Adding one edge to planar graphs makes crossing number hard

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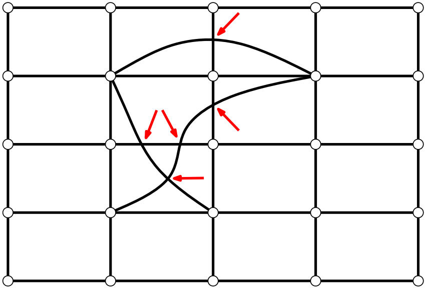
Crossing number

$cr(G) =$ minimum number of crossings over all drawings

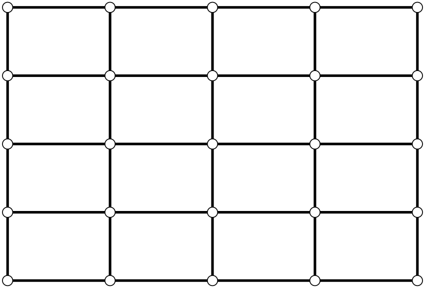
- vertices to points
- edges to curves
- edge-vertex incidence preserved
- no point in the interior of 3 edges
- no vertex in the interior of an edge



cr(G) $\geq$ 5

Planar graphs

G planar $\Leftrightarrow$ $cr(G) = 0$

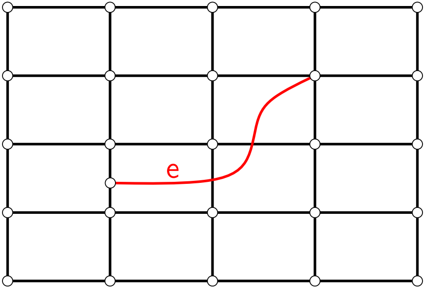


Crossing number: algorithmic results

- "is $cr(G) = 0$?" decidable in linear time [Hopcroft, Tarjan, '74]
- computing $cr(G)$ is NP-hard [Garey, Johnson '83]
  - for cubic graphs [Hliněný '06]
  - with rotation systems [Pelsmajer, Schaefer, Štefankovic '08]
- computing $cr(G)$ is FPT wrt $cr(G)$ [Grohe '04] [Kawarabayashi, Reed '07]
- $cr(G) + |V(G)|$ approximable within $O(\log^3 |V(G)|)$ [Even, Guha, Schieber '02]
- $f(\Delta)$-approximation algorithms for special graphs of max deg Δ [Hliněný, Salazar, Chimani, C., M.]

Near-planar graphs

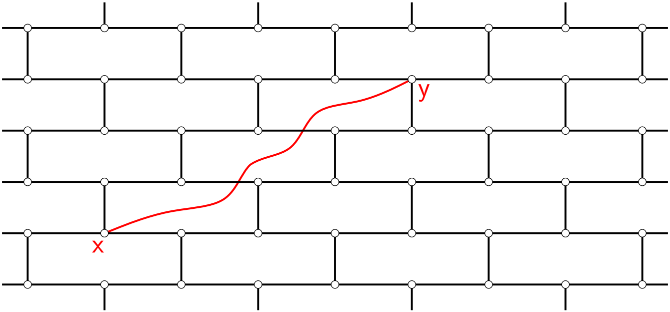
G near-planar if $G - e$ planar for some e.



- weak relaxation of planarity
- near-planar $\Rightarrow$ toroidal, apex

Near-planar – Previous work

- G planar, 3-connected, and 3-regular
  $\Rightarrow$ $cr(G + xy)$ is a distance in $(G - x - y)^*$. [Riskin '96]
  - draw $G - xy$ planarly and insert xy minimizing crossings.



- No extension to non-cubic graphs possible [Mohar '06]

Near-planar – Previous work II

- G near-planar with max degree Δ
  $\Rightarrow \lfloor \frac{\Delta}{2} \rfloor$-approximation to $cr(G)$ [Cabello, Mohar '08]
  - implies Riskin's result
  - improves previous Δ-approximation [Hliněný, Salazar '06]
  - number of edge-disjoint cycles separating x and y
  - number of vertex-disjoint cycles separating x and y
- G near-planar. Why do we approximate $cr(G)$?

Our new result

Theorem

Computing $cr(G)$ for near-planar graphs is NP-hard.

Our new result

Theorem

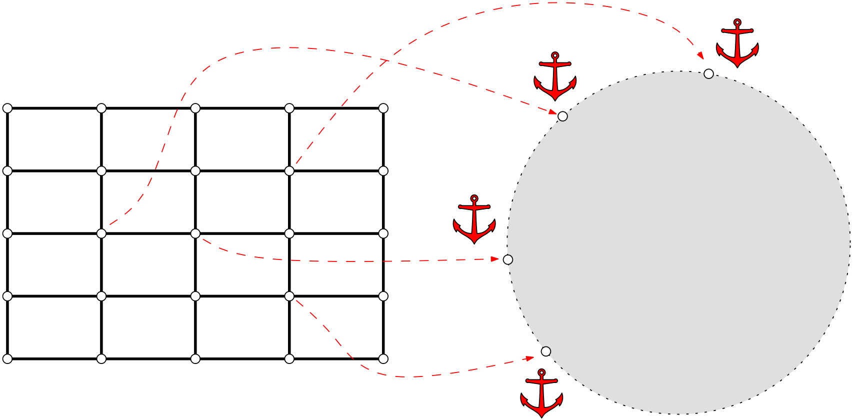
Computing $cr(G)$ for near-planar graphs is NP-hard.

- adding one edge messes up a lot
- we knew it for *weighted* crossing number
  - polynomial weights would be ok
- new reduction from SAT
  - previous reductions are from Linear Ordering
- new problem: anchored drawings

Tool: anchored drawings

- Ω a disk
- Anchored graph: graph G with assigned placements for a subset $A_G \subseteq V(G)$ of anchors on the boundary of Ω
- Anchored drawing: drawing in Ω extending the placement of A_G
- Anchored embedding: anchored drawing without crossings
- Anchored crossing number: minimize crossings

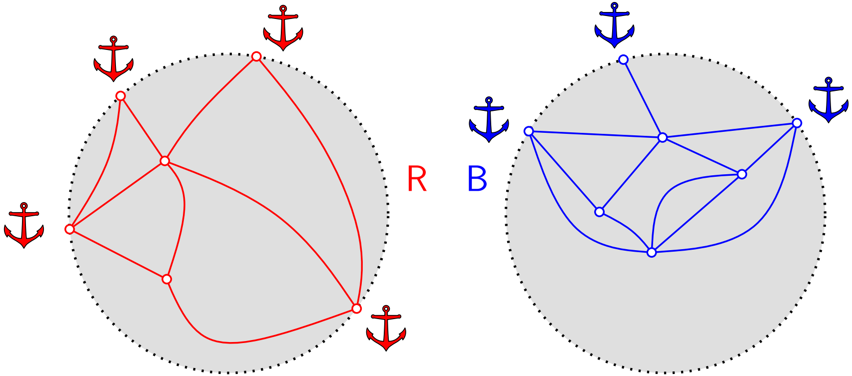
Tool: anchored drawings



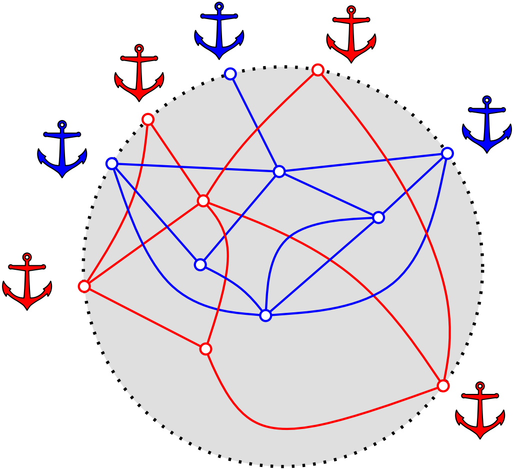
New problem: red-blue anchored drawings

- Ω a disk
- R an anchored embedded red graph in Ω
- B an anchored embedded blue graph in Ω
- anchored drawing D of $R \cup B$ in Ω
  - we may require $D_{|R}$ and/or $D_{|R}$ is an embedding
  - or same combinatorial embedding
- anchored crossing number of $R \cup B$
  - we may only count red-blue crossings

New problem: red-blue anchored drawings



New problem: red-blue anchored drawings



New theorem: red-blue anchored drawings

Theorem

It is NP-hard to compute the anchored crossing number of $R \cup B$.

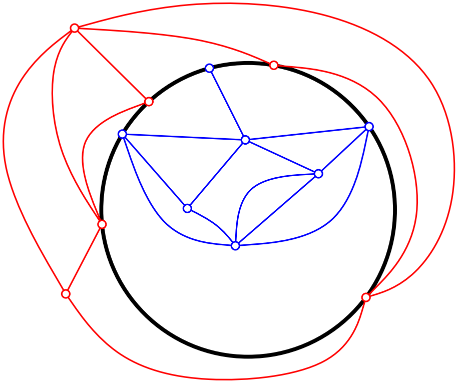
New theorem: red-blue anchored drawings

Theorem

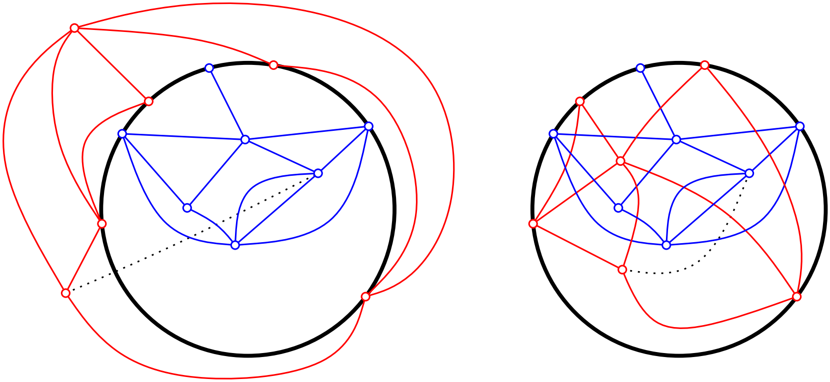
It is NP-hard to compute the anchored crossing number of $R \cup B$.

- also true if R and B disjoint
- also true if restricted to embeddings of R or B
- reduction from SAT

Why red-blue anchored drawings?



Why red-blue anchored drawings?

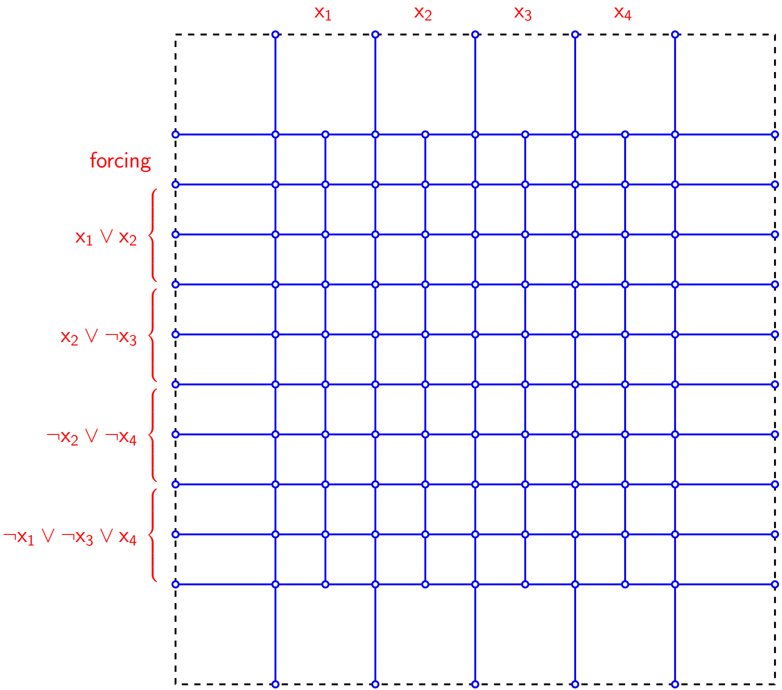


New theorem: red-blue anchored drawings

Theorem

It is NP-hard to compute the anchored crossing number of $R \cup B$.

- reduction from SAT
- proof by example
- we will use polynomial weights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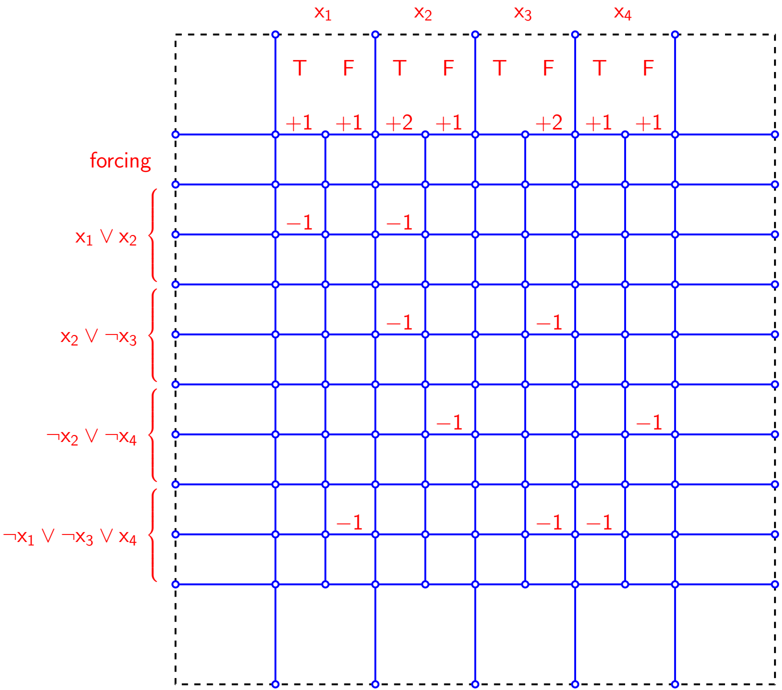
x_3
x_4
forcing
$x_1 \vee x_2$
$x_2 \vee \neg x_3$
$\neg x_2 \vee \neg x_4$
$\neg x_1 \vee \neg x_3 \vee x_4$



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forcing
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$x_2 \vee \neg x_3$
−1 −1
$\neg x_2 \vee \neg x_4$
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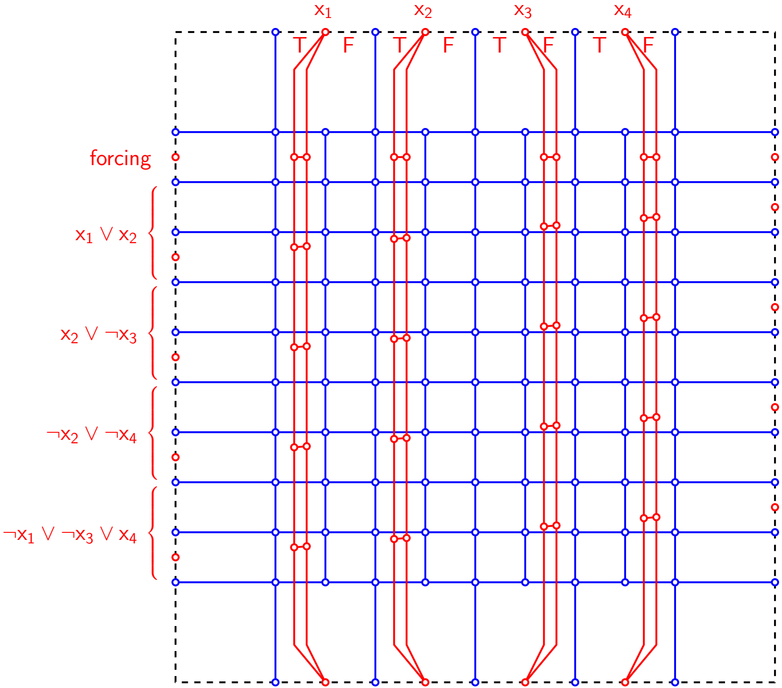
$\neg x_1 \vee \neg x_3 \vee x_4$
−1 −1 −1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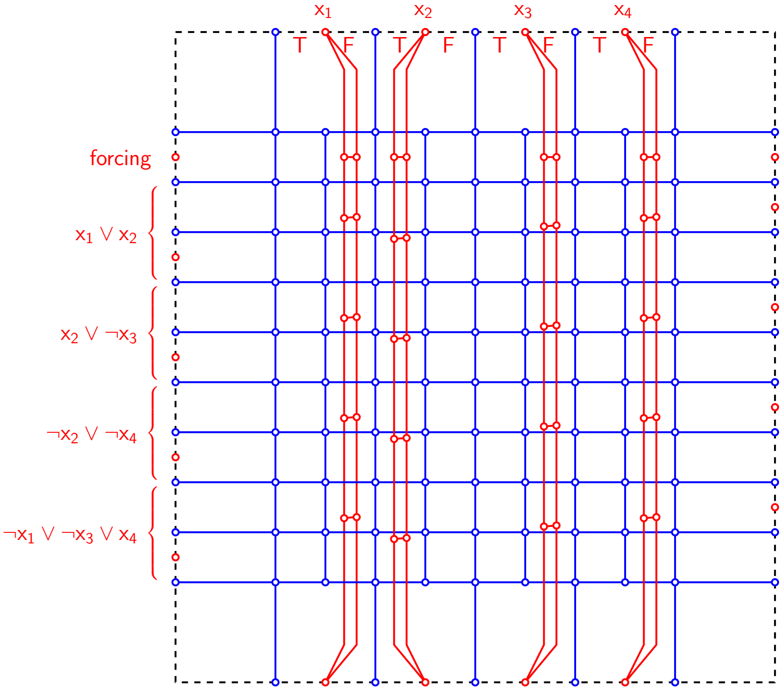
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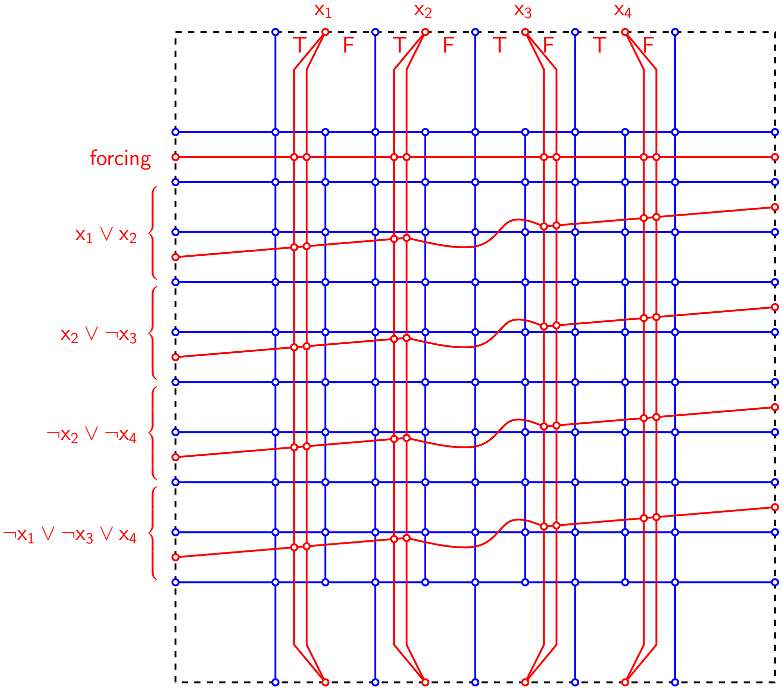

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forcing
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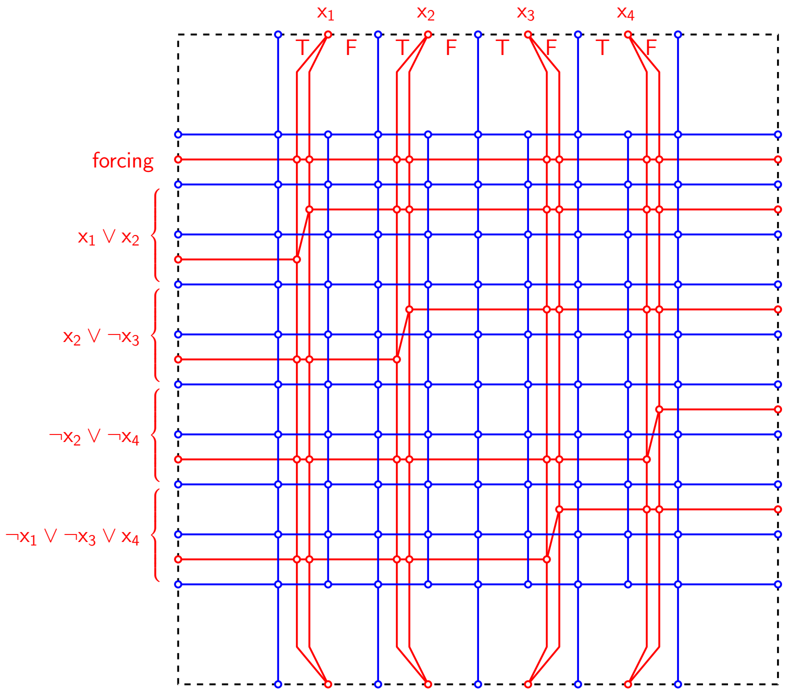
x1
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x3
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T
F
T
F
forcing
x1 ∨ x2
x2 ∨ ¬x3
¬x2 ∨ ¬x4
¬x1 ∨ ¬x3 ∨ x4

Conclusions

- Crossing numbers are hard. Any doubt?
- New proof of NP-hardness for crossing numbers.
  - reduction from SAT
  - works cubic graphs (Hliněný)
- New problem: anchored drawing in a disk.
  - approximation?
  - other surfaces ($\mathbb{P}^2$ is done)
- Crossing number
  - approximation?
  - bounded treewidth?

The end

- thanks
- thanks
- . . .