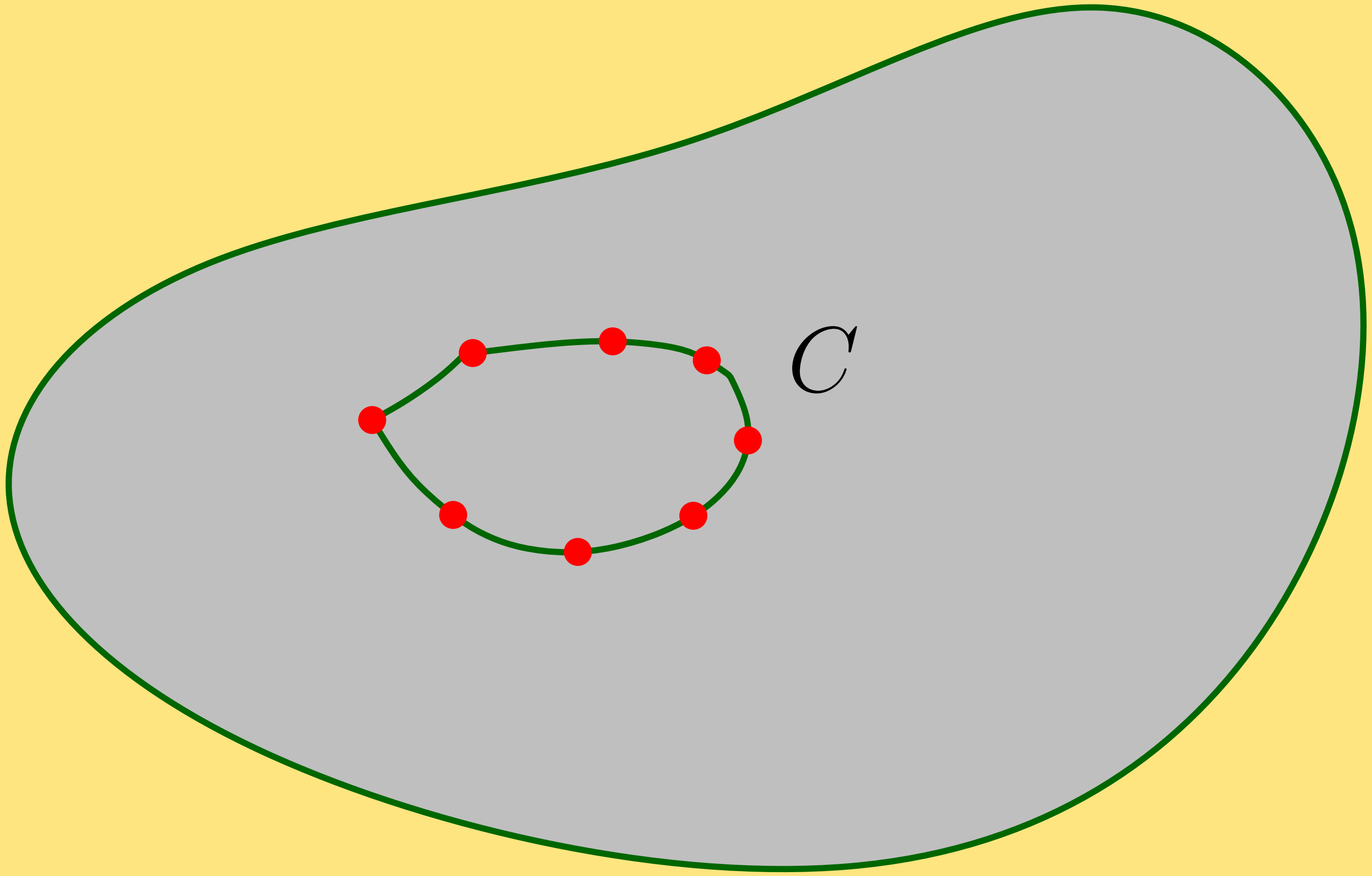


Thm: Data structure for queries: "$d_G(u, v)$ with $u \in F$?"
$O(n \log n)$ preprocessing time, $O(\log n)$ time per query.

Topological result $\begin{cases} \text{uses embedding} \\ \text{two shortest paths do not cross twice} \end{cases}$

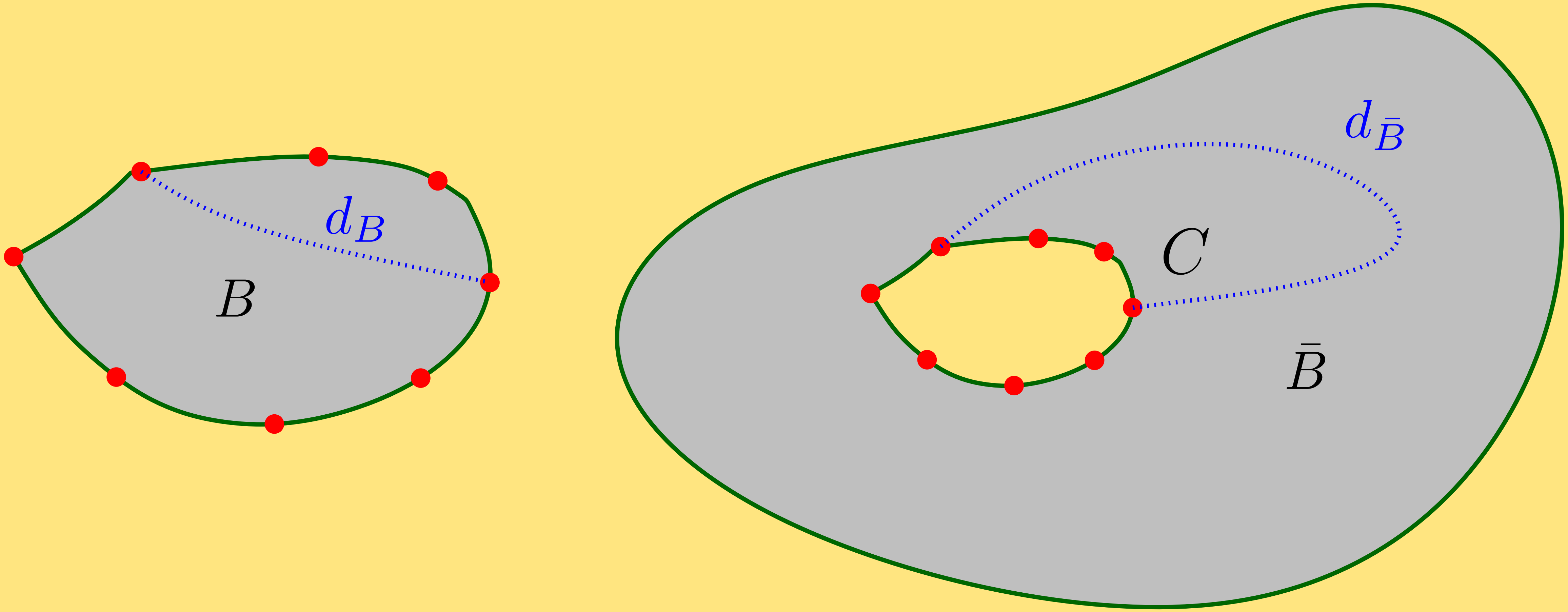
Distances within a cycle

Lem: pairwise distances in a cycle C in $O^*(n + |C|^3)$ time



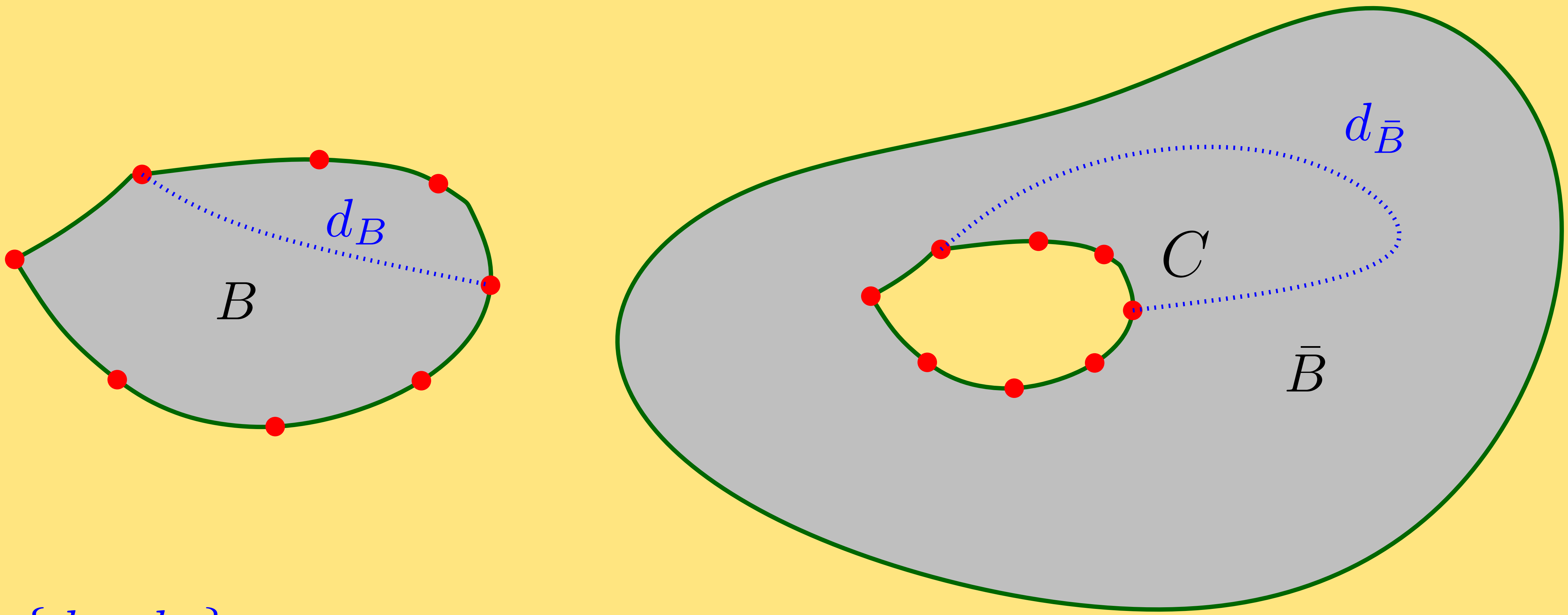
Distances within a cycle

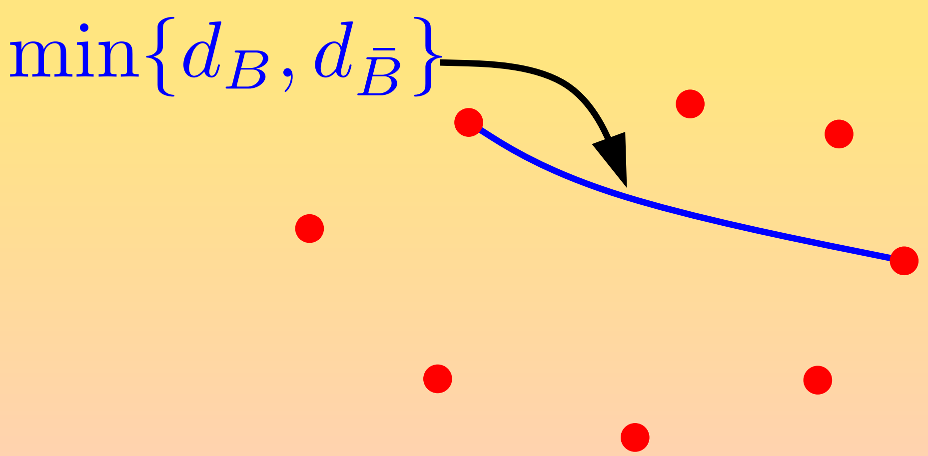
Lem: pairwise distances in a cycle C in $O^*(n + |C|^3)$ time



Distances within a cycle

Lem: pairwise distances in a cycle C in $O^*(n + |C|^3)$ time



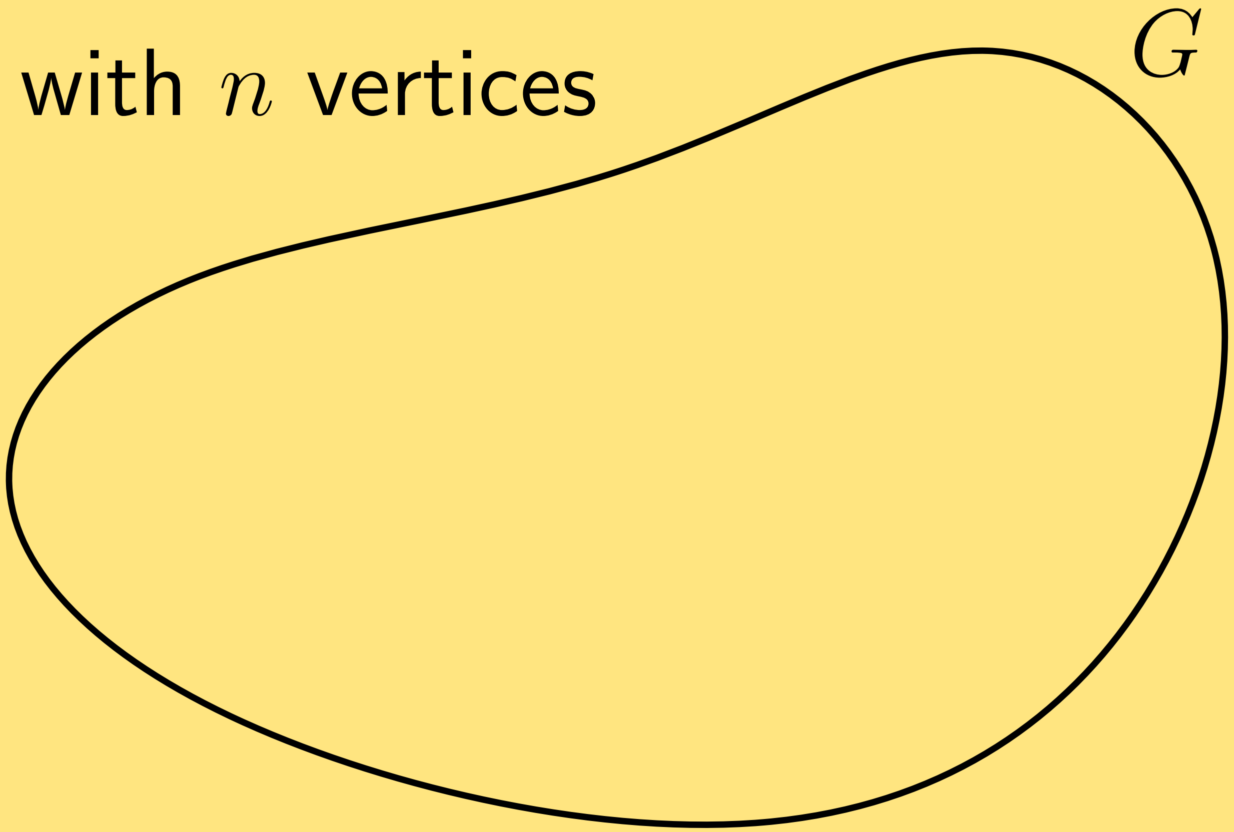


Construct $K_C \Rightarrow O^*(n + |C|^2)$ time

APSP in $K_C \Rightarrow O(|C|^3)$ time

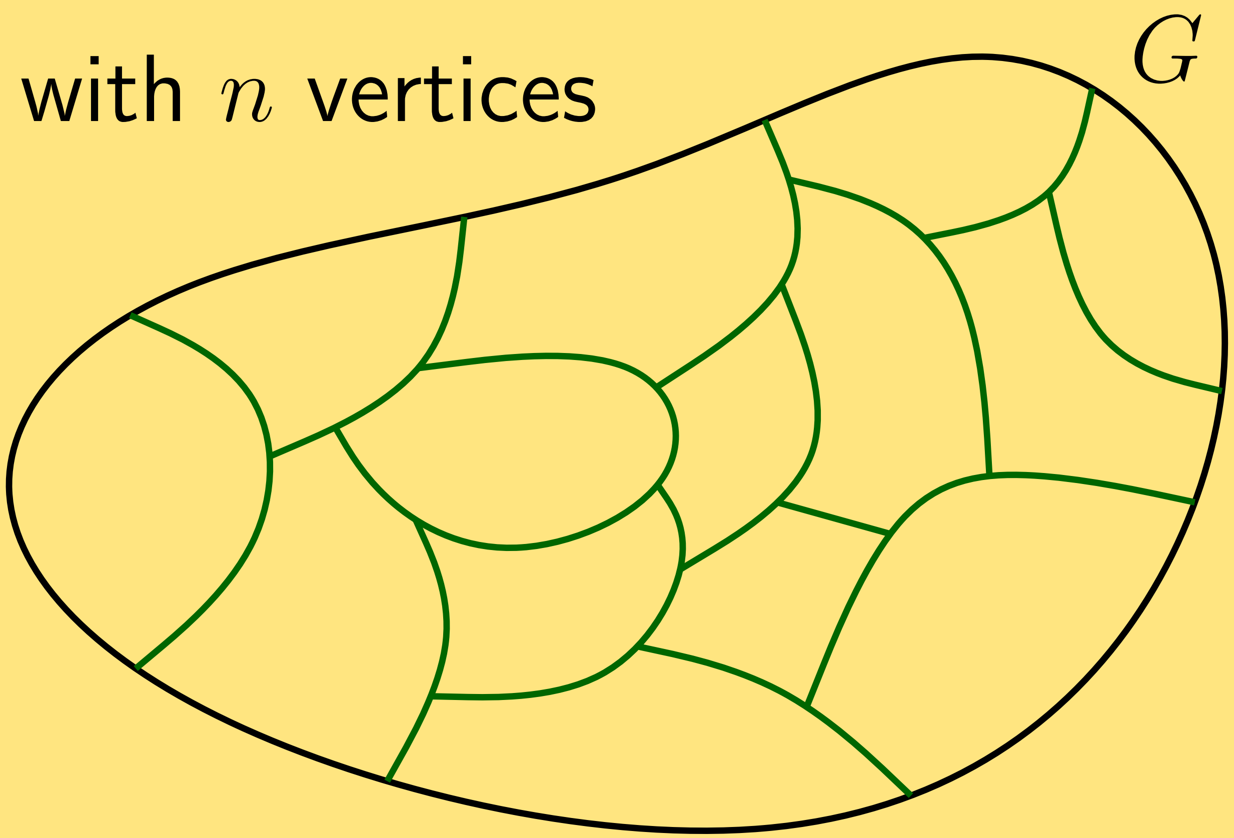
Toolbox: Frederickson + Miller

G a (3-conn) planar graph with n vertices
$r \in (0, n)$ a parameter



Toolbox: Frederickson + Miller

G a (3-conn) planar graph with n vertices
$r \in (0, n)$ a parameter

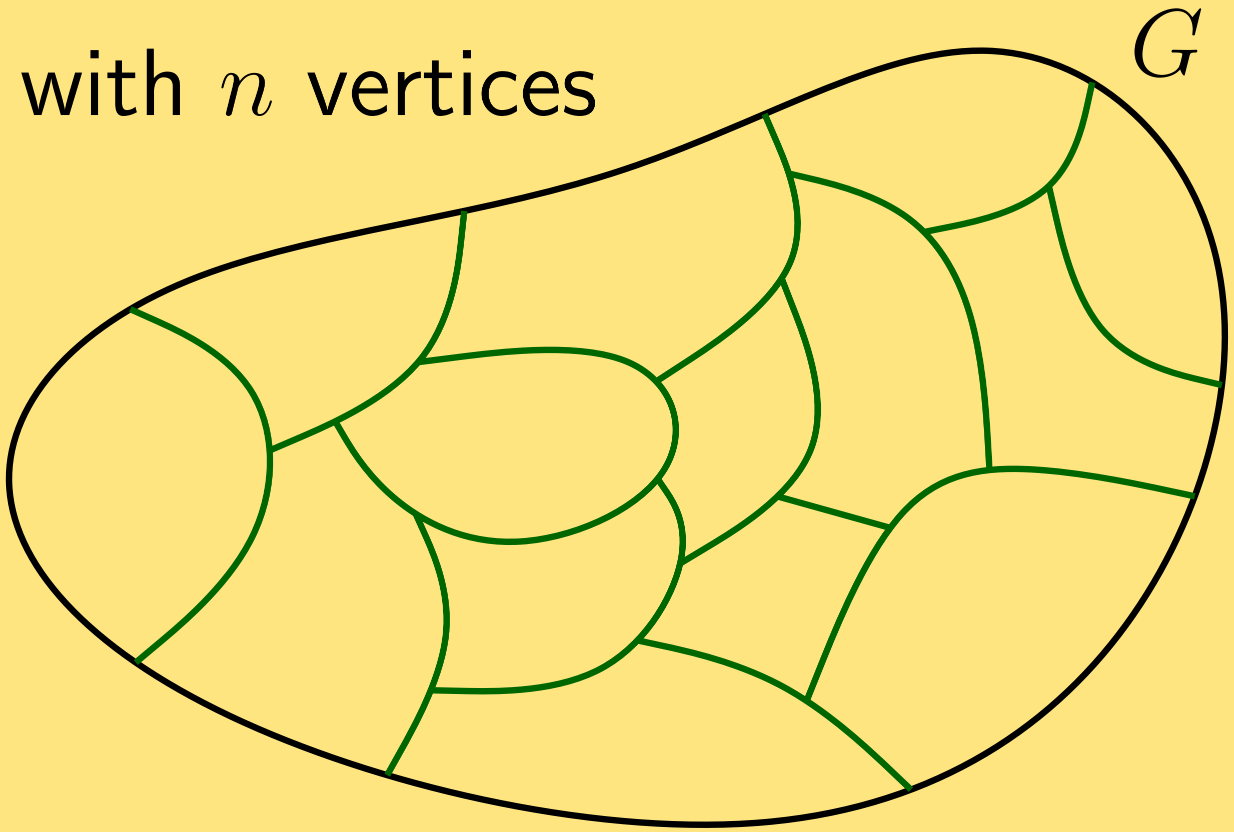


Thm: We can decompose G into n/r pieces, each piece with $O(r)$ vertices and a boundary cycle of $O(\sqrt{r})$ vertices

Structural result $\begin{cases} \text{separators} \\ \text{cycles} \end{cases}$

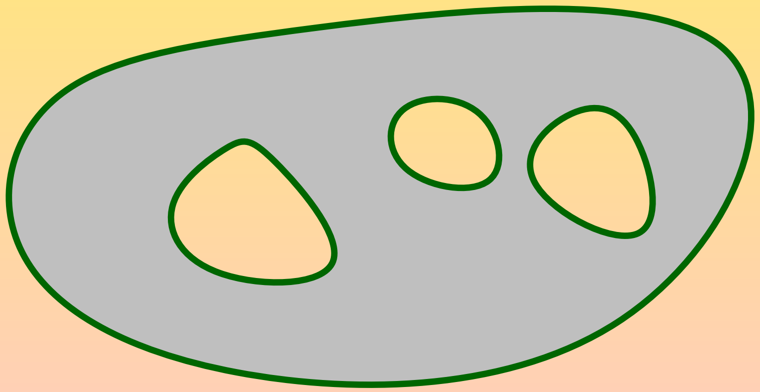
Toolbox: Frederickson + Miller

G a (3-conn) planar graph with n vertices
$r \in (0, n)$ a parameter



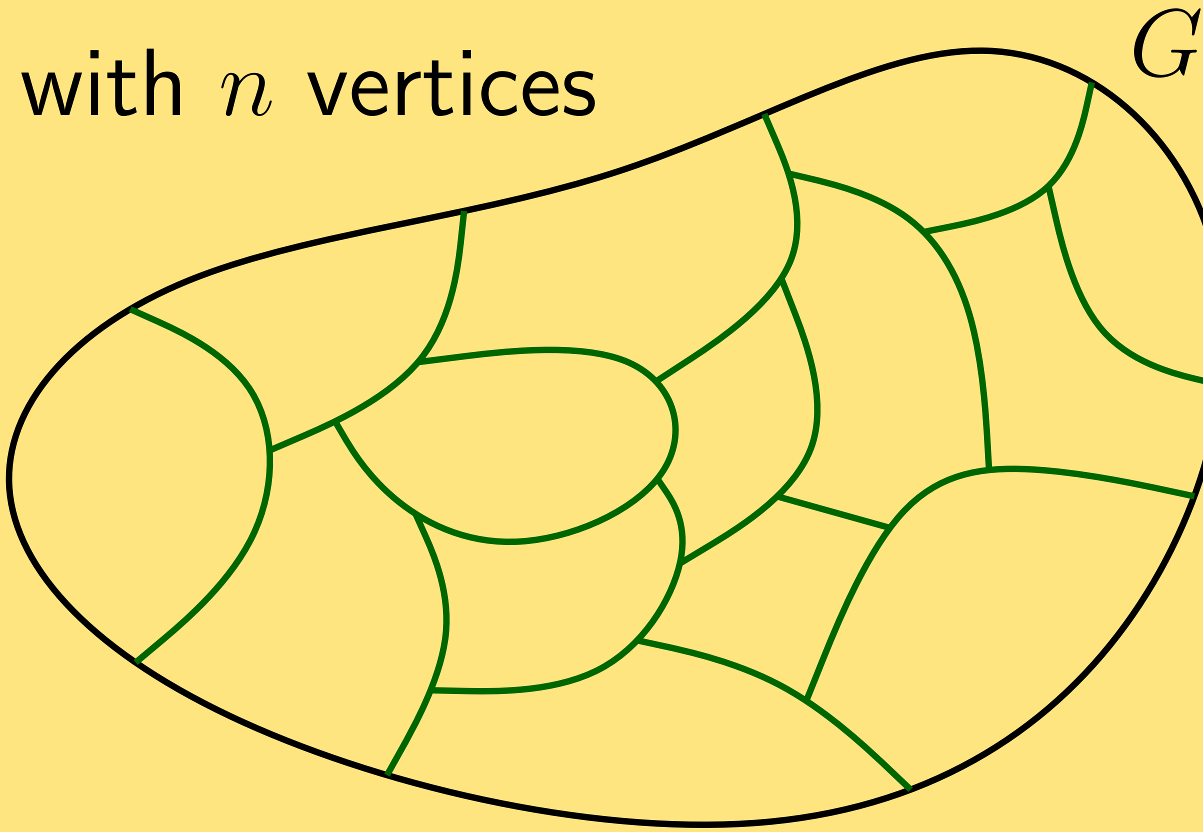
Thm: We can decompose G into n/r pieces, each piece with $O(r)$ vertices and a boundary cycle of $O(\sqrt{r})$ vertices

Warning!! A piece may have "complicated" boundary



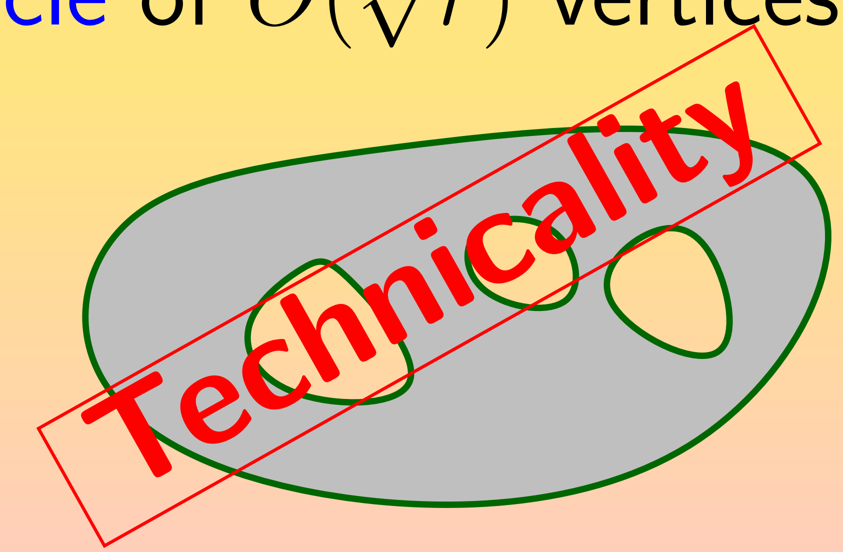
Toolbox: Frederickson + Miller

G a (3-conn) planar graph with n vertices
$r \in (0, n)$ a parameter



Thm: We can decompose G into n/r pieces, each piece with $O(r)$ vertices and a boundary cycle of $O(\sqrt{r})$ vertices

Warning!! A piece may have "complicated" boundary

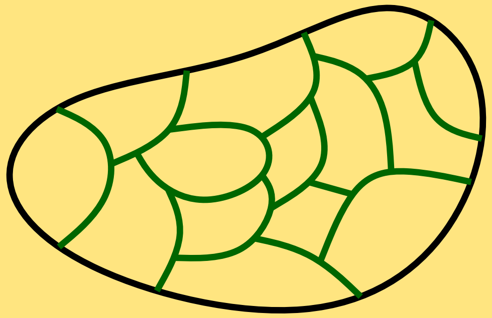


Main ideas for the new result

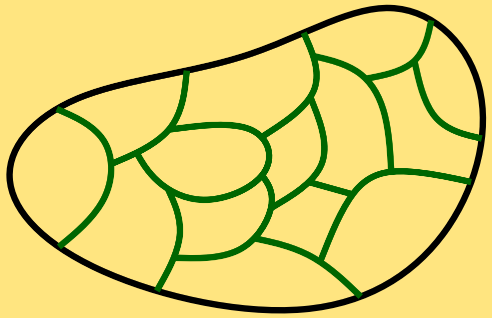
n/r pieces

$O(r)$ vertices

boundary cycle $O(\sqrt{r})$ vertices



Main ideas for the new result



n/r pieces

$O(r)$ vertices

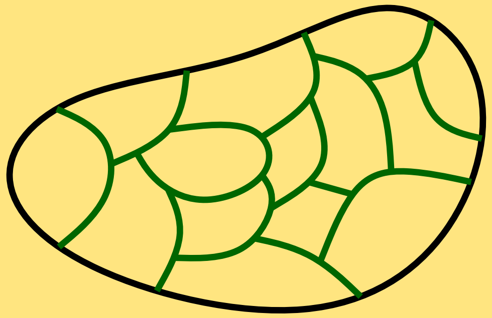
boundary cycle $O(\sqrt{r})$ vertices

pairwise distances in the boundary: $O^*(n + r^{3/2})$

Lem: pairwise distances in a cycle C in $O^*(n + |C|^3)$ time

Main ideas for the new result

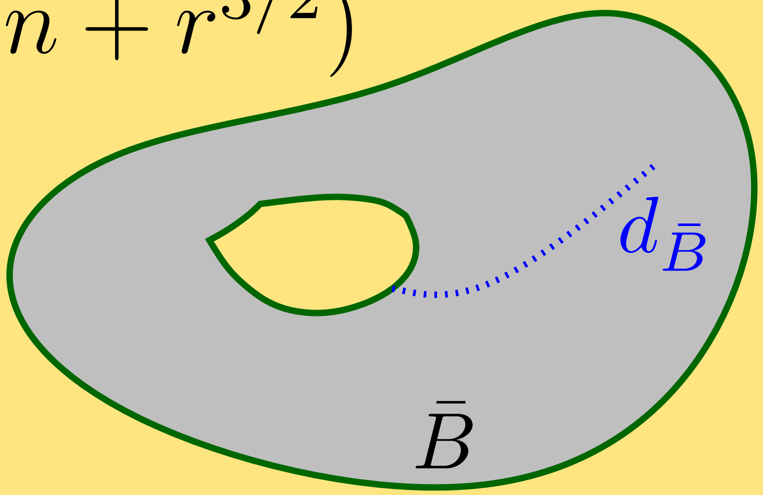
n/r pieces



$O(r)$ vertices

boundary cycle $O(\sqrt{r})$ vertices

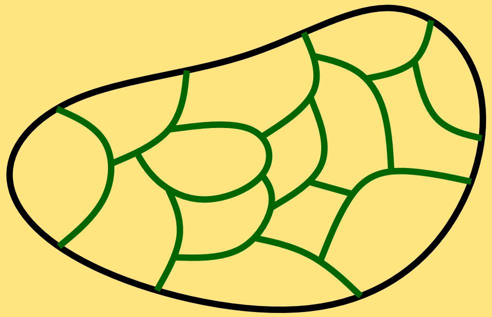
pairwise distances in the boundary: $O^*(n + r^{3/2})$



store Klein's DS for $\bar{B}$

Main ideas for the new result

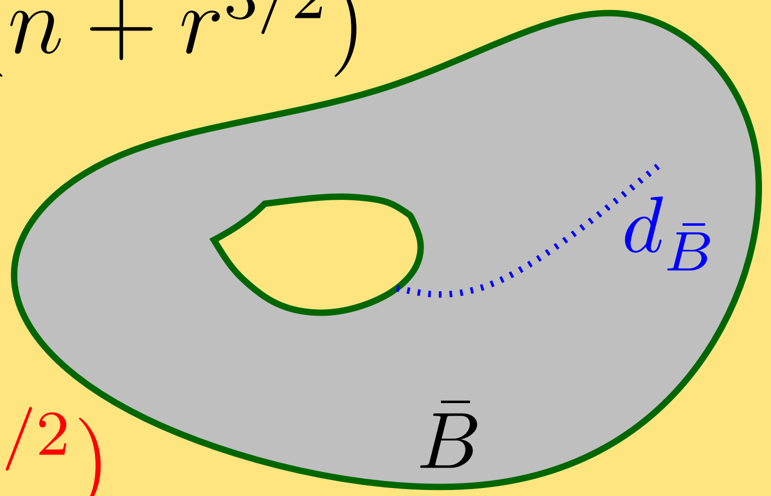
n/r pieces



$O(r)$ vertices

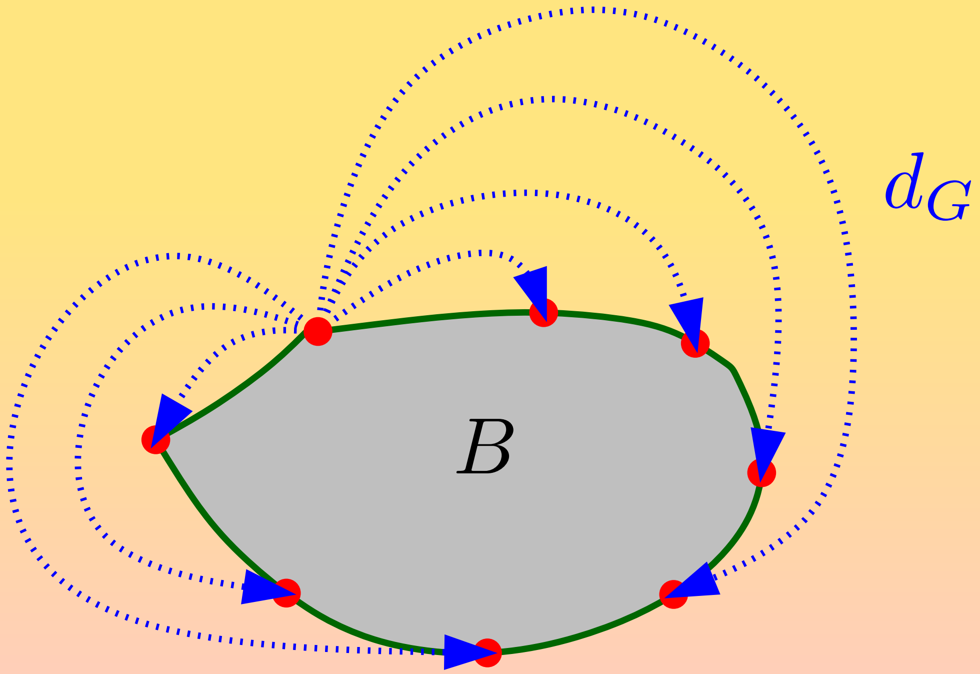
boundary cycle $O(\sqrt{r})$ vertices

pairwise distances in the boundary: $O^*(n + r^{3/2})$



store Klein's DS for $\bar{B}$

distances from boundary to piece: $O(r^{3/2})$

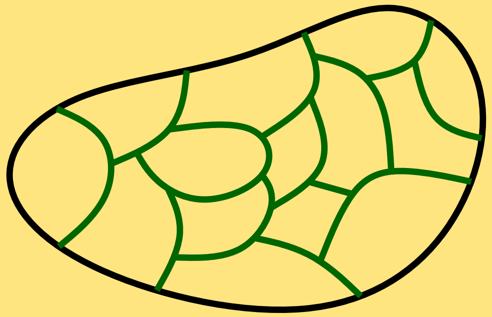


$SSSP$ from each boundary vertex

$O(\sqrt{r}) \times O(r)$

Main ideas for the new result

n/r pieces

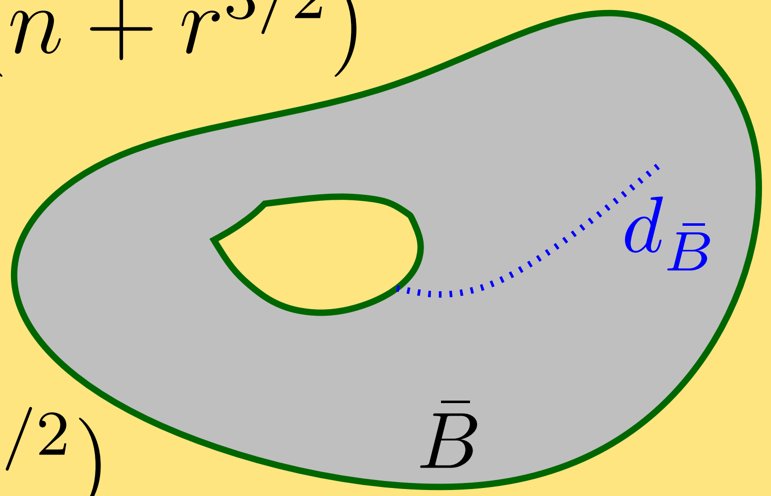


$O(r)$ vertices

boundary cycle $O(\sqrt{r})$ vertices

pairwise distances in the boundary: $O^*(n + r^{3/2})$

store Klein's DS for $\bar{B}$



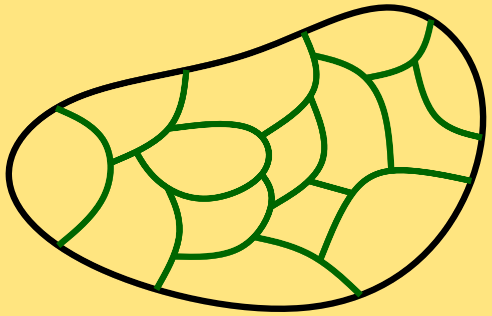
distances from boundary to piece: $O(r^{3/2})$

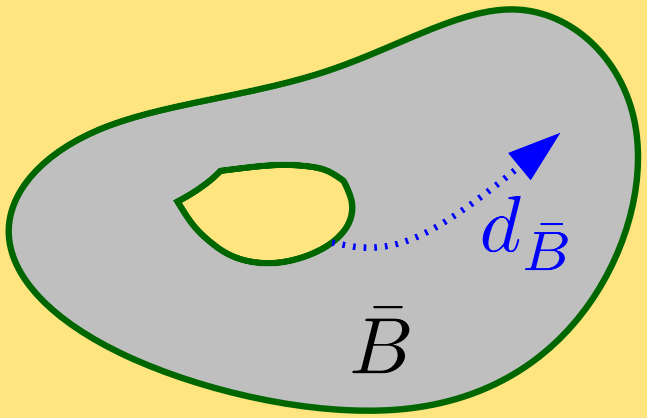
Total time: $\frac{n}{r} \times O^*(n + r^{3/2}) = O^*(n^2/r + nr^{1/2})$

At least $O^*(n^{4/3})$ time

Main ideas for the new result

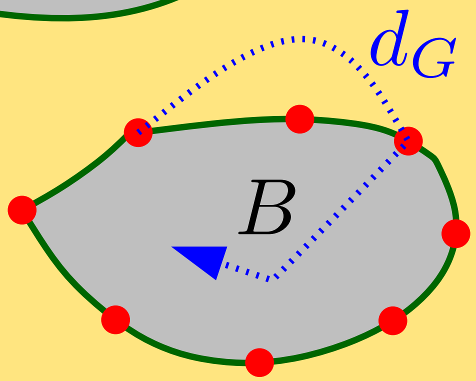
n/r pieces





store Klein's DS for $\bar{B}$

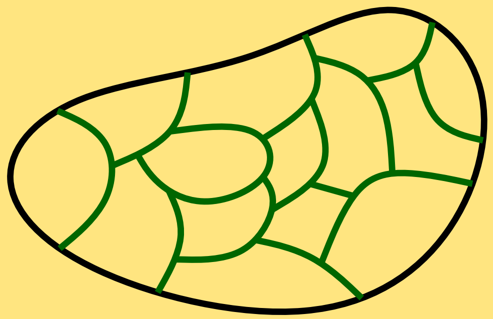
distances from boundary to piece: $O(r^{3/2})$



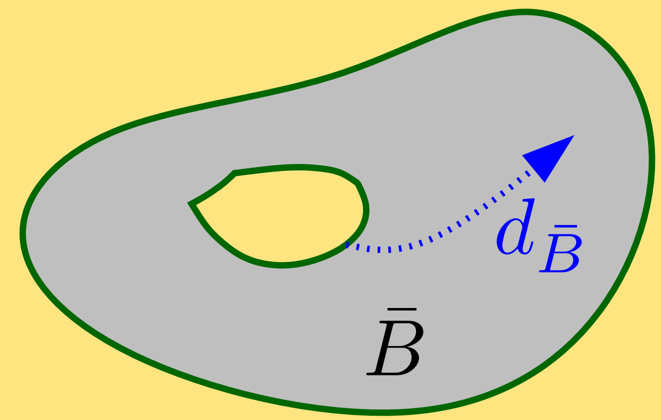
Total time: $O^*(n^2/r + nr^{1/2})$

Main ideas for the new result

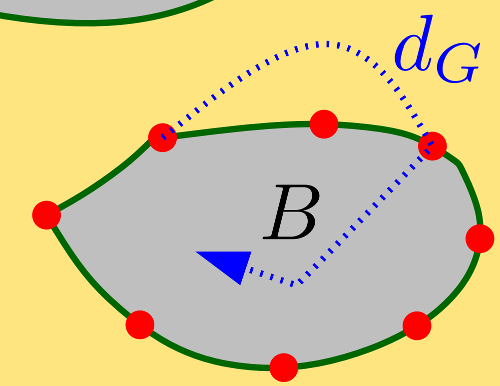
n/r pieces



store Klein's DS for $\bar{B}$

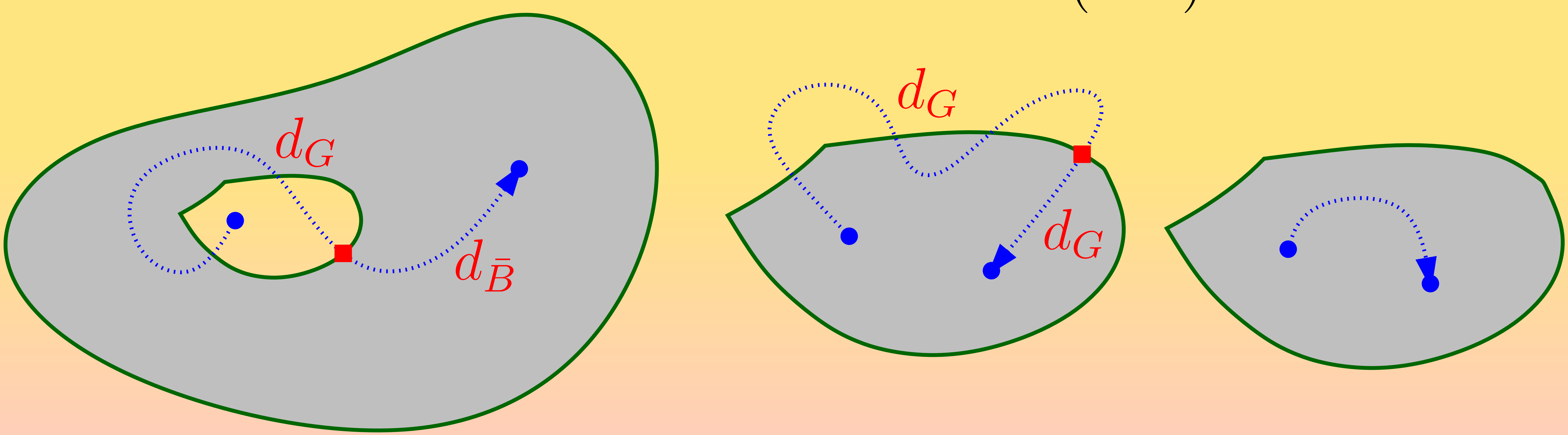


distances from boundary to piece: $O(r^{3/2})$



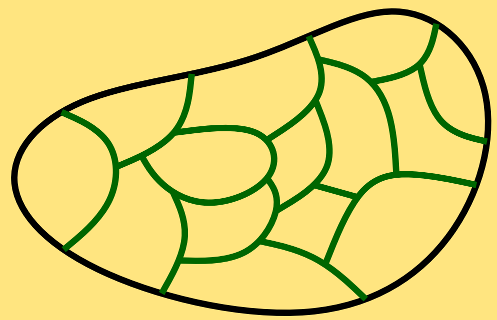
Total time: $O^*(n^2/r + nr^{1/2})$

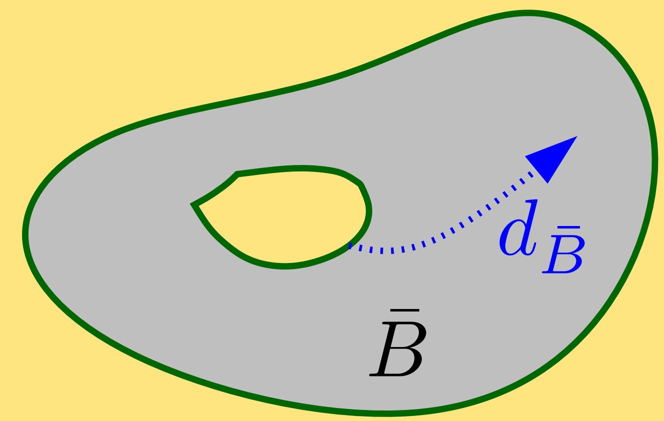
Lem: each distance can be answered in $O^*(r^{1/2})$



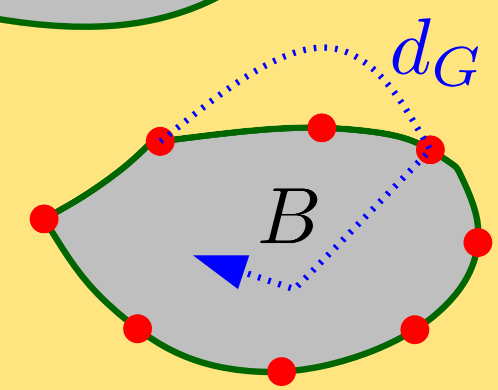
Main ideas for the new result

n/r pieces





store Klein's DS for $\bar{B}$



distances from boundary to piece: $O(r^{3/2})$

Total time: $O^*(n^2/r + nr^{1/2})$

Lem: each distance can be answered in $O^*(r^{1/2})$

$\Rightarrow O^*(n^2/r + nr^{1/2} + kr^{1/2})$ time in total.

Choose best r.

Summary

G planar graph with n vertices.

- k-many distances in $O^*(n^{2/3}k^{2/3} + n^{4/3})$
- improvement for $k \in (n^{5/6}, n^2/\log^6 n)$

Open problems:

- n-many distances in $O^*(n)$ time?
- Pairwise distances between $\sqrt{n}$ vertices?
- Does "off-linety" help?

Summary

G planar graph with n vertices.

- k-many distances in $O^*(n^{2/3}k^{2/3} + n^{4/3})$
- improvement for $k \in (n^{5/6}, n^2/\log^6 n)$

Open problems:

- n-many distances in $O^*(n)$ time?
- Pairwise distances between $\sqrt{n}$ vertices?
- Does ”off-linety” help?

THANKS!

Sergio Cabello